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A STRONG COMPARISON PRINCIPLE FOR POSITIVE SOLUTIONS OF DEGENERATE ELLIPTIC EQUATIONS

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Abstract. A *strong comparison principle* (SCP, for brevity) is obtained for nonnegative weak solutions $u \in W_0^{1,p}(\Omega)$ of the following class of quasilinear elliptic boundary value problems,

$$(P) \qquad -\operatorname{div}(\mathbf{a}(x,\nabla u)) - b(x,u) = f(x) \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega.$$

Here, $p \in (1,\infty)$ is a given number, Ω is a bounded domain in $\mathbb{R}^N$ with a connected C^2-boundary, $\mathbf{a}(x,\nabla u)$ and $b(x,u)$ are slightly more general than the functions $a_0(x)|\nabla u|^{p-2}\nabla u$ and $b_0(x)|u|^{p-2}u$, respectively, with $a_0 \geq \text{const} > 0$ and $b_0 \geq 0$ in $L^\infty(\Omega)$, and $0 \leq f \in L^\infty(\Omega)$. Validity of the SCP is investigated also in the case when $b_0 \leq 0$ depending upon whether $p \leq 2$ or $p > 2$. The methods of proofs are new.

1. Introduction. We consider the following quasilinear elliptic boundary value problem,

$$-\operatorname{div}(\mathbf{a}(x,\nabla u)) - b(x,u) = f(x) \text{ in } \Omega; \qquad u = 0 \text{ on } \partial\Omega. \tag{1}$$

Here, Ω is a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a connected C^2-manifold, $x = (x_1,\dots,x_N)$ is a generic point in Ω, and $u \in W_0^{1,p}(\Omega)$

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is the unknown function for $p \in (1, \infty)$. The quasilinear elliptic operator $(x, u) \mapsto \mathrm{div}(\mathbf{a}(x, \nabla u))$ is defined by

$$\mathrm{div}(\mathbf{a}(x, \nabla u)) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\, a_i(x, \nabla u(x)) \quad \text{ for } x \in \Omega \text{ and } u \in W_0^{1,p}(\Omega) \tag{2}$$

with values in $W^{-1,p'}(\Omega)$, the dual space of $W_0^{1,p}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. The components a_i of the vector $\mathbf{a} = (a_1, \ldots, a_N)$ are functions of x and $\eta = \nabla u \in \mathbb{R}^N$. The functions $a_i(x, \eta)$ and $b(x, u)$ are assumed to be such that $a_i \in C^0(\Omega \times \mathbb{R}^N) \cap C^1(\Omega \times (\mathbb{R}^N \setminus \{0\}))$ and b is a Carathéodory function, that is, measurable in $x \in \Omega$ for every $u \in \mathbb{R}$ and continuous in $u \in \mathbb{R}$ for a.e. $x \in \Omega$. In addition, we assume that $\mathbf{a}$ and b satisfy the following *ellipticity* and *growth conditions*: There exist some constants $\kappa \in [0, 1]$ and $\gamma, \Gamma \in (0, \infty)$ such that

$$a_i(x, 0) = 0; \quad i = 1, \ldots, N, \tag{3}$$

$$\sum_{i,j=1}^{N} \frac{\partial a_i}{\partial \eta_j}(x, \eta) \cdot \xi_i \xi_j \geq \gamma \cdot (\kappa + |\eta|)^{p-2} \cdot |\xi|^2, \tag{4}$$

$$\sum_{i,j=1}^{N} \Big| \frac{\partial a_i}{\partial \eta_j}(x, \eta) \Big| \leq \Gamma \cdot (\kappa + |\eta|)^{p-2}, \tag{5}$$

$$\sum_{i,j=1}^{N} \Big| \frac{\partial a_i}{\partial x_j}(x, \eta) \Big| \leq \Gamma \cdot (\kappa + |\eta|)^{p-2} \cdot |\eta|, \tag{6}$$

$$|b(x, u)| \leq \Gamma \cdot (\kappa + |u|)^{p-2} \cdot |u|, \tag{7}$$

for all $x \in \Omega$, all $\eta \in \mathbb{R}^N \setminus \{0\}$, all $\xi \in \mathbb{R}^N$, and all $u \in \mathbb{R}$. Finally, $f \in L^\infty(\Omega)$ is a given function with $f \geq 0$ a.e. in Ω.

Conditions (3) through (7) are motivated by the elliptic boundary value problem

$$-\Delta_p u = \lambda \psi_p(u) + f(x) \ \text{ in } \Omega; \qquad u = 0 \ \text{ on } \partial\Omega, \tag{8}$$

for the p-Laplacian defined by $\Delta_p u \stackrel{\mathrm{def}}{=} \mathrm{div}(|\nabla u|^{p-2} \nabla u)$, with $\psi_p(u) \stackrel{\mathrm{def}}{=} |u|^{p-2} u$ and the spectral parameter $\lambda \in \mathbb{R}$.

In this article we assume that b satisfies either of the following two conditions:

(**b1**) $b(x,u)$ is nondecreasing in u for $(x,u) \in \Omega \times \mathbb{R}$ and Problem (1) has a unique nonnegative weak solution $u \in W_0^{1,p}(\Omega)$.

(**b2**) $b(x,u)$ is locally Lipschitz continuous in u with $\frac{\partial b}{\partial u}(x,u) \leq 0$ for almost all $(x,u) \in \Omega \times (\mathbb{R} \setminus \{0\})$, and

$$\left| \frac{\partial b}{\partial u}(x,u) \right| \leq \begin{cases} \Gamma \cdot |u|^{p-2} & \text{if } 1 < p < 2; \\ \Gamma & \text{if } 2 \leq p < \infty, \end{cases} \tag{9}$$

holds for almost all $(x,u) \in \Omega \times (0, \varepsilon_0]$, with the same constant $\Gamma \in (0,\infty)$ as in (3)–(7) above and another constant $\varepsilon_0 > 0$.

Notice that both Conditions (**b1**) and (**b2**) are satisfied for $b(x,u) \equiv 0$ in $\Omega \times \mathbb{R}$.

We investigate the validity of the *strong comparison principle* (SCP, for brevity) for nonnegative weak solutions $u \in W_0^{1,p}(\Omega)$ to Problem (1). That is to say, let f and g be two functions from $L^\infty(\Omega)$ satisfying $0 \leq f \leq g$ and $f \not\equiv g$ in Ω. Assume that $u, v \in W_0^{1,p}(\Omega)$ are any weak solutions to the following boundary value problems, respectively,

$$-\operatorname{div}(\mathbf{a}(x,\nabla u)) - b(x,u) = f(x) \;\text{ in } \Omega; \qquad u = 0 \;\text{ on } \partial\Omega, \tag{10}$$

$$-\operatorname{div}(\mathbf{a}(x,\nabla v)) - b(x,v) = g(x) \;\text{ in } \Omega; \qquad v = 0 \;\text{ on } \partial\Omega. \tag{11}$$

We denote by $\nu \equiv \nu(x_0)$ the exterior unit normal to $\partial\Omega$ at $x_0 \in \partial\Omega$.

Problem. *Are the following inequalities valid for u and v,*

$$0 \leq u < v \;\text{ in } \Omega \quad \text{ and } \quad \frac{\partial v}{\partial \nu} < \frac{\partial u}{\partial \nu} \leq 0 \;\text{ on } \partial\Omega\,? \tag{12}$$

In the case when Problem (1) is replaced by its special form (8), in [4] the authors showed the SCP (12) under the restriction $0 \leq \lambda < \lambda_1$. Here, the number λ_1 is defined by

$$\lambda_1 \equiv \lambda_1(\Omega) \stackrel{\text{def}}{=} \inf \left\{ \int_\Omega |\nabla u|^p \, dx : \; u \in W_0^{1,p}(\Omega) \;\text{ with }\; \int_\Omega |u|^p \, dx = 1 \right\}.$$

It is well-known, cf. Anane [1, Théorème 1, p. 727], that $\lambda_1 > 0$, and λ_1 is the first eigenvalue of the negative Dirichlet p-Laplacian $-\Delta_p$ in Ω.

For $0 \leq \lambda < \lambda_1$ and $\Omega = (-a,a) \subset \mathbb{R}$, an open interval in the space dimension one, the inequalities (12) are proved in Fleckinger et al. [9, Prop.

4.1]. In the case of the regular Laplace operator Δ (that is, for $p = 2$), the inequalities (12) follow from the classical strong maximum and boundary point principles (due to E. Hopf), whenever $-\infty < \lambda < \lambda_1$. A number of uniqueness results for strong solutions of linear and semilinear boundary value problems involving the operator Δ follow (directly or indirectly) from (12), cf. Gilbarg and Trudinger [13, Chapt. 3] and Takáč [21].

In the case $p \neq 2$ and $N \geq 2$, the validity of (12) is still an open question, except for a few special cases mentioned above. In this article we give an *affirmative answer* to this question for $b(x, u)$ and p satisfying at least one of the following three hypotheses:

(i) Condition (**b1**) and $1 < p < \infty$, see Theorem 2.1 in Section 2.

(ii) Condition (**b2**), $1 < p < 2$, and $N = 1$, see Theorem 3.1 in Section 3.

(iii) Condition (**b2**), $1 < p < 2$, $N \geq 2$, and u, v are radially symmetric solutions in a ball, see Theorem 3.3 in Section 3.

