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ON A MODEL BOLTZMANN EQUATION WITHOUT ANGULAR CUTOFF

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Abstract. A model Boltzmann equation (see formulas (1.1.6) – (1.1.9) below) without Grad's angular cutoff assumption is considered. One proves: 1) the instantaneous smoothing in both position and velocity variables by the evolution semigroup associated to the Cauchy problem for this model; 2) the derivation of the analogue of the Landau-Fokker-Planck equation in the limit when grazing collisions prevail.

1. Introduction and main result.

1.1. Introduction. Let us first recall the Boltzmann equation of rarefied gases (see [5] for example),

$$\partial_t f + v \cdot \nabla_x f = \tilde{Q}_B(f)\,. \tag{1.1.1}$$

In this equation, the unknown $f \equiv f(t,x,v)$ is the density of particles which at time $t \geq 0$ and position $x \in \mathbf{R}^3$ have velocity $v \in \mathbf{R}^3$, and $\tilde{Q}_B$ is a quadratic operator acting only on the v-dependence of f which takes into account only the binary collisions in the gas. It reads

$$\tilde{Q_B}(f)(v) = \int_{v_*\in\mathbf{R}^3} \int_{\omega\in S^2} \Big\{ f(v + (\omega\cdot(v_* - v))\,\omega)\, f(v_* - (\omega\cdot(v_*-v))\,\omega) \\ - f(v)\, f(v_*) \Big\} B(|v-v_*|, |\tfrac{v-v_*}{|v-v_*|}\cdot\omega|)\, d\omega dv_*, \tag{1.1.2}$$

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where B is a cross section depending on the type of interactions between the particles of the gas.

In the case where interparticle forces are proportional to r^{-s} (where r is the distance between the two particles under consideration) one has

$$B(x,y) = |x|^{\frac{s-5}{s-1}}\,\beta_s(y), \tag{1.1.3}$$

$$\beta_s(y) \sim_{y\to 0} |y|^{-\frac{s+1}{s-1}}, \tag{1.1.4}$$

so that $\tilde{Q}_B$ is defined only as a singular integral operator.

The singularity of β_s, together with the dissipative character of $\tilde{Q}_B$ make it plausible that this operator behaves as some nonlinear diffusion operator, i.e. a nonlinear analogue (of some fractional power) of the Laplacian Δ acting on the velocity space.

On the contrary, when the singularity in β_s is removed, that is, when Grad's "angular cutoff assumption" is made (cf. [18]), in other words, when $B \in L^1_{loc}(\mathbf{R}^3 \times S^2)$, the collision integral behaves roughly as a bounded operator on functions of the velocity variable.

This fundamental difference can be seen on the evolution semigroup associated to the Cauchy problem for the space homogeneous Boltzmann equation, that is

$$\partial_t f = \tilde{Q}_B(f). \tag{1.1.5}$$

When β_s is of the form (1.1.4), one expects that the associated semigroup should be a smoothing operator for any positive value of the time variable. This has been established on various models or particular cases: see [7], [9], [10], [23], for the 2D Boltzmann equation, radially symmetric or not.

On the contrary, in the cutoff case, it is known (see [21], [28], [3]) that this semigroup does not regularize the initial data for any positive value of the time variable.

It is therefore expected that (when β_s has the form (1.1.4)), the space inhomogeneous Boltzmann operator $\tilde{Q}_B - v\cdot\nabla_x$ should be the nonlinear analogue of a hypoelliptic operator, by analogy with the linear Fokker-Planck operator $\Delta_v - v\cdot\nabla_x$ (see for example [19]). An indication of this could be that the evolution semigroup of the Boltzmann equation regularizes the initial data in both the position and velocity variable. This is a widely open question at the moment, even for the most elementary traditional simplified models of the Boltzmann equation. This is due to the side-effects of many additional difficulties (among which the absence of maximum principle and the effect of large velocities seem to be the most important). This paper is aimed at introducing the simplest possible nonlinear model of (inhomogeneous) Boltzmann type equation (with a singularity as in (1.1.4)), and at trying to prove in this context the expected smoothing properties.

This model is somehow a reduction of Kac's collision integral [20] to velocities of norm 1; it also is reminiscent of a model proposed in [4]. It reads

$$\partial_t f(t,x,v) + \cos(2\pi v)\,\partial_x f(t,x,v) = Q_b(f)(t,x,v)\,, \tag{1.1.6}$$

where the unknown is the number density $f \equiv f(t,x,v)$. Here, $t \geq 0$ is the time variable, the position variable is $x \in \mathbf{T}^1$, and $v \in \mathbf{T}^1$ parametrizes the velocity $\cos(2\pi v)$ of the particles. The collision operator is given (at the formal level) by

$$Q_b(\phi)(v) = \int_{-1/2}^{1/2}\int_{\mathbf{T}^1} [\phi(v+\theta)\,\phi(v'-\theta) - \phi(v)\,\phi(v')]\,b(\theta)\,d\theta dv'\,, \tag{1.1.7}$$

where b is an even function on $[-1/2, 1/2]$ with positive values.

As usual, the notation $Q_b(f(t,x,v))$ designates the function of the variable v defined by $v \mapsto Q_b(f(t,x,\cdot))(v)$, the variables t and x being parameters in the collision integral.

In order to mimic the behaviour of β_s in (1.1.4), we postulate that (for all $\theta \in [-1/2, 1/2]$),

$$C_0\,|\theta|^{-\gamma} \leq b(\theta) \leq C_1\,|\theta|^{-\gamma}\,, \tag{1.1.8}$$

where $C_0, C_1 > 0$ and $1 < \gamma < 3$. Hence the collision integral (1.1.7) is a nonlinear singular integral. Finally, we introduce the initial data

$$f(0,x,v) = f_0(x,v)\,, \quad (x,v) \in \mathbf{T}^1 \times \mathbf{T}^1\,. \tag{1.1.9}$$

In order to establish the smoothing effect of the evolution of (1.1.6) in all variables t, x and v, one must depart from the method used on the space homogeneous Boltzmann equation. Indeed, this method is based on applying the Fourier transform in the velocity variable: see [7, 9, 10], [25] and [11]. While the collision integral is well behaved under this transformation, the advection operator is not.

The present paper proposes a different strategy, where steps 2 and 3 are reminiscent of [22]:

1. use the entropy production to control (fractional) derivatives in the velocity variable of the number density;
2. write (1.1.6) in the form
$$\partial_t f(t,x,v) + \cos(2\pi v)\,\partial_x f(t,x,v) = h(t,x,v)\,, \tag{1.1.10}$$
where h is a singular integral with respect to the velocity variable, and apply the Velocity Averaging Method to obtain some smoothness on quantities of the form
$$\int_{\mathbf{T}^1} f(t,x,v')\,\chi(v,v')\,dv,$$
where $\chi \in C^\infty(\mathbf{T}^1 \times \mathbf{T}^1)$; finally, keep track of the dependence on χ of the norm of this average (in some Besov or Sobolev space);
3. replacing χ by suitable approximations of the identity in step 2, try to get some regularity on f in all variables t, x and v, by using the results in step 1.

There are obvious shortcomings in this method: it is not completely clear how to iterate in order to obtain that $f \in C^\infty(\mathbf{R}_+^* \times \mathbf{T}^1 \times \mathbf{T}^1)$ (whether this is true is not yet known, albeit very likely). As for the analogy with the Boltzmann equation, note that two of the difficulties quoted above are not treated: the set of velocities is bounded in our model, and, more important, the natural functional spaces for the true Boltzmann equation are L^1 or, at best, $L\log L$ spaces (and not L^∞ as in our model). In either L^1 or $L\log L$ spaces, the gain of smoothness by Velocity Averaging is marginal, and the meaning of the collision operator unclear. Thus, applying the method above to models more realistic than (1.1.6-7), and in particular to the Boltzmann equation, would certainly require new ideas and lead to tremendous technicalities.

Notice however that step 1 was recently achieved in the case of the Boltzmann equation (without angular cutoff) by P.-L. Lions: see [23]. An optimal regularity estimate, also based on the entropy production term, but best suited to space homogeneous problems has also been obtained recently by Villani [26].

Even on the simplified model (1.1.6) considered here, the regularity of the solution obtained by our method is very likely not optimal. It could be that some bootstrap procedure leads to better regularity; yet this would certainly lead to rather technical developments.

There are equally obvious advantages: the method seems robust in the sense that it rests on physically intrinsic arguments, like the entropy inequality. Even cavitation (i.e., the local vanishing of the density) does not wholly ruin the argument in step 1, be it in the case of our model (1.1.6-7) or in that of the Boltzmann equation without cutoff (see [23]). Notice however that cavitation can be dealt with very easily in the case of our model (1.1.6-7) — although better regularity can be obtained in the case where cavitation does not occur: see Proposition 4 below — while it gives rise to noticeable difficulties on more physical models, as can be seen on the example of the Landau-Fokker-Planck equation (see [22]).

