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# GLOBAL POSITIVE SOLUTION BRANCHES OF POSITONE PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

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**Abstract.** In this paper we consider a class of semilinear elliptic boundary value problems with nonlinear boundary conditions. The continuation method or the implicit function theorem is used to prove the existence of smooth branches of positive solutions. The characterization of the branches, and the uniqueness and asymptotic behavior of positive solutions are also studied by using some comparison principles with semilinear elliptic boundary value problems with linear boundary conditions.

**1. Introduction and results.** Let $D$ be a bounded domain of Euclidean space $\mathbf{R}^N$, $N \geq 2$, with $C^\infty$-boundary $\partial D$; its closure $\overline{D} = D \cup \partial D$ is an $N$-dimensional, compact $C^\infty$-manifold with boundary.

In this paper we consider the following semilinear elliptic boundary value problem:

$$\begin{cases} Au := (-\Delta + c(x))u = \lambda f(u) & \text{in } D, \\ Bu := \dfrac{\partial u}{\partial \mathbf{n}} + b(x)g(u) = 0 & \text{on } \partial D. \end{cases} \tag{$*$}_\lambda$$

Here

(1) $\Delta$ is the usual Laplacian $\sum_{i=1}^N \frac{\partial^2}{\partial {x_i}^2}$,
(2) $c$ is a real-valued $C^\infty$-function on $\overline{D}$ and $c > 0$ in $D$,
(3) $\lambda$ is a positive parameter,
(4) $f$ is a real-valued $C^1$-function on $[0, \infty)$,
(5) $b$ is a non-negative $C^\infty$-function on $\partial D$,
(6) $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ is the exterior unit normal of $\partial D$,
(7) $g$ is a non-negative $C^2$-function on $[0, \infty)$.

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The nonlinearity $f$ is assumed to be *forced*, namely

$$f(0) > 0. \tag{F.1}$$

The nonlinearity $g$ is assumed to satisfy the following *three* conditions:

$$g(0) = 0, \tag{G.1}$$

$$g'(t) > 0 \quad \text{for } t \geq 0, \tag{G.2}$$

$$g''(t) \leq 0 \quad \text{for } t > 0. \tag{G.3}$$

A function $u \in C^2(\overline{D})$ is referred to as a *solution* of $(*)_\lambda$ if it satisfies $(*)_\lambda$. Solutions $u$ which are positive everywhere in $D$ are referred to as *positive solutions*.

First we consider the case when the following further assumptions are imposed.

$$f(t) > 0 \quad \text{if } t > 0, \tag{F.2}$$

$$\left(\frac{f(t)}{g(t)}\right)' < 0 \quad \text{if } t > 0. \tag{FG.1}$$

From conditions (G.1)–(G.3) we find that $g(t)/t$ is non-increasing with respect to $t > 0$, which implies that the value $g_\infty$ defined below exists.

$$(0 \leq)\ \ g_\infty = \lim_{t \to \infty} \frac{g(t)}{t}\ \ (\leq g'(0)).$$

Since

$$\frac{f(t)}{t} = \frac{f(t)}{g(t)} \frac{g(t)}{t} \quad \text{for } t > 0,$$

condition (FG.1) ensures that $f(t)/t$ is strictly decreasing with respect to $t > 0$, which implies that the value $f_\infty$ defined below exists.

$$f_\infty = \lim_{t \to \infty} \frac{f(t)}{t}\ \ (\geq 0).$$

If $\beta$ is a non-negative number, then we denote by $\lambda_1(\beta)$ the first eigenvalue of the linear eigenvalue problem

$$\begin{cases} A\varphi = \mu\varphi & \text{in } D, \\ \frac{\partial \varphi}{\partial \mathbf{n}} + b\beta\varphi = 0 & \text{on } \partial D. \end{cases} \tag{1.1}$$

It follows from the well-known Krein–Rutman theorem (see [12]) that $\lambda_1(\beta)$ is positive and possesses only eigenfunctions of constant signs.

By using the super-sub-solution method for semilinear elliptic boundary value problems with nonlinear boundary conditions (see [2]), it is easily seen that the set of all $\lambda$ for which problem $(*)_\lambda$ has a positive solution contains the open interval $(0, \lambda_1(g_\infty)/f_\infty)$ if $f_\infty > 0$, and coincides with $(0, \infty)$ if $f_\infty = 0$.

Our purpose of this part is to give a more exact description of the set of positive solutions of $(*)_\lambda$, involving uniqueness results, by using the implicit function theorem and comparison arguments. In different classes of nonlinearities $f$ and $g$ from ours, uniqueness results for positive solutions can be found in [2, 6, 15].

The nonlinearity $g$ is said to satisfy *condition (G.4)* if $g(t) - g_\infty t$ is uniformly bounded above. Now we formulate our first main result.

**Theorem 1.** (i) *Assume that conditions* (F.1), (F.2), (G.1)–(G.3), *and* (FG.1) *are satisfied. If $f_\infty = 0$, then there exists a unique positive solution $u_\lambda$ of $(*)_\lambda$ for each $\lambda \in (0, \infty)$. More precisely, problem $(*)_\lambda$ possesses a unique $C^1$-branch $(\lambda, u_\lambda)$ of positive solutions parameterized by $\lambda \in (0, \infty)$, satisfying that*

$$\begin{aligned} &\|u_\lambda\|_{C(\overline{D})} \longrightarrow 0 \quad \text{as } \lambda \downarrow 0, \\ &\|u_\lambda\|_{C(\overline{D})} \longrightarrow \infty \quad \text{as } \lambda \to \infty. \end{aligned} \tag{1.2}$$

*Here $\|\cdot\|_{C(\overline{D})}$ is the norm of space $C(\overline{D})$ of continuous functions over $\overline{D}$.*

