Differential and Integral Equations Volume 13 (4–6) April–June 2000, pp. 707–720

PERTURBATION METHOD FOR A CLASS OF NONLINEAR DIRAC EQUATIONS

H. Ounaies
Département de mathématiques, Faculté des sciences de Monastir
Route de Kairouan, 5019 Monastir, Tunisia

(Submitted by: Reza Aftabizadeh)

Abstract. This paper deals with a class of nonlinear Dirac equations. We relate their solutions to those of nonlinear Schrödinger equation by a perturbation parameter. It is demonstrated that the nondegenerate solution (ground state) of Schrödinger equation generates solutions for Dirac equation.

1. Introduction. In this paper, we deal with a class of nonlinear Dirac equations in the case of an elementary fermion, which have the following form

$$i\sum_{j=0}^{3}\gamma^j\partial_j\psi - m\psi + \gamma^0\nabla F(\psi) \;=\; 0 \;\text{ in }\; \mathbb{R}^4 \tag{1.1}$$

where $\psi : \mathbb{R}^4 \to \mathbb{C}^4$, $\partial_j\psi = \frac{\partial}{\partial x_j}\psi$, m is a positive constant, $F : \mathbb{C}^4 \to \mathbb{R}$ models a nonlinear interaction and γ^j are the 4×4 Pauli-Dirac matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \quad \text{for} \quad k = 1,2,3$$

with

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that x_0 plays here the role of time. Throughout this paper we will assume that F satisfies

$$F(\psi) = \tfrac{1}{2}(G(\overline{\psi}\psi) + H(\overline{\psi}\gamma^5\psi)) \tag{1.2}$$

Accepted for publication December 1998.

where $\overline{\psi} = \gamma^0\psi,\ \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3;\ G, H \in C^1(\mathbb{R},\mathbb{R}) \cap C^2(\mathbb{R}^*,\mathbb{R}),\ G(0) = H(0) = 0$ and their derivatives $G' = g,\ H' = h$ verify

$$\left\{ \begin{array}{l} \text{there exists } \theta,\ 0 \le \theta \le 1, \quad \text{such that} \quad \text{for} \quad a \ge 0, \\ g(at) = a^\theta g(t) \ \text{ and } \quad h(at) = a^\theta h(t) \quad \forall\, t \in I\!R \end{array} \right. \tag{1.3}$$

the above condition implies

$$\text{there exists } \theta,\ 0 \le \theta \le 1, \ \text{ such that } \left\{ \begin{array}{l} |g(t)| \le c_1|t|^\theta, |h(t)| \le c_2|t|^\theta \\ |g'(t)| \le c_3|t|^{\theta-1}, |h'(t)| \le c_4|t|^{\theta-1} \end{array} \right. \tag{1.4}$$

where $c_1 = \sup\{|g(1)|, |g(-1)|\},\ c_2 = \sup\{|h(1)|, |h(-1)|\},\ c_3 = \theta c_1$ and $c_4 = \theta c_2$. Stationary states of the nonlinear Dirac equation are considered as particle-like solutions. These solutions are in some sense solitons which propagate without changing their shape. Stationary solutions are functions of the type $\psi(x_0, x) = e^{i\omega x_0}\phi(x)$, where by x we denote $(x_1, x_2, x_3) \in \mathbb{R}^3$. Under assumption (1.2), we remark that

$$F(e^{i\omega x_0}\phi) = F(\phi) \ \forall \ x_0 \in \mathbb{R} \tag{1.5}$$

so ϕ is a solution of the following stationary nonlinear Dirac equation

$$i\sum_{k=1}^{3} \gamma^k \partial_k \phi - m\phi + \omega\gamma^0\phi + \gamma^0\nabla F(\phi) = 0 \qquad \text{in} \quad \mathbb{R}^3\,. \tag{1.6}$$

The existence of solutions for equation (1.6) has been proved by [2], [3], [4] and [5] in the particular case when $F(\phi) = \frac{1}{2}G(\overline{\phi}\phi)$. By a shooting method, they yield an infinity of localized solutions. The above model corresponds to the so-called Soler-model and has been widely studied (see [6], [8]). In [7], Maria Esteban and Eric Séré have proven, by a variational method, the existence of an infinity of solutions in a more general case for F

$$F(\phi) = \tfrac{1}{2}(|\overline{\phi}\phi|^{\alpha_1} + b|\overline{\phi}\gamma^5\phi|^{\alpha_2})$$

with $b \neq 0$ and $1 < \alpha_1, \alpha_2 < \frac{3}{2}$. In this paper, we will prove the existence of solutions for equation (1.6) by a perturbation method. We improve the result proven in [7], since they consider $0 \le \theta < \frac{1}{2}$ and in our case we take $0 \le \theta < 1$. The advantage comes from the fact that we use a perturbation method and we work in $H^1(\mathbb{R}^3, \mathbb{C}^4)$.

We proceed as follows: We denote $\sigma p = i\sum_{j=1}^{3}\sigma^j\partial_j$. In the first section and similarly to reference [9], we use a rescaling argument to transform (1.6) in a perturbed system

$$\begin{cases} (\sigma p)\chi - \varphi + g(\varphi^2)\varphi + K_1(\epsilon,\varphi,\chi) = 0 \\ (\sigma p)\varphi + 2m\chi + K_2(\epsilon,\varphi,\chi) = 0 \end{cases} \tag{1.7}$$

where φ and $\chi : \mathbb{R}^3 \to \mathbb{C}^2$. In the second section, we relate the solutions of (1.7) to those of the non-linear Schrödinger equation

$$-\frac{\triangle\varphi}{2m} + \varphi - g(\varphi^2)\varphi = 0, \quad \chi = -\frac{(\sigma p)\varphi}{2m} \tag{1.8}$$

which are separable in spherical coordinates in the form

$$\begin{pmatrix}\varphi(x)\\ \chi(x)\end{pmatrix} = \begin{pmatrix} v(r) & \begin{pmatrix}1\\0\end{pmatrix} \\ iu(r) & \begin{pmatrix}\cos\theta\\ \sin\theta\, e^{i\Phi}\end{pmatrix}\end{pmatrix} \tag{1.9}$$

where $r = \|x\|$; θ and Φ are the spherical coordinates.