In Section 4 we give a counterexample (Example 4.1) to the SCP (12) for the boundary value problem (8) with $p > 2$ and $\lambda < 0$, where $-\lambda$ is large enough.

In Section 5 we return to Problem (8) and briefly discuss the case when $0 < \lambda < \lambda_1$ and the function $f(x)$ in Problem (1) has indefinite sign, see Remark 5.1. In analogy with the case $p = 2$, also for $p \neq 2$, the inequalities (12) imply a variety of uniqueness and nonexistence results for strong solutions of quasilinear boundary value problems involving the operator $\operatorname{div}(\mathbf{a}(x, \nabla u))$, cf. Clément, Manásevich and Mitidieri [3], Fleckinger et al. [8], [9, Theorem 2.1], and Guedda and Véron [14, Theorem 3.3, p. 896].

2. The Case of $b(x, u)$ Nondecreasing. Let Ω be a bounded domain in $\mathbb{R}^N$, where N is a positive integer. We denote by $\overline{\Omega}$ the closure of Ω in $\mathbb{R}^N$. For a pair of Lebesgue measurable functions $f, g : \Omega \to \mathbb{R}$, we write $f \leq g$ ($f \not\equiv g$, respectively) in Ω if and only if $f(x) \leq g(x)$ for a.e. $x \in \Omega$ ($f(x) \neq g(x)$ for all $x \in \Omega'$ from some set $\Omega' \subset \Omega$ of positive Lebesgue measure). If $N \geq 2$ we need to impose the following Hölder continuity condition on $\mathbf{a}(x, \eta)$:

($\mathbf{a}^\alpha$) For every compact set $K \subset \mathbb{R}^N \setminus \{0\}$, there exists a constant $\alpha \in (0, 1)$ such that $\partial a_i / \partial \eta_j \in C^\alpha(\overline{\Omega} \times K)$ holds for all $i, j = 1, \ldots, N$.

We have the following *strong comparison principle* for nonnegative weak solutions $u, v \in W_0^{1,p}(\Omega)$ of the partial differential equations (10) and (11), respectively. This theorem is our main result in this section.

Theorem 2.1. *Let Ω be either a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a connected C^2-manifold if $N \geq 2$, or a bounded open interval in $\mathbb{R}^1$ if $N = 1$, and let $1 < p < \infty$. Assume that* $\mathbf{a}$ *satisfies* Conditions (3) *through* (7)*, and if $N \geq 2$ then* ($\mathbf{a}^\alpha$) *as well, and b satisfies* Condition (**b1**). *Let $f, g \in L^\infty(\Omega)$ be such that $0 \leq f \leq g$ and $f \not\equiv g$ in Ω. Finally, assume that $u, v \in W_0^{1,p}(\Omega)$ are any nonnegative weak solutions of* Eqs. (10) *and* (11)*. Then the SCP* (12) *is valid.*

In the next few remarks we mention some related results which are known in the literature.

Remark 2.1. (i) In the special case of Problem (8) with $0 \leq \lambda < \lambda_1$, this result is due to Cuesta and Takáč [4]. If $0 \equiv f \leq g \not\equiv 0$ and $0 \equiv u \leq v \not\equiv 0$ in Ω, this is the strong maximum principle due to Tolksdorf [24, Prop. 3.2.1 and 3.2.2, p. 801] for $\mathbf{a}(x, \nabla u) \equiv \mathbf{a}(\nabla u)$ and Vázquez [26, Theorem 5, p. 200] for $\mathbf{a}(x, \nabla u) \equiv |\nabla u|^{p-2}\nabla u$. The proof given in Tolksdorf [24, p. 802] extends directly to our general case.

(ii) The following version of the SCP, which is considerably weaker than our Theorem 2.1, was shown in Tolksdorf [24, Prop. 3.3.2, p. 803]: *Let Ω' be a subdomain (that is, an open connected subset) of Ω, such that $|\nabla u(x)| \geq \delta > 0$ for every $x \in \Omega'$, where δ is a constant. Then $u < v$ in Ω'.* It is obvious that $x_0 \notin \overline{\Omega}'$ whenever the function u attains a local minimum or maximum at $x_0 \in \Omega$. Consequently $\Omega' \neq \Omega$.

(iii) Later, for Problem (8) with $\lambda = 0$, Guedda and Véron [14, Prop. 2.2, p. 888] proved the SCP (12) under the restriction that the set $\{x \in \Omega : f(x) = g(x)\}$ has empty interior. In our Theorem 2.1 above we allow f and g to coincide in Ω except for a set of positive Lebesgue measure (that is, $f \not\equiv g$ in Ω); but we need to impose a restriction on the set Ω, namely, that its boundary $\partial\Omega$ is connected.

In our *proof* of Theorem 2.1, we follow the same outline as in [4]. We will make use of the following three results, see Lemma 2.2 and Propositions 2.3 and 2.4 below, respectively:
(I) a *regularity* result due to DiBenedetto [6, Theorem 2, p. 829] and Tolksdorf [25, Theorem 1, p. 127] (interior regularity, shown independently), and to Lieberman [16, Theorem 1, p. 1203] (regularity near the boundary);

(II) a *weak comparison principle* due to Fleckinger et al. [10, Theorem 2] for $b(x,u)$ satisfying (**b1**), and to Tolksdorf [24, Lemma 3.1, p. 800] for $b(x,u)$ satisfying (**b2**);
(III) a strong comparison principle *near the boundary* shown in Fleckinger and Takáč [11, Prop. 2, p. 448] or [12, Prop. 5.1, p. 1238], or in Tolksdorf [24, Prop. 3.3.1 and 3.3.2, p. 803].

These three results can be stated as follows:

2.1. Preliminary Results. First, let us consider the Dirichlet problem

$$-\operatorname{div}(\mathbf{a}(x,\nabla u)) - b(x,u) = f(x) \ \text{ in } \Omega; \quad u = f' \ \text{ on } \partial\Omega. \tag{13}$$

We refer to Nečas [17, Chapt. 2, Sect. 5.4, p. 99] for a definition of the trace $u|_{\partial\Omega} \in W^{1-(1/p),p}(\partial\Omega)$ of a function $u \in W^{1,p}(\Omega)$. Unless stated otherwise, we assume that Ω is a bounded domain in $\mathbb{R}^N$ with a $C^{1,\alpha}$-boundary $\partial\Omega$, for some $\alpha \in (0,1)$, and $1 < p < \infty$. It is very convenient and often necessary to work with $C^{1,\beta}$-solutions of Problem (13). The following regularity result is due to Lieberman [16, Theorem 1, p. 1203] (and DiBenedetto [6] or Tolksdorf [25] for interior regularity):

Lemma 2.2. ([6], [16], [25]) *Let $u \in W^{1,p}(\Omega)$ be any weak solution of the Dirichlet problem* (13)*. In addition to Conditions* (3)–(7)*, assume that $0 \le f \in L^\infty(\Omega)$ with the norm $\|f\|_{L^\infty(\Omega)}$ and $0 \le f' \in C^{1,\alpha}(\partial\Omega)$ with the Hölder norm $|f'|_{1+\alpha;\partial\Omega}$. Let $C' \ge 0$ be any constant such that*

$$\|u\|_{W^{1,p}(\Omega)} \le C', \quad \|f\|_{L^\infty(\Omega)} \le C' \quad \text{and} \quad |f'|_{1+\alpha;\partial\Omega} \le C'.$$

Then there exist a constant $\beta \equiv \beta(\alpha,p,\gamma,\Gamma,N)$, $0 < \beta < 1$, depending solely upon α, p, γ, Γ and N, and another constant $C \equiv C(\alpha,p,\gamma,\Gamma,N,\Omega,C')$, $0 \le C < \infty$, depending solely upon α, p, γ, Γ, N, Ω and C' such that $u \in C^{1,\beta}(\overline{\Omega})$ with the Hölder norm $|u|_{1+\beta;\overline{\Omega}} \le C(\alpha,p,\gamma,\Gamma,N,\Omega,C')$.

Remark 2.2. In [6], [16], [25], Lemma 2.2 is proved under the stronger hypothesis that $\|u\|_{L^\infty(\Omega)} \le C'$. However, de Thélin [23, Théorème 1, p. 376] and Anane [2, Théorème A.1, p. 96] showed that the uniform boundedness hypothesis $\|u\|_{L^\infty(\Omega)} \le C'$ may be replaced by the weaker hypothesis $\|u\|_{W^{1,p}(\Omega)} \le C'$. Although their proofs are carried out for the case of $\mathbf{a}(x,\nabla u) \equiv |\nabla u|^{p-2}\nabla u$ only, they extend easily to our general case. Using Moser's iterative technique, they employ the growth condition (7) in an essential way.