(ii) *In addition to conditions* (F.1), (F.2), (G.1)–(G.3), *and* (FG.1), *assume that condition* (G.4) *is satisfied. If $f_\infty > 0$, then there exists a unique positive solution $u_\lambda$ of $(*)_\lambda$ for each $\lambda \in (0, \lambda_1(g_\infty)/f_\infty)$ and no positive solution for any $\lambda \in [\lambda_1(g_\infty)/f_\infty, \infty)$. More precisely, problem $(*)_\lambda$ possesses a unique $C^1$-branch $(\lambda, u_\lambda)$ of positive solutions parameterized by $\lambda \in (0, \lambda_1(g_\infty)/f_\infty)$, satisfying that*

$$\begin{aligned} &\|u_\lambda\|_{C(\overline{D})} \longrightarrow 0 \quad \text{as } \lambda \downarrow 0, \\ &\|u_\lambda\|_{C(\overline{D})} \longrightarrow \infty \quad \text{as } \lambda \uparrow \frac{\lambda_1(g_\infty)}{f_\infty}. \end{aligned} \tag{1.3}$$

Some examples of $f$ and $g$ satisfying the assumptions of Theorem 1 are given as follows:

(1) If constants $\sigma \geq 0$ and $\varepsilon > \sigma + 1/4$, then $f(t) = \exp[t/(1 + \varepsilon t)]$ and $g(t) = t/(1 + \sigma t)$ satisfy the assumptions of assertion (i) of Theorem 1.

This case is derived from a problem arising in chemical reactor theory, in which the steady-state temperature $u$ is governed by $(*)_\lambda$, $\lambda$ represents the heat evolution rate, $1/\varepsilon$ is parallel to the activation energy of reactions, $\exp[u/(1+\varepsilon u)]$ describes the Arrhenius rate law (cf. [4]) and $u/(1+\sigma u)$ is connected with Newtonian cooling law (cf. [14]). In this case $f_\infty = 0$, and $g_\infty = 0$ if $\sigma > 0$.

(2) $f(t) = (t+2)^2/(t+1)$ and $g(t) = t(t+2)/(t+1)$ satisfy the assumptions of assertion (ii) of Theorem 1. In this case, $f_\infty = g_\infty = 1$.

Secondly we consider the case when there exists a $t > 0$ such that $f(t) = 0$. More precisely, for the nonlinearity $f$ we assume here the following *two* conditions in place of conditions (F.1), (F.2) and (FG.1). There exists a constant $t_0 > 0$ such that

$$f(t)\begin{cases} > 0 & \text{if } 0 \le t < t_0, \\ = 0 & \text{if } t = t_0, \\ < 0 & \text{if } t_0 < t, \end{cases} \tag{F.3}$$

$$\left(\frac{f(t)}{g(t)}\right)' < 0 \quad \text{if } 0 < t < t_0. \tag{FG.2}$$

Similarly as in the previous case, the super-sub-solution method ensures that the set of all $\lambda$ for which there exists a positive solution of $(*)_\lambda$ coincides with the open interval $(0, \infty)$.

Our main purpose of the second part is to study the uniqueness and asymptotic behavior of positive solutions. Now we state our second main result.

**Theorem 2.** *Suppose that conditions* (F.3), (G.1)–(G.3) *and* (FG.2) *are satisfied. Then there exists a unique positive solution $u_\lambda$ of $(*)_\lambda$ for each $\lambda \in (0, \infty)$. More precisely, problem $(*)_\lambda$ possesses a unique $C^1$-branch $(\lambda, u_\lambda)$ of positive solutions parameterized by $\lambda \in (0, \infty)$, satisfying that*

$$\begin{aligned} \|u_\lambda\|_{C(\overline{D})} &\longrightarrow 0 \quad \textit{as } \lambda \downarrow 0. \\ \|u_\lambda\|_{C(\overline{D})} &\longrightarrow t_0 \quad \textit{as } \lambda \to \infty. \end{aligned} \tag{1.4}$$

We present examples of $f$ and $g$ satisfying the assumptions of Theorem 2 as follows:

$$f(t) = 1 - t^2 \quad \text{and} \quad g(t) = \frac{t}{1+\sigma t},$$

where $\sigma$ is a non-negative constant. In this case, $t_0 = 1$.

Elliptic problems with nonlinear boundary conditions have been studied by several authors. Cohen [6] considered linear elliptic problems with nonlinear boundary conditions led to by Newtonian cooling law, in which the monotone iteration method and the maximum principle are used to discuss the existence and uniqueness of positive solutions. Amann [2] considered general semilinear elliptic problems with nonlinear boundary conditions and established existence, uniqueness, multiplicity and non-existence theorems, in which one can see that the method of super-sub-solutions is applicable for the existence. Taira [19] studied the problem of nonlinear boundary conditions derived from the Yamabe problem in Riemannian geometry, where the implicit function theorem is used to obtain the global existence of $C^1$-branches of positive solutions. For the case where nonlinearity $f$ depends on the gradient of $u$, we refer to Inkmann [9], who established existence and multiplicity theorems based on super- and sub-solutions. We also refer to Pflüger [16] for variational approaches to the problem of nonlinear boundary conditions and refer to Pao [15] for a good survey of the problem of nonlinear boundary conditions.

In the case of linear boundary conditions, Amann [1] and Cohen and Laetsch [7] studied the similar problem under the conditions in Theorems 1 and 2, respectively (see also [3, 5, 10, 13, 18]).

The rest of this paper is organized as follows. Section 2 is devoted to the existence of positive solutions. Our approach based on the implicit function theorem, whose key lemma is Lemma 2.1, leads to the existence of a unbounded $C^1$-branch of positive solutions emanating from zero, in which in order to show the unboundedness we make good use of an *a priori* estimate for the solutions of the linear elliptic boundary value problems with the Neumann condition, having only an $L^p$-norm of the boundary term (see Lemma 2.2). In Section 3 we establish some comparison principles with semilinear elliptic problems with linear boundary conditions (see Proposition 3.1), which play an important role in the characterization of the solution branch. In Section 4 we characterize the asymptotic behavior of the solution branch, where one also makes good use of Lemma 2.2.

**2. Existence of positive solution branches.** This section is devoted to the problem of the existence of positive solutions of $(*)_\lambda$. First we prove the following.

**Proposition 2.1.** *Let conditions* (F.1), (G.1) *and* (G.2) *be satisfied. Then* *a unique branch of positive solutions of* $(*)_\lambda$ *emanates from* $(\lambda, u) = (0, 0)$.