A solution (φ_0,χ_0) of (1.8) which has the form (1.9), is unique under condition v real and positive; and it is called the ground state solution (see [1], [13], [14]).

The solutions of (1.7) are the vanishing points of a C^1 operator $D : \mathbb{R}\times X\times X \to Y\times Y$ where $X = H^1(\mathbb{R}^3,\mathbb{C}^2)$ and $Y = L^2(\mathbb{R}^3,\mathbb{C}^2)$. We denote by $D_{\varphi,\chi}(\epsilon,\varphi,\chi)$ the derivative of $D(\epsilon,.,.)$, and $N = KerD_{\varphi,\chi}(0,\varphi_0,\chi_0)$, where (φ_0,χ_0) is the ground state solution of (1.8). The fact that equation (1.7) is invariant under translations and rotations implies that $\dim \mathrm{N} \geq 6$. Thus, the nondegeneracy condition amounts to requiring that $\dim N = 6$ (see [15]).

The two following theorems are our main results: In a first step, we break the invariances of translations and rotations by working in the subspace $X_r \subset X\times X$, of the functions that verify (1.9). The application of the implicit function theorem yields the following result

Theorem 1.1. *Assume conditions* (1.3), (1.4), $0 \leq \theta < 1$, $h \equiv 0$ *and let* (φ_0,χ_0) *be the ground state solution of* (1.8), *then there exists* $\delta > 0$ *and a function* $\eta \in C((0,\delta),X_r)$ *such that* $\eta(0) = (\varphi_0,\chi_0)$ *and* $(\epsilon,\eta(\epsilon))$ *is a solution of* (1.7), *for* $0 \leq \epsilon < \delta$.

In a second step, we prove that near $(0, \varphi_0, \chi_0)$, $D(\epsilon, \varphi, \chi) = 0$ is equivalent to $D(\epsilon, \varphi, \chi) \in N$, and by the implicit function theorem, we give a more general result.

Theorem 1.2. *Assume the conditions* (1.3), (1.4), $0 \le \theta < 1$ *and let* (φ_0, χ_0) *be the ground state solution of* (1.8), *then there exist* $\delta > 0$, *a neighborhood* W_0 *of zero in* N, *and a function* $\eta \in C^1((0,\delta) \times W_0, N^\perp)$ *such that* $\eta(0,0) = (\varphi_0, \chi_0)$ *and* $(\epsilon, Q_1 + \eta(\epsilon, Q_1))$ *is a solution of* (1.7) *for* $Q_1 \in W_0$ *and* $0 \le \epsilon < \delta$.

2. Rescaling. The first step is to introduce the new variables (φ, χ) such that $\phi_1(x) = \alpha\varphi(\lambda x)$ and $\phi_2(x) = \beta\chi(\lambda x)$, where $\phi = (\phi_1, \phi_2)$ satisfies (1.6), and α, β, λ are constants to be chosen later. If g and h verify (1.3), it is easy to verify that (φ, χ) satisfy

$$\begin{cases} (\sigma p)\chi - \frac{\alpha}{\lambda\beta}(m-\omega)\varphi + \frac{\alpha^{2\theta+1}}{\lambda\beta} g(\varphi^2 - (\frac{\beta}{\alpha})^2\chi^2)\varphi + \frac{i}{\lambda}(\beta\alpha)^\theta h(i\varphi\chi - i\chi\varphi)\chi = 0 \\ \frac{\lambda\alpha}{\beta(m+\omega)}(\sigma p)\varphi + \chi + \frac{\alpha^{2\theta}}{m+\omega} g(\varphi^2 - (\frac{\beta}{\alpha})^2\chi^2)\chi - i\frac{\alpha(\alpha\beta)^\theta}{(m+\omega)\beta} h(i\varphi\chi - i\chi\varphi)\varphi = 0. \end{cases} \tag{2.1}$$

By adding the conditions $\alpha = \frac{\lambda\beta}{m-\omega}$, $\alpha^{2\theta+1} = \lambda\beta$, $\frac{\lambda\alpha}{\beta} = 1$, and $m - \omega \ge 0$ we obtain $\alpha^{2\theta} = m - \omega, \lambda\alpha = \beta$ and $\lambda = (m-\omega)^{\frac{1}{2}}$ This implies that $\beta = (m-\omega)^{\frac{\theta+1}{2\theta}}$.

Denoting $\epsilon = m - \omega$, (2.1) is equivalent to

$$\begin{cases} (\sigma p)\chi - \varphi + g(\varphi^2 - \epsilon\chi^2)\varphi + i\epsilon^{\frac{\theta+1}{2}} h(i(\varphi\chi - \chi\varphi))\chi = 0 \\ (\sigma p)\varphi + 2m\chi - \epsilon(1 - g(\varphi^2 - \epsilon\chi^2))\chi - i\epsilon^{\frac{\theta+1}{2}} h(i(\varphi\chi - \chi\varphi))\varphi = 0. \end{cases} \tag{2.2}$$

