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SOLUTIONS FOR A QUASILINEAR ELLIPTIC EQUATION WITH CRITICAL SOBOLEV EXPONENT AND PERTURBATIONS ON R^N

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Abstract. We consider the following quasilinear elliptic problem

$$\begin{cases} -\text{div}\,|\nabla u|^{p-2}\nabla u) + c|u|^{p-2}u = |u|^{p^*-2}u + f(x,u) + h(x) \\ u \in W^{1,p}(R^N), \quad N > p \geq 2, \end{cases} \tag{*}$$

where $c > 0$, $p^* = \frac{Np}{N-p}$, $h(x) \in W^{-1,\frac{p}{p-1}}(R^N)$ (i.e., the dual space of $W^{1,p}(R^N)$), $f(x,0) = 0$ and $f(x,u)$ is a lower-order perturbation of $|u|^{p^*-2}u$ in the sense that $\lim_{u\to\infty} \frac{f(x,u)}{|u|^{p^*-2}u} = 0$. It is well known that (*) has only a trivial solution if $f(x,u) \equiv h(x) \equiv 0$ by a Pohozaev type identity, but (*) has a nontrivial solution if there is a subcritical perturbation, e.g., $h(x) \equiv 0$ and $f(x,u) \not\equiv 0$. In this paper, we prove that (*) has at least two distinct solutions if there are two perturbations, i.e., $f(x,u) \not\equiv 0$ and $h(x) \not\equiv 0$ (inhomogeneous term) with $\|h\|$ small enough.

1. Introduction and the main results. In this paper, we study the existence of solutions to the following quasilinear elliptic problem

$$\begin{cases} -\text{div}\,|\nabla u|^{p-2}\nabla u) + c|u|^{p-2}u = |u|^{p^*-2}u + f(x,u) + h(x) \\ u \in W^{1,p}(R^N), \quad N > p > 2, \end{cases} \tag{1.1}$$

where $c > 0$, $p^* = \frac{Np}{N-p}$, $h(x) \in W^{-1,p'}(R^N)$ the dual space of $W^{1,p}(R^N)$, $p' = \frac{p}{p-1}$), $f(x,0) = 0$ and $f(x,u)$ is a lower-order perturbation of $|u|^{p^*-2}u$ in the sense that

$$\lim_{u\to\infty} \frac{f(x,u)}{|u|^{p^*-2}u} = 0.$$

This kind of problem occurs in many branches, see e.g., [2] [15] and the references therein.

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If $f(x,u) \equiv h(x) \equiv 0$, the existence and nonexistence results for problem (1.1) on bounded domains was studied by [9, 11] and [5] (for $p=2$), etc. For the entire space R^N, by using the Gauss formula and the same ideas as in [11] and [1], we can get the following Pohozaev type identity

$$\int_{R^N} |\nabla u|^p dx = \int_{R^N} |u|^{p^*} dx - \frac{cp^*}{p} \int_{R^N} |u|^p dx. \tag{1.2}$$

On the other hand, if u is a solution to (1.1) with $f(x,u) \equiv h(x) \equiv 0$ we mean that

$$\int_{R^N} |\nabla u|^p dx = \int_{R^N} |u|^{p^*} dx - c \int_{R^N} |u|^p dx. \tag{1.3}$$

So, it is easy to see from (1.2) and (1.3) that problem (1.1) has no nontrivial solution if $c > 0$ and $f(x,u) \equiv h(x) \equiv 0$.

However, if $h(x) \equiv 0$ but $f(x,u) \not\equiv 0$, by using concentration-compactness principle (see [13], [14]), [22] proved that (1.1) has a nontrivial solution. A natural question is whether (1.1) has more than one solutions if $f(x,u) \not\equiv 0$ and $h(x) \not\equiv 0$? Recently, there are many results related to this kind of problems. For example, Li in [12] proved that problem (1.1) with two perturbations has always a week solution. In [7], problem (1.1) in a bounded domain was studied and the existence of two solutions has obtained. Two solutions of the subcritical case of (1.1) on R^N has also been considered in [6], but in which

$$f(x,u) = \sum_{i=1}^{m} Q_i(x)|u|^{q_i-2}u.$$

There are also many works on (1.1) with $p=2$, see, e.g., [15, 2, 8, 19] and the references therein.

The aim of this paper is to use the variational methods to show that the more general problem (1.1) has also at least distinct two solutions. The main ideas of this paper are due to [6]. But here we encounter serious difficulties caused by the Sobolev exponent and the subcritical perturbation $f(x,u)$ is not a polynomial.

It is well known that the solutions of (1.1) are the critical points of the following variational functional defined on $W^{1,p}(R^N)$:

$$I(u) = \frac{1}{p}\int (|\nabla u|^p + c|u|^p) - \frac{1}{p^*}\int |u|^{p^*} - \int F(x,u) - \int hu, \tag{1.4}$$

where $F(x,u) = \int_0^u f(x,u)ds$ and from now on we omit "dx" and "R^N" in all integration if there is no other indications.

Now we give a sketch of how to look for two distinct critical points of (1.4). First, we consider a minimization of $I(u)$ constrained in a neighborhood of zero, by using Ekeland's variational principle [10], we can find a critical point of $I(u)$ which achieves its level (i.e., the local minimum of $I(u)$), moreover, this local minimum is negative, see Theorem 1.1. Next, around "zero", by the mountain pass theorem without (PS) condition [5], we can also get a critical point of $I(u)$ and its level is positive, (see Theorem 1.4). Because these two critical points are in different levels, they must be distinct if both of them achieve their levels. Unfortunately, it seems hard to prove that the critical point obtained by the mountain pass theorem [5] achieves its level. In this paper, by establishing some new comparisons between the levels of the above two critical points and then we can prove the two solutions are distinct.

Throughout this paper, we denote the dual space of $W^{1,p}(R^N)$ by $W^{-1,p'}(R^N)$ and its norm by $\|\cdot\|_*$; the norms of $L^p(R^N)$ and $W^{1,p}(R^N)$ are denoted by $\|\cdot\|_p$ and $\|\cdot\|$ respectively, that is,

$$\|u\|_p = [\int_{R^N} |u|^p dx]^{\frac{1}{p}}, \quad \text{if} \quad u \in L^p(R^N);$$

$$\|u\| = [\int_{R^N} (|\nabla u|^p + c|u|^p) dx]^{\frac{1}{p}}, \quad \text{if} \quad u \in W^{1,p}(R^N).$$

The conditions posed on $f(x,t)$ are as follows:

($f1$) $f(x,t) \in C(R^N \times R)$.

($f2$) $\frac{\partial(f(x,t)t)}{\partial t}$ is continuous with respect to $t \in R^1$.