**Proof.** In order to apply the implicit function theorem to our problem, we introduce the existence and uniqueness theorem in the framework of Sobolev spaces for the linear elliptic boundary value problem

$$\begin{cases} Au = w & \text{in } D, \\ \frac{\partial u}{\partial \mathbf{n}} + bu = \varphi & \text{on } \partial D. \end{cases} \tag{2.1}$$

Let $1 < p < \infty$. If $m$ is a non-negative integer, then we denote by $W^{m,p}(D)$ the usual Sobolev space with norm $\|\cdot\|_{W^{m,p}(D)}$ and write $L^p(D) = W^{0,p}(D)$. Then the following result is well-known.

**Theorem 2.1** (cf. [20])**.** *Let* $c > 0$ *in* $D$ *and let* $b \geq 0$ *on* $\partial D$*. Then the mapping*

$$\mathcal{A}\colon\ W^{2,p}(D) \longrightarrow L^p(D) \oplus W^{1-1/p,p}(\partial D), \qquad u \longmapsto \Big(Au,\ \frac{\partial u}{\partial \mathbf{n}} + bu\Big),$$

*is an algebraic and topological isomorphism for all* $1 < p < \infty$*. Here* $W^{1-1/p,p}(\partial D)$ *is the space of the boundary values* $\varphi$ *of functions* $v \in W^{1,p}(D)$*, which is a Banach space with respect to norm*

$$\|\varphi\|_{W^{1-1/p,p}(\partial D)} = \inf\left\{\|v\|_{W^{1,p}(D)} : v \in W^{1,p}(D),\ v|_{\partial D} = \varphi\right\}.$$

For $1 < p < \infty$ we define a nonlinear map $F$ associated with $(*)_\lambda$ as

$$\begin{aligned} F\colon\ \mathbf{R} \times W^{2,p}(D) &\longrightarrow L^p(D) \oplus W^{1-1/p,p}(\partial D) \\ (\lambda,\ u) &\longmapsto (Au - \lambda f(u),\ \frac{\partial u}{\partial \mathbf{n}} + bg(u)). \end{aligned}$$

For $\lambda > 0$ we observe that $u$ is positive in $D$ and $F(\lambda, u) = 0$ if and only if $u$ is a positive solution of $(*)_\lambda$, by using the well-known regularity argument with the following existence and uniqueness theorem for the solutions of (2.1) with $b \equiv 0$ in the framework of Hölder spaces.

**Theorem 2.2 (cf.** [8]**).** *Let* $c > 0$ *in* $D$*. Then the mapping*

$$\mathcal{A}\colon\ C^{2+\theta}(\overline{D}) \longrightarrow C^{\theta}(\overline{D}) \oplus C^{1+\theta}(\partial D), \qquad u \longmapsto \Big(Au,\ \frac{\partial u}{\partial \mathbf{n}}\Big),$$

*is an algebraic and topological isomorphism for all* $0 < \theta < 1$.

For non-negative functions $u \in W^{2,p}(D)$, $p > N$, the Fréchet derivative $F_u(\lambda, u)$ of $F$ with respect to $u$ is given as

$$F_u(\lambda,\, u)\colon W^{2,p}(D) \longrightarrow L^p(D) \oplus W^{1-1/p,p}(\partial D)$$
$$\varphi \longmapsto \Big(A\varphi - \lambda f'(u)\varphi,\ \frac{\partial \varphi}{\partial \mathbf{n}} + bg'(u)\varphi\Big),$$

since $f'(u) \in C(\overline{D})$ and $g'(u) \in C^1(\overline{D})$ by virtue of Sobolev's imbedding theorem. The following eigenvalue problem is useful for considering the injectivity of $F_u(\lambda,\, u)$.

$$\begin{cases} A\varphi - \lambda f'(u)\varphi = \mu(\lambda, u)\varphi & \text{in } D, \\ \frac{\partial \varphi}{\partial \mathbf{n}} + bg'(u)\varphi = 0 & \text{on } \partial D. \end{cases} \tag{2.2}$$

Because of condition (G.1), $u \equiv 0$ is a solution of $(*)_\lambda$ when $\lambda = 0$. We apply (2.2) to the case $(\lambda, u) = (0, 0)$ as follows.

$$\begin{cases} A\varphi = \mu(0,0)\varphi & \text{in } D, \\ \frac{\partial \varphi}{\partial \mathbf{n}} + bg'(0)\varphi = 0 & \text{on } \partial D. \end{cases}$$

It follows from condition (G.2) that the first eigenvalue $\mu_1(0, 0)$ of this eigenvalue problem is positive. This means that $F_u(0, 0)$ is injective, and hence bijective by virtue of a combination of Theorem 2.1 and the index theory of Fredholm operators. The implicit function theorem ensures the unique existence of a solution branch $(\lambda, u_\lambda)$ of $(*)_\lambda$ emanating from zero.

The derivative $u'_\lambda$ of $u_\lambda$ with respect to $\lambda$ satisfies

$$\begin{cases} Au'_\lambda - \lambda f'(u_\lambda)u'_\lambda = f(u_\lambda) & \text{in } D, \\ \frac{\partial u'_\lambda}{\partial \mathbf{n}} + bg'(u_\lambda)u'_\lambda = 0 & \text{on } \partial D. \end{cases}$$

Applying to the case $(\lambda, u_\lambda) = (0, 0)$, we have

$$\begin{cases} Au'_0 = f(0) & \text{in } D, \\ \frac{\partial u'_0}{\partial \mathbf{n}} + bg'(0)u'_0 = 0 & \text{on } \partial D. \end{cases}$$

Conditions (F.1) and (G.2) lead to the assertion that $u'_0$ is positive everywhere in $D$ by the strong maximum principle and boundary point lemma (cf. [17]). This assertion means that the solution branch $(\lambda, u_\lambda)$ for $\lambda > 0$ small is a set of positive solutions. □

Next we show that the solution branch is unbounded. For this the following result is essential.