Now consider the Dirichlet problems

$$-\operatorname{div}(\mathbf{a}(x,\nabla u)) - b(x,u) = f(x) \ \text{ in } \Omega; \qquad u = f' \ \text{ on } \partial\Omega, \tag{14}$$

$$-\operatorname{div}(\mathbf{a}(x,\nabla v)) - b(x,v) = g(x) \ \text{ in } \Omega; \qquad v = g' \ \text{ on } \partial\Omega. \tag{15}$$

Proposition 2.3. ([10], [24]) *Assume that $f \le g$ in $L^{p/(p-1)}(\Omega)$, $f' \le g'$ in $W^{1-(1/p),p}(\partial\Omega)$, and $u, v \in W^{1,p}(\Omega)$ are any weak solutions of the Dirichlet problems* (14) *and* (15)*, respectively. Then $u \le v$ holds almost everywhere in Ω, provided either of the following two additional conditions is valid:*
(a) *b satisfies* (**b1**)*, $0 \le f \le g$ in $L^\infty(\Omega)$, $f' \equiv g' \equiv 0$ on $\partial\Omega$, and $u, v \ge 0$ a.e. in Ω.*
(b) *$b(x, \bullet) : \mathbb{R} \to \mathbb{R}$ is nonincreasing for a.e. $x \in \Omega$.*

Proof. (b) Although this result is proved in [24, Lemma 3.1, p. 800] (or [12, Prop. 4.1, p. 1235]) only for the special case of $\mathbf{a}(x,\nabla u) \equiv \mathbf{a}(\nabla u)$, its proof extends directly to our general case without any major change, see Gilbarg and Trudinger [13, Proof of Theorem 10.7, p. 269]. For instance, the "ellipticity inequalities" in [25, Lemma 1, p. 129] (with an elementary proof) may be applied.

(a) We combine Part (b) for $b \equiv 0$ with the following monotone iteration procedure from [10, Theorem 2] where it was carried out only for the special case of the p-Laplacian.

Set $L^\infty_+(\Omega) \stackrel{\text{def}}{=} \{f \in L^\infty(\Omega) : f \ge 0 \text{ a.e. in } \Omega\}$. Given any $f \in L^\infty_+(\Omega)$, define the mapping $T_f : L^\infty_+(\Omega) \to L^\infty_+(\Omega)$ by $v \mapsto T_f v \stackrel{\text{def}}{=} w$, where $w \in W^{1,p}_0(\Omega)$ is the weak solution of

$$-\operatorname{div}(\mathbf{a}(x,\nabla w)) = b(x,v(x)) + f(x) \ \text{ in } \Omega; \quad w = 0 \ \text{ on } \partial\Omega. \tag{16}$$

Existence of this solution is guaranteed by the standard theory of monotone operators (see Deimling [5, Theorem 12.1, p. 117]) based on the "ellipticity inequalities" from [25, Lemma 1, p. 129], whereas uniqueness and positivity follow from Part (b) with $b \equiv 0$. Furthermore, we have $w \in L^\infty(\Omega)$ by Remark 2.2, and $w \in C^{1,\beta}(\overline{\Omega})$ by Lemma 2.2. The following properties of the mapping T_f are proved in Fleckinger and Takáč [12, Lemma 6.1, p. 1243], albeit for the special case of $\mathbf{a}(x,\nabla u) \equiv |\nabla u|^{p-2}\nabla u$ and $b(x,u) \equiv \lambda |u|^{p-2}u$ only. Their proof can be carried over directly to the present general case.

(i) $T_f : L^\infty_+(\Omega) \to L^\infty_+(\Omega)$ is continuous and maps bounded sets into relatively compact sets in $C^{1,\beta'}(\overline{\Omega})$, for some $\beta' \in (0,1)$.

(ii) The mapping $(v,f) \mapsto T_f v$ is monotone, that is, $v \le v^*$ and $f \le f^*$ in $L^\infty_+(\Omega)$ implies $T_f v \le T_{f^*} v^*$.

By our hypothesis (**b1**), Problem (1) has a unique nonnegative weak solution $u \in W_0^{1,p}(\Omega)$. Thus $u \in L^\infty_+(\Omega)$, by Remark 2.2 above. Consequently, the sequence $0 \le T_f 0 \le (T_f)^2 0 \le \cdots \le (T_f)^n 0 \le \cdots \le u$ converges in $C^{1,\beta'}(\overline{\Omega})$ to the smallest nonnegative weak solution $\underline{u} \in W_0^{1,p}(\Omega)$ of Problem (1). Hence $\underline{u} = u$, by uniqueness, and $(T_f)^n 0 \nearrow u$ in $C^{1,\beta'}(\overline{\Omega})$ as $n \to \infty$. Similarly $(T_{f^*})^n 0 \nearrow u^*$. Since also $(T_f)^n 0 \le (T_{f^*})^n 0$ for all $n \ge 1$, by $0 \le f \le f^*$, we arrive at $0 \le u \le u^*$ in Ω. □

For quasilinear boundary value problems (1) satisfying Conditions (3)–(7) with $p = 2$ and $b(x,\bullet) : \mathbb{R} \to \mathbb{R}$ nonincreasing for a.e. $x \in \Omega$, a weak comparison principle (WCP, for brevity) is proved in Gilbarg and Trudinger [13, Theorem 10.7, p. 268].

Next we consider the open δ-neighborhood $\Omega_\delta \subset \Omega$ of the boundary $\partial\Omega$,

$$\Omega_\delta = \{x \in \Omega : d(x) < \delta\} \quad \text{for } \delta > 0 \text{ small enough.} \tag{17}$$

As usual, $d(x) \stackrel{\text{def}}{=} \operatorname{dist}(x, \partial\Omega)$ denotes the distance from a point $x \in \Omega$ to the boundary $\partial\Omega$. From now on we assume that $\partial\Omega$ is a compact manifold of class C^2. Then, by [13, Lemma 14.16, p. 355] and its proof, we have $d \in C^2(\overline{\Omega_\delta})$, $\overline{\Omega_\delta}$ is C^1-diffeomorphic to $\partial\Omega \times [0,\delta]$ with $x \mapsto (x,0)$ for all $x \in \partial\Omega$, and $\Omega \setminus \Omega_\delta$ is C^1-diffeomorphic to $\overline{\Omega}$. Both diffeomorphisms are considered between manifolds with boundary of class C^2. Of course, they can be replaced by C^2-diffeomorphisms, by Hirsch [15, Theorem 3.5, p. 57].

Proposition 2.4. ([11], [12], [24]) *Let Ω be a bounded domain in $\mathbb{R}^N$ with a C^2-boundary $\partial\Omega$, and let $1 < p < \infty$. Assume that $\mathbf{a}$ satisfies* Conditions (3) *through* (7) *and* ($\mathbf{a}^\alpha$) *as well, and b satisfies* Condition (**b1**) *or* (**b2**)*. Let $f, g \in L^\infty(\Omega)$ satisfy $0 \le f \le g$ in Ω. Finally, assume that $u, v \in W_0^{1,p}(\Omega)$ are any nonnegative weak solutions of* Eqs. (10) *and* (11)*. Then, for every $\delta > 0$ small enough and for every connected component Σ of Ω_δ, we have either $u \equiv v$ in Σ or else*

$$u < v \quad \text{in } \overline{\Sigma} \setminus \partial\Omega \qquad \text{and} \qquad \frac{\partial u}{\partial \nu} > \frac{\partial v}{\partial \nu} \quad \text{on } \partial\Omega \cap \overline{\Sigma}. \tag{18}$$

We recall that $u, v \in C^{1,\beta}(\overline{\Omega})$ by Lemma 2.2.

Proof. We need to adapt the proof from [12, Prop. 5.1, p. 1238] to the present setting which is much more general. From Proposition 2.3 we obtain $u \le v$ in Ω. If the function b satisfies (**b1**), then we have

$$f^*(x) = b(x,u) + f(x) \le g^*(x) = b(x,v) + g(x) \quad \text{in } \Omega. \tag{19}$$

Also $f \not\equiv g$ implies $f^* \not\equiv g^*$ in Ω. Thus, we may replace the pair (f,g) by (f^*, g^*) and assume that $b \equiv 0$. We conclude that from now on it suffices to investigate the case when b satisfies (**b2**).

If $0 \equiv u \le v \not\equiv 0$ in Ω, then (12) is the strong maximum principle due to Tolksdorf [24, Prop. 3.2.1 and 3.2.2, p. 801]. His proof [24, p. 802] extends directly to our more general case. Consequently, we may assume that

$$0 < u \le v \;\text{ in } \Omega \quad \text{and} \quad 0 > \frac{\partial u}{\partial \nu} \ge \frac{\partial v}{\partial \nu} \;\text{ on } \partial\Omega. \tag{20}$$

Taking advantage of these inequalities, let us fix constants η and δ, both positive and small enough, such that

$$|(1-t)\nabla u + t\nabla v| \ge \eta, \quad (1-t)u + tv \le \varepsilon_0 \;\text{ throughout } \overline{\Omega_\delta} \text{ for all } t \in [0,1]. \tag{21}$$

Of course, we choose δ small enough so that $\overline{\Omega_\delta}$ be C^1-diffeomorphic to $\partial\Omega \times [0,\delta]$ as described above. Set $w \equiv v - u$; so $0 \le w \in C^{1,\beta}(\overline{\Omega})$ with $w = 0$ on $\partial\Omega$. Subtracting Eq. (10) from (11), we find out that w satisfies the following linear elliptic inequality in the sense of distributions in Ω_δ,

$$\begin{aligned} &- \operatorname{div}(\mathbf{A}(x)\nabla w) - B(x)w \stackrel{\text{def}}{=} \\ &- \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial w}{\partial x_j}\right) - B(x)w = g - f \ge 0 \quad \text{for } x \in \Omega_\delta. \end{aligned} \tag{22}$$

Here, the coefficients

$$a_{ij}(x) = \int_0^1 \frac{\partial a_i}{\partial \eta_j}\big(x, (1-t)\nabla u(x) + t\nabla v(x)\big)\, dt \tag{23}$$

belong to $C^{\alpha'}(\overline{\Omega_\delta})$, where $\alpha' = \alpha\beta \in (0,1)$, and form a uniformly elliptic operator in Ω_δ, by (4) and (5). The coefficient

$$B(x) = \int_0^1 \frac{\partial b}{\partial u}\big(x, (1-t)u(x) + tv(x)\big)\, dt \tag{24}$$

satisfies $B(x) \leq 0$ together with

$$|B(x)| \leq \begin{cases} C \cdot d(x)^{p-2} & \text{if } 1 < p < 2; \\ C & \text{if } 2 \leq p < \infty, \end{cases} \tag{25}$$

for almost every $x \in \Omega_\delta$, where $C \geq 0$ is a constant. The last estimate is deduced from (9) combined with $u, v \in C^{1,\beta}(\overline{\Omega})$ and $d \in C^2(\overline{\Omega_\delta})$.