($f3$)

$$\lim_{|t|\to 0} \frac{f(x,t)}{|t|^{p-2}t} = 0 \quad \text{uniformly in} \quad x \in R^N;$$

$$\lim_{|t|\to \infty} \frac{f(x,t)}{|t|^{p^*-2}t} = 0 \quad \text{uniformly in} \quad x \in R^N.$$

($f4$) There exists $\theta > 0$ if $N \geq p^2$; or $\theta > \frac{p(p^2-N)}{(N-p)(p-1)}$ if $p < N < p^2$ such that for all $x \in R^N$, $t \geq 0$,

$$0 \leq (p+\theta)F(x,t) \equiv (p+\theta)\int_0^t f(x,s)ds \leq tf(x,t).$$

($f5$) $\lim_{|x|\to+\infty} f(x,t) = \bar{f}(t)$ uniformly for $|t|$ bounded; $f(x,t) \geq \bar{f}(t) \geq 0$, $\forall x \in R^N$ and $\frac{\bar{f}(t)}{|t|^{p-2}t}$ is nondecreasing in $t \geq 0$.

($f6$) meas $\{x \in R^N : f(x,t) > \bar{f}(t)\} > 0$.

Corresponding to (1.4), for $u \in W^{1,p}(R^N)$ we define that

$$I^\infty(u) = \frac{1}{p}\int(|\nabla u|^p + c|u|^p) - \frac{1}{p^*}\int |u|^{p^*} - \int \bar{F}(u); \tag{1.5}$$

$$I_0(u) = \frac{1}{p}\int(|\nabla u|^p + c|u|^p) - \frac{1}{p^*}\int |u|^{p^*} - \int F(x,u), \tag{1.6}$$

where $\bar{F}(u) = \int_0^u \bar{f}(s)ds$, $F(x,u) = \int_0^u f(x,s)ds$.

Furthermore, we set

$$\Lambda = \{u \in W^{1,p}(R^N) \setminus \{0\} : < I^{\infty\prime}(u), u >= 0\}, \tag{1.7}$$

$$J^\infty = \begin{cases} \inf\{I^\infty(u) : u \in \Lambda\}, & \text{if } \Lambda \neq \phi \\ +\infty, & \text{if } \Lambda = \phi. \end{cases} \tag{1.8}$$

Remark 1.1. By a result of [22], if $(f1)$–$(f5)$ hold, then I^∞ satisfies $(PS)_c$ condition for any $c \in (0, \min\{J^\infty, \frac{1}{N}S^{\frac{N}{p}}\})$. Moreover, if $J^\infty < \frac{1}{N}S^{\frac{N}{p}}$, then J^∞ is achieved by some $\bar{u} \in W^{1,p}(R^N)$ such that

$$J^\infty = I^\infty(\bar{u}) = \sup_{t\geq 0} I^\infty(t\bar{u}), \tag{1.9}$$

where $S = \inf\{\|\nabla u\|_p^p : u \in W^{1,p}(R^N),\ \|u\|_{p^*} = 1\}$.

Our main results are that:

Theorem 1.1. *If $(f1), (f3)$ hold, then there exist $R > 0$ (small), $C = C(R, N, p, f) > 0$ and $u_1 \in W^{1,p}(R^N)$ such that for any $h \in W^{-1,p'}(R^N)$ with $\|h\|_* < C$ we have*

$$I(u_1) = c_1 := c_1(R) \overset{\Delta}{=} \inf_{u\in \overline{B}_R} I(u), \tag{1.10}$$

where $\overline{B}_R = \{u \in W^{1,p}(R^N) : \|u\| \leq R\}$ and $I(u)$ is defined by (1.4). Furthermore, u_1 is a solution of (1.1) and $c_1 \leq 0$, $c_1 \to 0$ as $\|h\|_* \to 0$.

This theorem was essentially shown in [12], but in [12] we do not know if u_1 achieves the local minimum c_1. In this paper we will prove it in Section 1.

Theorem 1.2. *Let $u_n \subset W^{1,p}(R^N)$ be a bounded $(PS)_c$ sequence of $I(u)$ and for some $u \in W^{1,p}(R^N)$, $u_n \overset{n}{\rightharpoonup} u$ weakly in $W^{1,p}(R^N)$, then u is a weak solution of (1.1). Moreover, either $u_n \to u$ in $W^{1,p}(R^N)$ i.e. $I_g(u_0) = c$; or, $c \geq I(u) + J^\infty$.*

This theorem is essentially a weaker version of the global compactness results, see[17] [20] for bounded domains, [3] [21] for $p = 2$.

Theorem 1.3. *Let I and I_0 be given by (1.4) (1.6), respectively. Then if $(f1)-(f6)$ hold, there is a $v_0 \in W^{1,p}(R^N)$ such that*

i)

$$\sup_{t\geq 0} I_0(tv_0) < \min\{J^\infty, \frac{1}{N}S^{\frac{N}{p}}\}. \tag{1.11}$$

ii) *There exist $\bar{t} > 0$ such that*

$$I_0(tv_0) < 0 \text{ if } t \geq \bar{t};\ I(tv_0) < 0 \text{ if } t \geq \bar{t} \text{ and } \|h\|_* \leq 1. \tag{1.12}$$

Furthermore, if we denote

$$\Gamma = \{\gamma \in C([0,1], W^{1,p}(R^N)) : \gamma(0) = 0, \gamma(1) = \bar{t}v_0\}, \tag{1.13}$$

$$c_2 = \inf_{\gamma\in\Gamma} \max_{u\in\gamma} I(u). \tag{1.14}$$

Thus, there is a positive constant C such that

$$c_2 < c_1 + \min\{J^\infty, \frac{1}{N} S^{\frac{N}{p}}\}, \tag{1.15}$$

if $\|h\|_* \le C$, where $c_1 < 0$ is given by (1.10)

Theorem 1.4. *If $(f1)$–$(f6)$ are satisfied and $h(x) \in W^{-1,p'}(R^N)$, then, there is a positive constant $C = C(N, P, f)$ such that* (1.1) *possesses at least two distinct solutions for all $h(x) \in W^{-1,p'}(R^N)$ with $\|h\| \le C$.*

2. Some Lemmas and the Proof of Theorem 1.1. The first lemma is a generalization of Brezis-Lieb Lemma [4], it was shown in [21] for $p = 2$ and in [6] for general $p > 1$, by using the mean theorem and Strauss Lemma [16]