**Lemma 2.1.** *For positive solutions $u$ of $(*)_\lambda$, the first eigenvalue $\mu_1(\lambda,u)$ of* (2.2) *is given as follows.*

$$\begin{aligned}\mu_1(\lambda,\, u)\int_D g(u)\varphi_0\, dx =&\lambda\int_D \left(g'(u)f(u)-f'(u)g(u)\right)\varphi_0\, dx\\ &+\int_D \left\{c\left(g(u)-ug'(u)\right)-g''(u)|\nabla u|^2\right\}\varphi_0\, dx,\end{aligned}$$

*where $\varphi_0$ denotes the eigenfunction, which can be chosen to be positive in $D$, corresponding to the first eigenvalue $\mu_1(\lambda,u)$.*

**Proof.** Let $u$ be a positive solution of $(*)_\lambda$. Green's formula gives

$$\int_D \left(A\varphi_0\cdot g(u)-A(g(u))\cdot\varphi_0\right)\, dx=\int_{\partial D}\big(g'(u)\frac{\partial u}{\partial \mathbf{n}}\varphi_0-\frac{\partial\varphi_0}{\partial\mathbf{n}}g(u)\big)d\sigma=0,$$

where $d\sigma$ denotes the surface element on $\partial D$. By a direct computation we have

$$A(g(u))=-g''(u)|\nabla u|^2+g'(u)Au+c(g(u)-ug'(u)).$$

It follows that

$$\begin{aligned}0&=\int_D A\varphi_0\cdot g(u)-A(g(u))\cdot\varphi_0\, dx\\ &=\int_D \left(\lambda f'(u)\varphi_0+\mu_1(\lambda,u)\varphi_0\right)g(u)\\ &\quad+g''(u)|\nabla u|^2\varphi_0-\lambda g'(u)f(u)\varphi_0-c(g(u)-ug'(u))\varphi_0\, dx\\ &=\lambda\int_D \left(f'(u)g(u)-g'(u)f(u)\right)\varphi_0\, dx+\mu_1(\lambda,u)\int_D g(u)\varphi_0\, dx\\ &\quad+\int_D g''(u)|\nabla u|^2\varphi_0+c\left(ug'(u)-g(u)\right)\varphi_0\, dx,\end{aligned}$$

which implies the assertion of Lemma 2.1. □

Now let us consider our problem under the assumptions of Theorem 1. We see that the formula in Lemma 2.1 tells us that $\mu_1(\lambda,u)>0$ for any positive solution $u$ of $(*)_\lambda$, since condition (FG.1) implies

$$g'(t)f(t)-f'(t)g(t)>0 \quad \text{for } t>0$$

and conditions (G.1)–(G.3) imply

$$g(t) - tg'(t) \geq 0 \quad \text{for } t \geq 0.$$

In order to show that the branch of positive solutions obtained in Proposition 2.1, which is denoted by $\Gamma$, is extended unboundedly in $(0, \infty) \times C(\overline{D})$, we need the following *a priori* $L^p$ estimate for the solutions of problem (2.1) with $b \equiv 0$, which has only the $L^p$ norm of the boundary term (see [2, Proposition 3.3]). Let $1 < p < \infty$. We denote by $L^p(\partial D)$ the space of all measurable functions $\varphi$ on $\partial D$ which is a Banach space with respect to norm

$$\|\varphi\|_{L^p(\partial D)} := \Big( \int_{\partial D} |\varphi|^p \, d\sigma \Big)^{1/p}.$$

**Lemma 2.2.** *Let $1 < p < \infty$. Then there exists a constant $C > 0$ such that, for every $u \in C^2(\overline{D})$, we have*

$$\|u\|_{W^{1,p}(D)} \leq C\big(\|Au\|_{L^p(D)} + \Big\|\frac{\partial u}{\partial \mathbf{n}}\Big\|_{L^p(\partial D)}\big).$$

Since $\mu_1(\lambda, u) > 0$ for any positive solution $u$ of $(*)_\lambda$, the implicit function theorem extends branch $\Gamma$, make it be parameterized by $\lambda$, and moreover all the solutions $u$ for $(\lambda, u) \in \Gamma$ are strictly positive in $D$ by the strong maximum principle.

Assume to the contrary that the branch is bounded in $(0, \infty) \times C(\overline{D})$. Letting $\Lambda = \sup\{\lambda > 0 : (\lambda, u_\lambda) \in \Gamma\}$, we have $0 < \Lambda < \infty$ and we can choose a sequence $(\lambda_j, u_{\lambda_j}) \in \Gamma$ such that $\lambda_j \to \Lambda$ and $\|u_{\lambda_j}\|_{C(\overline{D})}$ is bounded. Writing $u_{\lambda_j}$ simply as $u_j$, we use Lemma 2.2 to obtain that $\|u_j\|_{W^{1,p}(D)}$ is bounded uniformly with respect to $j \geq 1$ for each $1 < p < \infty$. Theorem 2.1 with $b \equiv 0$ ensures that so is $\|u_j\|_{W^{2,p}(D)}$, from which it follows that so is $\|u_j\|_{C^{1+\alpha}(\overline{D})}$ for each $0 < \alpha < 1$ by using the Sobolev imbedding theorem. Applying Theorem 2.2, we obtain that $\|u_j\|_{C^{2+\alpha}(\overline{D})}$ is bounded uniformly with respect to $j \geq 1$ for each $0 < \alpha < 1$. By the Ascoli-Arzelà theorem, we may have a function $\hat{u} \in C^2(\overline{D})$ such that $u_j \to \hat{u}$ in $C^2(\overline{D})$. We observe that $\hat{u}$ is non-negative in $\overline{D}$ and

$$\begin{cases} A\hat{u} = \Lambda f(\hat{u}) & \text{in } D, \\ \frac{\partial \hat{u}}{\partial \mathbf{n}} + bg(\hat{u}) = 0 & \text{on } \partial D. \end{cases}$$

Since condition (F.1) implies that $u \equiv 0$ is not a solution of $(*)_\lambda$ for any $\lambda > 0$, it follows that $\hat{u} \not\equiv 0$ in $\overline{D}$. By the strong maximum principle and boundary point lemma, we have $\hat{u} > 0$ in $\overline{D}$. If the implicit function theorem is applied to the case when $(\lambda, u) = (\Lambda, \hat{u})$ then the definition of $\Lambda$ becomes contradictory.

