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EIGENVALUES OF SINGULAR BOUNDARY VALUE PROBLEMS AND EXISTENCE RESULTS FOR POSITIVE RADIAL SOLUTIONS OF SEMILINEAR ELLIPTIC PROBLEMS IN EXTERIOR DOMAINS

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1. Introduction. In this paper, we consider the nonexistence, existence and multiplicity of positive radial solutions for semilinear elliptic problems of the form

$$\triangle u + \mu g(|x|) f(u(x)) = 0, \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\Omega = \{x \in \mathbf{R}^n : |x| > r_o\}$, $r_o > 0$, $n \geq 3$ and μ is a positive real parameter.

We introduce some terminology to facilitate the statement of propositions. We say that given problem (P) has *Prop A for solutions* if there exists $\mu_f > 0$ such that (P) has at least two solutions, at least one solution or none according to $0 < \mu < \mu_f$, $\mu = \mu_f$ or $\mu > \mu_f$, (P) has *Prop B for solutions* if there exists $\mu_f > 0$ such that (P) has at least one solution or none according to $0 < \mu < \mu_f$ or $\mu > \mu_f$, and we say that (P) has *Prop C for solutions* if (P) has a solution for all $\mu > 0$.

By virtue of the strong maximum principle and conditions $(H_1) \sim (H_2')$ below, solutions of problems that we consider throughout this paper are positive on the interior of their domains.

For Ω an annulus, problem (1.1),(1.2) has been studied by Bandle, Coffman and Marcus [1], Garazia [5], Lin [10], Santanilla [14], Nagasaki and Suzuki [11] and Pacard [13].

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Among them, Lin considered the problem when $g \equiv 1$ and proved that the problem has Prop A for positive radial solutions if $f > 0$ on $[0, \infty)$ and $f_\infty \equiv \lim_{u\to\infty} f(u)/u = \infty$, and the problem has Prop C for positive radial solutions if $f(0) = 0$, $f_0 \equiv \lim_{u\to 0^+} f(u)/u = 0$, and $f_\infty = \infty$.

Now let us give some conditions on g and f for precise description.

- (H_1) $g : [r_o, \infty) \to (0, \infty)$ is continuous and $\int_{r_o}^{\infty} rg(r)dr < \infty$.
- (H_2) $f : [0, \infty) \to [0, \infty)$ is continuous, strictly increasing and $f(0) = 0$.
- (H_2') $f : [0, \infty) \to (0, \infty)$ is continuous and nondecreasing.
- (H_3) $f_\infty \equiv \lim_{u\to\infty} \frac{f(u)}{u} = \infty$.

If Ω is an exterior domain, works related to problem (1.1), (1.2) include Noussair and Swanson [12], Bandle and Marcus [2] and Santanilla [15].

In particular, when g satisfies (H_1) and $f(u) = e^u$, Ha and Lee [7] proved that problem (1.1), (1.2) has Prop A for positive radial solutions and the solutions u satisfy

$$u(x) \to 0 \quad \text{as} \quad |x| \to \infty, \tag{1.3}$$

and Lee [9] proved that the problem has Prop C for positive radial solutions satisfying (1.3) if g satisfies (H_1), $f_0 = 0$ and $f_\infty = \infty$.

In the present work, we obtain two results for problem (1.1), (1.2). First, assume (H_1), (H_2) and (H_3), then for each $b > 0$, the problem has a weakened type of Prop A, say *W-Prop A* for positive radial solutions, namely there exist $0 < \mu_0 \le \mu_f$ such that the problem has at least two radial solutions, at least one radial solution or none according to $0 < \mu < \mu_0$, $\mu_0 \le \mu \le \mu_f$ or $\mu > \mu_f$, and the solutions u satisfy

$$u(x) \to b \quad \text{as } |x| \to \infty. \tag{1.4}$$

Second, assume (H_1), (H_2') and (H_3), then the problem has Prop A for positive radial solutions satisfying (1.3).

Since we are looking for radial solutions, we may reduce problem (1.1), (1.2), via suitable transformations to the following problems of ordinary differential equations

$$u''(t) + \mu q(t) f(u(t)) = 0, \quad 0 < t < 1 \tag{1.5}$$

$$u(0) = 0 = u(1), \quad \text{or} \tag{1.6}$$

$$u(0) = 0,\ u(1) = b, \tag{1.7}$$

where $b > 0$ and q is singular at 1.

Problem (1.5),(1.6) itself also has some references, for example, Choi [3] proved that the problem has Prop B for positive solutions when $f(u) = e^u$, and q is singular at 0 with certain growth restriction. Wong [16] and Dalmasso [4] gave more general conditions on f to prove that the problem has Prop B. Ha and Lee [7] generalized Choi's result that the problem has Prop A for positive solutions if $f(u) = e^u$, q is singular at 0 with $\int_0^1 sq(s)ds < \infty$. Zhang [17] proved that the problem has Prop C for positive solutions if $f(u) = u^p,\ 0 < p < 1$ and q is singular at 0 and 1 with $\int_0^1 s(1-s)q(s)ds < \infty$. And Lee [9] proved that the problem has Prop C for positive solutions if $f(u) = u^p,\ p > 1$ and q satisfies the same as in Zhang.