Lemma 2.1. *Let $\{u_n\} \subset W^{1,p}(R^N)$ be a bounded sequence such that for some $u_0 \in W^{1,p}(R^N)$,*

$$u_n \overset{n}{\rightharpoonup} u_0 \ \text{ weakly in } \ W^{1,p}(R^N),\ \ u_n \overset{n}{\to} u_0 \ \text{a.e. in } R^N$$

and $f(x,t)$ satisfies $(f1)$ and

$$\lim_{t\to 0} |\frac{f(x,t)}{t^{p-1}}| \le M \text{ for some } M > 0$$

$$\lim_{|t|\to\infty} |\frac{f(x,t)}{t^{\ell}}| = 0 \ \text{ uniformly in } \ x \in R^N,\ p < \ell \le p^*.$$

Then,

i)

$$\lim_{n\to\infty} [\int_{R^N} F(x,u_n)dx - \int_{R^N} F(x,u_0)dx - \int_{R^N} F(x,u_n - u_0)dx] = 0,$$

where $F(x,u) = \int_0^u f(x,s)ds$.

ii) *If*

$$\lim_{t\to 0} |\frac{\frac{d}{dt}(f(x,t)t)}{t^{p-1}}| \le M \text{ for some } M > 0;$$

$$\lim_{|t|\to\infty} |\frac{\frac{d}{dt}(f(x,t)t)}{t^{p^*-1}}| = 0 \text{ uniformly in } x \in R^N.$$

Then

$$\lim_{n\to\infty} [\int_{R^N} f(x,u_n)u_n dx - \int_{R^N} f(x,u_0)u_0 dx - \int_{R^N} f(x,u_n - u_0)(u_n - u_0)dx] = 0.$$

Lemma 2.2. *Assume* (f_1)–(f_3), (f_5) *hold and* $\{v_n\} \subset W^{1,p}(R^N)$ *is a sequence with*

$$v_n \rightharpoonup 0 \text{ weakly in } W^{1,p}(R^N). \tag{2.1}$$

Then,

$$\lim_{n\to\infty}[I_0(v_n) - I^\infty(v_n)] = 0; \tag{2.2}$$

$$\lim_{n\to\infty}[\langle I_0'(v_n), v_n\rangle - \langle I^{\infty\prime}(v_n), v_n\rangle] = 0. \tag{2.3}$$

Proof. This lemma is essentially due to [6]. For the sake of completeness, we give the proof here. By the definitions of (1.5) and (1.6), we see that

$$I_0(v_n) = I^\infty(v_n) + \int[\bar{F}(v_n) - F(x, v_n)]. \tag{2.4}$$

It is clear that to prove (2.2) we need only show that

$$\lim_{n\to\infty}\int|\overline{F}(v_n) - F(x, v_n)|dx = 0. \tag{2.5}$$

In fact, from (2.1) and Sobolev's embedding, we may assume that

$$v_n \xrightarrow{n} 0 \text{ strongly in } L^q_{loc}(R^N), \text{ for } p \le q < p^*. \tag{2.6}$$

Then, for any $\delta > 0, R > 0$,

$$\begin{aligned}\int_{R^N}|F(x, v_n) - \bar{F}(v_n)| &\le \int_{B_R}|F(x, v_n)| + \int_{B_R}|\bar{F}(v_n)| \\ &+ [\int_{\{ {|x|\ge R \atop |v_n|<\delta} \}} + \int_{\{ {|x|\ge R \atop \delta\le|v_n|<\delta^{-1}} \}} + \int_{\{ {|x|\ge R \atop |v_n|\ge\delta^{-1}} \}}]|F(x, v_n) - \bar{F}(v_n)|.\end{aligned} \tag{2.7}$$

However,

$$\int_{\{ {|x|\ge R \atop |v_n|<\delta} \}}|F(x, v_n) - \bar{F}(v_n)| \le \varepsilon_1(\delta)\int_{R^N}|v_n|^p, \tag{2.8}$$

where $\varepsilon_1(\delta) = \sup\limits_{\substack{0\le t<\delta \\ |x|\ge R}} \frac{|F(x,t)-\bar{F}(t)|}{|t|^p} \longrightarrow 0$ as $\delta \to 0^+$, by $(f3)$.

$$\int_{\{ {|x|\ge R \atop \delta\le|v_n|<\delta^{-1}} \}}|F(x, v_n) - \bar{F}(v_n)| \le \varepsilon(R)\delta^{-p}\int_{R^N}|v_n|^p, \tag{2.9}$$

where $\varepsilon(R) = \sup\limits_{\substack{|x|\ge R \\ \delta\le|t|<\delta^{-1}}} |F(x, t) - \bar{F}(t)| \longrightarrow 0$ as $R \to +\infty$ and $\delta > 0$ fixed, by $(f5)$.

$$\int_{\{ {|x|\ge R \atop |v_n|\ge\delta^{-1}} \}}|F(x, v_n) - \bar{F}(v_n)| \le \varepsilon_2(\delta)\int_{R^N}|v_n|^{p^*}, \tag{2.10}$$

where $\varepsilon_2(\delta) = \sup\limits_{\substack{|x|\ge R \\ |t|\ge\delta^{-1}}} \frac{|F(x,t)-\bar{F}(t)|}{|t|^{p^*}} \longrightarrow 0$ as $\delta \to 0$, by $(f3)$.

It follows from (2.7)–(2.10) that (2.5) holds. Thus (2.2) is proved by using (2.4) and (2.5). By the same way we can prove (2.3).

Lemma 2.3. *Let $\alpha > 0$, $\beta > 0$, $\gamma > 0$, then*

i) $\int_0^\infty \frac{r^\alpha dr}{(1+r^\beta)^\gamma} < +\infty$, *if* $\beta - 1 \le \alpha < \beta\gamma - 1$;

ii) *For $R > 0$ fixed, there are $K_1 > 0, K_2 > 0, K_3 > 0$ (which depend only on R) such that for $\varepsilon \to 0^+$,*

$$\int_0^R \frac{r^\alpha dr}{(\varepsilon + r^\beta)^\gamma} = \begin{cases} K_1, & \text{if} \quad \alpha > \beta\gamma - 1; \\ K_2 |\ell n \varepsilon| + O(1), & \text{if} \quad \alpha = \beta\gamma - 1; \\ K_3 \varepsilon^{\frac{\alpha+1}{\beta} - \gamma}, & \text{if} \quad \beta \le \alpha < \beta\gamma - 1. \end{cases}$$

Proof. i) It is obvious by noticing that

$$\int_0^\infty \frac{r^\alpha dr}{(1+r^\beta)^\gamma} = \frac{1}{\beta} \int_0^1 \frac{r^{\alpha-\beta+1} dr^\beta}{(1+r^\beta)^\gamma} + \int_1^\infty r^{\alpha-\beta\gamma} dr.$$

ii) If $\alpha > \beta\gamma - 1$, it is easy to see that

$$\lim_{\varepsilon \to 0^+} \int_0^R \frac{r^\alpha dr}{(\varepsilon + r^\beta)^\gamma} = \int_0^R r^{\alpha - \beta\gamma} dr = \frac{1}{1 + \alpha - \beta\gamma} R^{\alpha - \beta\gamma + 1} \triangleq K_1.$$