Secondly let us consider our problem under the assumptions of Theorem 2. For our purpose we need to get the *a priori* upper bounds for positive solutions.

Let $h$ be a real-valued $C^1$-function on $\mathbf{R}$. We consider the following semilinear elliptic problem with the linear boundary condition.

$$\begin{cases} Au = h(u) & \text{in } D, \\ \frac{\partial u}{\partial \mathbf{n}} + bu = 0 & \text{on } \partial D. \end{cases} \tag{2.3}$$

A function $\psi \in C^2(\overline{D})$ is called a *super-solution* of (2.3) if

$$\begin{cases} A\psi \geq h(\psi) & \text{in } D, \\ \frac{\partial \psi}{\partial \mathbf{n}} + b\psi \geq 0 & \text{on } \partial D. \end{cases}$$

*Sub-solutions* of (2.3) are defined similarly.

The following existence theorem based on super- and sub-solutions is known (see [1, Theorem 9.4]).

**Theorem 2.3.** *Assume that problem* (2.3) *has a sub-solution* $\psi$ *and a super-solution* $\phi$ *satisfying that* $\psi \leq \phi$ *in* $\overline{D}$. *Then we have a solution* $u \in C^2(\overline{D})$ *of* (2.3) *such that* $\psi \leq u \leq \phi$ *in* $\overline{D}$.

For our purpose it is necessary to consider the Neumann problem.

$$\begin{cases} Au = \lambda f(u) & \text{in } D, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial D. \end{cases} \tag{2.4}$$

For this problem we can state the following theorem concerning existence, uniqueness and asymptotic behavior of positive solutions. For the similar investigations we refer to [3, 13].

**Theorem 2.4.** *Let condition (F.3) be satisfied. In addition, we assume that* $f(t)/t$ *is strictly decreasing with respect to* $0 < t < t_0$. *Then problem* (2.4) *has a unique positive solution* $u_\lambda$ *for each* $\lambda > 0$. *Moreover* $u_\lambda$ *is of class* $C^1$, *strictly increasing with respect to* $\lambda > 0$ *and satisfies*

$$\|u_\lambda\|_{C(\overline{D})} \longrightarrow 0 \quad \text{as } \lambda \downarrow 0,$$
$$\|u_\lambda\|_{C(\overline{D})} \longrightarrow t_0 \quad \text{as } \lambda \to \infty.$$

**Proof.** The proof and that of assertion (ii) of [21, Theorem 2] are almost the same. Hence we give only a sketch of the proof.

Condition (F.3) ensures the *a priori* bounds that, for any positive solution $u$ of (2.4), we have

$$u \leq t_0 \ \text{ in } \ \overline{D}.$$

Since $\psi \equiv 0$ is a sub-solution and $\phi \equiv t_0$ is a super-solution, Theorem 2.3 ensures that problem (2.4) has at least one positive solution for every $\lambda > 0$. Since $f(t)/t$ is strictly decreasing with respect to $0 < t < t_0$, we have the uniqueness of positive solutions (cf. [7, Theorem 4.2]). As a result of the maximum principle, the unique positive solution $u_\lambda$ is strictly increasing with respect to $\lambda > 0$, so that we can put

$$\overline{t} = \lim_{\lambda \to \infty} \|u_\lambda\|_{C(\overline{D})} \ (\leq t_0).$$

Let $0 < \varepsilon < 1$. Since $f(t)/t$ is strictly decreasing with respect to $0 < t < t_0$, we have for $\lambda > 1/t_0$

$$\begin{aligned} &A\big((t_0 - \frac{1}{\lambda})\varepsilon\varphi_1(0)\big) - \lambda f\big((t_0 - \frac{1}{\lambda})\varepsilon\varphi_1(0)\big) \\ \leq &\big(\lambda_1(0) - \lambda\,\frac{f(t_0\varepsilon)}{t_0\varepsilon}\big)\big(t_0 - \frac{1}{\lambda}\big)\varepsilon\varphi_1(0) \quad \text{in } D, \end{aligned}$$

where $\varphi_1(\beta)$ is the eigenfunction associated with the first eigenvalue $\lambda_1(\beta)$ of (1.1), normalized as $\|\varphi_1(\beta)\|_{C(\overline{D})} = 1$, which can be chosen such that $\varphi_1(\beta) > 0$ in $\overline{D}$. It is observed that $(t_0 - 1/\lambda)\varepsilon\varphi_1(0)$ is a sub-solution if

$$\lambda > \max\{\frac{1}{t_0}, \frac{\lambda_1(0)\,t_0\varepsilon}{f(t_0\varepsilon)}\}.$$

By applying Theorem 2.3 and the uniqueness result, the unique positive solution $u_\lambda$ of (2.4) satisfies that, for any $\lambda$ large enough,

$$\big(t_0 - \frac{1}{\lambda}\big)\varepsilon \leq \|u_\lambda\|_{C(\overline{D})} \leq t_0.$$

As $\lambda \to \infty$, it follows that $t_0\varepsilon \leq \overline{t} \leq t_0$. Since $\varepsilon$ is arbitrary, it follows that $\overline{t} = t_0$. The proof of Theorem 2.4 is complete. $\square$

Combining Theorems 2.3 and 2.4, we obtain the *a priori* upper bounds for positive solutions of $(*)_\lambda$.

**Lemma 2.3.** *Let conditions (F.3), (G.1), (G.2) and (FG.2) be satisfied. Then we have, for any positive solution $u$ of $(*)_\lambda$,*

$$0 \le u < t_0 \text{ in } \overline{D}.$$

**Proof.** Let $u$ be a positive solution of $(*)_\lambda$. Then we have $\partial u/\partial \mathbf{n} \le 0$ on $\partial D$. This means that $u$ is a sub-solution of (2.4). It is easy to see that a sufficiently large constant $T_0$ such that $u < T_0$ is a super-solution of (2.4). Applying Theorem 2.3, we have a solution $v$ of (2.4) such that $u \le v$ in $\overline{D}$. Since we find from Theorem 2.4 that $v < t_0$ in $\overline{D}$, the assertion of this lemma follows. The proof of Lemma 2.3 is complete. □

Now it follows from Lemma 2.3 and condition (FG.2) that the first eigenvalue $\mu_1(\lambda, u)$ is positive for any positive solution $u$ of $(*)_\lambda$. By the same argument we conclude that the solution branch is extended boundedlessly by the implicit function theorem.