Putting

$$K_1(\epsilon, \varphi, \chi) = g(\varphi^2 - \epsilon\chi^2)\varphi - g(\varphi^2)\varphi + i\epsilon^{\frac{\theta+1}{2}} h(i(\varphi\chi - \chi\varphi))\chi$$

$$K_2(\epsilon, \varphi, \chi) = -\epsilon(1 - g(\varphi^2 - \epsilon\chi^2))\chi - i\epsilon^{\frac{\theta+1}{2}} h(i(\varphi\chi - \chi\varphi))\varphi,$$

then (2.2) may be written in the form

$$\begin{cases} (\sigma p)\chi - \varphi + g(\varphi^2)\varphi + K_1(\epsilon, \varphi, \chi) = 0 \\ (\sigma p)\varphi + 2m\chi + K_2(\epsilon, \varphi, \chi) = 0. \end{cases} \tag{2.3}$$

For $\epsilon = 0$, (2.3) becomes

$$\begin{cases} (\sigma p)\chi - \varphi + g(\varphi^2)\varphi = 0 \\ (\sigma p)\varphi + 2m\chi = 0 \end{cases} \tag{2.4}$$

but $\triangle = -\sigma p \circ \sigma p$ this yields the non-linear Schrödinger equation

$$\frac{\triangle\varphi}{2m} - \varphi + g(\varphi^2)\varphi = 0, \quad \chi = -\frac{(\sigma p)\varphi}{2m}\,. \tag{2.5}$$

Since we are trying to solve (2.3) near $\epsilon = 0$, we aim to obtain their solutions from the solutions of (2.5). For this we define three operators $A_i : \mathbb{R} \times X \times X \to Y$ for i= 1, 2 and $A : \mathbb{R} \times X \times X \to Y \times Y$ by

$$A_1(\epsilon, \varphi, \chi) = (\sigma p)\chi - \varphi + g(\varphi^2)\varphi + K_1(\epsilon, \varphi, \chi)$$

$$A_2(\epsilon, \varphi, \chi) = (\sigma p)\varphi + 2m\chi + K_2(\epsilon, \varphi, \chi)$$

$$A(\epsilon, \varphi, \chi) = (A_1(\epsilon, \varphi, \chi), A_2(\epsilon, \varphi, \chi)).$$

We must ensure a certain minimal regularity of the operators A_1, A_2 and A.

Lemma 2.1. *Under condition* (1.3), $A_1, A_2 \in C^1(\mathbb{R} \times X \times X, Y)$.

Proof. We begin with A_1. The same method can be applied to A_2 . If $(\epsilon, \varphi, \chi) \in \mathbb{R} \times X \times X$, then $(\sigma p)\chi - \varphi \in L^2$, $(\sigma p)\chi - \varphi$ is a linear part in Y so it is C^∞.

For the nonlinear part $g(\varphi^2)\varphi + K_1(\epsilon, \varphi, \chi)$, we have to prove first of all that it is well defined in L^2. By using (1.4), we remark that

$$g(\varphi^2)\varphi + K_1(\epsilon, \varphi, \chi) \leq C_\epsilon(|\varphi| + |\chi|)^{2\theta+1},$$

where C_ϵ is a real constant. Thus, it suffices to show that $(|\varphi| + |\chi|)^{2\theta+1} \in L^2$. This requires $0 \leq \theta \leq 1$, which is given by (1.4).

Let us prove now that the nonlinear part is C^1. By classical arguments, it is enough to show; that for $k_1, k_2 \in X$,

$$I_1(\epsilon, \varphi, \chi)k_1 = \frac{\partial}{\partial\varphi}(g(\varphi^2 - \epsilon\chi^2)\varphi + i\epsilon^{\frac{\theta+1}{2}} h(i\varphi\chi - i\chi\varphi)\chi)k_1 \ \in L^2$$
$$I_2(\epsilon, \varphi, \chi)k_2 = \frac{\partial}{\partial\chi}(g(\varphi^2 - \epsilon\chi^2)\varphi + i\epsilon^{\frac{\theta+1}{2}} h(i\varphi\chi - i\chi\varphi)\varphi)k_2 \ \in L^2.$$

We begin with I_1,

$$\begin{aligned} I_1(\epsilon, \varphi, \chi)k_1 &= g'(\varphi^2 - \epsilon\chi^2)\varphi^2 k_1 + g(\varphi^2 - \epsilon\chi^2)k_1 \\ &\quad - \epsilon^{\frac{\theta+1}{2}} h'(i\varphi\chi - i\chi\varphi)(< k_1, \chi > - < \chi, k_1 >)\chi\,. \end{aligned}$$

The condition (1.4), implies

$$\|I_1(\epsilon,\varphi,\chi)h\|_{L^2}^2 \leq d_\epsilon \int_{\mathbb{R}^3} (|\varphi|+|\chi|)^{4\theta}|k_1|^2\,dx,$$

where d_ϵ is a real constant. Let $p \geq 1$ and $q \geq 1$ such that $\frac{1}{p}+\frac{1}{q}=1$, the Hölder inequality yields

$$\|I_1(\epsilon,\varphi,\chi)h\|_{L^2}^2 \leq d_\epsilon(\int_{IR^3} (|\varphi|+|\chi|)^{4\theta p}\,dx)^{\frac{1}{p}}(\int_{IR^3} |k_1|^{2q}\,dx)^{\frac{1}{q}}.$$

We have to prove that $(|\varphi|+|\chi|)^{2\theta p} \in L^2$ and $h^q \in L^2$. If k_1, φ and $\chi \in X$ then k_1, φ and $\chi \in L^r$ for $2 \leq r \leq 6$ and so $h^q \in L^2$ for $1 \leq q \leq 3$.