Next, let Σ be any connected component of Ω_δ. In every subdomain Σ' of Σ with $\overline{\Sigma'} \cap \partial\Omega = \emptyset$, we apply the strong maximum principle from Gilbarg and Trudinger [13, Theorem 8.19, p. 198] to the linear elliptic inequality (22) considered in Σ'. We thus obtain either $u \equiv v$ in Σ or else $u < v$ in Σ. Finally, assume that $u < v$ in Σ. From the boundary point principle as shown in Finn and Gilbarg [7, Lemma 7, p. 31] we deduce that $-\frac{\partial u}{\partial \nu}(x_0) < -\frac{\partial v}{\partial \nu}(x_0)$ holds in an arbitrary boundary point $x_0 \in \partial\Omega \cap \overline{\Sigma}$. □

Remark 2.3. (i) Condition ($\mathbf{a}^\alpha$) is essential for our application of the boundary point principle from Finn and Gilbarg [7, Lemma 7, p. 31] to the inequality (22); see [7, Remarks, p. 35] for $N \geq 3$.

(ii) Notice that the growth estimate near the boundary (25), as a consequence of the growth condition (9) which in turn is motivated by Problem (8), is important also in the proof of the SCP in Fleckinger and Takáč [12, Eq. (5.14), p. 1240]. The case $2 \leq p < \infty$ can be treated by methods from Tolksdorf [24, Eq. (3.2), p. 802] as well, where the following stronger condition is assumed in place of (9),

$$\left| \frac{\partial b}{\partial u}(x,u) \right| \leq \Gamma \quad \text{for all } (x,u) \in \Omega \times (0, \varepsilon_0],$$

with some constants $\Gamma \in (0, \infty)$ and $\varepsilon_0 > 0$.

2.2. Proof of Theorem 2.1. To begin with, for $u, v \geq 0$ in Ω the WCP (Proposition 2.3(a) above) forces $u \leq v$ in Ω. Since the function b satisfies (**b1**), we have (19). Thus, we may assume $b \equiv 0$ by simply replacing $b(x,u) + f$ by f and $b(x,v) + g$ by g. Moreover, by Lemma 2.2, we have $u, v \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$.

Case $N \geq 2$**.** Next, as $\partial\Omega$ is connected, we deduce from Proposition 2.4 with $\Sigma = \Omega_\delta$ that precisely one of the following two mutually exclusive alternatives must be valid, for some $\delta > 0$:

(A1) $u \equiv v$ in Ω_δ; (A2) the SCP (18) holds in Ω_δ.

In the remaining part of this proof, we rule out the first alternative and show that the second one implies $u < v$ throughout Ω.

Alt. (A1): First, the version of the divergence theorem which we employ below is shown in Appendix A, Lemma A.1. For $0 < \eta < \delta$, η small enough, we apply the divergence theorem to Eqs. (10) and (11) over the domain $\Omega'_\eta = \Omega \setminus \overline{\Omega_\eta}$. We thus obtain

$$-\int_{\partial\Omega'_\eta} \mathbf{a}(x, \nabla u(x)) \cdot \nu(x)\, d\sigma(x) = \int_{\Omega'_\eta} f(x)\, dx, \tag{26}$$

$$-\int_{\partial\Omega'_\eta} \mathbf{a}(x, \nabla v(x)) \cdot \nu(x)\, d\sigma(x) = \int_{\Omega'_\eta} g(x)\, dx. \tag{27}$$

Since $u \equiv v$ in Ω_δ and $\partial\Omega'_\eta \subset \Omega_\delta$, the two surface integrals on the left-hand side in Eqs. (26) and (27) are equal. Therefore, we have

$$\int_{\Omega'_\eta} f(x)\, dx = \int_{\Omega'_\eta} g(x)\, dx.$$

Combined with $f \leq g$ in Ω, this equality forces $f \equiv g$ in Ω'_η. From $u \equiv v$ in Ω_δ, we obtain also $f \equiv g$ in Ω_δ. Thus, we arrive at $f \equiv g$ throughout $\Omega = \Omega'_\eta \cup \Omega_\delta$, a contradiction to our hypothesis $f \not\equiv g$ in Ω. We have ruled out Alt. (A1).

Alt. (A2): Again, choose η small enough, $0 < \eta < \delta$, so that $\partial\Omega_\eta$ is a C^2-manifold which is C^1-diffeomorphic to $\partial\Omega$. Set $\Omega'_\eta = \Omega \setminus \overline{\Omega_\eta}$. We have $u < v$ on $\partial\Omega'_\eta$, by the SCP (18) in Ω_δ. Since also $u, v \in C^{1,\beta}(\overline{\Omega})$, for some $\beta \in (0, 1)$, there exists a constant $c > 0$ such that $u + c \leq v$ on $\partial\Omega'_\eta$. Furthermore, we have

$$-\operatorname{div} \mathbf{a}(x, \nabla(u + c)) = -\operatorname{div} \mathbf{a}(x, \nabla u) = f \leq g = -\operatorname{div} \mathbf{a}(x, \nabla v) \quad \text{ in } \Omega'_\eta.$$

Hence, we may apply the WCP (Proposition 2.3) to the pair $u + c$ and v in Ω'_η, thus arriving at $u + c \leq v$ throughout Ω'_η. Thus, the SCP (12) is valid in the entire domain Ω.

Case $N = 1$**.** We may take $\Omega = (-R, R) \subset \mathbb{R}^1$, where $0 < R < \infty$. As usual, we write $' \equiv d/dx$. Thus, as we now assume $b \equiv 0$, the boundary value problem (1) becomes

$$-(a(x, u'))' = f(x) \ \text{ for } x \in (-R, R); \qquad u(-R) = u(R) = 0. \tag{28}$$

We have replaced the vector **a** by the scalar a, which is a function of x and $\eta = u' \in \mathbb{R}^1$. Let $\alpha(x, \bullet) : \mathbb{R} \to \mathbb{R}$ denote the inverse function of $a(x, \bullet) : \mathbb{R} \to \mathbb{R}$, for each fixed $x \in [-R, R]$. Notice that, by Conditions (3) and (4), the function $\alpha(x, \bullet)$ exists and is strictly monotone increasing in $\mathbb{R}$ with $\alpha(x, 0) = 0$. Applying two-fold integration to Problem (28), we arrive at

$$u(x) = u(x_0) + \int_{x_0}^{x} \alpha\left(y, a(x_0, u'(x_0)) - \int_{x_0}^{y} f(z)\, dz\right) dy \tag{29}$$

for all $x, x_0 \in [-R, R]$. Now suppose that the SCP (12) is false. Then we deduce from Proposition 2.4 that the function $w = v - u : [-R, R] \to \mathbb{R}$ must attain its zero minimum at an interior point $x_0 \in (-R, R)$. Hence, $u(x_0) = v(x_0)$ and $u'(x_0) = v'(x_0) \equiv \eta_0$. We set $a_0 = a(x_0, \eta_0) \in \mathbb{R}$. Inserting these two equalities and $u(\pm R) = v(\pm R) = 0$ into Eq. (29) for u and v, respectively, we obtain

$$\begin{aligned} &\int_{x_0}^{x} \alpha\left(y, a_0 - \int_{x_0}^{y} f(z)\, dz\right) dy \\ &= \int_{x_0}^{x} \alpha\left(y, a_0 - \int_{x_0}^{y} g(z)\, dz\right) dy \quad \text{for } x = \pm R. \end{aligned} \tag{30}$$

As the function $\alpha(y, \bullet)$ is strictly monotone increasing in $\mathbb{R}$ and $f \le g$ in $(-R, R)$, Eq. (30) forces $f = g$ a.e. in $(-R, R)$, a contradiction. Theorem 2.1 is proved. □

3. The Case $1 < p \le 2$ **and** $\partial b / \partial u \le 0$**.** Here we prove the strong comparison principle for the elliptic boundary value problem (1) in the following two cases: **1.** for the space dimension $N = 1$; and **2.** for a radially symmetric problem in a ball ($N \ge 2$).