1.2. Main results. We begin with a precise functional definition of Q_b.

Proposition 1. *Let b satisfy* (1.1.8). *Then the operator Q_b defined by* (1.1.7) *is a continuous* (*nonlinear*) *operator from $C^2(\mathbf{T}^1)$ to $C^0(\mathbf{T}^1)$ which extends as a continuous operator from $L^1(\mathbf{T}^1)$ to $\mathcal{D}'(\mathbf{T}^1)$.*

As a corollary, it is now possible to define a solution f of (1.1.6)–(1.1.9) in the sense of distributions, as soon as $0 \le f_0 \in L^\infty(\mathbf{T}^1 \times \mathbf{T}^1)$ and $f \ge 0 \in L^\infty(\mathbf{R}_+ \times \mathbf{T}^1 \times \mathbf{T}^1) \cap C^0(\mathbf{R}_+; \mathcal{D}'(\mathbf{T}^1 \times \mathbf{T}^1))$.

A slightly stronger notion of solutions will be needed in the sequel, that of entropic solutions, defined below:

Definition 1. Let b satisfy (1.1.8) and $f_0 \ge 0 \in L^\infty(\mathbf{T}^1 \times \mathbf{T}^1)$. An entropic solution of (1.1.6)–(1.1.9) is a function $f \ge 0 \in L^\infty(\mathbf{R}_+^* \times \mathbf{T}^1 \times \mathbf{T}^1) \cap C(\mathbf{R}^+; \mathcal{D}'(\mathbf{T}^1 \times \mathbf{T}^1))$ satisfying (1.1.6) – (1.1.9) in the sense of distributions as well as the following entropy relation: for all $T > 0$,

$$
\begin{aligned}
&\tfrac{1}{2} \iint_{\mathbf{T}^1 \times \mathbf{T}^1} |f(T,x,v)|^2 \, dxdv \\
&+ \tfrac{1}{2} \int_0^T \int_{\mathbf{T}^1} \rho_f(t,x) \left(\iint_{\mathbf{T}^1 \times \mathbf{T}^1} |f(t,x,v+\theta) - f(t,x,v)|^2 \, b(\theta) \, d\theta dv \right) dxdt \\
&\le \tfrac{1}{2} \iint_{\mathbf{T}^1 \times \mathbf{T}^1} |f_0(x,v)|^2 \, dxdv \, . \qquad (1.2.1)
\end{aligned}
$$

The main result in this paper is the following:

Theorem A. *Let b satisfy* (1.1.8) *and $f_0 \geq 0 \in L^\infty(\mathbf{T}^1 \times \mathbf{T}^1)$. The Cauchy problem* (1.1.6)–(1.1.9) *admits an entropic solution $f \in H^{s(\gamma)-\epsilon}_{loc}(\mathbf{R}^*_+ \times \mathbf{T}^1 \times \mathbf{T}^1)$ for all $\epsilon > 0$ with*

$$s(\gamma) = \frac{\gamma - 1}{2\,(\gamma+1)\,(\gamma+3)}. \tag{1.2.2}$$

If $f_0 \geq R_0$ a.e. for some $R_0 > 0$, the value in the right hand side of (1.2.2) *can be replaced by the better regularity index*

$$s(\gamma) = \frac{\gamma - 1}{2\,(\gamma+1)^2}. \tag{1.2.3}$$

For example, the typical case $\gamma = 2$ leads to respectively $f \in H^{1/30}$ and $f \in H^{1/18}$: hence the smoothing effect is rather weak.

The problem (1.1.6) – (1.1.9) also admits an analogue of the Landau-Fokker-Planck asymptotics (see [24] as well as [8], [6], [26], [2]) in the limit when grazing collisions prevail — in the present case, when b is concentrated near $\theta = 0$.

In the case of the true Boltzmann equation, this limit has been proved only in particular situations: the case of the linearized inhomogeneous equation is considered in [8], while the spatially homogeneous equation is considered in [2] and [26]. The good features of our model allow to state a very general convergence theorem. Observe the specific scaling (1.2.4) below, aimed at zooming on the grazing collisions in the model collision integral (1.1.7): it is the analogue in the case of our model (1.1.6-7) of the scaling appearing in [8] and [26].

Theorem B. *Let $0 \leq f_0 \in L^\infty(\mathbf{T}^1 \times \mathbf{T}^1)$, and f^ε be an entropic solution of eq.* (1.1.6)–(1.1.9) *with b replaced by b_ε defined by*

$$b_\varepsilon(\theta) = \begin{cases} \varepsilon^{-3}\, b(\frac{\theta}{\epsilon}) & \text{if} \quad |\theta| \leq \frac{\varepsilon}{2}, \\ 0 & \text{if} \quad |\theta| \geq \frac{\varepsilon}{2}, \end{cases} \tag{1.2.4}$$

and where b satisfies (1.1.8). *Then there exists a subsequence (still denoted by f^ε) converging in $L^\infty(\mathbf{R}^*_+ \times \mathbf{T}^1 \times \mathbf{T}^1)$ weak-$*$ to a solution f of*

$$\partial_t f(t,x,v) + \cos(2\pi v)\partial_x f(t,x,v) = C\rho_f(t,x)\partial_v^2 f(t,x,v) \tag{1.2.5}$$

in the sense of distributions, where

$$C = \int_{-1/2}^{1/2} \theta^2 b(\theta) d\theta \,, \qquad \rho_f(t,x) = \int_{v \in \mathbf{T}^1} f(t,x,v)\, dv. \tag{1.2.6}$$

The outline of the paper is as follows: Section 2 reviews the precise definition of Q_b as a singular integral, thus leading in particular to Proposition 1. Different useful expressions of Q_b are also presented. Section 3 deals with the entropy relation, thereby leading to the existence of entropic solutions, and to step 1 in the method above. Section 4 establishes various results from the theory of Velocity Averaging which are crucial in proving step 2. The end of the proof of Theorem A (that is, step 3) belongs to Section 5, while the Landau-Fokker-Planck approximation (Theorem B) is established in Section 6.

2. The collision integral. To the function b satisfying (1.1.8) is associated a distribution of order 2 on $\mathbf{T}^1$, denoted by $PV(b)$ (the principal value of b) and defined for all $g \in C^2([-1/2, 1/2])$ by

$$< PV(b); g >= \int_{-1/2}^{1/2} \tfrac{1}{2} \Big[g(\theta) + g(-\theta) - 2\, g(0) \Big]\, b(\theta)\, d\theta \,. \tag{2.1}$$

With this definition at hand, Proposition 1 can be proved. In fact, we prove the slightly more precise

Lemma and Definition 1. *For all $\phi \in C^2(\mathbf{T}^1)$ and all $v \in \mathbf{T}^1$, consider the functions $F_v[\phi]$ defined by*

$$F_v[\phi](\theta, v') = \phi(v+\theta)\, \phi(v'-\theta) - \phi(v)\, \phi(v') \,, \tag{2.2}$$

and $G_v[\phi]$ defined by

$$G_v[\phi](\theta) = \phi(v+\theta) - \phi(v) \,. \tag{2.3}$$

1. Then

$$< PV(b) \otimes 1; F_v[\phi] >= \rho_\phi \;< PV(b); G_v[\phi] > \,, \tag{2.4}$$

with

$$\rho_\phi = \int_{\mathbf{T}^1} \phi(v)\, dv \,. \tag{2.5}$$

2. *The formula*

$$Q_b(\phi)(v) =< PV(b) \otimes 1; F_v[\phi] > , \quad \forall v \in \mathbf{T}^1 \tag{2.6}$$

defines a continuous operator $Q_b : C^2(\mathbf{T}^1) \to C^0(\mathbf{T}^1)$.

3. *The continuous operator* $Q_b : C^2(\mathbf{T}^1) \to C^0(\mathbf{T}^1)$ *extends as a continuous operator* $Q_b : L^1(\mathbf{T}^1) \to \mathcal{D}'(\mathbf{T}^1)$ *defined as follows: for all* $\phi \in L^1(\mathbf{T}^1)$ *and* $\chi \in C^2(\mathbf{T}^1)$,

$$< Q_b(\phi); \chi >= \rho_\phi \int_{\mathbf{T}^1} \phi(v) < PV(b); G_v[\chi] > dv \,. \tag{2.7}$$

Proof. Let $(X_n)_{n\geq 0} \in \mathbf{R}_+^{\mathbf{N}}$ be an increasing sequence converging to $+\infty$, and let $b_n(\theta) = \inf(b(\theta), X_n)$. Now, for all $n \in \mathbf{N}$, $b_n \in L^1([-1/2, 1/2])$ is even (because b itself is even) and for all $\phi \in L^1(\mathbf{T}^1)$ one has, by symmetrizing the integrand of (1.1.7) in the variable θ:

$$Q_{b_n}(\phi)(v) = \tfrac{1}{2} \int_{\mathbf{T}^1} \int_{-1/2}^{1/2} \Big\{ \phi(v+\theta)\, \phi(v'-\theta) + \phi(v-\theta)\, \phi(v'+\theta) - \, 2\, \phi(v)\, \phi(v') \Big\}\, b_n(\theta)\, d\theta dv' \,. \tag{2.8}$$