For the first member, $(|\varphi|+|\chi|)^{2\theta p} \in L^2$ is equivalent to $2 \leq 4\theta p \leq 6$, this gives $\frac{1}{2}(1-\frac{1}{q}) \leq \theta \leq \frac{3}{2}(1-\frac{1}{q})$, hence, $0 \leq \theta \leq 1$. This condition is ensured by (1.4). The same argument is available for I_2. Furthermore, I_1 and I_2 are continuous; thus the proof. □

We finish this section with a result concerning the smoothness of the vanishing points of A. If $A(\epsilon,\varphi,\chi)=0$, then (ϵ,φ,χ) is called a weak solution. However, if $0 \leq \theta < 1$, (ϵ,φ,χ) is a classical solution of (2.3), in fact:

Lemma 2.2. *Let $(\epsilon,\varphi,\chi) \in \mathbb{R}\times X\times X$ satisfy $A(\epsilon,\varphi,\chi)=0$, then for $0 \leq \theta < 1$*

$$\varphi,\chi \in \bigcap_{2\leq q<+\infty} W^{2,q}$$

and (ϵ,φ,χ) is a classical solution of (2.3); whereas if $\epsilon=0$, then (φ,χ) satisfy (2.5) classically.

Before proving the above lemma, we give the following useful results

Lemma 2.3. *We define the operator $V : X\times X \to Y\times Y$, by*

$$V(\varphi,\chi) = \begin{pmatrix} \sigma p)\chi-\varphi \\ (\sigma p)\varphi+2m\chi \end{pmatrix},$$

then V is an isomorphism of $X\times X$ onto $L^2\times L^2$.

Proof. First we prove that V is one to one. Let $(\varphi,\chi) \in X\times X$ such that $V(\varphi,\chi)=0$. Then

$$\sigma p\chi=\varphi, \qquad \sigma\varphi+2m\chi=0.$$

This implies

$$\sigma p\circ\sigma p\chi+2m\chi=0.$$

We multiply by χ both sides of the above equation, therefore we obtain

$$\int_{\mathbb{R}^3} |\nabla\chi|^2 + 2m\chi^2 \, dx = 0$$

which gives $\chi = 0$ and $\varphi = 0$.

Secondly, let us prove now that for $f(f_1, f_2) \in L^2 \times L^2$, there exists $(\varphi, \chi) \in X \times X$ such that $V(\varphi, \chi) = f$. We denote by $\tilde{f}$ the Fourier transform of f and we take the Fourier transform in the equation $V(\varphi, \chi) = f$, we obtain

$$M(x)\tilde{\chi} - \tilde{\varphi} = \tilde{f}_1, \qquad M(x)\tilde{\varphi} + 2m\tilde{\chi} = \tilde{f}_2,$$

where $M(x) = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}$. We remark that $M(x)^2 = \|x\|^2 I_2$, this yields

$$\tilde{\chi} = \frac{\tilde{f}_2 + M(x)\tilde{f}_1}{\|x\|^2 + 2m}, \quad \tilde{\varphi} = \frac{-2m\tilde{f}_1 + M(x)\tilde{f}_2}{(\|x\|^2 + 1)2m}.$$

We get χ and φ by applying the inverse Fourier transform. It is easy to see that $\chi$and $\varphi \in X$. Thus the proof. □

Lemma 2.4. *Let $S : \mathbb{R} \to \mathbb{R}$ be of C^1-class and S(0)=0, $|S'(x)| \leq c|x|^{\alpha-1}$, $\alpha > 0$, then $\varphi \in W^{1,p}$ implies $S(\varphi) \in W^{1,\frac{p}{\alpha}}$* (see [11]).

We define two C^1 operators S and $D : \mathbb{R} \times X \times X \to Y \times Y$, by

$$S(\epsilon, \varphi, \chi) = \begin{pmatrix} K_1(\epsilon, \varphi, \chi) + g(\varphi^2)\varphi \\ K_2(\epsilon, \varphi, \chi) \end{pmatrix}$$

and

$$D(\epsilon, \varphi, \chi) = V(\varphi, \chi) + S(\epsilon, \varphi, \chi).$$

We can now prove Lemma 2.2.

Proof of Lemma 2.2. Let $(\varphi, \chi) \in X \times X$ be such that $A(\epsilon, \varphi, \chi) = 0$. By Lemma 2.3, this is equivalent to $(\varphi, \chi) = V^{-1}(S(\epsilon, \varphi, \chi))$. By (1.4), we have $|S(\epsilon, \varphi, \chi)| \leq k_\epsilon(|\varphi| + |\chi|)^{2\theta+1}$, where k_ϵ is a real constant. Since $\varphi, \chi \in L^2 \bigcap L^6$ and $0 \leq \theta < 1$, we obtain $S(\epsilon, \varphi, \chi) \in L^2 \bigcap L^{p_0}$ for $p_0 > 2$. According to Lemma 2.3, φ and $\chi \in W^{1,p_0} \bigcap X$, the Sobolev embedding injection implies $\varphi, \chi \in L^2 \bigcap L^{P_0} \bigcap L^{q_0}$ for $q_0 > 6$. A bootstrop argument yields $\varphi, \chi \in \bigcap_{2 \leq q < +\infty} W^{1,q}$. So by Lemma 2.4 we obtain $S(\epsilon, \varphi, \chi) \in \bigcap_{2 \leq q < +\infty} W^{1,\frac{q}{\theta}}$, this gives $S(\epsilon, \varphi, \chi) \in \bigcap_{2 \leq q < +\infty} W^{1,q}$. Coming back now to the first equation $A(\epsilon,\varphi,\chi)=0$, we conclude $\varphi,\chi\in\bigcap_{2\le q<+\infty}W^{2,q}$, hence $\varphi,\chi\in C^1(\mathbb{R}^3,\mathbb{C}^2)$, (see [10]) for $\varepsilon=0$, (φ,χ) is a classical solution of (2.4), then $\varphi,\chi\in\bigcap_{2\le q<+\infty}W^{2,q}\bigcap C^2(\mathbb{R}^3,\mathbb{C}^2)$ and consequently (φ,χ) satisfy (2.5). $\square$