If $\alpha = \beta\gamma - 1$, since

$$\int_0^R \frac{r^\alpha dr}{(\varepsilon + r^\beta)^\gamma} = \int_0^{R\varepsilon^{-\frac{1}{\beta}}} \frac{\rho^\alpha}{(1+\rho^\beta)^\gamma} d\rho, \ \lim_{\varepsilon \to 0^+} \frac{\int_0^{R\varepsilon^{-\frac{1}{\beta}}} \frac{\rho^\alpha}{(1+\rho^\beta)^\gamma} d\rho}{\int_0^{R\varepsilon^{-\frac{1}{\beta}}} \frac{1}{1+\rho} d\rho} = 1,$$

and

$$\int_0^{R\varepsilon^{-\frac{1}{\beta}}} \frac{1}{1+\rho} d\rho = \ell n |1 + R\varepsilon^{-\frac{1}{\beta}}| + O(1), \text{ as } \varepsilon \to 0^+.$$

Then,

$$\int_0^R \frac{r^\alpha dr}{(\varepsilon + r^\beta)^\gamma} = K_2 |\ln \varepsilon| + O(1), \text{ as } \varepsilon \to 0^+.$$

If $\beta - 1 < \alpha < \beta\gamma - 1$, by the argument of i),

$$\int_0^R \frac{r^\alpha dr}{(\varepsilon + r^\beta)^\gamma} = \varepsilon^{\frac{\alpha+1}{\beta} - \gamma} \int_0^{R\varepsilon^{-\frac{1}{\beta}}} \frac{\rho^\alpha d\rho}{(1+\rho^\beta)^\gamma} = K_3 \varepsilon^{\frac{\alpha+1}{\beta} - \gamma}.$$

Proof of Theorem 1.1. By using Ekeland's variational principle and the second concentration-compactness Lemma [14], we can prove that, see [12] for the details, there exist $\{u_n\} \subset \bar{B}_{R_0}$ and $u_1 \in \bar{B}_{R_0}$ such that

$$I(u_n) \xrightarrow{n} c_1, \quad I'(u_n) \xrightarrow{n} 0 \text{ in } W^{-1,p'}(R^N); \tag{2.11}$$

$$u_n \overset{n}{\rightharpoonup} u_1 \text{ weakly in } W^{1,p}(R^N), \ u_n \overset{n}{\to} u_1 \text{ a.e. in } R^N, \tag{2.12}$$

$$\nabla u_n \xrightarrow{n} \nabla u_1 \text{ a.e. in } R^N. \tag{2.13}$$

Moreover, u_1 is a weak solution of (1.1), that is,

$$< I'(u_1), \varphi >= 0, \ \forall \varphi \in W^{1,p}(R^N). \tag{2.14}$$

Since, by (2.11) and by $(f4)$,

$$I(u_n) = \int [\frac{1}{p} f(x, u_n)u_n - F(x, u_n)] + (\frac{1}{p} - \frac{1}{p^*}) \int |u_n|^{p^*} + (\frac{1}{p} - 1) \int hu_n,$$
$$\frac{1}{p} f(x, u_n)u_n - F(x, u_n) \geq 0.$$

Then, (2.12) and Fatou's lemma give that

$$c_1 = \liminf_{n \to +\infty} I(u_n) \tag{2.15}$$
$$\geq \int [\frac{1}{p} f(x, u_1)u_1 - F(x, u_1)] + (\frac{1}{p} - \frac{1}{p^*}) \int |u_1|^{p^*} + (\frac{1}{p} - 1) \int hu_1. \tag{2.16}$$

On the other hand, by the definition of c_1 and $u_1 \in \bar{B}_{R_0}$, then $I(u_1) \geq c_1$, i.e.,

$$I(u_1) = \frac{1}{p} \int |\nabla u_1|^p + c|u_1|^p - \frac{1}{p^*} \int |u_1|^{p^*} - \int F(x, u_1) - \int hu_1 \geq c_1. \tag{2.17}$$

Taking $\varphi = u_1$ in (2.14) and substitute it into (2.17), then

$$\begin{aligned} I(u_1) = &\int [\frac{1}{p} f(x, u_1)u_1 - F(x, u_1)] \\ &+ (\frac{1}{p} - \frac{1}{p^*}) \int |u_1|^{p^*} + (\frac{1}{p} - 1) \int hu_1 \geq c_1. \end{aligned} \tag{2.18}$$

Combining (2.16) and (2.18), (1.10) holds.

Since $0 \in \bar{B}_{R_0}, c_1 \leq I(0) = 0$. By $(f2)$ and $(f3)$, we know that for any $\varepsilon > 0$, there is a $C_\varepsilon > 0$ such that

$$|f(x, t)| \leq \varepsilon |t|^{p-1} + C_\varepsilon |t|^{p^*-1}. \tag{2.19}$$

Letting $\varepsilon = \frac{c}{2}$, then by (1.4) (2.19) and Sobolev embedding and Young's inequality, we have

$$\begin{aligned} I(u) &\geq \frac{1}{2p} \|u\|^p - C\|u\|^{p^*} - \|h\|_* \|u\| \\ &\geq (\frac{1}{2p} - \eta)\|u\|^p - C\|u\|^{p^*} - C_\eta \|h\|_*^{\frac{p}{p-1}}. \end{aligned} \tag{2.20}$$

Now, if we choose $\eta = \frac{1}{4p}$ and $u \in \bar{B}_{R_0}$ for $R_0 > 0$ small enough, it is clear that $I(u) \geq 0$ for $u \in \bar{B}_{R_0}$ and $\|h\|_*$ small enough, i.e. $c_1 \geq 0$ as $\|h\|_*$ small, this implies that $c_1 \to 0$ as $\|h\|_* \to 0$.

3. Proofs of Theorem 1.2 to Theorem 1.4.

Proof of Theorem 1.2. By the assumptions of this theorem we may assume that there is an $u_0 \in W^{1,p}(R^N)$ such that

$$u_n \xrightarrow{n} u_0 \text{ weakly in } W^{1,p}(R^N), \tag{3.1}$$

$$< I'(u_n), u_n > \xrightarrow{n} 0; \tag{3.2}$$

$$I'(u_n) \xrightarrow{n} c. \tag{3.3}$$

Then, by the results of [12] (see Section 2),

$$\nabla u_n \xrightarrow{n} \nabla u_0 \text{ a.e. in } R^N; \tag{3.4}$$

$$< I'(u_0), \varphi >= 0, \forall \varphi \in W^{1,p}(R^N). \tag{3.5}$$

Letting $v_n = u_n - u_0$ and by (3.1) (3.4) that

$$v_n \overset{n}{\rightharpoonup} 0 \text{ weakly in } W^{1,p}(R^N); \ \nabla v_n \xrightarrow{n} 0 \text{ a.e. in } R^N.$$

Since $(f1)$ $(f2)$, it follows from Lemma 2.1 and the definition of I_0 (see (1.6)) that

$$o(1) + c = I(u_n) = I(u_0) + I(v_n) = I(u_0) + I_0(v_n); \tag{3.6}$$

$$\begin{aligned} o(1) =< I'(u_n), u_n > &=< I'(u_0), u_0 > + < I'(v_n), v_n > + o(1) \\ &=< I_0(v_n), v_n > + o(1). \end{aligned} \tag{3.7}$$

Now, there are two cases to be considered.