Also, one can apply Fubini's theorem and integrate first in the variable v' in (2.8). This gives

$$Q_{b_n}(\phi)(v) = \tfrac{1}{2}\rho_\phi \int_{-1/2}^{1/2} \Big[\phi(v+\theta) + \phi(v-\theta) - 2\, \phi(v) \Big]\, b_n(\theta)\, d\theta \,. \tag{2.9}$$

In other words, for all $n \in \mathbf{N}$, one has

$$Q_{b_n}(\phi)(v) =< PV(b_n) \otimes 1; F_v[\phi] >= \rho_\phi < PV(b_n); G_v[\phi] > \,. \tag{2.10}$$

Now, for all $g \in C^2([-1/2, 1/2])$, $g(\theta) + g(-\theta) - 2\, g(0) = O_{\theta\sim 0}(|\theta|^2)$, so that

$$\theta \mapsto \Big[g(\theta) + g(-\theta) - 2\, g(0) \Big]\, b(\theta) \, \in L^1(\mathbf{T}^1) \,. \tag{2.11}$$

Taking $\phi \in C^2(\mathbf{T}^1)$, one sees, by applying (2.11) to $g(\theta) = \phi(v+\theta)\, \phi(v'-\theta)$, and to $g(\theta) = \phi(v+\theta)$, that each side of (2.4) is well-defined. Letting

$n \to +\infty$ in (2.10) establishes, by dominated convergence, equality (2.4) as well as the relation

$$Q_b(\phi)(v) = \lim_{n\to+\infty} Q_{b_n}(\phi)(v) = \rho_\phi < PV(b); G_v[\phi] > \ . \tag{2.12}$$

As for point 2, observe that the continuity is made obvious by formula (2.12). Now for point 3: if $\chi \in C^2(\mathbf{T}^1)$, one obviously has

$$\begin{aligned}&\int_{\mathbf{T}^1} \chi(v)\, Q_{b_n}(\phi)(v)\, dv \\ &= \rho_\phi \int_{\mathbf{T}^1}\int_{-1/2}^{1/2} \tfrac{1}{2}\,\phi(v)\left[\chi(v-\theta)+\chi(v+\theta)-2\,\chi(v)\right] b_n(\theta)\, d\theta dv\,. \end{aligned}\tag{2.13}$$

Letting $n \to +\infty$ and applying the first equality in (2.12) results in (2.7).

An immediate consequence of (2.7) is that the collision integral (1.2) conserves the total number of particles:

Corollary 1. *For all $\phi \in L^1(\mathbf{T}^1)$, one has*

$$< Q_b(\phi); 1 >= 0\,. \tag{2.14}$$

Proof. Apply (2.7) with $\chi = 1$.

No other quantity (such as momentum, energy, etc.) is conserved in this model. This will become clear in Section 3, where we prove that $Q_b(\phi) = 0$ only when ϕ is a constant.

Next we proceed to another way of defining Q_b, as the second derivative (in the velocity variable) of a nonsingular integral operator. The following proposition can be viewed as yet another definition of the collision integral (1.1.7); we shall not use it specifically until Section 6.

Notice that it is also possible to write Boltzmann's collision integral as a divergence (with respect to the v variable); this idea can be found in §41 of [24] and is a possible starting point of the derivation of Landau's collision operator from Boltzmann's collision integral. The computations in [24] have recently been put on more mathematical footing by Villani [27].

Proposition 2. *For all $r \in \mathbf{R}_+$ and $z \in \mathbf{T}^1$, consider the expression*

$$A(r,z) = \tfrac{1}{2}\,(r - |z|)_+ \tag{2.15}$$

(where, for all $z \in \mathbf{T}^1$, $|z| \in [0,1/2]$ designates the geodesic distance to 0*). Let b satisfy* (1.1.8) *and consider the function $B_b : \mathbf{R} \to [0,+\infty]$ defined by*

$$B_b(z) = \int_{-1/2}^{1/2} A(|\theta|, z)\, b(\theta)\, d\theta\,, \quad \forall z \in \mathbf{T}^1\,. \tag{2.16}$$

Then,

1. *for all $z \in \mathbf{T}^1$, one has*

$$0 \le B_b(z) \le D(1 + |z|^{\gamma-2}) \tag{2.17}$$

   for some $D > 0$ (depending only on γ, C_0, C_1),

2. *for all $\phi \in L^1(\mathbf{T}^1)$, one has* (*compare with* [24])

$$Q_b(\phi) = \partial_v^2 S_b(\phi)\,, \qquad \text{where} \qquad S_b(\phi) = \rho_\phi\, \phi * B_b\,. \tag{2.18}$$

Proof. One has $A(|\theta|, z) \le \frac{1}{2}\, |\theta|\, \mathbf{1}_{|z| \le |\theta|}$ so that

$$\int_{-1/2}^{1/2} b_n(\theta)\, A(|\theta|, z)\, d\theta \le \int_0^{1/2} \theta\, \mathbf{1}_{|z| \le \theta}\, b(\theta)\, d\theta \le D + \frac{D}{|z|^{\gamma-2}}\,. \tag{2.19}$$

This proves 1. Now, integrating by parts twice shows that, for all $\theta \in [-1/2, 1/2]$ and all $\chi \in C^2(\mathbf{T}^1)$:

$$\begin{aligned} &\tfrac{1}{2}\big[\chi(v-\theta) + \chi(v+\theta) - 2\chi(v)\big] \\ &= -\tfrac{1}{2} \int_{\mathbf{T}^1} \big(\mathbf{1}_{[v-|\theta|, v]}(w) - \mathbf{1}_{[v, v+|\theta|]}(w)\big)\partial_w \chi(w)\, dw \\ &= \int_{\mathbf{T}^1} A(|\theta|, v-w)\, \partial_w^2 \chi(w)\, dw. \end{aligned} \tag{2.20}$$

Notice that, for all $v \in \mathbf{T}^1$ and all $\theta \in \mathbf{R}$, expressions like $v - \theta$ are to be understood as the image of v under the translation by $-\theta$ (which obviously induces a map on $\mathbf{T}^1$ viewed as $\mathbf{R}/\mathbf{Z}$). On the other hand, intervals like $[v-\theta, v]$ are to be understood as arcs on the circle of length 1 identified to $\mathbf{T}^1$.

Applying (2.10) and (2.20) shows that, for all $n \in \mathbf{N}$ and all $\phi \in L^1(\mathbf{T}^1)$:

$$< Q_{b_n}(\phi); \chi > = \rho_\phi \iint_{\mathbf{T}^1\times\mathbf{T}^1} \phi(v)\, B_{b_n}(v-w)\, \partial_w^2\chi(w)\, dwdv$$
$$= \rho_\phi \int_{\mathbf{T}^1} \partial_w^2\chi(w)\, (B_{b_n} * \phi)(w)\, dw\,. \tag{2.21}$$

Here, the symbol $*$ denotes the convolution is the sense of $\mathbf{T}_1$. Since $1 < \gamma < 3$,

$$\int_{\mathbf{T}^1} \frac{C_0}{\text{dist }(v,w)^{\gamma-2}} dw < +\infty\,. \tag{2.22}$$

Next take the limit as $n \to +\infty$ in (2.21): applying the dominated convergence theorem with (2.17) and (2.22) leads to

$$< Q_b(\phi); \chi >= \rho_\phi \int_{\mathbf{T}^1} \partial_v^2\chi(v)\, (B_b * \phi)(v)\, dv\,, \tag{2.23}$$

which proves (2.18).

3. The entropy relation. This section is devoted to an analogue of Boltzmann's H-theorem for the model (1.1.6) – (1.1.9).

The following simple lemma reflects once again the fact that the collision cross section b is even:

Lemma and Definition 2. *For all ϕ and $\psi \in C^2(\mathbf{T}^1)$,*

$$\int_{\mathbf{T}^1} < PV(b); G_v[\phi] > \psi(v) dv = \iint_{[-1/2,1/2]\times\mathbf{T}^1} G_v[\phi](\theta) G_v[\psi](\theta) b(\theta)\, d\theta dv\,. \tag{3.0}$$

This formula extends naturally the definition of the left hand side of (3.0) *to the case where ϕ and $\psi \in C^1(\mathbf{T}^1)$ only. In particular Q_b extends as a mapping from $C^1(\mathbf{T}^1)$ into distributions of order* 1 *on $\mathbf{T}^1$ and verifies, for all ϕ, $\psi \in C^1(\mathbf{T}^1)$:*

$$< Q_b(\phi); \psi >= -\tfrac{1}{2}\rho_\phi \iint_{[-1/2,1/2]\times\mathbf{T}^1} [\phi(v+\theta)-\phi(v)][\psi(v+\theta)-\psi(v)] b(\theta)\, d\theta dv. \tag{3.1}$$

Specializing (3.1) *to the case where $\phi = \psi$ leads to the following extended definition: for all $\phi \in L^2(\mathbf{T}^1)$, the notation $\rho_\phi^{-1} < Q_b(\phi); \phi >$ designates the following element of $[0, +\infty]$:*

$$\tfrac{1}{2} \iint_{\mathbf{T}^1\times\mathbf{T}^1} |\phi(v+\theta) - \phi(v)|^2\, b(\theta)\, d\theta dv\,.$$