3. Solutions "generated" by Schrödinger solutions. In this section, we try to ensure that a solution $\phi_0=(\varphi_0,\chi_0)$ of (2.5) in the form (1.9) with v is real and positive, can generate a local branch of solutions of (2.3). ϕ_0 is called the ground state and has an important property as we will see later. We linearize the operator D on (φ,χ) around $(0,\phi_0)$: $D_{\varphi,\chi}(0,\phi_0)(h,k)=(V_{\varphi,\chi}(\phi_0)+S_{\varphi,\chi}(0,\phi_0))(h,k)$, where

$$V_{\varphi,\chi}(\phi_0)(h,k)=\begin{pmatrix}(\sigma p)k-h\\(\sigma p)h+2mk\end{pmatrix}$$

$$S_{\varphi,\chi}(0,\phi_0)(h,k)=\begin{pmatrix}g(\varphi_0^2)h+g'(\varphi_0^2)(\varphi_0 h+\overline{\varphi_0 h})\varphi_0\\0\end{pmatrix}.$$

The simplest approach to find solutions of (2.3) near the ground state is to show that $D_{\varphi,\chi}(0,\phi_0)$ is an isomorphism; hence, the implicit function theorem can be applied. However, this approach cannot be applied since $D_{\varphi,\chi}(0,\phi_0)$ is not invertible on $X\times X$, and ϕ_0 is not an isolated point for $D(0,Q)$ in $X\times X$, where $Q=(\varphi,\chi)$. In fact, defining the translations $T_{i,t}$, for i= 1,3 by $T_{1,t}Q=Q(.+t,.,.)$, $T_{2,t}Q=Q(.,.+t,.)$, $T_{3,t}Q=Q(.,.,.+t)$ for $t\in\mathbb{R}$, we see that $D(0,T_{i,t}\phi_0)=0\quad\forall\, x\in\mathbb{R}^3$. Furthermore, $t\to T_{i,t}\phi_0$, is continuously differentiable since $\partial_i\phi_0\in X\times X$ (Lemma 2.2) and $\frac{d}{dt}T_{i,t}\phi_0=T_{i,t}\partial_i\phi_0$, hence

$$\frac{d}{dt}D(0,T_{i,t}\phi_0)=D_{\varphi,\chi}(0,T_{i,t}\phi_0)T_{i,t}\partial_i\phi_0$$

for all $t\in\mathbb{R}$ and in particular, as we already know, $\partial_i\phi_0\in\ker D_{\varphi,\chi}(0,\phi_0)$.

Remark 3.1. It is easy to note that

$$\frac{d}{dt}T_{i,t}\phi_0(0)=\partial_i\phi_0=\begin{pmatrix}\partial_i v(r) & \begin{pmatrix}1\\0\end{pmatrix}\\ -\frac{\sigma p}{2m}\partial_i\varphi_0 & \end{pmatrix}\quad\text{for}\quad 1\le i\le 3.$$

Furthermore, this symmetry caused by the invariance of translations is not the only one. There is another symmetry due to the invariance of rotations in the complex plans. In fact, denoting the elements of su(2), (the Lie algebra of SU(2)):

$$A_\theta = \begin{pmatrix} i\theta_1 & -\theta_2 + i\theta_3 \\ \theta_2 + i\theta_3 & -i\theta_1 \end{pmatrix} \quad , \quad \theta_i \in \mathbb{R} \quad \text{for} \quad 1 \le i \le 3.$$

We see that $D(0, B_\theta \phi_0) = 0$, under the condition $\theta_1^2 + \theta_2^2 + \theta_3^2 = 1$ where

$$B_\theta = \begin{pmatrix} A_\theta & 0 \\ 0 & A_\theta \end{pmatrix}$$

hence, for $\theta^1 = (0, \theta_2, \theta_3)$, $\frac{d}{d\theta_1} D(0, B_\theta \phi_0)(\theta^1) = D_{\varphi,\chi}(0, \phi_0) \frac{d}{d\theta_1} B_\theta \phi_0(\theta^1) = 0$, for $\theta^2 = (\theta_1, 0, \theta_3)$, $\frac{d}{d\theta_2} D(0, B_\theta \phi_0)(\theta^2) = D_{\varphi,\chi}(0, \phi_0) \frac{d}{d\theta_2} B_\theta \phi_0(\theta^2) = 0$, for $\theta^3 = (\theta_1, \theta_2, 0)$, $\frac{d}{d\theta_3} D(0, B_\theta \phi_0(\theta^3) = D_{\varphi,\chi}(0, \phi_0) \frac{d}{d\theta_3} B_\theta \phi_0(\theta^3) = 0$ and in particular, we have $\frac{d}{d\theta_k} B_\theta \phi_0(\theta^k) \in \ker D_{\varphi,\chi}(0, \phi_0)$ for $1 \le k \le 3$. Furthermore, $\frac{d}{d\theta_1} A_\theta \varphi_0 = \begin{pmatrix} iv \\ 0 \end{pmatrix}$, $\frac{d}{d\theta_2} A_\theta \varphi_0 = \begin{pmatrix} 0 \\ v \end{pmatrix}$ and $\frac{d}{d\theta_3} A_\theta \varphi_0 = \begin{pmatrix} 0 \\ iv \end{pmatrix}$.