As we indicated in the introductory comment for problem (1.1), (1.2), we shall prove that problem (1.5), (1.6) has Prop A if f satisfies (H_2'), (H_3) and q is singular at 0 and/or 1 with a suitable integrability condition, and problem (1.5), (1.7) has W-Prop A if f satisfies (H_2), (H_3) and q satisfies the same condition. These extend the results of Choi, Wong and Ha and Lee, and provide results we expect for problem (1.1), (1.2). Our techniques of proofs for problem (1.5), (1.6) or (1.5), (1.7) mainly use the method of upper and lower solutions and the fixed point index theory.

2. Preliminaries. Let us consider

$$\triangle u + \mu g(|x|)f(u(x)) = 0, \qquad \text{in } |x| > r_o \tag{1.1}$$

$$u = 0, \qquad \text{if } |x| = r_o, \tag{2.1}$$

$$u \to 0, \qquad \text{as } |x| \to \infty, \tag{2.2}$$

$$u \to b > 0, \qquad \text{as } |x| \to \infty, \tag{2.3}$$

where $r_o > 0$ and $n \geq 3$.

We are concerned with radial solutions, thus for the radial variable $r = |x|$, we write (1.1)$\sim$(2.3) as

$$u''(r) + \frac{n-1}{r}u'(r) + \mu g(r)f(u(r)) = 0,$$
$$u(r_o) = 0,\ u(r) \to 0 \qquad \text{as } r \to \infty$$
$$u(r_o) = 0,\ u(r) \to b > 0 \qquad \text{as } r \to \infty.$$

Setting $s = r^{2-n},\ v(s) = u(r)$, we write this problem as

$$v''(s) + \frac{\mu}{(n-2)^2} s^{\frac{-2(n-1)}{n-2}} g(s^{\frac{-1}{n-2}})f(v(s)) = 0,$$
$$u(r_o^{2-n}) = 0 = u(0)$$
$$u(r_o^{2-n}) = 0,\ u(0) = b.$$

Let $t = (r_o^{2-n} - s)/r_o^{2-n},\ z(t) = v(s)$, then we can rewrite the problem as

$$\begin{aligned} &z''(t) + \mu q(t) f(z(t)) = 0,\\ &z(0) = 0 = z(1)\\ &z(0) = 0,\ z(1) = b, \end{aligned}$$

where $q(t) = \frac{r_o^2}{(n-2)^2}(1-t)^{\frac{-2(n-1)}{n-2}} g\big(r_o(1-t)^{\frac{-1}{n-2}}\big)$. Thus by (H_1), $q : [0,1) \to (0,\infty)$ is continuous and singular at 1 satisfying $\int_0^1 (1-s)q(s)ds < \infty$, and problem (1.1), (2.1), (2.2) and (1.1), (2.1), (2.3) with condition (H_1) are reduced to problem (1.5), (1.6) and (1.5), (1.7) respectively with q and conditions on q described above.

We now present a theorem on upper and lower solutions for the singular problems we are dealing with. Consider the problem

$$\begin{aligned} &u''(t) + F(t, u(t)) = 0,\\ &u(0) = a,\ u(1) = b, \end{aligned} \tag{2.4}$$

where $F : D \to \mathbf{R}$ is a continuous function and $D \subset (0,1) \times \mathbf{R}$. A solution $u(\cdot)$ of (2.4) means a function $u \in C([0,1],\mathbf{R}) \cap C^2((0,1),\mathbf{R})$ such that $(t, u(t)) \in D$ for all $t \in (0,1)$ and $u''(t) + F(t, u(t)) = 0$ for all $t \in (0,1)$ with $u(0) = a$ and $u(1) = b$.

Definition 1. $\alpha \in C([0,1],\mathbf{R}) \cap C^2((0,1),\mathbf{R})$ is called a *lower solution* of (2.4) if $(t, \alpha(t)) \in D$ for all $t \in (0,1)$ and

$$\begin{aligned} &\alpha''(t) + F(t, \alpha(t)) \ge 0,\ t \in (0,1)\\ &\alpha(0) \le a,\ \alpha(1) \le b. \end{aligned}$$

We also define an *upper solution* $\beta \in C([0,1],\mathbf{R}) \cap C^2((0,1),\mathbf{R})$ if β satisfies the reverse of the above inequalities.

If α and $\beta \in C([0,1],\mathbf{R})$ are such that $\alpha(t) \le \beta(t)$, for all $t \in [0,1]$, we define the set $D_\alpha^\beta = \{(t,x) \in (0,1) \times \mathbf{R} : \alpha(t) \le x \le \beta(t)\}$. The following is a fundamental theorem of the method of upper and lower solutions for problem (2.4) due to Habets and Zanolin [8].

Theorem 0. *Let α and β be a lower and an upper solution for* (2.4) *such that*

(a_1) $\alpha(t) \le \beta(t)$ *for all* $t \in [0,1]$

(a_2) $D_\alpha^\beta \subset D$.

Assume also that there is a function $h \in C((0,1),(0,\infty))$ *such that*

(a_3) $|F(t,x)| \le h(t)$, *for all* $(t,x) \in D_\alpha^\beta$ *and*

(a_4) $\int_0^1 s(1-s)h(s)ds < \infty$.