Proof. Formula (3.0) is recovered from (3.1) and (2.4-6) if $\rho_\phi > 0$ everywhere, which can be ensured by considering $\phi + C$ in the place of ϕ. It suffices to prove (3.1) in the case where $\phi = \psi \in C^2(\mathbf{T}^1)$, which corresponds to the classical H Theorem. The general case follows by density and polarization. Since b is even, for all $\phi \in C^1(\mathbf{T}^1)$ and all $n \geq 0$, formula (2.9) shows that

$$Q_{b_n}(\phi)(v) = \rho_\phi \int_{-1/2}^{1/2} [\phi(v+\theta) - \phi(v)]\, b_n(\theta)\, d\theta\,. \tag{3.2}$$

Hence,

$$\begin{aligned}\int_{\mathbf{T}^1} \phi(v)\ Q_{b_n}(\phi)(v)\, dv &= \rho_\phi \int_{\mathbf{T}^1} \int_{-1/2}^{1/2} [\phi(v+\theta) - \phi(v)]\, \phi(v)\, b_n(\theta)\, d\theta dv \\ &= \rho_\phi \int_{\mathbf{T}^1} \int_{-1/2}^{1/2} [\phi(v) - \phi(v-\theta)]\, \phi(v-\theta)\, b_n(\theta)\, d\theta dv \\ &= \rho_\phi \int_{\mathbf{T}^1} \int_{-1/2}^{1/2} [\phi(v) - \phi(v+\theta)]\, \phi(v+\theta)\, b_n(\theta)\, d\theta dv \\ &= -\tfrac{1}{2}\, \rho_\phi \int_{\mathbf{T}^1} \int_{-1/2}^{1/2} [\phi(v+\theta) - \phi(v)]^2\, b_n(\theta)\, d\theta dv\,. \end{aligned} \tag{3.3}$$

Now, since $\phi \in C^2(\mathbf{T}^1)$, $|\phi(v+\theta) - \phi(v)|^2 = O_{\theta\sim 0}(\theta^2)$ so that $\theta \mapsto |\phi(v+\theta) - \phi(v)|^2\, b(\theta) \in L^1(\mathbf{T}^1)$ thanks to assumption (1.1.8) on b. By dominated convergence,

$$\begin{aligned}\int_{\mathbf{T}^1} \phi(v)\, Q_b(\phi)(v)\, dv &= \lim_{n\to+\infty} \int_{S^1} \phi(v)\, Q_{b_n}(\phi)(v)\, dv \\ &= -\tfrac{1}{2}\rho_\phi \int_{\mathbf{T}^1} \int_{-1/2}^{1/2} [\phi(v+\theta) - \phi(v)]^2\, b(\theta)\, d\theta dv\,,\end{aligned}$$

which proves (3.1).

A well-known consequence of the H-theorem in the case of the classical Boltzmann equation is that the only nonnegative integrable number densities for which the collision integral vanishes are local Maxwellian distributions. The analogous result for (1.1.6) – (1.1.9) is the following

Corollary 2. *Let $\phi \in L^1(\mathbf{T}^1)$ be such that $Q_b(\phi) = 0$. Then ϕ is equal to a constant a.e..*

Proof. If $\rho_\phi = 0$, then $\phi = 0$ a.e. and the theorem is proved. If $\rho_\phi \neq 0$, $\rho_\phi^{-1} Q_b(\phi) = 0$. Let $Z \in C_0^\infty(\mathbf{R})$ be a nonnegative even function supported in $[-1,1]$ and denote, for all $\epsilon \in]0,1/2[$, $\zeta_\epsilon(v) = \sum_{k\in\mathbf{Z}} \frac{1}{\epsilon} Z\left(\frac{v+k}{\epsilon}\right)$. By (2.7), for all $\epsilon \in]0,1/2[$,

$$\rho_\phi^{-1}\, Q_b(\phi) * \zeta_\epsilon = 0 = \rho_\phi^{-1}\, Q_b(\phi * \zeta_\epsilon)\,. \tag{3.4}$$

Since $\phi * \zeta_\epsilon \in C^\infty(\mathbf{T}^1)$, (3.4) shows that for all $\epsilon \in]0,1/2[$, $\phi * \zeta_\epsilon = C_\epsilon$, where C_ϵ is a constant. But

$$C_\epsilon = \int_{\mathbf{T}^1} \phi * \zeta_\epsilon(v)\, dv = \int_{\mathbf{T}^1} \phi(v)\, dv,$$

which shows that C_ϵ is in fact independent of ϵ: hence, for all $\epsilon \in]0,1/2[$,

$$\phi * \zeta_\epsilon = C \tag{3.5}$$

where C is a constant. As $\epsilon \to 0$, the left side of (3.5) converges vaguely to ϕ. Therefore $\phi = C$ as a measure on $\mathbf{T}^1$, that is to say a.e.

Note that as announced in section 2, this proves that the only conserved quantity is the mass.

We do not know whether all L^∞ solutions of (1.1.6) – (1.1.9) in the sense of distributions necessarily satisfy an entropy inequality. However, by truncating the collision cross section b, we prove that there exist entropic solutions to (1.1.6) – (1.1.9) for any bounded nonnegative initial data.

Proposition 3. *Let* $0 \le f_0 \in L^\infty(\mathbf{T}^1 \times \mathbf{T}^1)$*, and* b *satisfy* (1.1.8). *Then there exists an entropic solution of* (1.1.6)–(1.1.9) *such that*

$$0 \le f(t,x,v) \le \|f_0\|_{L^\infty}\,, \quad \textit{a.e. on } \mathbf{R}_+ \times \mathbf{T}^1 \times \mathbf{T}^1. \tag{3.6}$$

If moreover $f_0 \ge R_0$ *a.e. for some* $R_0 > 0$*, then* $f(t,x,v) \ge R_0$ *for a.e.* $(t,x,v) \in \mathbf{R}_+ \times \mathbf{T}^1 \times \mathbf{T}^1$.

Proof. Consider first the model equation (1.1.6) with b replaced by its truncation b_n as in the proof of Lemma and Definition 1. To begin with, (2.9) holds for all $\phi \in L^1(\mathbf{T}^1)$ (by the density of $C^1(\mathbf{T}^1)$ in $L^1(\mathbf{T}^1)$). This can be recast as:

$$Q_{b_n}(\phi)(v) = \rho_\phi \int_{-1/2}^{1/2} \phi(v+\theta)\, b_n(\theta)\, d\theta - \|b_n\|_{L^1} \rho_\phi\, \phi(v) \tag{3.7}$$

for all $\phi \in L^1(\mathbf{T}^1)$. In addition, we have:

Lemma 1. *Let* $0 \le f_0 \in L^\infty(\mathbf{T}^1 \times \mathbf{T}^1)$*. For all* $n \in \mathbf{N}$*, there exists a solution* $f^n \in L^\infty(\mathbf{R}_+^* \times \mathbf{T}^1 \times \mathbf{T}^1) \cap C(\mathbf{R}_+; L^1(\mathbf{T}^1 \times \mathbf{T}^1))$ *to the problem*

$$(\partial_t f^n + \cos(2\pi v)\,\partial_x f^n)(t,x,v) + \|b_n\|_{L^1}\rho_{f^n}(t,x)\,f^n(t,x,v) = \rho_{f^n}(t,x)\int_{-1/2}^{1/2} f^n(t,x,v+\theta)\,b_n(\theta)\,d\theta\,; \tag{3.8}$$

$$f^n(0,x,v) = f_0(x,v)\,, \quad (x,v) \in \mathbf{T}^1 \times \mathbf{T}^1\,. \tag{3.9}$$

It satisfies

$$0 \le f^n(t,x,v) \le \|f_0\|_{L^\infty}\,, \quad \text{a.e. on } \mathbf{R}_+^* \times \mathbf{T}^1 \times \mathbf{T}^1. \tag{3.10}$$

Moreover, if $f_0 \ge R_0$ *a.e. for some* $R_0 > 0$*, then* $f^n(t,\cdot) \ge R_0$ *a.e. for all* $t > 0$*.*

The proof of Lemma 1 is classical and deferred until after that of Proposition 3.

The standard Velocity Averaging lemmas (Cf. [12] for example) together with estimates (2.17), (2.18) imply that the sequence f^n converges (possibly after extraction of a subsequence) to f in $L^\infty(\mathbf{R}_+^* \times \mathbf{T}^1 \times \mathbf{T}^1)$ weak * while ρ_{f^n} converges to ρ_f a.e..

Then, for all $\chi \in C^2(\mathbf{R}_+^* \times \mathbf{T}^1 \times \mathbf{T}^1)$, the quantity $< PV(b_n), G_v(\chi) >$ converges a.e. (in t,x,v) to $< PV(b), G_v(\chi) >$, so that $Q_{b_n}(f^n)$ converges to $Q_b(f)$ in the sense of distributions, and f is a solution in the sense of distributions of (1.1.6).

By the uniform estimates in $L^\infty(\mathbf{R}_+^* \times \mathbf{T}^1 \times \mathbf{T}^1)$ on f^n, (3.6) and the last affirmation of Proposition 3 are clear.