Case 1: If $v_n \xrightarrow{n} 0$ strongly in $W^{1,p}(R^N)$, that is, $u_n \xrightarrow{n} u_0$ strongly in $W^{1,p}(R^N)$ and $I(u_0) = \lim_{n\to+\infty} I(u_n)$.

Case 2: If $v_n \overset{n}{\not\to} 0$, we need to show that $c \geq I(u_0) + J^\infty$. In fact, in this case, we may assume that for some $\ell > 0$,

$$\|v_n\| \xrightarrow{n} \ell. \tag{3.8}$$

Since (3.6) (3.7) and Lemma 2.2, we see that

$$c = I(u_0) + I^\infty(v_n) + o(1), \tag{3.9}$$

$$< I_0'(v_n), v_n >=< I^{\infty\prime}(v_n), v_n >= o(1), \tag{3.10}$$

Let

$$\mu_n =< I^{\infty\prime}(v_n), v_n >= \int |\nabla v_n|^p + c|v_n|^p - \int |v_n|^{p^*} - \int \bar{f}(v_n)v_n, \tag{3.11}$$

and (3.10) implies that $\mu_n \to 0$ as $n \to +\infty$. Furthermore, by (2.19), (3.8) and (3.10) we know that for some $\alpha > 0$,

$$\lim_n \int |v_n|^{p^*} \geq \alpha > 0. \tag{3.12}$$

By (3.9), to prove $c \geq I(u_0) + J^\infty$ we need only to show that

$$I^\infty(v_n) \geq J^\infty + o(1). \tag{3.13}$$

If $v_n \in \Lambda$ (See (1.7)), by the definition of J^∞, (3.13) holds. In general, for $t > 0$, since

$$< I^{\infty\prime}(tv_n), tv_n >= t^p \int (|\nabla v_n|^p + c|v_n|^p) - t^{p^*} \int |v_n|^{p^*} - \int \bar{f}(tv_n)tv_n.$$

If we can find a sequence $\{t_n\}$ with $t_n > 0$, $t_n \to 1$ as $n \to +\infty$ and

$$< I^\infty(t_n v_n), t_n v_n >= 0, \text{ i.e. } \ t_n v_n \in \Lambda. \tag{3.14}$$

Then,

$$I^\infty(v_n) = I^\infty(t_n v_n) + \circ(1) \geq J^\infty + \circ(1), \tag{3.15}$$

this gives (3.13).

To prove (3.14), we let

$$g(t) = t^p \int (|\nabla v_n|^p + c|v_n|^p) - t^{p^*} \int |v_n|^{p^*} - \int \bar{f}(tv_n)tv_n. \tag{3.16}$$

Since for any $\varepsilon \in (0,1)$, there is $S_0 > 0$ such that

$$\bar{f}(s) \geq -\varepsilon |s|^{p^*-1}, \quad \forall s \geq S_0.$$

Then, for any fixed n and if $t > 0$ large enough, (3.12) and (3.16) give that

$$g(t) \leq t^p \int (|\nabla v_n|^p + c|v_n|^p) - t^{p^*} \int |v_n|^{p^*} + \varepsilon t^{p^*} \int |v_n|^{p^*} < 0. \tag{3.17}$$

On the other hand, by (2.19) and (3.16) we see that for any $\varepsilon \in (0, c)$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} g(t) &\geq t^p \int (|\nabla v_n|^p + c|v_n|^p) - t^{p^*} \int |v_n|^{p^*} - \varepsilon t^p \int |v_n|^p - C_\varepsilon t^{p^*} \int |v_n|^{p^*} \\ &\geq t^p (1 - \frac{\varepsilon}{c}) \|v_n\|^p - (1 + C_\varepsilon) t^{p^*} \|v_n\|_{p^*}^{p^*} \\ &\geq t^p (1 - \frac{\varepsilon}{c}) \|v_n\|^p - S(1 + C_\varepsilon) t^{p^*} \|v_n\|^{p^*}, \text{ by Sobolev inequality} \\ &> 0, \text{ as } t \to 0. \end{aligned} \tag{3.18}$$

So, by (3.17) and (3.18), the mean theorem implies that for any fixed n, we can find a $t_n > 0$ such that $g(t_n) = 0$, that is,

$$< I^\infty(t_n v_n), t_n v_n >= t_n^p \int (|\nabla v_n|^p + c|v_n|^p) - t_n^{p^*} \int |v_n|^{p^*} - t_n \int \bar{f}(t_n v_n) v_n = 0. \tag{3.19}$$

Now, we may assume that

$$t_n \longrightarrow a > 0. \tag{3.20}$$

and we claim that $a = 1$. In fact, by (3.19),

$$\lim_{n\to+\infty} \int (|\nabla v_n|^p + c|v_n|^p) - a^{p^*-p} \lim_{n\to+\infty} \int |v_n|^{p^*} = a^{1-p} \lim_{n\to+\infty} \int \bar{f}(a v_n) v_n. \tag{3.21}$$

Combining (3.10) (3.21) and noticing that $(f5)$,

$$\begin{aligned} (1 - a^{p^*-p}) \lim_{n\to+\infty} \int |v_n|^{p^*} &= - \lim_{n\to+\infty} [\int \bar{f}(v_n) v_n - a^{1-p} \int \bar{f}(a v_n) v_n] \\ &= - \lim_{n\to+\infty} [\frac{\bar{f}(v_n)}{|v_n|^{p-2} v_n} - \frac{\bar{f}(a v_n)}{|a v_n|^{p-2} a v_n}] |v_n|^p = \begin{cases} \geq 0, & \text{if } a \geq 1, \\ \leq 0, & \text{if } a \leq 1, \end{cases} \end{aligned}$$

this implies that $(1 - a^{p^*-p}) \lim_{n\to+\infty} \int |v_n|^{p^*} \equiv 0$, that is, $a = 1$ by (3.12). So, by (3.14) and (3.15) that (3.13) holds, then Case 2 is proved. □

We turn now to prove Theorem 1.3.