Then (2.4) *has at least one solution* u *such that*

$$\alpha(t) \le u(t) \le \beta(t), \quad \textit{for all } t \in [0,1].$$

Remark. It is not hard to see that if we assume, instead of (a_4), the condition $\int_0^1 sh(s)ds < \infty$, then the solution u, we find, belongs to $C^1((0,1])$. Similarly, if $\int_0^1 (1-s)h(s)ds < \infty$, then $u \in C^1([0,1))$.

3. Existence. In the remaining sections, we consider the eigenvalues of problem (1.5),(1.6) or (1.5),(1.7). Since the arguments are generally the same for arbitrary nonnegative boundary values in the problems, in what follows, we consider

$$u''(t) + \mu q(t) f(u(t)) = 0, \tag{1.5}$$

$$u(0) = a \ge 0, \ u(1) = b \ge 0 \tag{3.1}$$

$$u(0) = a \ge 0, \ u(1) = b > a. \tag{3.2}$$

We assume the following conditions:

(b_1) $q : (0,1) \to (0,\infty)$ is continuous and $\int_0^1 s(1-s)q(s)ds < \infty$.

(b_2) $f : [a,\infty) \to [0,\infty)$ is continuous, $f(0) = 0$ and $f(u) > 0$ for $u > 0$.

(b_2') $f : [a,\infty) \to (0,\infty)$ is continuous.

(b_3) $I_b = \sup_{c \in (b,\infty)} \int_0^c \frac{du}{\sqrt{F(c)-F(u)}} < \infty$, where $F(u) = \int_0^u f(s)ds$.

Remark. We assume, for convenience, $b \ge a$ in (3.1). However, in general, it is enough to assume $b \ge 0$.

The theorem for existence is:

Theorem 1. *Assume* (b_1), (b_2) *and* (b_3), *then problem* (1.5), (3.2) *has Prop B for positive solutions. And assume* (b_1), (b_2') *and* (b_3), *then problem* (1.5), (3.1) *also has Prop B for positive solutions.*

We need a lemma to prove Theorem 1. Let us consider

$$u''(t) + kf(u(t)) = 0, \tag{3.3}$$

$$u(0) = A \ge 0, \ u(1) = B \ge 0, \tag{3.4}$$

$$u(0) = A \ge 0, \ u(1) = B > 0, \tag{3.5}$$

where k is a positive real parameter.

Lemma 1. *If f satisfies (b_2) and (b_3), then problem* $(3.3), (3.5)$ *has no solution for $k \geq 2{I_B}^2+1$. Similarly, if f satisfies (b_2') and (b_3), then problem* $(3.3), (3.4)$ *has no solution for $k \geq 2{I_B}^2+1$.*

We notice that the condition (b_3) for problem (3.3), (3.4) or (3.3), (3.5) is precisely given as

$$I_B = \sup_{c\in(B,\infty)} \int_0^c \frac{du}{\sqrt{F(c)-F(u)}} < \infty,$$

where B is in (3.4) or (3.5).

Proof. Let $k \geq 2I_B^2+1$ and suppose that $(3.3), (3.5)$ has a solution $u(t)$, then by the maximum principle, u is convex up and attains its maximum at only one point in $[0,1]$. Let u attain its maximum at $t_o \in (0,1)$, so let $u_o = u(t_o) = \max_{t\in[0,1]} u(t)$. Multiplying both sides of (3.3) by $2u'$ and integrating from t_o to t, we get

$$u'^2(t) = -2k\int_{t_o}^t f(u(s))u'(s)ds = 2k[F(u_o)-F(u(t))].$$

For $t \leq t_o$, $u'(t) \geq 0$, thus $u'(t) = \sqrt{2k[F(u_o)-F(u(t))]}$. Integrating from 0 to t_o,

$$t_o = \int_A^{u_o} \frac{du}{\sqrt{2k[F(u_o)-F(u)]}}.$$

Similarly, $u'(t) \leq 0$, for $t \geq t_o$ and we get

$$1-t_o = \int_B^{u_o} \frac{du}{\sqrt{2k[F(u_o)-F(u)]}}.$$

Adding two equations

$$\begin{aligned}\sqrt{2k} &= \int_A^{u_o} \frac{du}{\sqrt{F(u_o)-F(u)}} + \int_B^{u_o} \frac{du}{\sqrt{F(u_o)-F(u)}} \\ &\leq 2\int_0^{u_o} \frac{du}{\sqrt{F(u_o)-F(u)}}. \end{aligned} \tag{3.6}$$

Thus $k \leq 2I_B^2$. This contradicts to $k \geq 2I_B^2+1$. We also get contradictions in the similar fashion, when u attains its maximum at $t=1$. The proof for (3.3),(3.4) follows the same way and the proof is complete. □

Now we prove Theorem 1 for problem (1.5), (3.2).

Proof of Theorem 1. The condition (b_1) and the continuity of f imply (a_3) and (a_4) in Theorem 0, thus it suffices to find suitable upper and lower solutions satisfying (a_1) for the existence of solutions of (1.5), (3.2).