3.1. The Space Dimension $N = 1$**.** Fixing $N = 1$ throughout this paragraph, we may take $\Omega = (-R, R) \subset \mathbb{R}^1$, where $0 < R < \infty$. Recall that $' \equiv d/dx$. Thus, the boundary value problem (1) becomes

$$\begin{cases} -\left(a(x, u')\right)' - b(x, u) = f(x) & \text{for } x \in (-R, R); \\ u(-R) = u(R) = 0. \end{cases} \tag{31}$$

We have replaced the vector **a** by the scalar a, which is a function of x and $\eta = u' \in \mathbb{R}^1$. Conditions (3) through (7) reduce to the following ones: There exist some constants $\kappa \in [0,1]$ and $\gamma, \Gamma \in (0,\infty)$ such that

$$a(x,0) = 0, \tag{32}$$

$$\gamma \cdot (\kappa + |\eta|)^{p-2} \leq \frac{\partial a}{\partial \eta}(x,\eta) \leq \Gamma \cdot (\kappa + |\eta|)^{p-2}, \tag{33}$$

$$\left|\frac{\partial a}{\partial x}(x,\eta)\right| \leq \Gamma \cdot (\kappa + |\eta|)^{p-2} \cdot |\eta|, \tag{34}$$

$$|b(x,u)| \leq \Gamma \cdot (\kappa + |u|)^{p-2} \cdot |u|, \tag{35}$$

for all $x \in (-R,R)$, all $\eta \in \mathbb{R} \setminus \{0\}$, and all $u \in \mathbb{R}$.

We have the following SCP for the weak solutions $u, v \in W_0^{1,p}(-R,R)$ of the differential equations (10) and (11), respectively, with $N = 1$. This theorem complements our main result, Theorem 2.1.

Theorem 3.1. *Let $f, g \in L^\infty(-R,R)$ satisfy $0 \leq f \leq g$ with $f \not\equiv g$ in $(-R,R)$, where $0 < R < \infty$. Let $1 < p \leq 2$, and let b satisfy* Condition (**b2**) *and also $\partial b/\partial u \in L^\infty_{\mathrm{loc}}((-R,R) \times (0,\infty))$. Assume that $u, v \in W_0^{1,p}(-R,R)$ are any weak solutions of* Eqs. (10) *and* (11) *with $N = 1$. Then we have the SCP*

$$\begin{cases} 0 \leq u(x) < v(x) & \text{for all } x \in (-R,R); \\ v'(R) < u'(R) \leq 0 \leq u'(-R) < v'(-R). \end{cases} \tag{36}$$

Remark 3.1. When $N = 1$, for two-point boundary value problems similar to Problem (1) but with *other* than Dirichlet boundary conditions, the SCP (36) is studied in Walter [27, Sect. 5]. There, a theorem and a counterexample covering the problem

$$\begin{cases} -(\psi_p(u'))' - \lambda\psi_p(u) \leq -(\psi_p(v'))' - \lambda\psi_p(v) \ \text{ in } (-1,1); \\ u(x) \leq v(x) \ \text{ for } x = \pm 1, \end{cases} \tag{37}$$

are presented.

The main idea of our *proof* of Theorem 3.1 is to apply Prüfer's transformation to the boundary value problem (31). We denote by $\alpha(x,\bullet) : \mathbb{R} \to \mathbb{R}$ the inverse function of $a(x,\bullet) : \mathbb{R} \to \mathbb{R}$, for each fixed $x \in [-R,R]$, and set

$$\beta_f(x,u_1) = -b(x,u_1) - f(x) \quad \text{for } (x,u_1) \in [-R,R] \times \mathbb{R}. \tag{38}$$

Thus, Problem (31) is equivalent to

$$u_1' = \alpha(x, u_2), \quad u_2' = \beta_f(x, u_1) \ \text{ for } -R < x < R; \quad u_1(-R) = u_1(R) = 0. \tag{39}$$

This is a system of two coupled first-order differential equations for the unknown pair of functions $(u_1, u_2) : [-R, R] \to \mathbb{R}^2$. We have obtained this system by first making the substitutions $u_1 = u$ and $u_2 = a(x, u')$ in Eq. (31) (which yields $u_2' = \beta_f(x, u_1)$) and then calculating the unknown value of $u_1' = u'$ from the equation $u_2 = a(x, u_1')$ (which yields $u_1' = \alpha(x, u_2)$). Notice that, by Conditions (32) and (33), the function $\alpha(x, u_2)$ exists with

$$\frac{\partial \alpha}{\partial u_2}(x, u_2) = \big(\frac{\partial a}{\partial \eta}(x, \alpha(x, u_2))\big)^{-1} \quad \text{for all } (x, u_2) \in [-R, R] \times (\mathbb{R} \setminus \{0\}).$$

In particular, we have

$$\alpha(x, 0) = 0 \ \text{ and } \ \alpha(x, u_2)u_2 > 0 \ \text{ if } u_2 \neq 0, \tag{40}$$

$$\Gamma^{-1} \cdot (\kappa + |\alpha(x, u_2)|)^{2-p} \leq \frac{\partial \alpha}{\partial u_2}(x, u_2) \leq \gamma^{-1} \cdot (\kappa + |\alpha(x, u_2)|)^{2-p}, \tag{41}$$

for all $x \in (-R, R)$ and all $u_2 \in \mathbb{R} \setminus \{0\}$.

Since $1 < p \leq 2$, Conditions (40) and (41) above imply that the function $\alpha(x, u_2)$ is uniformly Lipschitz continuous with respect to u_2 for $(x, u_2) \in [-R, R] \times [-\varrho, \varrho]$, whenever $\varrho \in (0, \infty)$. Moreover, we have

$$\frac{p-1}{\Gamma} \leq \big|\frac{\partial}{\partial u_2}(\kappa + |\alpha(x, u_2)|)^{p-1}\big| \leq \frac{p-1}{\gamma} \text{ for } (x, u_2) \in [-R, R] \times (\mathbb{R} \setminus \{0\}).$$

Let now $x_0 \in (-R, R)$ be given as an initial point for the initial value problem corresponding to System (39),

$$\begin{cases} u_1' = \alpha(x, u_2) & \text{for } x_0 < x < R; \quad u_1(x_0) = u_{1,0}, \\ u_2' = \beta_f(x, u_1) & \text{for } x_0 < x < R; \quad u_2(x_0) = u_{2,0}, \end{cases} \tag{42}$$

where the initial data $(u_{1,0}, u_{2,0})$ are given in $\mathbb{R}^2$ such that $u_{1,0} > 0$. By our conditions (41) and $\partial b / \partial u \in L^\infty_{\text{loc}}((-R, R) \times (0, \infty))$, the right-hand side of System (42) is locally Lipschitz continuous with respect to u_1 and u_2

for $x \in (-R, R)$ and $(u_1, u_2) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$. It is also *cooperative* for $x \in (-R, R)$ and $(u_1, u_2) \in (\mathbb{R} \setminus \{0\})^2$:

$$\frac{\partial \alpha}{\partial u_2}(x, u_2) \geq 0 \quad \text{and} \quad \frac{\partial \beta_f}{\partial u_1}(x, u_1) \geq 0,$$

see Smith [20, Sect. 3.1, pp. 32–34]. Consequently, we may combine the local existence and uniqueness theory with the Müller-Kamke theorem to obtain the following lemma (which follows easily from results in [20, Sect. 3.1, pp. 32–34]):

Lemma 3.2. *Let $0 \leq f \in L^\infty(-R, R)$, $1 < p \leq 2$, and let b satisfy $0 \leq -\partial b/\partial u \in L^\infty_{\mathrm{loc}}((-R, R) \times (0, \infty))$. Then there exists a point $x_1 \equiv x_1(f, u_{1,0}, u_{2,0})$ in the interval (x_0, R), depending upon f and $(u_{1,0}, u_{2,0})$, such that* System (42) *(for $x_0 < x < x_1$) has a unique weak solution $(u_1, u_2) \in [W^{1,\infty}(x_0, x_1)]^2$ satisfying $u_1 > 0$ in $[x_0, x_1]$.*

Furthermore, System (42) *possesses the following* monotonicity *property: In addition, let $g \in L^\infty(-R, R)$ and $(v_{1,0}, v_{2,0}) \in \mathbb{R}^2$ be such that $f \leq g$ in $(-R, R)$ together with $0 < v_{1,0} \leq u_{1,0}$ and $v_{2,0} \leq u_{2,0}$. Assume that $(v_1, v_2) \in [W^{1,\infty}(x_0, x_1)]^2$ is the (unique) weak solution of the system*

$$\begin{cases} v_1' = \alpha(x, v_2) & \text{for } x_0 < x < R; \quad v_1(x_0) = v_{1,0}, \\ v_2' = \beta_g(x, v_1) & \text{for } x_0 < x < R; \quad v_2(x_0) = v_{2,0}, \end{cases} \tag{43}$$

satisfying $v_1 > 0$ in $[x_0, x_1]$, where $x_1 \equiv x_1(g, v_{1,0}, v_{2,0})$. Then we have

$$v_1(x) \leq u_1(x) \quad \textit{and} \quad v_2(x) \leq u_2(x) \qquad \textit{for all } x \in [x_0, x_1^*],$$

where $x_1^* \in (x_0, R)$ is defined by $x_1^* = \min\{x_1(f, u_{1,0}, u_{2,0}), x_1(g, v_{1,0}, v_{2,0})\}$.