Remark 3.2. We can write

$$\frac{d}{d\theta_k} B_\theta \phi_0(\theta^k) = \begin{pmatrix} \frac{d}{d\theta_k} A_\theta \varphi_0(\theta^k) \\ -\frac{(\sigma p)}{2m} \frac{d}{d\theta_k} A_\theta \varphi_0(\theta^k) \end{pmatrix}, \quad \text{for} \quad 1 \le k \le 3.$$

The symmetries will be broken and the implicit function theorem will be applied if we use functions which have the form (1.9), as we will prove in theorem (3.1). Let us try to set more information on $\ker D_{\varphi,\chi}(0, \phi_0)$. If $D_{\varphi,\chi}(0, \phi_0)(h, k) = 0$, then by using the same argument already used in lemma 2.2, we obtain

$$\begin{cases} \frac{\triangle h_1}{2m} - h_1 + g(v^2) h_1 + g'(v^2) v^2 (h_1 + \overline{h_1}) = 0 \\ \frac{\triangle h_2}{2m} - h_2 + g(v^2) h_2 = 0, \quad k = -\frac{(\sigma p) h}{2m} \end{cases} \tag{3.1}$$

where $h = (h_1, h_2) \in X$ and $k \in X$. Denoting $h_1 = h_1^r + i h_1^j$ and $h_2 = h_2^r + i h_2^j$, the above system will be

$$\begin{cases} \frac{\triangle h_1^r}{2m} - h_1^r + g(v^2) h_1^r + 2g'(v^2) v^2 h_1^r = 0 \\ \frac{\triangle h_1^j}{2m} - h_1^j + g(v^2) h_1^j = 0, \quad \frac{\triangle h_2^r}{2m} - h_2^r + g(v^2) h_2^r = 0 \\ \frac{\triangle h_2^j}{2m} - h_2^j + g(v^2) h_2^j = 0, \quad k = -\frac{(\sigma p) h}{2m}. \end{cases} \tag{3.2}$$

According to [1], the subspace of solutions for the first equation of (3.2) is spanned by $\{\partial_i v, 1 \leq i \leq 3\}$. The subspace of solutions for the second, the third and the one before the last are spanned by $\{v\}$.

Remark 3.3. The above result and Remarks 3.1, 3.2 imply $\operatorname{Ker} D_{\varphi,\chi}(0,\phi_0)$ is spanned by $(\frac{d}{d\theta_k} B_\theta \phi_0(\theta^k), \frac{d}{dt} T_{i,t}\phi_0(0))_{1\leq i,k\leq 3}$.

In the first step we solve problem (2.3) in the subspace $X_r \subset X \times X$, of the functions that verify (1.9).

A. Radial case. Denoting $L^2_r = \{(h,k) \in Y \times Y$, verifying (1.9)$\}$ and defining the operator D^1 from $\mathbb{R} \times X_r \times X_r$ to $L^2_r \times L^2_r$ as $D^1(\epsilon, Q) = D(\epsilon, Q)$ where $Q = (\varphi, \chi)$. We state our first result

Theorem 3.1. *Let the conditions* (1.3), (1.4), $0 \leq \theta < 1$ *and* $h \equiv 0$*, then there exists* $\delta > 0$ *and a function* $\eta \in C^1((0,\delta), X_r)$ *such that* $\eta(0) = \phi_0$ *and* $D^1(\epsilon, \eta(\epsilon)) = 0$ *for* $0 \leq \epsilon < \delta$.

Proof. Since $D^1(0,\phi_0) = 0$, we have to prove that $D^1_{\varphi,\chi}(0,\phi_0)$ is an isomorphism of $X_r \times X_r$ onto $L^2_r \times L^2_r$. $D^1_{\varphi,\chi}(0,\phi_0)(h,k) = 0$, then (h,k) verifies (3.2). According to [1], the unique solution of (3.2), in $X_r \times X_r$ is $h \equiv 0$ and $k \equiv 0$. So $D^1_{\varphi,\chi}(0,\phi_0)$ is one to one in $X_r \times X_r$. Furthermore, $D^1_{\varphi,\chi}(0,\phi_0)$ is a sum of a compact operator and an isomorphism. In fact, $D^1_{\varphi,\chi}(0,\phi_0)=V_{\varphi,\chi}(\phi_0) + S_{\varphi,\chi}(0,\phi_0)$; $V_{\varphi,\chi}(\phi_0)$ is an isomorphism of $X_r \times X_r$ onto $L^2_r \times L^2_r$ and $S_{\varphi,\chi}(0,\phi_0)$ is a compact operator since $v(r) = O(e^{-r^2})$ (see [13], [10]) and $g(v^2) + 2g'(v^2)v^2 \to 0$, when $|x| \to +\infty$. Hence $D^1_{\varphi,\chi}(0,\phi_0)$ is an isomorphism of $X_r \times X_r$ onto $L^2_r \times L^2_r$. The implicit function theorem yields: there exists $\delta > 0$, and a function $\eta \in C((0,\delta), X_r)$ such that $\eta(0) = \phi_0$ and $D^1(\epsilon, \eta(\epsilon)) = 0$ for $0 \leq \epsilon < \delta$. $\square$

Remark 3.4. The argument of regularity used in Lemma 2.2 implies that $(\epsilon, \eta(\epsilon))$ is a classical solution of (2.3), for $0 \leq \epsilon < \delta$.

B. Non radial case. We denote by $N = \operatorname{Ker} D_{\varphi,\chi}(0,\phi_0)$, $J=\operatorname{Im} D_{\varphi,\chi}(0,\phi_0)$, $N^\perp=\{(h,k) \in X \times X,$ such that $< (h,k), (\varphi,\chi) >_{X\times X}= 0 \ \ \forall\, (\varphi,\chi) \ \in N\}$, $M=\{(h,k) \in Y \times Y,$ such that $< (h,k), (\varphi,\chi) >_{L^2\times L^2}= 0 \ \ \forall\, (\varphi,\chi) \ \in N\}$ and $\{R^j_{(t_k,\theta)},\ 1 \leq k \leq 3,\ j = 1,2$ such that $R^1_{(t_k,\theta)} = T_{k,t}$ and $R^2_{(t_k,\theta)} = B_\theta\}$ the group of the actions that leave invariant equation (1.7) and that yields the subspace N. The later subspace is spanned by $\{E^1_k, E^2_k, 1 \leq k \leq 3\}$, where$E^1_k = \frac{d}{dt} R^1_{(t_k,\theta)}(\phi_0)(0) = \frac{d}{dt} T_{k,t}\phi_0(0)$ and$E^2_k = \frac{d}{d\theta_k} R^2_{(t_k,\theta)}(\phi_0)(\theta^k) = \frac{d}{d\theta_k} B_\theta \phi_0(\theta^k)$ (see Remark 3.3).