It remains to prove the entropy condition (1.2.1). Let $0 \le \chi \in C_0^\infty(\mathbf{R})$ and denote $\chi_\epsilon(t) = \epsilon^{-1}\chi(t/\epsilon)$. The solution f^n is extended to negative values of t by the value 0. Multiplying (3.8) by $(\chi_\epsilon \otimes \zeta_\epsilon \otimes \zeta_\epsilon) * f^n$ (where ζ_ϵ is defined in the proof of Corollary 2), integrating in all variables and letting $\epsilon \to 0$ leads, for all $T > 0$, to

$$\tfrac{1}{2}\iint_{\mathbf{T}^1\times\mathbf{T}^1} |f^n(T,x,v)|^2\,dxdv + \tfrac{1}{2}\int_0^T\!\!\int_{\mathbf{T}^1} \rho_{f^n}(t,x)\Big(\iint_{\mathbf{T}^1\times\mathbf{T}^1} |f^n(t,x,v+\theta) - f^n(t,x,v)|^2\,b_m(\theta)\,d\theta dv\Big)dxdt$$

$$\leq \tfrac{1}{2}\iint_{\mathbf{T}^1\times\mathbf{T}^1} |f^n(T,x,v)|^2\,dxdv$$
$$+\tfrac{1}{2}\int_0^T\int_{\mathbf{T}^1}\rho_{f^n}(t,x)\Big(\iint_{\mathbf{T}^1\times\mathbf{T}^1}|f^n(t,x,v+\theta)-f^n(t,x,v)|^2\,b_n(\theta)\,d\theta dv\Big)dxdt$$
$$=\tfrac{1}{2}\iint_{\mathbf{T}^1\times\mathbf{T}^1}|f_0(x,v)|^2\,dxdv\,, \tag{3.11}$$

for all $m \leq n \in \mathbf{N}$ according to the proof of Lemma 3.1. Since ρ_{f^n} converges a.e. towards ρ_f, we get, by convexity and weak convergence, (the integer m being fixed while $n \to +\infty$) that for all $T > 0$,

$$\tfrac{1}{2}\iint_{\mathbf{T}^1\times\mathbf{T}^1}|f(T,x,v)|^2\,dxdv$$
$$+\tfrac{1}{2}\int_0^T\int_{\mathbf{T}^1}\rho_f(t,x)\Big(\iint_{\mathbf{T}^1\times\mathbf{T}^1}|f(t,x,v+\theta)-f(t,x,v)|^2\,b_m(\theta)\,d\theta dv\Big)dx\,dt$$
$$\leq\tfrac{1}{2}\iint_{\mathbf{T}^1\times\mathbf{T}^1}|f_0(x,v)|^2\,dxdv\,. \tag{3.12}$$

Letting then $m \to +\infty$ gives the entropy inequality (1.2.1).

Proof of Lemma 1. The quickest route to this result is to generate for each $n \in \mathbf{N}$ a sequence $(f^n_m)_{m\in\mathbf{N}}$ by the following iteration procedure:

$$f^n_0 = 0, \quad (\text{ or } \ f^n_0 = R_0 \ \text{ if } \ f_0 \geq R_0 > 0 \ a.e.) \tag{3.13}$$

and, for all $m \geq 1$:

$$\partial_t f^n_m(t,x,v) + \cos(2\pi v)\,\partial_x f^n_m(t,x,v) + \rho_{f^n_{m-1}}(t,x) f^n_m(t,x,v)$$
$$= \rho_{f^n_{m-1}}(t,x)\int_{-1/2}^{1/2} f^n_{m-1}(t,x,v+\theta)\,b_n(\theta)\,d\theta\,; \tag{3.14}$$
$$f^n_m(0,x,v) = f_0(x,v)\,, \quad (x,v)\in\mathbf{T}^1\times\mathbf{T}^1\,. \tag{3.15}$$

Now (3.14) can be solved explicitly and it is easy to show that $(f^n_m)_{m\in\mathbf{N}}$ converges pointwise to a limit denoted by f^n. The convergence also holds in L^1 (by dominated convergence). Taking the limit as $m \to +\infty$ in (3.14) – (3.15) leads to the announced result. An easy induction argument shows that, if $f_0 \geq R_0$ a.e., then $f^n_m \geq R_0$ a.e. for all $m \geq 0$.

The entropy inequality (1.2.1) provides a regularity estimate in the v variable for all entropic solutions to (1.1.6)–(1.1.9). The main difficulty is to take into account cavitation, i.e., zones where the density might vanish. In the case of our model (1.1.6-7), we know from Proposition 3 that cavitation cannot occur unless it is already present in the initial data. This property is not shared by the true Boltzmann equation, and [23] finds a way around this by proving Besov regularity in the v variable on $\sqrt{f}$ instead of f itself. In the case of our model (1.1.6-7), cavitation is treated by the following simple argument:

Proposition 4. *Let* $0 \leq f_0 \in L^\infty(\mathbf{T}^1 \times \mathbf{T}^1)$*, b satisfy (1.1.8) and f be an entropic solution of* (1.1.6)–(1.1.9). *For all* $T > 0, \epsilon > 0$*, there exists a constant C (depending on* $T, ||f_0||_{L^\infty}, \gamma, C_0, C_1, \epsilon$*) such that*

$$\int_0^T \int_{\mathbf{T}^1} \int_{\mathbf{T}^1} \int_{-1/2}^{1/2} |f(t,x,v+\theta) - f(t,x,v)|^2 \, |\theta|^{-1-\frac{\gamma-1}{2}+\frac{\epsilon}{2}} \, d\theta dx dv dt \leq C \, .$$

If moreover $f_0 \geq R_0$ *a.e. for some* $R_0 > 0$*, then*

$$\int_0^T \int_{\mathbf{T}^1} \int_{\mathbf{T}^1} \int_{-1/2}^{1/2} |f(t,x,v+\theta) - f(t,x,v)|^2 \, |\theta|^{-\gamma} \, d\theta dx dv dt \leq C \, ,$$

where C depends on the previous parameters and R_0.

Proof. The case when $f_0 \geq R_0$ a.e. for some $R_0 > 0$ is a simple consequence of the entropy inequality (1.2.1). In the general case, one has to isolate the points where cavitation can appear. Estimate (1.2.1) gives

$$\begin{aligned}
&\int_0^T \int_{\mathbf{T}^1} \int_{\mathbf{T}^1} \int_{-1/2}^{1/2} |f(t,x,v+\theta) - f(t,x,v)|^2 \, |\theta|^{-1-\frac{\gamma-1}{2}+\frac{\epsilon}{2}} \, d\theta dx dv dt \\
&\leq \int_{\mathbf{T}^1} \int_{-1/2}^{1/2} \left(\int \int_{\rho_f(t,x) \geq |\theta|^{\frac{\gamma-1}{2}+\frac{\epsilon}{2}}} |f(t,x,v+\theta) - f(t,x,v)|^2 \, dx dt \right) \\
&\qquad\qquad |\theta|^{-1-\frac{\gamma-1}{2}+\frac{\epsilon}{2}} \, d\theta dv \\
&+ \int_{\mathbf{T}^1} \int_{-1/2}^{1/2} \left(\int \int_{\rho_f(t,x) \leq |\theta|^{\frac{\gamma-1}{2}+\frac{\epsilon}{2}}} \left\{ 2|f(t,x,v+\theta)|^2 + 2|f(t,x,v)|^2 \right\} dx dt \right) \\
&\qquad\qquad |\theta|^{-1-\frac{\gamma-1}{2}+\frac{\epsilon}{2}} \, d\theta dv
\end{aligned}$$

$$\leq \int_0^T \int_{\mathbf{T}^1} \int_{\mathbf{T}^1} \int_{-1/2}^{1/2} \rho(t,x) \, |f(t,x,v+\theta) - f(t,x,v)|^2 \, |\theta|^{-\gamma} \, d\theta dx dv dt$$
$$+ \int_0^T \int_{\mathbf{T}^1} \int_{-1/2}^{1/2} ||f||_{L^\infty} \, \mathbf{1}_{\{\rho(t,x) \leq |\theta|^{\frac{\gamma-1}{2}+\frac{\epsilon}{2}}\}} \, 4\, \rho(t,x) \, |\theta|^{-1-\frac{\gamma-1}{2}+\frac{\epsilon}{2}} \, d\theta dx dt$$
$$\leq C_0^{-1} \, ||f_0||^2_{L^2(\mathbf{T}^1 \times \mathbf{T}^1)} + 4 \, ||f||_{L^\infty} \, T \int_{-1/2}^{1/2} |\theta|^{-1+\epsilon} \, d\theta,$$

whence the desired result follows.

Proposition 4 provides the regularity estimate in the v-variable which is precisely step 1 in the method described in subsection 1.1.

4. Velocity averaging. In this section, we return to the classical estimates of the Velocity Averaging method first introduced in [1], [15], [14]. The goal is to keep track of the dependence of the estimates (in Sobolev spaces) for velocity averages like

$$\int_{\mathbf{T}^1} f(t,x,v') \, \chi(v,v') \, dv' \tag{4.0}$$

on the norms of derivatives of the smooth function χ. This can be done with the original methods of proof in the references quoted above.