Proof of Theorem 3.1. Again, the WCP (Proposition 2.3) forces $0 \leq u \leq v$ in $(-R, R)$. Moreover, we have $u, v \in C^{1,\beta}([-R, R])$ for some $\beta \in (0, 1)$, by Lemma 2.2. Next, by Proposition 2.4 with $\Sigma = (-R, -R + \delta)$ or $\Sigma = (R - \delta, R)$, precisely one of the following two mutually exclusive alternatives must be valid:

(A1) $u \equiv v$ in $(-R, -R + \delta)$ or $(R - \delta, R)$, for some $\delta \in (0, R)$;

(A2) the SCP (18) holds in $(-R, -R + \delta) \cup (R - \delta, R)$, for some $\delta \in (0, R)$.

In the remaining part of this proof, we rule out the first alternative and show that the second one implies $u < v$ throughout $(-R, R)$.

Alt. (A1): Without loss of generality, we may assume $u \equiv v$ in $(-R, -R+\delta)$. We fix any $x_0 \in (-R, -R+\delta)$ and apply Lemma 3.2 with $u_{i,0} = v_{i,0} = u_i(x_0) = v_i(x_0)$ for $i = 1, 2$. Thus, we obtain $v(x) \le u(x)$ for every $x \in [x_0, R)$. Since also $u \equiv v$ in $(-R, x_0]$ and $u \le v$ in $[x_0, R)$, we arrive at $u \equiv v$ throughout $(-R, R)$, a contradiction to our hypothesis $f \not\equiv g$ in $(-R, R)$. We have ruled out Alt. (A1).

Alt. (A2): Assume that the SCP (36) is not valid in the interval $[-R+\delta, R-\delta]$, that is, $u(x_0) = v(x_0)$ for some $x_0 \in (-R, R)$. The function $u - v : [-R, R] \to \mathbb{R}$ being nonpositive, it attains its zero maximum at the point $x_0 \in (-R, R)$. Hence, we must have also $u'(x_0) = v'(x_0)$. Applying Lemma 3.2 as above, we obtain $v(x) \le u(x)$ for every $x \in [x_0, R)$. It follows that $u \equiv v$ in $[x_0, R)$. This is a contradiction since this alternative assumes $u(x) < v(x)$ for every $x \in (R-\delta, R)$, for some $\delta \in (0, R)$. We conclude that $u(x) < v(x)$ for every $x \in (-R, R)$, and thus, (36) is valid. Theorem 3.1 is proved.

3.2. A Radially Symmetric Problem in a Ball ($N \ge 2$). Throughout this paragraph we assume that the boundary value problem (1) is radially symmetric in a ball $\Omega = \{x \in \mathbb{R}^N : |x| < R\}$, where $N \ge 2$ and $0 < R < \infty$. That is to say,

$$\mathbf{a}(x, \nabla u(x)) \equiv a(|x|, u'(|x|)) \frac{x}{|x|} \quad \text{for } x \in \Omega \setminus \{0\} \text{ and } u \in W_0^{1,p}(\Omega), \tag{44}$$

where $u(x) \equiv u(r)$ is a radially symmetric function of $r = |x|$, for $0 \le r \le R$, and similarly, $b(x, u) \equiv b(r, u)$ and $f(x) \equiv f(r)$. Again, we write $' \equiv d/dr$. Thus, the boundary value problem (1) becomes

$$\begin{cases} -(a(r, u'))' - \frac{N-1}{r} a(r, u') - b(r, u) = f(r) & \text{for } 0 < r < R; \\ u'(0) = u(R) = 0. \end{cases} \tag{45}$$

In this sense, we use the scalar a in place of the vector $\mathbf{a}$, where a is a function of r and $\zeta = u' \in \mathbb{R}$.

Notice that Eq. (44) combined with $\zeta = u'(r)$ and $\eta = \nabla u(x) = \zeta r^{-1} x$

entails

$$\begin{aligned} a(r,\zeta) &= \mathbf{a}\left(x,\zeta\frac{x}{r}\right)\cdot\frac{x}{r}, & (46)\\ \frac{\partial a}{\partial \zeta}(r,\zeta) &= \sum_{i,j=1}^{N}\frac{\partial a_i}{\partial \eta_j}\left(x,\zeta\frac{x}{r}\right)\cdot\frac{x_i}{r}\cdot\frac{x_j}{r}, & (47)\\ \frac{\partial a}{\partial r}(r,\zeta) &= \sum_{i,j=1}^{N}\frac{\partial a_i}{\partial x_j}\left(x,\zeta\frac{x}{r}\right)\cdot\frac{x_i}{r}\cdot\frac{x_j}{r}, & (48) \end{aligned}$$

for all $x \in \Omega\setminus\{0\}$ and all $\zeta\in\mathbb{R}$. Consequently, Conditions (3) through (7) reduce to the following ones: There exist some constants $\kappa\in[0,1]$ and $\gamma,\Gamma\in(0,\infty)$ such that

$$\begin{aligned} a(r,0) &= 0, & (49)\\ \gamma\cdot(\kappa+|\zeta|)^{p-2} \le \frac{\partial a}{\partial\zeta}(r,\zeta) &\le \Gamma\cdot(\kappa+|\zeta|)^{p-2}, & (50)\\ \left|\frac{\partial a}{\partial r}(r,\zeta)\right| &\le \Gamma\cdot(\kappa+|\zeta|)^{p-2}\cdot|\zeta|, & (51)\\ |b(r,u)| &\le \Gamma\cdot(\kappa+|u|)^{p-2}\cdot|u|, & (52) \end{aligned}$$

for all $r\in(0,R)$, all $\zeta\in\mathbb{R}\setminus\{0\}$, and all $u\in\mathbb{R}$.

We have the following SCP for the radially symmetric solutions $u,v\in W_0^{1,p}(\Omega)$ of the differential equations (10) and (11), respectively, with radially symmetric entries. This theorem complements our main result, Theorem 2.1.

Theorem 3.3. *Let $f,g\in L^\infty(0,R)$ satisfy $0\le f\le g$ with $f\not\equiv g$ in $(0,R)$, where $0<R<\infty$. Let $1<p\le 2$, and let b satisfy* Condition (**b2**) *and also $\partial b/\partial u\in L^\infty_{\mathrm{loc}}([0,R)\times(0,\infty))$. Assume that $u,v\in W_0^{1,p}(\Omega)$ are any radially symmetric solutions of* Eqs. (10) *and* (11) *with radially symmetric entries. Then we have the SCP*

$$0\le u(r)<v(r)\ \ \text{for all } r\in[0,R) \quad\text{and}\quad v'(R)<u'(R)\le 0. \tag{53}$$

Remark 3.2. The SCP (53) for radially symmetric solutions in a ball is studied in Reichel and Walter [19, Theorem 5, p. 64]. More precisely, they consider the uniqueness question for the initial value problem

$$\begin{cases} -r^{-\alpha}(r^\alpha|u'|^{p-2}u')'-b(r,u)=f(r) & \text{for } r_0<r<r_1;\\ u(r_0)=u_0,\ u'(r_0)=u_0'. \end{cases} \tag{54}$$

Here, $\alpha \geq 0$ and $0 \leq r_0 < r_1 < \infty$ are given numbers. From their uniqueness result [19, Theorem 4, p. 57] they derive (53). Our proof of (53) below hinges upon a similar uniqueness argument as well.

Again, as in §3.1, we *prove* Theorem 3.3 by rewriting the boundary value problem (45) in the following equivalent form:

$$u_1' = \alpha(r, u_2), \quad u_2' = -\frac{N-1}{r} u_2 + \beta_f(r, u_1) \tag{55}$$

for $0 < r < R$; $u_1(R) = u_2(0) = 0$. Here, $\alpha(r, \bullet) : \mathbb{R} \to \mathbb{R}$ denotes the inverse function of $a(r, \bullet) : \mathbb{R} \to \mathbb{R}$, for each fixed $r \in [0, R]$, and

$$\beta_f(r, u_1) = -b(r, u_1) - f(r) \quad \text{ for } (r, u_1) \in [0, R] \times \mathbb{R}. \tag{56}$$

This is a system for the unknown pair of functions $(u_1, u_2) : [0, R] \to \mathbb{R}^2$. The substitutions $u_1 = u$ and $u_2 = a(r, u')$ in Eq. (45) are the same as in §3.1. Notice that, by Conditions (49) and (50), the function $\alpha(r, u_2)$ exists with

$$\frac{\partial \alpha}{\partial u_2}(r, u_2) = \Big(\frac{\partial a}{\partial \eta}(r, \alpha(r, u_2)) \Big)^{-1} \quad \text{ for all } (r, u_2) \in [0, R] \times (\mathbb{R} \setminus \{0\}).$$

In order to be able to treat System (55) conveniently in much the same way as System (39), we extend the functions $\alpha(\bullet, u_2)$, $\beta_f(\bullet, u_1)$, and the unknowns u_1, u_2 from $[0, R]$ to $[-R, R]$ as either even or odd functions, for all $r \in [0, R]$ and $w_1, w_2 \in \mathbb{R}$: $\alpha(-r, w_2) = -\alpha(r, -w_2)$, $\beta_f(-r, w_1) = \beta_f(r, w_1)$, $u_1(-r) = u_1(r)$, $u_2(-r) = -u_2(r)$. The singular coefficient in the second equation in System (55) is extended by $-(N-1)/r$ for $r \neq 0$.