To this end, we first introduce that for any $\varepsilon > 0$,

$$u_\varepsilon(x) = \frac{b \varepsilon^{\frac{N-p}{p^2}}}{(\varepsilon + |x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}}, \tag{3.22}$$

where $b > 0$ is a constant. By the result of [18],

$$\|\nabla u_\varepsilon\|_p^p = \|u_\varepsilon\|_{p^*}^{p^*} = S^{\frac{N}{p}}, \tag{3.23}$$

where S is given by Remark 1.1.

Next, for $R > 0$, we choose $\varphi(x) \in C_0^\infty$ such that $\varphi(x) \equiv 1$ if $|x| \le \frac{R}{2}$; $\varphi(x) \equiv 0$ if $|x| \ge R$ and let

$$\psi_\varepsilon(x) = \varphi(x) u_\varepsilon(x), \tag{3.24}$$

then we have the following estimates:

$$\begin{aligned}\int_{R^N} |\nabla \psi_\varepsilon|^p dx &= \int_{R^N} |\varphi \nabla u_\varepsilon + u_\varepsilon \nabla \varphi|^p dx \\ &= \int_{R^N} |\nabla u_\varepsilon|^p dx + \int_{\frac{R}{2} \le |x| \le R} [|\varphi \nabla u_\varepsilon + u_\varepsilon \nabla \varphi|^p - |\nabla u_\varepsilon|^p] dx \\ &= S^{\frac{N}{p}} + O(\varepsilon^{\frac{N-p}{p}}), \text{ by (3.23);}\end{aligned} \tag{3.25}$$

$$\int_{R^N} |\psi_\varepsilon|^{p^*} dx = \int_{R^N} |u_\varepsilon|^{p^*} dx + \int_{\frac{R}{2} \le |x| \le R} [|\varphi u_\varepsilon|^{p^*} - |u_\varepsilon|^{p^*}] dx \tag{3.26}$$

$$= S^{\frac{N}{p}} + O(\varepsilon^{\frac{N}{p}}), \text{ by (3.23)} \tag{3.27}$$

$$\begin{aligned}\int_{R^N} |\psi_\varepsilon|^p dx &= \int_{|x| \le \frac{R}{2}} |u_\varepsilon|^p dx + \int_{\frac{R}{2} \le |x| \le R} |\varphi u_\varepsilon|^p dx \\ &= \omega_{N-1} b \varepsilon^{\frac{N-p}{p}} \int_0^{\frac{R}{2}} \frac{r^{N-1}}{[\varepsilon + r^{\frac{p}{p-1}}]^{N-p}} dr + O(\varepsilon^{\frac{N-p}{p}}) \\ &= \begin{cases} K_1 \varepsilon^{\frac{N-p}{p}} + O(\varepsilon^{\frac{N-p}{p}}), & \text{if } p < N < p^2; \\ K_2 \varepsilon^{p-1} |\ell n \varepsilon| + O(\varepsilon^{p-1}), & \text{if } N = p^2; \\ K_3 \varepsilon^{p-1} + O(\varepsilon^{\frac{N-p}{p}}), & \text{if } N > p^2 \ge \sqrt{2}; \end{cases}\end{aligned} \tag{3.28}$$

by Lemma 2.3 ii), where ω_{N-1} is the measure of a unit ball in R^N.

Furthermore, for I_0 defined by (1.6), we have the following lemma.

Lemma 3.1. *For any $\varepsilon \in (0,1)$, let $\psi_\varepsilon(x)$ be given by (3.24), if $(f1)$–$(f3)$ hold, then there exists $t_\varepsilon > 0$ such that*

$$I_0(t_\varepsilon \psi_\varepsilon) = \sup_{t \ge 0} I_0(t \psi_\varepsilon). \tag{3.29}$$

Moreover, there are two positive constants K_1, K_2 (independent on ε) such that

$$0 < K_1 \le t_\varepsilon \le K_2 < +\infty. \tag{3.30}$$

Proof. For any fixed $\varepsilon > 0$, by the definition of I_0, it is easy to see that $I_0(t\psi_\varepsilon) > 0$ for $t > 0$ small enough and $I_0(t\psi_\varepsilon) < 0$ for $t > 0$ large enough, so (3.29) holds. By Sobolev's inequality, we know that

$$||t_\varepsilon \psi_\varepsilon||_{p^*}^{p^*} \leq C\|t_\varepsilon \psi_\varepsilon\|^p = C(\|\nabla \psi_\varepsilon\|_p^p + \|\psi_\varepsilon\|_p^p)t_\varepsilon^p,$$

that is,

$$t_\varepsilon^{p^*-p} \leq C\frac{\|\nabla \psi_\varepsilon\|_p^p + \|\psi_\varepsilon\|_p^p}{\|\psi_\varepsilon\|_{p^*}^{p^*}} \leq A < +\infty, \ \forall \varepsilon \in (0,1), \tag{3.31}$$

by using (3.25) (3.26) and (3.28), where A does not depend on ε.

Now, we may assume that

$$t_\varepsilon \longrightarrow \ell \text{ as } \varepsilon \to 0. \tag{3.32}$$

Obviously, $0 \leq \ell < +\infty$. But we claim that there exists $\alpha > 0$ such that

$$\ell \geq \alpha > 0. \tag{3.33}$$

In fact, by (3.29), $\frac{d}{dt}|_{t=t_\varepsilon} I_0(t\psi_\varepsilon) = 0$, that is

$$\frac{\int (|\nabla \psi_\varepsilon|^p + c|\psi_\varepsilon|^p)}{\int |\psi_\varepsilon|^{p^*}} - t_\varepsilon^{p^*-p} - \frac{\int f(x, t_\varepsilon \psi_\varepsilon)\psi_\varepsilon}{t_\varepsilon^{p-1}\int |\psi_\varepsilon|^{p^*}} = 0, \tag{3.34}$$

by letting $\varepsilon \to 0$ in (3.34) and using (3.25)–(3.28) (3.32), we have

$$1 - \ell^{p^*-p} \leq \lim_{\varepsilon \to 0} |\frac{\int f(x, t_\varepsilon \psi_\varepsilon)\psi_\varepsilon}{t_\varepsilon^{p-1}\int |\psi_\varepsilon|^{p^*}}|. \tag{3.35}$$

On the other hand, by (2.19) with $\varepsilon = 1$, there is $C > 0$ such that

$$\begin{aligned} 1 - \ell^{p^*-p} &\leq \lim_{\varepsilon \to 0} |\frac{Ct_\varepsilon^{p^*-1}\int |\psi_\varepsilon|^{p^*} + t_\varepsilon^{p-1}\int |\psi_\varepsilon|^p}{t_\varepsilon^{p-1}\int |\psi_\varepsilon|^{p^*}}| \\ &= \lim_{\varepsilon \to 0} |\frac{t_\varepsilon^{p^*-p} C\int |\psi_\varepsilon|^{p^*} + \int |\psi_\varepsilon|^p}{\int |\psi_\varepsilon|^{p^*}}| \\ &= C\ell^{p^*-p}, \text{ by using (3.25)–(3.28) (3.32),} \end{aligned}$$

i.e. $(1 + C)\ell^{p^*-p} \geq 1$,then $\ell^{p^*-p} \geq \bar{C}$, hence (3.33) holds and (3.30) is proved by noticing (3.31) (3.33).