In order to deal with equations of the type (1.1.10) (specifically, to be able to treat fractional derivatives in v in the right–hand side), we use the method of [12] and [16], and adapt the computations to our case.

In the sequel, we say that $f(z) << g(z)$ (when f, g are two real-valued functions of z) if there exists some constant $C > 0$ (independent of z) such that $f(z) \leq C \, g(z)$.

Let us first establish the following technical result.

Lemma 2. *For all x and $y \in \mathbf{R}$,*

$$I_{x,y} = \int_{\mathbf{T}^1} \mathbf{1}_{|x+y\cos(2\pi v)| \leq 1} \, dv << (1 + |x|^2 + |y|^2)^{-1/4} \, ; \tag{4.1}$$

$$J_{x,y} = \int_{\mathbf{T}^1} \mathbf{1}_{|x+y\cos(2\pi v)| > 1} \, \frac{dv}{|x + y\cos(2\pi v)|^2} << (1 + |x|^2 + |y|^2)^{-1/4} \, . \tag{4.2}$$

Proof. For 1, set $w = |y| \cos(2\pi v)$.

$$I_{x,y} = \tfrac{1}{\pi} \int_{-1+|x|}^{1+|x|} \mathbf{1}_{|w| \leq |y|} \, \frac{dw}{\sqrt{|y|^2 - |w|^2}} \leq \tfrac{1}{\pi} \, \mathbf{1}_{|x| \leq |y|+1} \int_{\sup(|y|-2, -|y|)}^{|y|} \frac{dw}{\sqrt{|y|^2 - |w|^2}}$$

$$\leq \tfrac{1}{\pi}\,\mathbf{1}_{|x|\leq|y|+1}\int_{\sup(1-2/|y|,-1)}^{1}\frac{d\lambda}{\sqrt{1-\lambda^2}} << \mathbf{1}_{|x|\leq|y|+1}\ \inf(1,|y|^{-1/2})$$

$$<< \mathbf{1}_{|x|\leq|y|+1}\,(1+|y|^2)^{-1/4} << (1+|x|^2+|y|^2)^{-1/4}\,.$$

As for 2, set $\lambda=\cos(2\pi v)$:

$$J_{x,y}=\tfrac{1}{\pi}\int_{-1}^{1}\frac{1}{|x+\lambda y|^2}\,\mathbf{1}_{|x+\lambda y|>1}\,\frac{d\lambda}{\sqrt{1-\lambda^2}}\,. \tag{4.3}$$

If $|x|\geq 2\,|y|$, $\forall\lambda\in[-1,1]$, one has $|x+\lambda y|\geq\frac{1}{2}\,|x|$ and $|x+\lambda y|\geq 1$ implies $|x|\geq\frac{1}{2}$. In which case, (4.3) shows that

$$J_{x,y}\leq\tfrac{1}{\pi}\int_{-1}^{1}\frac{4}{|x|^2}\,\mathbf{1}_{|x|>1/2}\,\frac{d\lambda}{\sqrt{1-\lambda^2}} << (1+x^2)^{-1} << (1+x^2+y^2)^{-1}$$

$$\leq(1+x^2+y^2)^{-1/4}\,. \tag{4.4}$$

It remains to deal with the case when $x=-py$ with $|p|<2$: one has

$$J_{x,y}=\frac{1}{|y|^2}\int_{-1}^{1}\frac{\mathbf{1}_{|\lambda-p|\geq|y|^{-1}}}{|\lambda-p|^2}\frac{d\lambda}{\sqrt{1-\lambda^2}}$$

and by symmetry, only the case of $p\geq 0$ and $y>0$ will matter. Moreover,

$$J_{x,y}\leq\int_{-1}^{1}\frac{d\lambda}{\sqrt{1-\lambda^2}},$$

so that it suffices to prove that $J_{x,y} << |y|^{-1/2}$ for $|y|\geq 2$. In order to establish this estimate, one distinguishes four different cases: $p\in[0,1-y^{-1}]$, $p\in[1-y^{-1},1]$, $p\in[1,1+y^{-1}]$ and finally $p\in[1+y^{-1},2]$. The last three cases are quite easy to treat; to cut short, we only consider the first. Using the inequalities $0\leq p\leq 1-\frac{1}{y}$ and $y\geq 2$, one sees that

$$J_{x,y}\leq\frac{1}{y^2}\int_{p+y^{-1}}^{1}\frac{d\lambda}{(\lambda-p)^2\sqrt{1-\lambda}}$$

$$+\frac{\sqrt{2}}{y^2}\int_{-1/2}^{p-y^{-1}}\frac{d\lambda}{(p-\lambda)^2\sqrt{1-\lambda}}+\frac{4}{y^2}\int_{-1}^{-1/2}\frac{d\lambda}{\sqrt{1-\lambda^2}}\,. \tag{4.5}$$

The third integral in the right side of (4.5) being trivial, it remains to estimate the first and the second. This is done by changing the variable λ into $u = \sqrt{1-\lambda}$; thus, the second integral for example becomes

$$\int_{-1/2}^{p-y^{-1}} \frac{d\lambda}{(p-\lambda)^2\sqrt{1-\lambda}} = \int_{\sqrt{1+y^{-1}-p}}^{\sqrt{3/2}} \frac{2du}{(1-p-u^2)^2}$$

$$\leq \tfrac{1}{2(1-p)} \int_{\sqrt{1+y^{-1}-p}}^{\sqrt{3/2}} \tfrac{du}{(\sqrt{1-p}-u)^2} \leq \frac{y}{2\sqrt{1-p}}\Big(1+\sqrt{1+\frac{1}{y(1-p)}}\Big) \leq \tfrac{1+\sqrt{2}}{2} y\sqrt{y}.$$

Then the contribution of the second term in the right hand side of (4.5) is $O(y^{-1/2})$; a similar computation shows that the first term is of exactly the same order.

We proceed next to stating the main result in this section; it is an amplification of the Velocity Averaging results of [12] and of the Appendix of [16]. We first need the following:

Notation 1.

1. *For* $\alpha \in]0,2[$*, the following seminorm will be used in the sequel:*

$$\|h\|_{2,\alpha} = \left(\int_{\mathbf{T}^1}\int_{-1/2}^{1/2} |h(w+\theta)-h(w)|^2\,|\theta|^{-1-\alpha}\,d\theta dw\right)^{1/2}, \quad (4.6)$$

$$\|h\|_{\infty,2,\alpha} = \left(\sup_{w\in\mathbf{T}^1}\int_{-1/2}^{1/2} |h(w+\theta)-h(w)|^2\,|\theta|^{-1-\alpha}\,d\theta\right)^{1/2}.$$

2. *For all* $f \in L^1(\mathbf{T}^1)$ *and all* $\phi \in C^1(\mathbf{T}^1)$*,*

$$< f >_\phi = \int_{\mathbf{T}^1} f(v)\,\phi(v)\,dv\,. \quad (4.7)$$

3. *The notation* $\hat{f}(\tau,\xi,v)$ *designates the Fourier transform of* f *in the variables* t *and* x *(*v *being a parameter).*

Proposition 5. *Let* $f \in L^2(\mathbf{R}\times\mathbf{T}^1\times\mathbf{T}^1)$ *and let* g *satisfy, for some* $\alpha \in]0,2[$*:*

$$\int_{\mathbf{R}\times\mathbf{T}^1} (\|g(t,x,\cdot)\|_{2,\alpha})^2 dtdx < +\infty\,. \quad (4.8)$$

Assume that

$$(\partial_t + \cos(2\pi v)\,\partial_x) f(t,x,v) = < PV(\beta); G_v[g(t,x,\cdot)] > \tag{4.9}$$

in the sense of distributions, for some β *verifying* $C_0\theta^{-1-\alpha} \le \beta(\theta) \le C_1\theta^{-1-\alpha}$, *for some* $C_0, C_1 > 0$, $\alpha \in]0,2[$. *Then, for all* $\epsilon > 0$, *there exists* $C(\epsilon)$ *such that, for all* $\phi \in C^1(\mathbf{T}^1)$, $\tau \in \mathbf{R}$ *and* $\xi \in \mathbf{Z}$*:*

$$\begin{aligned} &| < \hat f >_\phi (\tau,\xi)|^2 \\ << C(\epsilon)\,\|\phi\|^2_{L^\infty} (\tau^2+\xi^2)^{-\frac{1}{2(2+\alpha+\epsilon)}} &\Big[\|\hat g(\tau,\xi,\cdot)\|^2_{2,\alpha} + \|\hat f(\tau,\xi,\cdot)\|^2_{L^2(\mathbf{T}^1)}\Big] \\ &+ \|\phi\|^2_{\infty,2,\alpha} \|\hat g(\tau,\xi,\cdot)\|^2_{2,\alpha} (\tau^2+\xi^2)^{-\frac{1+2(\alpha+\epsilon)}{2(2+\alpha+\epsilon)}} . \end{aligned} \tag{4.10}$$