Let now r_0 and r_1, respectively, with $-R < r_0 < r_1 < R$, be given as initial and terminal points for the initial value problem corresponding to System (55) extended to $r \in [-R, R]$. Because the second equation in this system contains the singular coefficient $-(N-1)/r$, we assume that $0 \notin (r_0, r_1)$ and rewrite the system in the following integral form,

$$\begin{aligned} u_1(r) &= u_1(r_0) + \int_{r_0}^{r} \alpha(\varrho, u_2(\varrho))\, d\varrho, \\ r^{N-1} u_2(r) &= r_0^{N-1} u_2(r_0) - \int_{r_0}^{r} \varrho^{N-1} [b(\varrho, u_1(\varrho)) + f(\varrho)]\, d\varrho, \end{aligned} \tag{57}$$

for all $r \in [r_0, r_1]$. Here, $(u_1(r_0), u_2(r_0)) \in \mathbb{R}^2$ are considered to be the initial data with $u_1(r_0) > 0$; if $r_0 = 0$, then we must take $u_2(0) = 0$. Again, the right-hand side of System (57) (both integrands) is locally Lipschitz continuous with respect to u_1 and u_2 for $\varrho \in [r_0, r_1)$ and $(u_1, u_2) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$. It is also *cooperative* for $\varrho \in [r_0, r_1)$ and $(u_1, u_2) \in (\mathbb{R} \setminus \{0\})^2$:

$$\frac{\partial \alpha}{\partial u_2}(\varrho, u_2) \geq 0 \quad \text{and} \quad \frac{\partial \beta_f}{\partial u_1}(\varrho, u_1) \geq 0,$$

see Smith [20, Sect. 3.1, pp. 32–34]. Consequently, we may combine the local existence and uniqueness theory with the Müller-Kamke theorem to obtain the corresponding analogue of Lemma 3.2 for System (57) in place of System (42) ([20, Sect. 3.1, pp. 32–34]).

Proof of Theorem 3.3. In analogy with our proof of Theorem 3.1, we obtain $0 \leq u \leq v$ in $(-R, R)$ and $u, v \in C^{1,\beta}([-R, R])$ for some $\beta \in (0, 1)$. Next, by Proposition 2.4 with $\Sigma = \{x \in \mathbb{R}^N : R - \delta < |x| < R\}$, precisely one of the following two mutually exclusive alternatives must be valid:

(A1) $u \equiv v$ in $(-R, -R + \delta) \cup (R - \delta, R)$, for some $\delta \in (0, R)$;

(A2) the SCP (18) holds in $(-R, -R + \delta) \cup (R - \delta, R)$, for some $\delta \in (0, R)$.

Again, we rule out the first alternative and show that the second one implies $u < v$ throughout $(-R, R)$.

Alt. (A1): We fix any $r_0 \in (-R, -R+\delta)$, set $r_1 = 0$, and apply Lemma 3.2 with $u_i(r_0) = v_i(r_0)$ for $i = 1, 2$. Thus, we obtain $v(r) \leq u(r)$ for every $r \in [r_0, 0)$. Since also $u \equiv v$ in $(-R, r_0]$ and $u \leq v$ in $[r_0, 0)$, we arrive at $u \equiv v$ throughout $(-R, 0)$. By symmetry, it follows that $u \equiv v$ throughout $(0, R)$, a contradiction to our hypothesis $f \not\equiv g$ in $(0, R)$. We have ruled out Alt. (A1).

Alt. (A2): Assume that the SCP (53) is not valid in the interval $[-R + \delta, R - \delta]$, that is, $u(r_0) = v(r_0)$ for some $r_0 \in (-R, R)$. The function $u - v : [-R, R] \to \mathbb{R}$ being nonpositive, it attains its zero maximum at the point $r_0 \in (-R, R)$. Hence, we must have also $u'(r_0) = v'(r_0)$. By symmetry, we may assume $r_0 \in [0, R)$; if $r_0 = 0$, then we have $u'(0) = v'(0) = 0$. Applying Lemma 3.2 as above, we obtain $v(r) \leq u(r)$ for every $r \in [r_0, R)$. It follows that $u \equiv v$ in $[r_0, R)$. This is a contradiction since this alternative assumes $u(r) < v(r)$ for every $r \in (R - \delta, R)$. We conclude that $u(r) < v(r)$ for every $r \in [0, R)$, and thus, (53) is valid. Theorem 3.3 is proved.

4. The Case $p > 2$ and $\partial b/\partial u < 0$. Below we present a counterexample to the strong comparison principle for the elliptic boundary value problem (8) with $p > 2$ and $\lambda \le \lambda_p$, where $-\lambda_p > 0$ is a sufficiently large constant depending upon p.

Example 4.1. Let $\Omega = \{x \in \mathbb{R}^N : |x| < 1\}$ be the open unit ball of dimension $N \ge 1$, $p > 2$ and $\lambda < 0$. Again, set $r = |x|$; so $0 \le r \le 1$. For every number θ satisfying $0 < \theta < \infty$, we define the functions

$$u_\theta(x) \stackrel{\text{def}}{=} 1 - r^\theta \qquad \text{and} \tag{58}$$

$$f_\theta(x) \stackrel{\text{def}}{=} [(p-1)(\theta-1) - 1 + N]\theta^{p-1}\, r^{(p-1)(\theta-1)-1} - \lambda(1-r^\theta)^{p-1} \tag{59}$$

of $x \in \overline{\Omega}$. It is obvious that u_θ and f_θ satisfy $u_\theta, f_\theta \in C^1(\overline{\Omega})$ with $u_\theta > 0$ in Ω and $f_\theta > 0$ in $\overline{\Omega}$, together with $\psi_p(\nabla u_\theta) \equiv |\nabla u_\theta|^{p-2}\nabla u_\theta \in [C^1(\overline{\Omega})]^N$ and

$$-\Delta_p u_\theta - \lambda u_\theta^{p-1} = f_\theta(x) \ \text{ in } \Omega; \qquad u_\theta = 0 \ \text{ on } \partial\Omega. \tag{60}$$

We observe that $u_\theta(0) = 1$, and $u_{\theta_1}(x) < u_{\theta_2}(x)$ whenever $0 < r < 1$ and $0 < \theta_1 < \theta_2 < \infty$.

We claim that, given any two numbers θ_1 and θ_2 satisfying $\frac{p}{p-2} \le \theta_1 < \theta_2 < \infty$, there exists a constant $\lambda_p \equiv \lambda_p(\theta_1, \theta_2) < 0$ such that also $f_{\theta_1}(x) < f_{\theta_2}(x)$ for $0 < r \le 1$ and every λ with $-\infty < \lambda \le \lambda_p$. Consequently, the SCP (12) is *violated* for the boundary value problem (60).

To *prove* our claim, it suffices to verify that there exists a constant $\lambda_p \equiv \lambda_p(\theta_1, \theta_2) < 0$ with the following property: For all θ and λ satisfying $\theta_1 \le \theta \le \theta_2$ and $-\infty < \lambda \le \lambda_p$, we have

$$\partial_\theta f_\theta(x) > 0 \quad \text{ for } 0 < r \le 1. \tag{61}$$

Notice that the partial derivative $\partial_\theta f_\theta \equiv \partial f_\theta/\partial\theta$ exists,

$$\begin{aligned}(p-1)^{-1}\, \partial_\theta f_\theta = {} & [p(\theta-1)+N]\theta^{p-2}\, r^{(p-1)(\theta-1)-1} \\ & + [p(\theta-1)+N-\theta]\theta^{p-1}\, r^{(p-1)(\theta-1)-1}\, \ln r \\ & + \lambda(1-r^\theta)^{p-2}\, r^\theta\, \ln r \quad \text{ for } x \in \overline{\Omega},\end{aligned} \tag{62}$$

and thus $\partial_\theta f_\theta \in C^0(\overline{\Omega})$. We can verify the inequality (61) as follows.

First, let us define the number

$$\ell_p(\theta) \stackrel{\text{def}}{=} \frac{1}{\theta} + \frac{1}{p(\theta-1)+N-\theta} > 0 \quad \text{ for } 1 < \theta < \infty.$$

Second, we consider the following two cases for $x \in \overline{\Omega}$:

(a) Case $e^{-\ell_p(\theta)} \le r \le 1$. Then

$$[p(\theta-1)+N]\theta^{p-2} + [p(\theta-1)+N-\theta]\theta^{p-1}\ \ln r \ge 0.$$

Therefore, the sum of the first and second summands on the right-hand side in Eq. (62) is zero for $r = e^{-\ell_p(\theta)}$ and positive for $e^{-\ell_p(\theta)} < r \le 1$. Since the third summand there is positive for $e^{-\ell_p(\theta)} \le r < 1$ and zero for $r = 1$, we conclude that (61) is valid whenever $e^{-\ell_p(\theta)} \le r \le 1$. So far, the spectral parameter $\lambda < 0$ is arbitrary.