Consider

$$u''(t) + q(t) = 0, \qquad u(0) = a,\ u(1) = b.$$

It is easy to see that the solution $\beta(t)$ is given by

$$\beta(t) = a + (b-a)t + \int_0^1 G(t,s)q(s)ds,$$

where $G(t,s)$ is the Green function corresponding to the problem explicitly written by

$$G(t,s) = \begin{cases} s(1-t) & \text{for } 0 \le s \le t \\ t(1-s) & \text{for } t \le s \le 1. \end{cases}$$

Let $M_\beta = \max_{t\in[0,1]} f(\beta(t))$, then $M_\beta > 0$ and

$$\beta''(t) + \mu q(t) f(\beta(t)) = q(t)(\mu f(\beta(t)) - 1) \le 0,$$

for $\mu \le \frac{1}{M_\beta}$. This shows that $\beta(t)$ is an upper solution of (1.5), (3.2). On the other hand, $\alpha(t)$, the straight line connecting $(0,a)$ and $(1,b)$ is obviously a lower solution of (1.5), (3.2) and $\alpha(t) \le \beta(t)$ for all $t \in [0,1]$. Thus by Theorem 0, (1.5),(3.2) has a positive solution for $0 < \mu \le \frac{1}{M_\beta}$.

Let $\mu_1 > 0$ be fixed and assume that (1.5), (3.2) has a positive solution u_1 for μ_1, then for all $\mu \in (0,\mu_1)$, (1.5), (3.2) also has a solution for μ. Indeed, u_1 is an upper solution and $\alpha(t)$, the straight line connecting $(0,a)$ and $(1,b)$ is a lower solution smaller than u_1.

Let $\mu_f = \sup\{\mu > 0 : (1.5), (3.2) \text{has a positive solution for } \mu\}$, then $\mu_f \ge \frac{1}{M_\beta}$. We claim $\mu_f < \infty$. Suppose not, then we may choose a sequence of parameters (μ_n) with $\mu_n \to \infty$ such that for each n, (1.5), (3.2) has a positive solution u_n. Consider the following equation on the interval $[\frac{1}{4}, \frac{3}{4}]$,

$$\begin{aligned} &y''(t) + \mu_n q_o f(y(t)) = 0, \quad t \in [\tfrac{1}{4}, \tfrac{3}{4}] \\ &y(\tfrac{1}{4}) = u_n(\tfrac{1}{4}),\ y(\tfrac{3}{4}) = u_n(\tfrac{3}{4}), \end{aligned} \tag{3.7}$$

where $q_o = \min_{\frac{1}{4}\le t\le\frac{3}{4}} q(t)$. Then u_n is an upper solution and $\alpha_n(t)$ the straight line connecting $\left(\frac{1}{4}, u_n(\frac{1}{4})\right)$ and $\left(\frac{3}{4}, u_n(\frac{3}{4})\right)$ is a lower solution of (3.7).

Thus for each n, (3.7) has a positive solution. Since $\lim_{n\to\infty}\mu_n = \infty$, the above conclusion contradicts to Lemma 1 and the proof is done. $\square$

We may prove the theorem for problem (1.5),(3.1) analogously. In fact, problem (1.5),(3.1) and (1.5),(3.2) are similar in the method of upper and lower solutions point of view, since the problems have neither the trivial solution nor straight lines connecting their boundaries as solutions.

4. Multiplicity. Under the assumptions of Theorem 1, we know that there exists a positive real number μ_f such that (1.5), (3.1) ((1.5), (3.2)) has at least one positive solution for $\mu \in (0, \mu_f)$ and none for $\mu > \mu_f$. In this section, the existence of the second positive solution for $\mu \in (0, \mu_f)$ will be proved, under additional conditions on f, by using fixed point index arguments. The first step in this direction is to prove *a priori* boundedness of possible solutions for (1.5), (3.1) or (1.5), (3.2). Let

$$I(c) = \int_0^c \frac{du}{\sqrt{F(c) - F(u)}}, \quad \text{and } F(u) = \int_0^u f(s)ds.$$

Then the following lemma is crucial for the *a priori* estimate.

Lemma 2. *Assume* (H_2) *and* (H_3). *Then for given* $b > 0$,

$$I(c) > 0,\ \forall c \geq b, \quad I_b = \sup_{c\in(b,\infty)} I(c) < \infty \quad \textit{and} \quad \lim_{c\to\infty} I(c) = 0.$$

Similarly, assume (H_2') *and* (H_3). *Then for given* $b \geq 0$,

$$I(c) > 0,\ \forall c > b, \quad I_b = \sup_{c\in(b,\infty)} I(c) < \infty, \quad \textit{and} \quad \lim_{c\to\infty} I(c) = 0.$$

Proof. Assume (H_2) and (H_3).

$$I(c) = \int_0^c \frac{du}{\sqrt{F(c) - F(u)}} = \int_0^1 \frac{dx}{\sqrt{\frac{F(c)-F(cx)}{c^2}}}, \quad \text{taking } u = cx.$$

Since f is increasing,

$$\frac{F(c) - F(cx)}{c^2} = \int_x^1 \frac{f(cs)}{c} ds \leq \frac{f(c)}{c}(1 - x).$$

Thus

$$I(c) \geq \sqrt{\frac{c}{f(c)}} \int_0^1 \frac{dx}{\sqrt{1-x}} = 2\sqrt{\frac{c}{f(c)}}. \tag{4.1}$$