Proof. Let $\chi \in C^\infty(\mathbf{R})$ be even and satisfy

$$0 \le \chi \le 1\,, \quad \chi_{|[0,\frac{1}{2}]} = 0\,, \quad \chi_{|[1,+\infty[} = 1\,, \quad \|\chi'\|_{L^\infty} \le 3\,. \tag{4.11}$$

Writing (4.9) in the Fourier variables $\xi \in \mathbf{Z}$ and $\tau \in \mathbf{R}$, we get for all $\eta > 0$,

$$i(\tau + \cos(2\pi v)\,\xi) \hat f(\tau,\xi,v) = < PV(\beta); G_v[\hat g(\tau,\xi,\cdot)] >, \tag{4.12}$$

so that, by formula (3.0) and some obvious density argument

$$\begin{aligned} < \hat f >_\phi (\tau,\xi) &= \int_{\mathbf{T}^1} \big[1 - \chi\big(\frac{\tau + \cos(2\pi v)\,\xi}{\eta}\big)\big] \hat f(\tau,\xi,v)\phi(v)dv \\ &+ \int_{\mathbf{T}^1}\int_{-1/2}^{1/2} G_v[\hat g(\tau,\xi,\cdot)](\theta) G_v\big[v \mapsto \frac{\phi(v)\,\chi\big(\frac{\tau+\cos(2\pi v)\,\xi}{\eta}\big)}{i(\tau+\cos(2\pi v)\,\xi)}\big]\beta(\theta)d\theta dv\,. \end{aligned}$$

Therefore

$$\begin{aligned} | < \hat f >_\phi (\tau,\xi)|^2 \le 2\,\|\phi\|^2_{L^\infty} &\int_{\mathbf{T}^1} |\hat f(\tau,\xi,v)|^2 dv \int_{\mathbf{T}^1} \mathbf{1}_{|\tau+\cos(2\pi v)\,\xi| \le \eta}\, dv \\ &+ 2\,C_1\,\|\hat g(\tau,\xi,\cdot)\|^2_{2,\alpha} \\ \times \int_{\mathbf{T}^1}\int_{-1/2}^{1/2} \Big| &\frac{\phi(v+\theta)\chi\left(\frac{\tau+\cos(2\pi(v+\theta))\,\xi}{\eta}\right)}{i(\tau+\cos(2\pi(v+\theta))\,\xi)} - \frac{\phi(v)\,\chi\big(\frac{\tau+\cos(2\pi v)\,\xi}{\eta}\big)}{i(\tau+\cos(2\pi v)\,\xi)}\Big|^2 \frac{d\theta dv}{|\theta|^{1+\alpha}} \end{aligned} \tag{4.14}$$

$$\leq 2\,\|\phi\|^2_{L^\infty}\|\hat f(\tau,\xi,\cdot)\|^2_{L^2}\,I_{\frac{\tau}{\eta},\frac{\xi}{\eta}}$$

$$+2C_1\,\|\hat g(\tau,\xi,\cdot)\|^2_{2,\alpha}\int_{\mathbf{T}^1}\int_{-1/2}^{1/2}|\phi(v+\theta)-\phi(v)|^2\left|\frac{\chi\left(\frac{\tau+\cos(2\pi v)\,\xi}{\eta}\right)}{i(\tau+\cos(2\pi v)\,\xi)}\right|^2\frac{d\theta dv}{|\theta|^{1+\alpha}}$$

$$+\,2\,C_1\,\|\hat g(\tau,\xi,\cdot)\|^2_{2,\alpha}\|\phi\|^2_{L^\infty}$$

$$\times\int_{\mathbf{T}^1}\int_{-1/2}^{1/2}\left|\frac{\chi\left(\frac{\tau+\cos(2\pi(v+\theta))\,\xi}{\eta}\right)}{i(\tau+\cos(2\pi(v+\theta))\,\xi)}-\frac{\chi\left(\frac{\tau+\cos(2\pi v)\,\xi}{\eta}\right)}{i(\tau+\cos(2\pi v)\,\xi)}\right|^2\frac{d\theta dv}{|\theta|^{1+\alpha}} \qquad (4.15)$$

$$\leq 2\,\|\phi\|^2_{L^\infty}\,\|\hat f(\tau,\xi,\cdot)\|^2_{L^2}\,I_{\frac{\tau}{\eta},\frac{\xi}{\eta}}+2\,C_1\,\|\hat g(\tau,\xi,\cdot)\|^2_{2,\alpha}\,\|\phi\|^2_{\infty,2,\alpha}\,\frac{2}{\eta^2}\,J_{\frac{2\tau}{\eta},\frac{2\xi}{\eta}}$$

$$+\,2\,C_1\,\|\hat g(\tau,\xi,\cdot)\|^2_{2,\alpha}\,\|\phi\|^2_{L^\infty}\,\frac{1}{\eta^2}\,\mathcal{T}\,, \qquad (4.16)$$

where

$$\mathcal{T}=\int_{\mathbf{T}^1}\int_{-1/2}^{1/2}\left|S\left(\frac{\tau+\cos(2\pi(v+\theta))\,\xi}{\eta}\right)-S\left(\frac{\tau+\cos(2\pi v)\,\xi}{\eta}\right)\right|^2\frac{d\theta dv}{|\theta|^{1+\alpha}} \qquad (4.17)$$

with the notation $S(x)=\chi(x)/x$ for all $x\in\mathbf{R}$. Then

$$\mathcal{T}\leq\int_{\mathbf{T}^1}\int_{-1/2}^{1/2}\left|S\left(\frac{\tau+\cos(2\pi(v+\theta))\,\xi}{\eta}\right)-S\left(\frac{\tau+\cos(2\pi v)\,\xi}{\eta}\right)\right|^{2-\alpha-\epsilon}$$

$$\times\left|\frac{2\,\pi\,\xi\,\theta}{\eta}\int_0^1 S'\left(\frac{\tau+\cos(2\pi(v+l\theta))\,\xi}{\eta}\right)\sin(2\pi(v+l\theta))\,dl\right|^{\alpha+\epsilon}\frac{d\theta dv}{|\theta|^{1+\alpha}} \qquad (4.18)$$

$$<<\left|\frac{\xi}{\eta}\right|^{\alpha+\epsilon}\int_{-1/2}^{1/2}\left[\int_{\mathbf{T}^1}\left(\left|S\left(\frac{\tau+\cos(2\pi(v+\theta))\,\xi}{\eta}\right)\right|^2\right.\right.$$

$$\left.\left.+\left|S\left(\frac{\tau+\cos(2\pi v)\,\xi}{\eta}\right)\right|^2\right)dv\right]\frac{d\theta}{|\theta|^{1-\epsilon}}$$

$$+\left|\frac{\xi}{\eta}\right|^{\alpha+\epsilon}\int_{-1/2}^{1/2}\int_{\mathbf{T}^1}\int_0^1\left|S'\left(\frac{\tau+\cos(2\pi(v+l\theta))\,\xi}{\eta}\right)\right|^2 dl dv\frac{d\theta}{|\theta|^{1-\epsilon}} \qquad (4.19)$$

$$\leq C(\epsilon)\left|\frac{\xi}{\eta}\right|^{\alpha+\epsilon}\int_{\mathbf{T}^1}(|S|^2+|S'|^2)\left(\frac{\tau+\cos(2\pi v)\,\xi}{\eta}\right)dv. \qquad (4.20)$$

Clearly,

$$|S(r)|^2 + |S'(r)|^2 \leq \frac{\chi(r)}{r^2} + \frac{2|\chi'(r)|^2}{r^2} + \frac{2\chi(r)}{r^4} \leq \mathbf{1}_{r\geq 1/2}\ \frac{27}{r^2}\,. \tag{4.21}$$

Using (4.21) in (4.20) leads to

$$\mathcal{T} \leq C(\epsilon)\Big|\frac{\xi}{\eta}\Big|^{\alpha+\epsilon} J_{\frac{2\tau}{\eta},\frac{2\xi}{\eta}}\,. \tag{4.22}$$

Collecting estimates (4.16) and (4.22) leads to

$$\begin{aligned} | < \hat{f} >_\phi (\tau,\xi)|^2 &\leq 2\,\|\phi\|^2_{L^\infty}\|\hat{f}(\tau,\xi,\cdot)\|^2_{L^2}\, I_{\frac{\tau}{\eta},\frac{\xi}{\eta}} \\ &+ 4\,C_1\,\|\hat{g}(\tau,\xi,\cdot)\|^2_{2,\alpha}\,\|\phi\|^2_{\infty,2,\alpha}\,\frac{1}{\eta^2}\,J_{\frac{2\tau}{\eta},\frac{2\xi}{\eta}} \\ &+ 2\,C(\epsilon)\,\|\hat{g}(\tau,\xi,\cdot)\|^2_{2,\alpha}\,\|\phi\|^2_{L^\infty}\,\frac{1}{\eta^2}\Big|\frac{\xi}{\eta}\Big|^{\alpha+\epsilon} J_{\frac{2\tau}{\eta},\frac{2\xi}{\eta}}\,. \end{aligned} \tag{4.23}$$