Proof of Theorem 1.3. We prove part i) by the following two cases.

Case 1): $J^\infty \leq \frac{1}{N}S^{\frac{N}{p}}$. In this case, we may choose $v_0 = \bar{u}$ ($\bar{u}$ given by (1.9)).

In fact, by $(f3)$ and the definition of I_0 (see (1.6)), we see that there is $t_0 > 0$ such that

$$\sup_{t\geq 0} I_0(t\bar{u}) = I_0(t_0\bar{u}). \tag{3.36}$$

Since $(f6)$, we may assume that (if necessary, replacing $\bar{u}(x)$ by $\bar{u}(x+y)$ for an appropriate $y \in R^N$):

$$I_0(t_0\bar{u}) < I^\infty(t_0\bar{u}). \tag{3.37}$$

Thus, it follows from (1.9) (3.36) and (3.37) that

$$\sup_{t\geq 0} I_0(t\bar{u}) = I_0(t_0\bar{u}) < I^\infty(t_0\bar{u}) \leq \sup_{t\geq 0} I^\infty(t\bar{u}) = J^\infty,$$

then by using $J^\infty < \frac{1}{N}S^{\frac{N}{p}}$, we know (1.11) is true.

Case 2): $J^\infty > \frac{1}{N}S^{\frac{N}{p}}$. In this case, we take $v_0 = t_\varepsilon\psi_\varepsilon$, where ψ_ε and t_ε are given by (3.24) and Lemma 3.1, respectively.

Since $(f5)$, $I_0(t_\varepsilon\psi_\varepsilon) \leq I^\infty(t_\varepsilon\psi_\varepsilon)$ and by Lemma 3.1 that

$$\sup_{t\geq 0} I_0(t\psi_\varepsilon) = I_0(t_\varepsilon\psi_\varepsilon) \leq I^\infty(t_\varepsilon\psi_\varepsilon).$$

Then (1.11) is proved if we know that

$$I^\infty(t_\varepsilon\psi_\varepsilon) < \frac{1}{N}S^{\frac{N}{p}}. \tag{3.38}$$

We turn now to prove (3.38). By the definition of I^∞ and (3.25)–(3.28),

$$\begin{aligned}
I^\infty(t_\varepsilon\psi_\varepsilon) =& \frac{t_\varepsilon^p}{p}\int(|\nabla\psi_\varepsilon|^p + c|\psi_\varepsilon|^p) - \frac{t_\varepsilon^{p^*}}{p^*}\int|\psi_\varepsilon|^{p^*} - \int\bar{F}(t_\varepsilon\psi_\varepsilon) \\
=& \left(\frac{t_\varepsilon^p}{p} - \frac{t_\varepsilon^{p^*}}{p^*}\right)S^{\frac{N}{p}} + O(\varepsilon^{\frac{N-p}{p}}) + O(\varepsilon^{\frac{N}{p}}) - \int\bar{F}(t_\varepsilon\psi_\varepsilon) \\
&+ \begin{cases} C_1\varepsilon^{\frac{N-p}{p}} + O(\varepsilon^{\frac{N-p}{p}}), & \text{if } p < N < p^2 \\ C_2\varepsilon^{p-1}|\ell n\varepsilon| + O(\varepsilon^{p-1}), & \text{if } p^2 = N \\ C_3\varepsilon^{p-1} + O(\varepsilon^{\frac{N-p}{p}}), & \text{if } N > p^2 \geq \sqrt{2} \end{cases} \\
\leq& \frac{1}{N}S^{\frac{N}{p}} - \int\bar{F}(t_\varepsilon\psi_\varepsilon) \\
&+ \begin{cases} C_1\varepsilon^{\frac{N-p}{p}} + O(\varepsilon^{\frac{N-p}{p}}), & \text{if } p < N < p^2 \\ C_2\varepsilon^{p-1}|\ell n\varepsilon| + O(\varepsilon^{p-1}), & \text{if } p^2 = N \\ C_3\varepsilon^{p-1} + O(\varepsilon^{\frac{N-p}{p}}), & \text{if } N > p^2 \geq \sqrt{2}. \end{cases}
\end{aligned} \tag{3.39}$$

On the other hand, by $(f4)$ $(f5)$, there is $C > 0$ such that

$$\bar{F}(t) \geq Ct^{p+\theta}, \text{ if } t \geq 1. \tag{3.40}$$

Choosing $\varepsilon > 0$ small enough such that

$$R\varepsilon^{-\frac{p-1}{p}} \geq (\varepsilon^{-\frac{p-1}{p}}K - 1)^{\frac{p-1}{p}}, \tag{3.41}$$

where $K = b^{\frac{p}{N-p}} K_1^{\frac{p}{N-p}}$, K_1 given by (3.30). Then,

$$\frac{bt_\varepsilon \varepsilon^{-\frac{N-p}{p}\cdot\frac{p-1}{p}}}{(1 + r^{\frac{p}{p-1}})^{\frac{N-p}{p}}} \geq 1 \text{ if } 0 < r \leq (\varepsilon^{-\frac{p-1}{p}}K - 1)^{\frac{p-1}{p}}. \tag{3.42}$$

By using (3.40) (3.41)) and (3.42) we have

$$\begin{aligned}
\int_{R^N} \bar{F}(t_\varepsilon \psi_\varepsilon)dx &= \varepsilon^{\frac{p-1}{p}N} \int_0^{R\varepsilon^{-\frac{p-1}{p}}} \bar{F}\Big(\frac{bt_\varepsilon \varepsilon^{\frac{N-p}{p}\cdot\frac{1-p}{p}}}{(1 + r^{\frac{p}{p-1}})^{\frac{N-p}{p}}}\Big) r^{N-1} dr \\
&\geq \varepsilon^{\frac{p-1}{p}N} \int_0^{(\varepsilon^{-\frac{p-1}{p}}K-1)^{\frac{p-1}{p}}} \bar{F}\Big(\frac{bt_\varepsilon \varepsilon^{\frac{N-p}{p}\cdot\frac{1-p}{p}}}{(1 + r^{\frac{p}{p-1}})^{\frac{N-p}{p}}}\Big) r^{N-1} dr, \text{ by (3.41)} \\
&\geq C\varepsilon^{p-1-\frac{(N-p)(p-1)}{p^2}\theta} \int_0^{(\varepsilon^{-\frac{p-1}{p}}K-1)^{\frac{p-1}{p}}} \frac{r^{N-1}dr}{(1 + r^{\frac{p}{p-1}})^{\frac{N-p}{p}(p+\theta)}},
\end{aligned} \tag{3.43}$$

by (3.40) (3.42).