For given $b > 0$ and $c > b$,

$$I(c) = \int_0^b \frac{du}{\sqrt{F(c)-F(u)}} + \int_b^c \frac{du}{\sqrt{F(c)-F(u)}}$$

and for $u \in [b, c]$, there exists $\xi \in (u, c)$ such that

$$F(c) - F(u) = f(\xi)(c-u) \geq f(b)(c-u),$$

$$\int_b^c \frac{du}{\sqrt{F(c)-F(u)}} \leq \frac{1}{\sqrt{f(b)}} \int_b^c \frac{du}{\sqrt{c-u}} \leq \frac{2\sqrt{c}}{\sqrt{f(b)}}.$$

Thus

$$I(c) \leq \eta_b + \frac{2\sqrt{c}}{\sqrt{f(b)}}, \quad \text{for } c > b, \tag{4.2}$$

where $\eta_b = \int_0^b \frac{du}{\sqrt{F(c)-F(u)}}$. Moreover, (H_3) also implies that for sufficiently large c, there exists $M_c > 0$ such that $M_c \to \infty$ as $c \to \infty$ and

$$\frac{f(cs)}{cs} \geq M_c,$$

for all $s \in [1/2, 1)$. Thus

$$\frac{F(c)-F(cx)}{c^2} = \int_x^1 \frac{f(cs)}{c} ds \geq \frac{M_c}{2}(1-x^2),$$

for all $x \geq 1/2$, and

$$I(c) \leq 2\int_{1/2}^1 \frac{dx}{\sqrt{\frac{F(c)-F(cx)}{c^2}}} \leq \frac{2\sqrt{2}}{\sqrt{M_c}} \int_{1/2}^1 \frac{dx}{\sqrt{1-x^2}} \leq \frac{\sqrt{2}\pi}{\sqrt{M_c}}, \tag{4.3}$$

for sufficiently large c. Therefore, by (4.1)∼(4.3), we get $I_b < \infty$ and $I(c) \to 0$ as $c \to \infty$. We can easily see that there is no point $c \geq b$ such that $I(c) = 0$, by (4.1).

Similarly, assume (H_2') and (H_3), then without loss of generality, for $a = 0 = b$, $F(c)-F(u) = f(\xi)(c-u)$, for some $\xi \in (u,c)$. Since f is nondecreasing, $F(c) - F(u) \geq f(0)(c-u)$, and thus,

$$I(c) \leq \frac{1}{\sqrt{f(0)}} \int_0^c \frac{du}{\sqrt{c-u}} = \frac{2\sqrt{c}}{\sqrt{f(0)}}, \tag{4.4}$$

for all $c > 0$. (4.1) and (4.3) are valid under the assumptions (H_2') and (H_3) too. Therefore by (4.3) and (4.4), $I(0) = 0$, $I_0 < \infty$ and $I(c) \to 0$ as $c \to \infty$. We also see that there is no point $c > 0$ such that $I(c) = 0$, by (4.1).

Lemma 3. *Assume* (H_2) *and* (H_3). *Let* $k > 0$ *be fixed and assume that* (3.3), (3.5) *has a positive solution for* k, *then there exists* $M(k) > 0$ *such that for all* $k^* \geq k$ *and for all possible positive solutions* u *of* (3.3), (3.5) *for* k^*, *one has*

$$\|u\|_\infty < M(k).$$

This result is also valid for problem (3.3), (3.4) *when we assume* (H_2') *and* (H_3).

Proof. Let k be fixed and assume that (3.3), (3,5) (or (3.3), (3.4)) has a positive solution. Let $k^* \geq k$ and u be a positive solution of the problem for k^*. Then by (3.6),

$$\frac{1}{I(u_o)} < \sqrt{\frac{2}{k}},$$

where $u_o = \max_{t\in[0,1]} u(t)$. Thus by Lemma 2, u_o is bounded above and its upper bound depends on k, but not on k^*. The proof is complete.

Lemma 4. *Assume* (b_1), (H_2) *and* (H_3). *Let* $\mu > 0$ *be fixed and assume that* $(1.5), (3.2)$ *has a positive solution for* μ, *then there exists* $M_\mu > 0$ *such that for all* $\mu^* \geq \mu$ *and for all possible positive solutions* u *of* $(1.5), (3.2)$ *for* μ^*, *one has*

$$\|u\|_\infty < M_\mu.$$

This result is also valid for problem $(1.5), (3.1)$ *when we assume* $(b_1), (H_2')$ *and* (H_3).

Proof. Let μ be given and assume that (1.5), (3.2) (or (1.5), (3.1)) has a positive solution at μ. Suppose that the conclusion is not true, then we may choose a sequence (μ_n), not necessarily distinct, and positive solution u_n of the problem for each μ_n such that $\mu_n \geq \mu$ and $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Consider problem (3.7) again

$$y''(t) + \mu_n q_o f(y(t)) = 0, \quad t \in [\frac{1}{4}, \frac{3}{4}] \tag{3.7}$$
$$y(\frac{1}{4}) = u_n(\frac{1}{4}), \ y(\frac{3}{4}) = u_n(\frac{3}{4}),$$

where $q_o = \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} q(t)$. By similar arguments in the proof of Theorem 1, for each μ_n, (3.7) has a positive solution y_n. Since $\|u_n\|_\infty \to \infty$ as $n \to \infty$, $\|y_n\|_\infty \to \infty$ as $n \to \infty$ and this contradicts Lemma 3. $\square$

We now state the theorem for multiplicity.