**3. Comparison principles.** The purpose of this section is to establish some comparison principles which will be used in the next section. As a preliminary, we introduce the following existence and uniqueness theorem for the semilinear elliptic problem with the linear boundary condition.

$$\begin{cases} Au = \lambda f(u) & \text{in } D, \\ \frac{\partial u}{\partial \mathbf{n}} + b\beta u = 0 & \text{on } \partial D, \end{cases} \tag{$(**)_\lambda$}$$

where $\beta$ is a non-negative constant. We remark that the following assertion (i) is given in [1, Theorems 25.2 and 25.3] and assertion (ii) can be obtained in the same manner as Theorem 2.4.

**Theorem 3.1.** (i) *Let conditions (F.1) and (F.2) be satisfied. If $f(t)/t$ is strictly decreasing with respect to $t > 0$, then problem $(**)_\lambda$ has a unique positive solution $u_\lambda$ for every $0 < \lambda < \lambda_1(\beta)/f_\infty$ and no positive solution for any $\lambda \ge \lambda_1(\beta)/f_\infty$. Moreover, $u_\lambda$ is of class $C^1$, strictly increasing with respect to $0 < \lambda < \lambda_1(\beta)/f_\infty$ and satisfies*

$$\|u_\lambda\|_{C(\overline{D})} \longrightarrow 0 \quad \text{as } \lambda \downarrow 0, \tag{3.1}$$

$$\|u_\lambda\|_{C(\overline{D})} \longrightarrow \infty \quad \text{as } \lambda \uparrow \lambda_1(\beta)/f_\infty. \tag{3.2}$$

*Here $\lambda_1(\beta)$ is the first eigenvalue of* (1.1).

(ii) *Let condition (F.3) be satisfied. If $f(t)/t$ is strictly decreasing with respect to $0 < t < t_0$, then problem $(**)_\lambda$ has a unique positive solution $u_\lambda$ for each $\lambda > 0$. Moreover $u_\lambda$ is of class $C^1$, strictly increasing with respect to $\lambda > 0$ and satisfies both condition (3.1) and the condition*

$$\|u_\lambda\|_{C(\overline{D})} \longrightarrow t_0 \quad \text{as } \lambda \to \infty. \tag{3.3}$$

The next proposition is essential for the characterization of the solution branch.

**Proposition 3.1.** *Let either conditions* (F.1), (F.2), (G.1)-(G.3) *and* (FG.1) *or* (F.3), (G.1)–(G.3) *and (FG.2) be satisfied. Then we have the following two assertions.*

(i) *Assume that problem $(**)_\lambda$ with $\beta = g'(0)$ has a positive solution $v_\lambda$. If $u$ is a positive solution of $(*)_\lambda$, then we have $v_\lambda \leq u$ in $\overline{D}$.*

(ii) *Assume that there exists a positive solution $w_\lambda$ of $(**)_\lambda$ with $\beta = g_\infty$. If $u$ is a positive solution of $(*)_\lambda$, then we have $u \leq w_\lambda$ in $\overline{D}$.*

**Proof.** First of all, it follows from Theorem 3.1 that the positive solutions $v_\lambda$ and $w_\lambda$ are unique.

First Theorem 2.3 is used to prove assertion (i). We find that a positive solution $u$ of $(*)_\lambda$ is a super-solution of $(**)_\lambda$ with $\beta = g'(0)$. Indeed,

$$0 = \frac{\partial u}{\partial \mathbf{n}} + bg(u) \leq \frac{\partial u}{\partial \mathbf{n}} + bg'(0)u \quad \text{on } \partial D,$$

since $g(t) \leq g'(0)t$ for $t \geq 0$ by means of conditions (G.1)–(G.3). Meanwhile, it is easy to see that $\phi \equiv 0$ is a sub-solution of $(**)_\lambda$ with $\beta = g'(0)$, since $f(0) > 0$. By using Theorem 2.3, there exists a solution $v$ of $(**)_\lambda$ with $\beta = g'(0)$ such that

$$0 \leq v \leq u \quad \text{in } \overline{D}.$$

By the strong maximum principle we have $v > 0$ in $\overline{D}$. Under the conditions in this proposition, we obtain the uniqueness of positive solutions of $(**)_\lambda$. Hence we have $v \equiv v_\lambda$, so that

$$v_\lambda \leq u \quad \text{in } \overline{D}.$$

Next we prove assertion (ii). First we consider the case when conditions (F.1), (F.2), (G.1)–(G.3) and (FG.1) are satisfied. We see that a positive solution $u$ of $(*)_\lambda$ is a sub-solution of $(**)_\lambda$ with $\beta = g_\infty$, since

$$\frac{\partial u}{\partial \mathbf{n}} + bg_\infty u \leq \frac{\partial u}{\partial \mathbf{n}} + bg(u) = 0 \quad \text{on } \partial D.$$

Now we construct a super-solution of $(**)_\lambda$ with $\beta = g_\infty$. Here we consider only the case $f_\infty > 0$, however the construction in the case $f_\infty = 0$ can be obtained similarly. Since problem $(**)_\lambda$ with $\beta = g_\infty$ has a positive solution, the parameter $\lambda$ should be such that

$$\lambda < \frac{\lambda_1(g_\infty)}{f_\infty},$$

which implies that there exists a constant $0 < \delta < 1$ such that

$$\lambda < \delta \, \frac{\lambda_1(g_\infty)}{f_\infty}.$$

Let

$$\delta_0 = (1-\delta)\lambda_1(g_\infty) > 0.$$

Then, for the constant $\delta_0$ we can take a constant $d > 0$ such that

$$\lambda f(t) \leq (\lambda f_\infty + \delta_0)t + d \quad \text{for } t \geq 0.$$

We consider the following linear problem in order to construct the super-solutions.