(b) Case $0 < r \le e^{-\ell_p(\theta)}$. From now on, we need to restrict θ to the interval $\frac{p}{p-2} \le \theta < \infty$. Notice that this is equivalent to $(p-1)(\theta-1)-1 \ge \theta$. Then

$$\begin{aligned} &[p(\theta-1)+N-\theta]\theta^{p-1}\ r^{(p-1)(\theta-1)-1-\theta} + \lambda(1-r^\theta)^{p-2} \\ &\le [p(\theta-1)+N-\theta]\theta^{p-1} + \lambda\left(1-e^{-\theta\ell_p(\theta)}\right)^{p-2} \le 0 \end{aligned}$$

provided $\lambda \le \lambda'_p(\theta)$, where

$$-\lambda'_p(\theta) \stackrel{\text{def}}{=} \frac{p(\theta-1)+N-\theta}{\left(1-e^{-\theta\ell_p(\theta)}\right)^{p-2}}\ \theta^{p-1} > 0.$$

Therefore, the sum of the second and third summands on the right-hand side in Eq. (62) is positive for $0 < r \le e^{-\ell_p(\theta)}$. Since the first summand there is positive as well, we conclude that (61) is valid also for all $0 < r \le e^{-\ell_p(\theta)}$, provided $\lambda \le \lambda'_p(\theta)$.

As both $\ell_p(\theta)$ and $\lambda'_p(\theta)$ are continuous functions of $\theta \in \left[\frac{p}{p-2}, \infty\right)$, we conclude that there exists a constant $\lambda_p \equiv \lambda_p(\theta_1,\theta_2) < 0$ such that the inequality (61) holds for all θ and λ satisfying $\theta_1 \le \theta \le \theta_2$ and $-\infty < \lambda \le \lambda_p$. The claim is proved.

5. Discussion. If $0 < \lambda < \lambda_1$ and the functions $f \le g$ in $L^\infty(\Omega)$ have indefinite sign, then even the WCP stated in Proposition 2.3 cannot be valid. This is an easy consequence of the following remark about the nonuniqueness of a weak solution to Problem (8) in the case when $0 < \lambda < \lambda_1$ and the function $f(x)$ has indefinite sign.

Remark 5.1. For $0 < \lambda < \lambda_1$, $p \neq 2$, and the domain $\Omega = (-1,1) \subset \mathbb{R}^1$, it is possible to construct simple examples of the function $f \in L^\infty(\Omega)$ (with indefinite sign) such that Problem (8) exhibits multiple solutions. In fact, for $2 < p < \infty$, this nonuniqueness was shown in del Pino, Elgueta and Manásevich [18, Eq. (5.26), p. 12]. For $1 < p < 2$, it was shown in Fleckinger et al. [10, Example 2].

In the case of Problem (54), a comparison principle closely related to our Lemma 3.2 above is proved in Reichel and Walter [19, Theorem 3, p. 51]. This interesting result replaces the uniqueness hypothesis by imposing strict comparison inequalities on the initial data. Nevertheless, for $N = 1$ (or radially symmetric solutions in a ball, respectively), the SCP (36) (or (53)) is equivalent to the uniqueness of solution for the initial value problem (42) (or (57)) with the initial data $u_{1,0} > 0 = u_{2,0}$, for all $x, x_0 \in (-R, R)$ with $|x - x_0| < \delta$. Here, $\delta > 0$ is a sufficiently small number. This can be easily seen from Tolksdorf's version of the SCP [24, Prop. 3.3.2, p. 803] (cf. Remark 2.1(ii) above).

A. Appendix: The Divergence Theorem. Although a number of various versions of the *divergence theorem* for strongly or weakly differentiable vector fields appear in the literature, see for instance Temam [22, Chapt. I, Theorem 1.2, p. 9] and Ziemer [28, Theorem 5.8.2, p. 248], we have been unable to find the following one for merely continuous vector fields:

Lemma A.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with a C^2-boundary $\partial\Omega$. Assume that $\mathbf{a} : \overline{\Omega} \to \mathbb{R}^N$ satisfies $\mathbf{a} \in [C^0(\overline{\Omega})]^N$ and $\operatorname{div} \mathbf{a} = f \in L^1(\Omega)$ in the sense of distributions in Ω. Then we have*

$$\int_{\partial\Omega} \mathbf{a}(x) \cdot \nu(x)\, d\sigma(x) = \int_\Omega f(x)\, dx. \tag{63}$$

As usual, we denote by $\nu \equiv \nu(x_0) \in \mathbb{R}^N$ the exterior unit normal to $\partial\Omega$ at $x_0 \in \partial\Omega$, and by $d\sigma(x_0)$ the surface measure on $\partial\Omega$. Notice that the relation $\operatorname{div} \mathbf{a} = f$ with $f \in L^1(\Omega)$ means

$$-\int_\Omega \mathbf{a} \cdot \nabla\varphi\, dx = \int_\Omega f\varphi\, dx \quad \text{for every } \varphi \in C_0^1(\Omega). \tag{64}$$

Here, $C_0^1(\Omega)$ denotes the set of all functions from $C^1(\Omega)$ that have compact support contained in Ω. Furthermore, the equality (64) remains valid for every $\varphi \in W_0^{1,q}(\Omega)$, $N < q < \infty$, as $W_0^{1,q}(\Omega)$ is the closure of $C_0^1(\Omega)$ in the Sobolev space $W^{1,q}(\Omega)$.

Proof. First, let us consider $d(x) \stackrel{\text{def}}{=} \text{dist}(x, \partial\Omega)$, the distance from a point $x \in \Omega$ to the boundary $\partial\Omega$. We denote by Ω_δ the open δ-neighborhood of the boundary $\partial\Omega$ in Ω, $\Omega_\delta = \{x \in \Omega : d(x) < \delta\}$ for $\delta > 0$ small enough. Since $\partial\Omega$ is a compact manifold of class C^2, making use of [13, Lemma 14.16, p. 355] and its proof, we obtain $d \in C^2(\overline{\Omega_\delta})$, and $\overline{\Omega_\delta}$ is C^1-diffeomorphic to $\partial\Omega \times [0, \delta]$ with $x \mapsto (x, 0)$ for all $x \in \partial\Omega$. This diffeomorphism is considered between manifolds with boundary of class C^2. It can be replaced by a C^2-diffeomorphism, see Hirsch [15, Theorem 3.5, p. 57]. Observe that the restriction $\nu = -(\nabla d)|_{\partial\Omega}$ of the C^1-vector field $-\nabla d$ to $\partial\Omega$ yields the exterior unit normal ν on $\partial\Omega$; we have $|\nabla d(x_0)| = 1$ for all $x_0 \in \partial\Omega$.

Next, given any $\eta \in (0, \delta)$, define the test function

$$\varphi_\eta(x) = \begin{cases} \eta^{-1} d(x) & \text{if } x \in \Omega_\eta \cup \partial\Omega; \\ 1 & \text{if } x \in \Omega \setminus \Omega_\eta. \end{cases}$$

Hence $0 \le \varphi_\eta \le 1$ in $\overline{\Omega}$, $\varphi_\eta \in W_0^{1,q}(\Omega)$ for $N < q < \infty$, and

$$\nabla\varphi_\eta(x) = \begin{cases} \eta^{-1} \nabla d(x) & \text{if } x \in \Omega_\eta \cup \partial\Omega; \\ \mathbf{0} & \text{if } x \in \Omega \setminus \overline{\Omega_\eta}, \end{cases}$$

by Gilbarg and Trudinger [13, Theorem 7.8, p. 153]. Inserting $\varphi = \varphi_\eta$ into Eq. (64), we arrive at

$$-\eta^{-1} \int_{\Omega_\eta} \mathbf{a}(x) \cdot \nabla d(x)\, dx = -\int_{\Omega_\eta} f(x)\, (1 - \eta^{-1} d(x))\, dx + \int_\Omega f(x)\, dx \quad (65)$$

whenever $0 < \eta < \delta$. In order to compute the limit of the integral on the left-hand side in Eq. (65) as $\eta \to 0+$, we introduce the mapping $\mathbf{h} : \partial\Omega \times [0, \delta] \to \overline{\Omega_\delta}$ defined by $\mathbf{h}(x_0, t) = x_0 - t\nu(x_0)$ for $x_0 \in \partial\Omega$ and $t \in [0, \delta]$. From the proof of [13, Lemma 14.16, p. 355] we deduce that $\mathbf{h}$ is a C^1-diffeomorphism of $\partial\Omega \times [0, \delta]$ onto $\overline{\Omega_\delta}$ with the Jacobian determinant $J(x_0, t)$ satisfying $|J(x_0, t)| \to 1$ as $t \to 0+$, uniformly for $x_0 \in \partial\Omega$. Consequently, we can perform a substitution of variables in Eq. (65) followed by Fubini's theorem, thus arriving at

$$
\begin{aligned}
&-\eta^{-1}\int_0^\eta \left[\int_{\partial\Omega} \mathbf{a}(\mathbf{h}(x_0,t))\cdot\nabla d(\mathbf{h}(x_0,t))\,|J(x_0,t)|\,d\sigma(x_0)\right] dt \\
&= -\int_{\Omega_\eta} f(x)\,(1-\eta^{-1}d(x))\,dx + \int_\Omega f(x)\,dx \qquad (66)
\end{aligned}
$$

whenever $0 < \eta < \delta$. Finally, letting $\eta \to 0+$ and using the mean value theorem for continuous functions, we obtain the divergence theorem (63) as desired.

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