Theorem 2. *Assume* $(b_1), (H_2)$ *and* (H_3), *then problem* (1.5), (3.2) *has W-Prop A for positive solutions. And assume* $(b_1), (H_2')$ *and* (H_3), *then problem* (1.5), (3.1) *has Prop A for positive solutions.*

Let us consider the following problem,

$$u''(t) + \mu q(t) f(u(t)) = 0, \tag{P_μ^λ}$$
$$u(0) = a + \lambda, \ u(1) = b + \lambda.$$

Under assumptions of $(b_1), (H_2)$ and (H_3), we obtain by Lemma 2 and Theorem 1 that for each $\lambda > 0$, there exists $\mu_\lambda > 0$ such that problem (P_μ^λ) has at least one solution for $0 < \mu < \mu_\lambda$. For given $0 \leq \bar{\lambda} < \lambda$, let $\mu < \mu_{\lambda}$ and if u_λ is a solution of (P_μ^λ). Then u_λ is an upper solution and constant $\bar{\lambda}$ is a lower solution of $(P_\mu^{\bar{\lambda}})$ respectively. Thus $(P_\mu^{\bar{\lambda}})$ has a solution and this implies $\mu_\lambda \leq \mu_{\bar{\lambda}}$. Therefore, μ_f appeared in the proof of Theorem 1 satisfies $\mu_\lambda \leq \mu_f$ for all $\lambda > 0$ and in particular,

$$0 < \mu_0 \triangleq \lim_{\lambda \to 0^+} \mu_\lambda \leq \mu_f,$$

and we have the following Lemma.

Lemma 5. *Assume* $(b_1), (H_2)$ *and* (H_3) *and let* $\mu \in (0, \mu_0)$. *Then there exists* $\lambda > 0$ *such that* (P_μ^λ) *has a positive solution.*

We notice that the fact about $\mu_0 = \mu_f$ is not known at this point of time. We obtain a similar lemma for problem (1.5), (3.1). The proof of the following proposition is simply done by uniform continuity of f.

Proposition 1 ([7]). *Let $f : [0,\infty) \to (0,\infty)$ be continuous and nondecreasing. Let μ and μ_o, $0 < \mu < \mu_o$, and $M > 0$ be given. Then there exist $\bar{\mu} \in (\mu, \mu_o)$ and $\nu_o \in (0,1)$ such that*

$$\mu f(u+\nu) < \bar{\mu} f(u),$$

for all $u \in [0, M]$ and all $\nu \in (0, \nu_o)$.

Lemma 6. *Assume $(b_1), (H_2')$ and (H_3). Let $\mu \in (0, \mu_f)$ be given, where μ_f is in Theorem 1. Then there exists $\lambda > 0$ such that (P_μ^λ) has a positive solution.*

Proof. Let $\mu \in (0, \mu_f)$ be given, and let u be a positive solution of (1.5), (3.1) at μ known to exist by Theorem 1. For μ, μ_f and M_μ given in Lemma 4, we may choose $\bar{\mu} \in (\mu, \mu_f)$ and $\lambda > 0$ by Proposition 1, such that

$$\mu f(u+\lambda) < \bar{\mu} f(u),$$

for all $u \in [0, M_\mu]$. We know by Theorem 1 that (1.5), (3.1) has a positive solution $\bar{u}$ for $\mu = \bar{\mu}$ which, without loss of generality, satisfies $u(t) \le \bar{u}(t)$ for all $t \in [0,1]$. It also satisfies by lemma 4 that $a \le \bar{u}(t) < M_\mu$ on $[0,1]$. Let $\bar{u}_\lambda(t) = \bar{u}(t) + \lambda$, then u is a lower solution of (P_μ^λ). On the other hand,

$$\begin{aligned} \bar{u}_\lambda''(t) + \mu q(t) f(\bar{u}_\lambda(t)) &= \bar{u}''(t) + \mu q(t) f(\bar{u}(t) + \lambda) \\ &= q(t)\big[\mu f\big(\bar{u}(t) + \lambda\big) - \bar{\mu} f\big(\bar{u}(t)\big)\big] \le 0 \end{aligned}$$

and $\bar{u}_\lambda(0) = a + \lambda$, $\bar{u}_\lambda(1) = b + \lambda$. Thus $\bar{u}_\lambda$ is an upper solution of (P_μ^λ) and obviously $u(t) < \bar{u}_\lambda(t)$ on $[0,1]$. Therefore (P_μ^λ) has a solution between u and $\bar{u}_\lambda$ and the proof is complete. □

We set up an operator equation for fixed point arguments. It is well known that problem (1.5), (3.1) (or (1.5), (3.2)) is equivalent to the integral equation

$$u(t) = a + (b-a)t + \mu \int_0^1 G(t,s) q(s) f(u(s)) ds,$$

where $G(t,s)$ is written in the proof of Theorem 1 and consequently, it is equivalent to the fixed point equation

$$u = Tu$$

in $X = C([0,1])$, where $T : X \to X$ is given by

$$Tu(t) = a + (b-a)t + \mu \int_0^1 G(t,s)q(s)f(u(s))ds.$$

It is well known that T is completely continuous on the cone of nonnegative functions in X (see [4]). We define a cone K in X by

$$K = \{u \in X | u(t) \geq 0,\ t \in [0,1] \ \min_{t \in [\frac{1}{4}, \frac{3}{4}]} u(t) \geq \frac{1}{4} \|u\|_\infty\}.$$

We can easily check $T(K) \subset K$. The following two lemmas are well known and crucial in our arguments, see Guo and Lakshmikantham [5] for proofs and further discussion of the fixed point index.