Lemma 3.1. $\phi_0 \in N^\perp$.

Proof. We know that $\|R^j_{(t_k,\theta)}(\phi_0)\|^2_{X\times X} = \|\phi_0\|^2_{X\times X}$, then

$$\frac{d}{d(\theta_k,t_k)}\|R^j_{(t_k,\theta)}(\phi_0)\|^2_{X\times X} = 0.$$

This yields $<\phi_0, E^j_k>_{X\times X} = 0$, for $1\leq k\leq 3$, $1\leq j\leq 2$, so $\phi_0\in N^\perp$. □

Lemma 3.2. *There is a neighborhood of* $(0,\phi_0)$ *in* $\mathbb{R}\times X\times X$, *such that the following statements are equivalent:*

i) $D(\epsilon,\varphi,\chi) = 0$

ii) $D(\epsilon,\varphi,\chi)\in N$.

Proof. Since $D_{\varphi,\chi}(0,\phi_0)$ is a Fredholm operator and $X\subset Y$, then $J = M$ and we can split $Y\times Y = M\oplus N$. The equation $D(\epsilon,\varphi,\chi)\in N$ means that for some $(\alpha_k,\beta_k)_{1\leq k\leq 3}\in\mathbb{R}^6$, we have

$$V(\varphi,\chi) + S(\epsilon,\varphi,\chi) = \sum_{k=1}^{3}(\alpha_k E^1_k + \beta_k E^2_k). \tag{3.3}$$

Let $Q = (\varphi,\chi)$ near ϕ_0 in $X\times X$, and denoting by

$$Q^1_i = \frac{\partial}{\partial t}R^1_{(t_i,\theta)}(Q)(0) = \frac{d}{dt}T_{i,t}Q(0)$$

$$Q^2_i = \frac{\partial}{\partial\theta_i}R^2_{(t_i,\theta)}(Q)(\theta^i) = \frac{d}{d\theta_i}B_\theta Q(\theta^i)$$

we can suppose that $<(Q^1_i,Q^2_i),(E^1_k,E^2_k)>_{X\times X} = 0$ for $i\neq k$. We multiply (3.3) by Q^1_i or by Q^2_i at both sides and we take the scalar product in $L^2\times L^2$.

$$<(V(\epsilon,Q)+S(\epsilon,Q)),Q^j_i>_{L^2\times L^2} = \sum_{k=1}^{3}<(\alpha_k E^1_k + \beta_k E^2_k),Q^j_i>_{L^2\times L^2}$$

for $j = 1,2$. This can be written as follows:

$$<i\gamma^0\gamma^k\partial_k Q, Q^j_i>_{L^2\times L^2} + \int_{\mathbb{R}^3}\frac{1}{2}\frac{\partial}{\partial_j}(\mathcal{A}R^j_{(t_i,\theta)}(Q))(0,\theta^i)\,dx$$
$$+\int_{\mathbb{R}^3}\frac{\partial}{\partial_j}F(R^j_{(t_i,\theta)}Q_\epsilon)(0,\theta^i)\,dx = <\alpha_i E^j_i + \beta_i E^2_i, Q^j_i>_{L^2\times L^2}$$

where $\partial_1 = \partial t$, $\partial_2 = \partial\theta_i$, $Q_\epsilon = (\varphi, \sqrt{\epsilon}\chi)$, $\mathcal{A} = \begin{pmatrix} -I_2 & 0 \\ 0 & (2m-\epsilon)I_2 \end{pmatrix}$. It is easy to see that the second and the third terms of the left side vanish, so this equation becomes

$$< i\gamma^0\gamma^k\partial_k Q, Q_i^j >_{L^2\times L^2} = < \alpha_i E_i^1 + \beta_i E_i^2, Q_i^j >_{L^2\times L^2}$$

this is equivalent to

$$\epsilon_{i,j} \begin{pmatrix} < E_{i,1}^j, \varphi_i^j >_{L^2} \\ < \chi_i^j, E_{i,2}^j >_{L^2} \end{pmatrix} = \begin{pmatrix} < \chi, (\sigma p)\varphi_i^j >_{L^2} \\ - < \chi, (\sigma p)\varphi_i^j >_{L^2} \end{pmatrix}$$

where $\epsilon_{i,1} = \alpha_i$, $\epsilon_{i,2} = \beta_i$, $E_i^1 = (E_{i,1}^1, E_{i,2}^1)$, $E_i^2 = (E_{i,1}^2, E_{i,2}^2)$. By adding the two terms on the right side, we obtain

$$\alpha_i(< E_{i,1}^1, \varphi_i^j >_{L^2} + < \chi_i^j, E_{i,2}^1 >_{L^2}) = 0$$

and $\beta_i(< E_{i,1}^2, \varphi_i^j >_{L^2} + < \chi_i^j, E_{i,2}^2 >_{L^2}) = 0$, $1 \le i \le 3$. If Q is close to ϕ_0 in $X \times X$, then $< E_{i,1}^J, \varphi_i^j >_{L^2} + < \chi_i^j, E_{i,2}^J >_{L^2}$ is strictly positive; and (α_i, β_i) must be zero for $1 \le i \le 3$. So, the desired result is achieved. □

We can now state our main result:

Theorem 3.2. *Let the conditions* (1.3), (1.4) *and* $0 \le \theta < 1$, *then there exist* $\delta > 0$, *a neighborhood* W_0 *of zero in* N, *and a function* $\eta \in C^1((0,\delta) \times W_0, N^\perp)$ *such that* $\eta(0,0) = \phi_0$ *and* $D(\epsilon, Q_1 + \eta(\epsilon, Q_1)) = 0$ *for* $0 \le \epsilon < \delta$ *and* $Q_1 \in W_0$.