It follows from (3.43) and Lemma 2.3 i) that

a): If $p < N < p^2$, $\theta > \frac{p(p^2-N)}{(N-p)(p-1)}$ and $\alpha = (p-1) - \frac{(N-p)(p-1)}{p^2}\theta - \frac{N-p}{p} < 0$, then

$$\begin{aligned}
\varepsilon^{-\frac{N-p}{p}} \int_{R^N} \bar{F}(t_\varepsilon \psi_\varepsilon) \geq C\varepsilon^{\alpha} \int_0^{(\varepsilon^{-\frac{p-1}{p}}K-1)^{\frac{p-1}{p}}} \frac{r^{N-1}dr}{(1 + r^{\frac{p}{p-1}})^{\frac{N-p}{p}(p+\theta)}} \\
\longrightarrow +\infty, \text{ as } \varepsilon \longrightarrow 0;
\end{aligned} \tag{3.44}$$

b): If $N = p^2, \theta > 0$, then

$$\begin{aligned}
&(\varepsilon^{p-1}|\ell n\varepsilon|)^{-1} \int_{R^N} \bar{F}(t_\varepsilon \psi_\varepsilon)dx \\
&\geq C\varepsilon^{-\frac{(p-1)^2}{p}} |\ell n\varepsilon|^{-1} \int_0^{(\varepsilon^{-\frac{p-1}{p}}K-1)^{\frac{p-1}{p}}} \frac{r^{N-1}dr}{(1 + r^{\frac{p}{p-1}})^{(p-1)(p+\theta)}} \\
&\longrightarrow +\infty, \text{ as } \varepsilon \longrightarrow 0;
\end{aligned} \tag{3.45}$$

c): If $N > p^2, \theta > 0$, then

$$\begin{aligned}
&\varepsilon^{1-p}\int_{R^N}\bar{F}(t_\varepsilon\psi_\varepsilon)dx \\
&\geq C\varepsilon^{-\frac{(N-p)(p-1)}{p^2}\theta}\int_0^{(\varepsilon^{-\frac{p-1}{p}}K-1)^{\frac{p-1}{p}}}\frac{r^{N-1}dr}{(1+r^{\frac{p}{p-1}})^{\frac{N-p}{p}(p+\theta)}} \qquad (3.46)\\
&\quad\longrightarrow +\infty,\ \text{as } \varepsilon\longrightarrow 0.
\end{aligned}$$

Combining (3.39) (3.44) (3.45) (3.46)), we see that (3.38) is true, so Case 2) and then part i) is proved.

Finally, we prove part ii). Indeed, by (1.11) there exists an $\varepsilon_0 > 0$ such that

$$\sup_{t\geq 0} I_0(tv_0) < \min\{J^\infty, \frac{1}{N}S^{\frac{N}{p}}\} - \varepsilon_0. \qquad (3.47)$$

For this $\varepsilon_0 > 0$, by Theorem 1.1, there is $M > 0$ such that $|c_1| < \frac{\varepsilon_0}{2}$ as $||h||_* \leq M$. Let $u \in \gamma_0 \stackrel{\triangle}{=} \{t\bar{t}v_0 : 0 \leq t \leq 1\}$ and $||h||_* < \frac{\varepsilon_0}{2\bar{t}||v_0||}$, then

$$|I(u) - I_0(u)| = |\int hu\,dx| \leq \bar{t}|\int hv_0dx| \leq \bar{t}||v_0||||h||_* < \frac{\varepsilon_0}{2}.$$

Taking $C = \min\{M, \frac{\varepsilon_0}{2\bar{t}||v_0||}\}$, then if $||h||_* \leq C$ we have

$$|c_1| < \frac{\varepsilon_0}{2} \text{ and } I(u) < I_0(u) + \frac{\varepsilon_0}{2}, \quad \forall u \in \gamma_0. \qquad (3.48)$$

Now by (3.47) (3.48), for c_2 given by (1.14), we have

$$\begin{aligned}
c_2 &\leq \sup_{u\in\gamma_0} I(u) \leq \sup_{u\in\gamma_0} I_0(u) + \frac{\varepsilon_0}{2} \leq \sup_{t\geq 0} I_0(tv_0) + \frac{\varepsilon_0}{2} \\
&< \min\{J^\infty, \frac{1}{N}S^{\frac{N}{p}}\} - \frac{\varepsilon_0}{2} < J^\infty + c_1.
\end{aligned}$$

this means (1.15), and the proof of Lemma 3.1 is completed.

Proof of Theorem 1.4. Combining Theorem 1.1 and (1.15), there exists $M_1 > 0$ such that (1.1) has a solution $u_1 \in W^{1,p}(R^N)$ with $c_1 = I(u_1) < 0$ and (1.15) holds.

On the other hand, by (2.20) we know that there exist $\rho > 0, M_2 > 0$ such that

$$I(u)|_{\partial B_\rho} > 0, \text{ if } \|h\|_* \leq M_2.$$

Then by the definition (1.14), $c_2 > 0$.

Noticing (1.12)–(1.14), it follows from the mountain pass theorem without (PS) Condition [5], we can find $\{u_n\} \subset W^{1,p}(R^N)$ such that

$$I(u_n) \xrightarrow{n} c_2 > 0; \quad I'(u_n) \xrightarrow{n} 0,$$

and $\{u_n\}$ is bounded in $W^{1,p}(R^N)$ by $(f4)$. Then we may assume that there is some $u_2 \in W^{1,p}(R^N)$ such that

$$u_n \overset{n}{\rightharpoonup} u_2 \text{ weakly in } W^{1,p}(R^N).$$

By Theorem 1.2, we know that u_2 is also a weak solution of (1.1) and

$$\text{either} \;\; c_2 = I(u_2); \quad \text{or} \quad c_2 \geq I(u_2) + J^\infty.$$

Letting $||h||_* \leq \min\{M_1, M_2\}$, then u_1, u_2 are solutions of (1.1), moreover,

$$c_1 = I(u_1) < 0; \;\; c_2 = I(u_2) > 0 \text{ or } \;\; c_2 \geq I(u_2) + J^\infty.$$

Thus, it is clear that $u_1 \not\equiv u_2$ by using (1.15).

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