Lemma 7. *Let X be a Banach space, K a cone in X and Ω bounded open in X. Let $0 \in \Omega$ and $T : K \cap \bar{\Omega} \to K$ be condensing. Suppose that $Tx \neq \nu x$, for all $x \in K \cap \partial\Omega$ and all $\nu \geq 1$. Then*

$$i(T, K \cap \Omega, K) = 1.$$

Lemma 8. *Let X be a Banach space and K a cone in X. For $r > 0$, define $K_r = \{x \in K : \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is a compact map such that $Tx \neq x$ for $x \in \partial K_r$. If $\|x\| \leq \|Tx\|$, for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 0.$$

Proof of Theorem 4. We first prove for problem (1.5), (3.2), so assume $(b_1), (H_2)$ and (H_3). Let $\mu \in (0, \mu_0)$ be given, then by Lemma 5, there exists $\lambda > 0$ such that (P_μ^λ) has a positive solution u_λ. Let $\Omega = \{u \in X : -M_\mu < u(t) < u_\lambda(t),\ t \in [0,1]\}$, where M_μ is given in Lemma 4, then Ω is bounded open in X, $0 \in \Omega$ and $T : K \cap \bar{\Omega} \to K$ is condensing, since it is completely continuous. Let $u \in K \cap \partial\Omega$, then there exists $t_o \in [0,1]$ such that $u(t_o) = u_\lambda(t_o)$ and

$$\begin{aligned}
0 \leq Tu(t_o) &= a + (b-a)t_o + \int_0^1 \mu G(t_o,s)q(s)f(u(s))ds \\
&< a + \lambda + (b-a)t_o + \int_0^1 \mu G(t_o,s)q(s)f(u_\lambda(s))ds \\
&= u_\lambda(t_o) \leq \nu u_\lambda(t_o) = \nu u(t_o),
\end{aligned}$$

for all $\nu \geq 1$. Thus $Tu \neq \nu u$, for all $u \in K \cap \partial\Omega$ and all $\nu \geq 1$ and by Lemma 3,

$$i(T, K \cap \Omega, K) = 1.$$

Next, let us choose $M > 0$ such that

$$\frac{M}{4} \int_{\frac{1}{4}}^{\frac{3}{4}} \mu G(\frac{1}{2}, s) ds > 1,$$

and let $q_o = \min_{t \in [\frac{1}{4}, \frac{3}{4}]} q(t)$, then by (H_3), we may choose $R_1 > 0$ such that $q_o f(u) \geq Mu$, for all $u \geq R_1$. Let $R = \max\{M_\mu, 4R_1, \|u_\lambda\|_\infty\}$, then by Lemma 4, $Tu \neq u$ for $u \in \partial K_R$. Furthermore, if $u \in \partial K_R$, then

$$\min_{t \in [\frac{1}{4}, \frac{3}{4}]} u(t) \geq \frac{1}{4} \|u\|_\infty \geq R_1.$$

Thus $q(t)f(u(t)) \geq q_o f(u(t)) \geq Mu(t)$, for all $t \in [\frac{1}{4}, \frac{3}{4}]$ and

$$\begin{aligned} Tu(\frac{1}{2}) &= a + \frac{(b-a)}{2} + \int_0^1 \mu G(\frac{1}{2}, s) q(s) f(u(s)) ds \\ &\geq \frac{(a+b)}{2} + \int_{\frac{1}{4}}^{\frac{3}{4}} \mu G(\frac{1}{2}, s) q(s) f(u(s)) ds \\ &\geq \frac{(a+b)}{2} + M \int_{\frac{1}{4}}^{\frac{3}{4}} \mu G(\frac{1}{2}, s) u(s) ds \\ &\geq \frac{(a+b)}{2} + \frac{M}{4} \int_{\frac{1}{4}}^{\frac{3}{4}} \mu G(\frac{1}{2}, s) \|u\|_\infty ds \\ &> \frac{(a+b)}{2} + \|u\|_\infty > \|u\|_\infty. \end{aligned}$$

Therefore $\|Tu\|_\infty \geq \|u\|_\infty$ and by Lemma 5,

$$i(T, K_R, K) = 0.$$

Consequently by the additivity of the fixed point index,

$$0 = i(T, K_R, K) = i(T, K \cap \Omega, K) + i(T, K_R \setminus \overline{K \cap \Omega}, K).$$

Since $i(T, K \cap \Omega, K) = 1$, $i(T, K_R \setminus \overline{K \cap \Omega}, K) = -1$ and thus, T has a fixed point on $K \cap \Omega$ and another on $K_R \setminus \overline{K \cap \Omega}$.

Finally, to show the existence of a positive solution at $\mu = \mu_f$, choose an increasing sequence (μ_n) such that $\mu_n \to \mu_f$ and (1.5), (3.2) has a positive solution, say u_n for each μ_n. Then by Lemma 4 and Arzela-Ascoli Theorem, (u_n) has a subsequence converging to $u \in C[0,1]$. Writing (1.5), (3.2) in integral form and applying Lebesgue Convergence Theorem, we can easily show that u is a solution of (1.5), (3.2) for μ_f. The proof for (1.5), (3.1) follows exactly the same way for the parameter $\mu \in (0, \mu_f)$ and the proof is complete. □

If $q \in C(0,1]$ or $q \in C[0,1)$, then Theorem 2 is still valid modifying (b_1) suitably. We state the facts as the following Theorem without proof.