Proof. We split $\mathbb{R}\times X \times X = \mathbb{R}\times(N \oplus N^\perp)$ and $Y \times Y = N \oplus J$, then D will be considered as an operator of $\mathbb{R}\times N \times N^\perp$ onto $N \times J$. Let P the orthogonal projection of $N \times J$ onto J. We define the operator $P \circ D : \mathbb{R}\times N \times N^\perp \to J$, $P\circ D$ is C^2, $\phi_0 \in N^\perp$ and $D(0,0,\phi_0) = 0$. Moreover, if we denote $Q \in \mathbb{R}\times N \times N^\perp$, $Q = (\epsilon, Q_1, Q_2)$ and $(P\circ D)_{Q_2}$ the derivative of $P \circ D(\epsilon, Q_1, .)$, then $(P \circ D)_{Q_2}(0,0,\phi_0)$ is an isomorphism of $N^\perp$ onto J, so by the implicit function theorem, the equation $P \circ D(\epsilon, Q_1, Q_2) = 0$, determines $Q_2 \in N^\perp$ in terms of the remaining variables. These are ϵ and the component (α_i, β_i) of Q_1 in N. By this, we mean $Q_1 = \sum_{k=1}^3(\alpha_k E_k^1 + \beta_k E_k^2)\ \forall\ Q_1 \in N$. So the equation $P \circ D(\epsilon, Q) = 0$ can be solved in a neighborhood of $(0,0,\phi_0)$ as follows: there exist $\delta > 0$, a neighborhood W_0 of zero in $\mathbb{R}^6$ and $\eta \in C((0,\delta) \times W_0, N^\perp)$ such that $\eta(0,0) = \phi_0$ and $P \circ D(\epsilon, \alpha_i, \beta_i, \eta(\epsilon, \alpha_i, \beta_i)) = 0$, for $0 \le \epsilon < \delta$ and

$(\alpha_i, \beta_i)_{1 \le i \le 3} \in W_0$. Apart from that, $P \circ D(\epsilon, \alpha_i, \beta_i, \eta(\epsilon, \alpha_i, \beta_i)) = 0$, means $D(\epsilon, \alpha_i, \beta_i, \eta(\epsilon, \alpha_i, \beta_i)) \in N$, so Lemma 2.3 gives the desired result. □

Acknowledgments. The author would like to thank Ivar Ekeland for his encouragements. The author would like also to thank E. Séré and M. Esteban for fruitful and stimulating discussions. He should acknowledge the hospitality he found in the Université de Paris- Dauphine.

REFERENCES

[1] M.I. Weinstein, *Modulational stability of ground states of nonlinear Schrödinger equations,* SIAM J. Math Anal., 16 (1985).

[2] M. Balabane, T. Cazenave, A. Douady, and F. Merle, *Existence of excited states for a nonlinear Dirac field,* Comm. Math. Phys., 119 (1988), 153–176.

[3] M. Balabane, T. Cazenave, and L. Vazquez, *Existence of standing waves for Dirac fields with singular nonlinearities,* Comm. Math. Phys., 133 (1990), 53–74.

[4] T. Cazenave, *On the existence of stationary states for classical nonlinear Dirac fields,* In “Hyperbolic systems and Mathematical Physics,” Textos e Notas 4, CMAF, Lisbonne, (1989).

[5] T. Cazenave and L. Vazquez, *Existence of localized solutions for a classical nonlinear Dirac fields,* Comm. Math. Phys., 105 (1986), 35–47.

[6] A.F. Rañada, *Classical nonlinear Dirac fields models of extended particles,* In “Quantum theory, groups, fields and particles,” (editor A.O. Barut) Reild, Amesterdam, Reild (1982).

[7] M. Esteban and E. Séré, *Stationary states of nonlinear Dirac equation: Variational approach,* Comm. Math. Phys., 171 (1995), 323–350.

[8] M. Soler, Phys. Rev. D1 (1970), 2766–2769.

[9] C.A. Stuart, *Bifurcation of homoclinic orbits and bifurcation from the essential spectrum,* SIAM. J. Math. Anal., 20 (1989), 1145-1171.

[10] C.A. Stuart, *A global branch of solutions to a semilinear equation on an unbounded interval,* Proc. Royal. Soc Edinbergh., 101 (1985), 273–282.

[11] H. Brézis, “Analyse Fonctionnelle,” Théorie et applications, Edition Masson . Paris 1983.

[12] D. Gilbarg and N.S. Trudinger, “Elliptic Partial Differential Equations of Second Order,” Springer-Verlag (1983).

[13] W.A. Strauss, *Existence of solitary waves in higher dimensions,* Comm. Math. Phys., 55 (1977), 149–162.

[14] C.V. Coffman, *Uniqueness of the Ground state solution for* $\triangle u - u + u^3 = 0$ *and a variational charaterisation of other solutions,* Arch. Rational Mech. Math. Anal., 46 (1972), 81–95.

[15] R.J. Magnus, *On perturbation of a translationaly invariant differential equation,* Proc. Royal. Soc. Edinburgh., 110 (1988), 1–25.