$$\begin{cases} Au = (\lambda f_\infty + \delta_0)u + d & \text{in } D, \\ \frac{\partial u}{\partial \mathbf{n}} + b g_\infty u = 0 & \text{on } \partial D. \end{cases}$$

As a result of the positivity lemma [11, Theorem 2.16], this problem has exactly one positive solution $\psi$, since

$$\lambda f_\infty + \delta_0 < \delta\lambda_1(g_\infty) + (1-\delta)\lambda_1(g_\infty) = \lambda_1(g_\infty).$$

Thus the function $\psi$ is a super-solution of $(**)_\lambda$ with $\beta = g_\infty$, and moreover for any constant $C > 1$ the function $C\psi$ is a super-solution, since $f(t)/t$ is strictly decreasing with respect to $t > 0$. Since constants $C > 1$ can be taken to be so large that $u \leq C\psi$ in $\overline{D}$, Theorem 2.3 ensures that there exists a solution $w$ of $(**)_\lambda$ with $\beta = g_\infty$ such that

$$u \leq w \leq C\psi \quad \text{in } D.$$

From the uniqueness of positive solutions of $(**)_\lambda$ it follows that $w \equiv w_\lambda$. This leads to the assertion

$$u \leq w_\lambda \quad \text{in } \overline{D}.$$

Finally we verify assertion (ii) in the case when conditions (F.3), (G.1)–(G.3) and (FG.2) are satisfied. In this case, Lemma 2.3 tells us that a positive solution $u$ of $(*)_\lambda$ satisfies that $u < t_0$ in $\overline{D}$. We observe that the $u$ is a sub-solution and $\phi \equiv t_0$ is a super-solution of $(**)_\lambda$ with $\beta = g_\infty$. By applying Theorem 2.3 and the uniqueness of positive solutions, problem $(**)_\lambda$ with $\beta = g_\infty$ has a solution $w$ such that $u \le w \le t_0$ in $\overline{D}$ and $w$ coincides with $w_\lambda$. This implies that $u \le w_\lambda$ in $\overline{D}$. The proof of Proposition 3.1 is now complete. □

**4. Characterization.** We have obtained in Section 2 that there exists a constant $0 < \overline{\lambda} \le \infty$ such that problem $(*)_\lambda$ has a unique unbounded $C^1$-branch of positive solutions parameterized by $0 < \lambda < \overline{\lambda}$ which emanates from zero. This section is devoted to the characterization of the branch by using the comparison principles established in Section 3.

**End of proof of Theorem 1.** In the case that $f_\infty = 0$, it follows from Proposition 3.1 and assertions (3.2) with $\beta = g'(0)$, $g_\infty$, that the critical value $\overline{\lambda}$ is equal to $\infty$ and assertion (1.2) holds.

Assume to the contrary that the uniqueness does not hold. Then by using the continuation method just as in Section 2 and applying Proposition 3.1 and Theorem 3.1, there should be distinct two branches of positive solutions emanating from zero, which is a contradiction.

Next we consider the case that $f_\infty > 0$. The same argument as in the previous case ensures

$$\frac{\lambda_1(g_\infty)}{f_\infty} \le \overline{\lambda} \le \frac{\lambda_1(g'(0))}{f_\infty}.$$

Moreover we shall prove

$$\overline{\lambda} = \frac{\lambda_1(g_\infty)}{f_\infty}. \tag{4.1}$$

Assume to the contrary that

$$\overline{\lambda} > \frac{\lambda_1(g_\infty)}{f_\infty}. \tag{4.2}$$

For $(\lambda, u_\lambda), 0 < \lambda < \overline{\lambda}$, on the branch, we have by Green's formula

$$\int_D \big(Au_\lambda \cdot \varphi_1(g_\infty) - A\varphi_1(g_\infty)\cdot u_\lambda\big)\, dx = \int_{\partial D} \big(\frac{\partial \varphi_1(g_\infty)}{\partial \mathbf{n}}\, u_\lambda - \frac{\partial u_\lambda}{\partial \mathbf{n}}\, \varphi_1(g_\infty)\big)\, d\sigma.$$

It follows that

$$\int_D \Big(\frac{\lambda f(u_\lambda)}{u_\lambda} - \lambda_1(g_\infty)\Big) u_\lambda\, \varphi_1(g_\infty)\, dx = \int_{\partial D} b\big(g(u_\lambda) - g_\infty u_\lambda\big)\varphi_1(g_\infty)\, d\sigma,$$

Since $f(t)/t$ is strictly decreasing with respect to $t > 0$ and condition (G.4) is satisfied, a constant $C > 0$ independent of $\lambda$ can be chosen such that

$$(\lambda f_\infty - \lambda_1(g_\infty)) \int_D u_\lambda\, \varphi_1(g_\infty)\, dx \le C,$$

Assumption (4.2) ensures that a constant $C' > 0$ can be chosen independently of $\lambda$ close to $\overline{\lambda}$ such that

$$\int_D u_\lambda\, \varphi_1(g_\infty)\, dx \le C'.$$

Let $v_\lambda = u_\lambda / \|u_\lambda\|_{C(\overline{D})}$ and then we have

$$\|u_\lambda\|_{C(\overline{D})} \int_D v_\lambda\, \varphi_1(g_\infty)\, dx \le C'. \tag{4.3}$$

The function $v_\lambda$ satisfies

$$\begin{cases} A v_\lambda = \frac{\lambda f(u_\lambda)}{\|u_\lambda\|_{C(\overline{D})}} & \text{in } D, \\ \frac{\partial v_\lambda}{\partial \mathbf{n}} = -\frac{b g(u_\lambda)}{\|u_\lambda\|_{C(\overline{D})}} & \text{on } \partial D. \end{cases}$$

Applying Lemma 2.2, for every $p \in (N, \infty)$ there exists a constant $C > 0$ such that for any $\lambda$ near $\overline{\lambda}$

$$\|v_\lambda\|_{W^{1,p}(D)} \le C.$$

As a combination of the Sobolev imbedding theorem and Ascoli-Arzelà theorem, we may obtain that there exists a function $\hat{v} \in C(\overline{D})$ such that

$$v_\lambda \longrightarrow \hat{v} \quad \text{in } C(\overline{D}) \quad \text{as } \lambda \uparrow \overline{\lambda}. \tag{4.4}$$