Theorem 3. *If q is singular at 1 and satisfies*

(b_1') $q : [0,1) \to (0,\infty)$ *is continuous and* $\int_0^1 (1-s)q(s)ds < \infty$.

Then problem (1.5), (3.2) *has W-Prop A assuming* (H_2) *and* (H_3).

Problem (1.5), (3.1) *has Prop A assuming* (H_2') *and* (H_3). *Similarly, if q is singular at 0 and satisfies*

(b_1'') $q : (0,1] \to (0,\infty)$ *is continuous and* $\int_0^1 sq(s)ds < \infty$.

Then problem (1.5), (3.2) *has W-Prop A assuming* (H_2) *and* (H_3).

Problem (1.5), (3.1) *has Prop A assuming* (H_2') *and* (H_3).

Example 1. Let us consider the problem

$$u''(t) + \mu q(t)(u(t)^p + \epsilon) = 0, \tag{4.5}$$

$$u(0) = 0 = u(1), \tag{1.6}$$

where $p > 1$ and $\epsilon > 0$. Let $f(u) = u^p + \epsilon$, then f satisfies (H_2') and (H_3), thus by Theorem 2, problem (4.5), (1.6) has Prop A if q satisfies (b_1).

Similarly, for $b > 0$ and $p > 1$, the following problem

$$u''(t) + \mu q(t)u(t)^p = 0, \qquad u(0) = 0,\ u(1) = b$$

has W-Prop A for positive solutions if q satisfies (b_1). It is interesting to notice that the problem, for $p > 1$,

$$u''(t) + \mu q(t)u(t)^p = 0, \qquad u(0) = 0 = u(1)$$

has Prop C if q satisfies (b_1) (see [9]).

If the coefficient function q in (1.5) is regular, then condition (b_1) is not necessary and we obtain a similar result as Theorem 2. The proof follows on the lines of previous one.

Theorem 4. *Let $q \in C([0,1],(0,\infty))$. If f satisfies (H_2) and (H_3), then problem (1.5), (3.2) has W-Prop A for positive solutions.*

Similarly, if f satisfies (H_2') and (H_3), then problem (1.5), (3.1) has Prop A for positive solutions.

We conclude this section describing the aim of this paper for problem (1.1)∼(2.3). Consider

$$\triangle u + \mu g(|x|) f(u(x)) = 0, \quad \text{in } |x| > r_o \tag{1.1}$$

$$u = a \geq 0, \quad \text{if } |x| = r_o, \tag{2.1}$$

$$u \to b \geq 0, \quad \text{as } |x| \to \infty, \tag{2.2}$$

$$u \to b > a, \quad \text{as } |x| \to \infty, \tag{2.3}$$

where $r_o > 0$ and $n \geq 3$. We know that problem (1.1)∼(2.3) can be transformed to problem (1.5)∼(3.2) below.

$$z''(t) + \mu q(t) f(z(t)) = 0, \tag{1.5}$$

$$z(0) = a \geq 0, \ z(1) = b \geq 0 \tag{3.1}$$

$$z(0) = a \geq 0, \ z(1) = b > a, \tag{3.2}$$

where q is given by

$$q(t) = \frac{r_o^2}{(n-2)^2}(1-t)^{\frac{-2(n-1)}{n-2}} g\big(r_o(1-t)^{\frac{-1}{n-2}}\big).$$

We notice that condition (H_1) corresponds to condition (b_1') in Theorem 3 via the transformations we used. It is also interesting to notice that problem (1.5) is either regular or singular depending on the asymptotic behavior of coefficient function g in (1.1), so we need to consider the following two cases:

Case I. $\lim_{r\to\infty} r^{2(n-1)} g(r) < \infty$. In this case, $\lim_{t\to 1} q(t) < \infty$ so that q can be extended continuously on $[0,1]$ and the problem becomes regular. Thus applying Theorem 4, we obtain desired result described below.

Case II. $\lim_{r\to\infty} r^{2(n-1)} g(r) = \infty$. In this case, $\lim_{t\to 1} q(t) = \infty$ and the problem becomes singular. To apply Theorem 3, we need to add condition (H_1). Since function g in Case I also satisfies (H_1), we can unify both cases as the following corollary.

Corollary 1. *Assume (H_1), (H_2) and (H_3), then problem (1.1), (2.1), (2.3) has W-Prop A for positive radial solutions.*

Similarly, assume (H_1), (H_2') *and* (H_3), *then problem* (1.1), (2.1), (2.2) *has Prop A for positive radial solutions.*

Example 2. Assume (H_1) throughout the following examples. For $b > 0$, problem (1.1), (2.1), (2.3) has W-Prop A for positive radial solutions when $f(u) = u^p,\ p > 1$, $f(u) = u^p e^u,\ p > 0$ or $f(u) = u^q + u^p,\ 0 < q < 1,\ p > 1$. Problem (1.1), (2.1), (2.2) has Prop A for positive radial solutions when $f(u) = e^u - u$.

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