Since $\|v_\lambda\|_{C(\overline{D})} = 1$, we see that

$$\hat{v} \not\equiv 0 \text{ and } \ge 0 \quad \text{in } \overline{D}. \tag{4.5}$$

Assertions (4.4) and (4.5) implies that

$$\int_D v_\lambda \, \varphi_1(g_\infty) \, dx \longrightarrow \int_D \hat{v} \, \varphi_1(g_\infty) \, dx > 0 \quad \text{as } \lambda \uparrow \overline{\lambda}. \tag{4.6}$$

Since $\|u_\lambda\|_{C(\overline{D})}$ tends to infinity as $\lambda \uparrow \overline{\lambda}$, assertions (4.3) and (4.6) yield a contradiction when $\lambda \uparrow \overline{\lambda}$. Hence we have assertion (4.1) and assertion (1.3) has been proved.

Finally we discuss the uniqueness and non-existence in the case that $f_\infty > 0$. The uniqueness of positive solutions holds for $\lambda \in (0, \lambda_1(g_\infty)/f_\infty)$. Otherwise, by using Proposition 3.1 and assertion (i) of Theorem 3.1, the implicit function theorem yields distinct two branches emanating from zero. This is a contradiction. Meanwhile, there is no positive solution of $(*)_\lambda$ for any $\lambda \geq \lambda_1(g_\infty)/f_\infty$. Indeed, if problem $(*)_\lambda$ has a positive solution $u_\lambda$ for some $\lambda \geq \lambda_1(g_\infty)/f_\infty$, then it follows from (1.3) that there exists $\mu \in (0, \lambda_1(g_\infty)/f_\infty)$ such that the positive solution $u_\mu$ of $(*)_\mu$ satisfies that $u_\mu \not\leq u_\lambda$ in $\overline{D}$. Since $u \equiv 0$ is a sub-solution of $(*)_\mu$, and since $u_\lambda$ is a super-solution of $(*)_\mu$, where super- and sub-solutions of $(*)_\lambda$ are defined in the same way as in (2.3), [2, Theorem 2.1] shows that there exists a solution $\tilde{u}_\mu$ of $(*)_\mu$ such that $0 \leq \tilde{u}_\mu \leq u_\lambda$ in $\overline{D}$. We note that $\tilde{u}_\mu$ is positive in $D$ by the strong maximum principle. However this contradicts the uniqueness of positive solutions of $(*)_\lambda$ for $\lambda \in (0, \lambda_1(g_\infty)/f_\infty)$, since $u_\mu \neq \tilde{u}_\mu$. The proof of Theorem 1 is now complete. □

**End of proof of Theorem 2.** By using assertions (3.3) with $\beta = g'(0)$, $g_\infty$, it follows from Proposition 3.1 that the critical value $\overline{\lambda}$ is equal to $\infty$ and we have assertion (1.4). The uniqueness can be shown just as in the proof of Theorem 1. The proof of Theorem 2 is now complete. □

## REFERENCES

[1] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), 620–709.

[2] H. Amann, *Nonlinear elliptic equations with nonlinear boundary conditions*, In "New Developments in differential equations" (Eckhaus, W., ed.), Math. Studies, Vol. 21, North–Holland, Amsterdam, 1976.

[3] A. Ambrosetti and P. Hess, *Positive solutions of asymptotically linear elliptic eigenvalue problems*, J. Math. Anal. Appl. **73** (1980), 411–422.

[4] T. Boddington, P. Gray, and G.C. Wake, *Criteria for thermal explosions with and without reactant consumption*, Proc. R. Soc. London A. **357** (1977), 403–422.

[5] D.S. Cohen, *Positive solutions of a class of nonlinear eigenvalue problems*, J. Math. Mech. **17** (1967), 209–215.

[6] D.S. Cohen, *Generalized radiation cooling of a convex solid*, J. Math. Anal. Appl. **35** (1971), 503–511.

[7] D.S. Cohen and T.W. Laetsch, *Nonlinear boundary value problems suggested by chemical reactor theory*, J. Differential Equations **7** (1970), 217–226.

[8] D. Gilbarg and N.S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, New York Berlin Heidelberg Tokyo, 1983.

[9] F. Inkmann, *Existence and multiplicity theorems for semilinear elliptic equations with nonlinear boundary conditions*, Indiana Univ. Math. J. **31** (1982), 213–221.

[10] H.B. Keller and D.S. Cohen, *Some positone problems suggested by nonlinear heat generation*, J. Math. Mech. **16** (1967), 1361–1376.

[11] M.A. Krasnosel'skii, Positive solutions of operator equations, P. Noordhoff, Groningen, 1964.

[12] M.G. Krein and M.A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Amer. Math. Soc. Transl. **10** (1962), 199–325.

[13] P.L. Lions, *On the existence of positive solutions of semilinear elliptic equations*, SIAM Rev. **24** (1982), 441–467.

[14] W.R. Mann and F. Wolf, *Heat transfer between solids and gases under nonlinear boundary conditions*, Quart. Appl. Math. **9** (1951), 163–184.

[15] C.V. Pao, Nonlinear parabolic and elliptic equations, Plenum, New York, London, 1992.

[16] K. Pflüger, *On indefinite nonlinear Neumann problems*, In "The Conference Proceedings of the ISAAC-Conference, 97" (to appear).

[17] M.H. Protter and H.F. Weinberger, Maximum principles in differential equations, Prentice - Hall, Englewood Cliffs, New Jersey, 1967.

[18] R.B. Simpson and D.S. Cohen, *Positive solutions of nonlinear elliptic eigenvalue problems*, J. Math. Mech. **19** (1970), 895–910.

[19] K. Taira, *The Yamabe problem and nonlinear elliptic boundary value problems*, J. Differential Equations **122** (1995), 316–372.

[20] M. Taylor, Pseudodifferential operators, Princeton Univ. Press, Princeton, 1981.

[21] K. Umezu, *Positive solutions of a forced nonlinear elliptic boundary value problem*, J. Math. Soc. Japan **51** (1999) (to appear).