Now it suffices to appeal to Lemma 2,

$$| < \hat{f} >_\phi (\tau,\xi)|^2 << \eta^{1/2}(\xi^2+\tau^2)^{-1/4}\Big[\|\phi\|^2_{L^\infty}\|\hat{f}(\tau,\xi,\cdot)\|^2_{L^2} \tag{4.24}$$

$$+\frac{1}{\eta^2}\|\hat{g}(\tau,\xi,\cdot)\|^2_{2,\alpha}\|\phi\|^2_{\infty,2,\alpha} + C(\epsilon)\frac{1}{\eta^2}\frac{(\xi^2+\tau^2)^{\frac{\alpha+\epsilon}{2}}}{|\eta|^{\alpha+\epsilon}}\|\hat{g}(\tau,\xi,\cdot)\|^2_{2,\alpha}\|\phi\|^2_{L^\infty}\Big].$$

Choosing

$$\eta = (\xi^2+\tau^2)^{\frac{\alpha+\epsilon}{2(2+\alpha+\epsilon)}} \tag{4.25}$$

leads to

$$| < \hat{f} >_\phi (\tau,\xi)|^2 << (\xi^2+\tau^2)^{-\frac{1}{2(2+\alpha+\epsilon)}}\,\|\phi\|^2_{L^\infty}\Big[\|\hat{f}(\tau,\xi,\cdot)\|^2_{L^2}$$

$$+\,\|\hat{g}(\tau,\xi,\cdot)\|^2_{2,\alpha}\Big] + (\xi^2+\tau^2)^{-\frac{1+2(\alpha+\epsilon)}{2(2+\alpha+\epsilon)}}\,\|\phi\|^2_{2,\alpha}\,\|\hat{g}(\tau,\xi,\cdot)\|^2_{\infty,2,\alpha}\,. \tag{4.26}$$

Remark 1. Clearly, Proposition 5 can be generalized to dimensions higher than one, to spaces other than L^2 (Besov spaces for examples, by repeating the same method on a dyadic decomposition) etc. We have not sought the maximum generality, but just the statement that fits the problem of interest here.

5. Proof of Theorem A. We begin with a proposition which achieves what is prescribed in step 3 of the method in Section 1.

Proposition 6. *Let $f \in L^2(\mathbf{R} \times \mathbf{T}^1 \times \mathbf{T}^1)$ be such that, for some $\delta \in]0,2[$:*

$$\int_{\mathbf{R}} \int_{\mathbf{T}^1} \|f(t,x,\cdot)\|^2_{2,\delta}\, dx\, dt < +\infty \tag{5.1}$$

and there exists $C > 0$, $\alpha \in]0,2[$, β_1, $\beta_2 \in \mathbf{R}^*_+$ and nonnegative functions h_1, $h_2 \in L^1(\mathbf{R}^2)$ such that, for all $\phi \in C^1(\mathbf{T}^1)$,

$$|<\hat{f}>_\phi (\tau,\xi)|^2$$
$$\leq C\left[\|\phi\|^2_{L^\infty}(\tau^2+\xi^2)^{-\beta_1} h_1(\tau,\xi) + \|\phi\|^2_{\infty,2,\alpha}(\tau^2+\xi^2)^{-\beta_2} h_2(\tau,\xi)\right]. \tag{5.2}$$

Then, one has

$$\int_{\mathbf{R}\times\mathbf{T}^1\times\mathbf{T}^1} \mathbf{1}_{\tau^2+\xi^2\geq 1}\,(\tau^2+\xi^2)^{\inf(\beta_1\frac{\delta}{\delta+2},\beta_2\frac{\delta}{\alpha+\delta+2})}|\hat{f}(\tau,\xi,v)|^2 d\tau d\xi dv < +\infty. \tag{5.3}$$

Proof. One has

$$\int_{\mathbf{T}^1} |\hat{f}(\tau,\xi,v)|^2\, dv << \int_{\mathbf{T}^1} \left| \int_{\mathbf{T}^1} \hat{f}(\tau,\xi,w)\, s(w-v)\, dw \right|^2 dv$$
$$+ \int_{\mathbf{T}^1} \left| \int_{\mathbf{T}^1} [\hat{f}(\tau,\xi,v) - \hat{f}(\tau,\xi,v+w)]\, s(w)\, dw \right|^2 dv \tag{5.4}$$

for all $s \in L^1(\mathbf{T}^1)$ such that $s \geq 0$ a.e. and $\int_{\mathbf{T}^1} s(v) dv = 1$. By assumption (5.2)

$$\int_{\mathbf{T}^1} |\hat{f}(\tau,\xi,v)|^2\, dv << (\tau^2+\xi^2)^{-\beta_1} h_1(\tau,\xi) \int_{\mathbf{T}^1} \|s(\cdot - v)\|^2_{L^\infty}\, dv$$
$$+ (\tau^2+\xi^2)^{-\beta_2} h_2(\tau,\xi) \int_{\mathbf{T}^1} \|s(\cdot - v)\|^2_{2,\alpha}\, dv$$
$$+ \int_{\mathbf{T}^1} \left| \int_{\mathbf{T}^1} [\hat{f}(\tau,\xi,v) - \hat{f}(\tau,\xi,v+w)]\, s(w)\, dw \right|^2 dv$$
$$<< (\tau^2+\xi^2)^{-\beta_1} h_1(\tau,\xi)\, \|s\|^2_{L^\infty} + (\tau^2+\xi^2)^{-\beta_2} h_2(\tau,\xi)\, \|s\|^2_{\infty,2,\alpha}$$
$$+ \int_{\mathbf{T}^1} \int_{\mathbf{T}^1} |\hat{f}(\tau,\xi,v) - \hat{f}(\tau,\xi,v+w)|^2\, s(w)\, dw dv\,. \tag{5.5}$$

Next we choose a particular type of function s. Specifically, we choose a family $(s_\eta)_{\eta>0}$ of functions in $L^1(\mathbf{T}^1)$ such that $s_\eta \geq 0$ a.e. and $\int_{\mathbf{T}^1} s_\eta(v)\,dv = 1$ for all $\eta > 0$, verifying, in addition, the following estimates (as $\eta \to 0$):

$$s_\eta(\theta) << \theta^{-1-\delta}\,\eta^\delta\,, \tag{5.6}$$

$$\|s_\eta\|^2_{L^\infty} << \eta^{-2}, \tag{5.7}$$

$$(\|s_\eta\|_{\infty,2,\alpha})^2 << C(\epsilon)\,\eta^{-\alpha-2}\,. \tag{5.8}$$

The construction of such a family $(s_\eta)_{\eta>0}$ is postponed until after the proof of Proposition 6. Then

$$\int_{\mathbf{T}^1} |\hat{f}(\tau,\xi,v)|^2\,dv << \eta^{-2}(\tau^2+\xi^2)^{-\beta_1}h_1(\tau,\xi) + \eta^{-\alpha-2}(\tau^2+\xi^2)^{-\beta_2}h_2(\tau,\xi) + \eta^\delta\,(\|\hat{f}(\tau,\xi,\cdot)\|_{2,\alpha})^2\,. \tag{5.9}$$

When $\tau^2+\xi^2 \geq 1$, optimizing in η leads to

$$\int_{\mathbf{T}^1} |\hat{f}(\tau,\xi,v)|^2 dv << (\tau^2+\xi^2)^{-\inf(\beta_1\frac{\delta}{\delta+2},\beta_2\frac{\delta}{\alpha+\delta+2})} \times \Big(h_1(\tau,\xi) + h_2(\tau,\xi) + (\|\hat{f}(\tau,\xi,\cdot)\|_{2,\alpha})^2\Big)\,. \tag{5.10}$$

The proof is complete modulo the construction of the family $(s_\eta)_{\eta>0}$.

Construction of the family $(s_\eta)_{\eta>0}$. We introduce the function $t_\eta(v) = \big(\sup(|v|,\eta)\big)^{-1-\delta}$, for all $v \in]-1/2, 1/2[$ and $\eta > 0$ (small enough), and we extend it by periodicity to get an element of $L^\infty(\mathbf{T}^1)$. First, observe that $||t_\eta||_{L^1(\mathbf{T}^1)} = (2+\frac{2}{\delta})\,\eta^{-\delta} + O(1)\,,\ ||t_\eta||_{L^\infty(\mathbf{T}^1)} = \eta^{-1-\delta}$. Also, since $t_\eta(x) = t_1(\frac{x}{\eta})\,\eta^{-1-\delta}$ one has, for all $x \in \mathbf{R}, \eta > 0$,

$$\begin{aligned}\|t_\eta\|^2_{\infty,2,\alpha} &= \sup_{v\in\mathbf{T}^1}\int_{-1/2}^{1/2} |t_\eta(v+\theta) - t_\eta(v)|^2\,|\theta|^{-1-\alpha}\,d\theta \\ &\leq \sup_{v\in\mathbf{R}}\int_{-\infty}^{+\infty} |t_\eta(v+\theta)-t_\eta(v)|^2\,|\theta|^{-1-\alpha}\,d\theta \\ &\leq \eta^{-2-2\,\delta-\alpha}\sup_{y\in\mathbf{R}}\int_{-\infty}^{+\infty} |t_1(y+\tau)-t_1(y)|^2\,|\tau|^{-1-\alpha}\,d\tau << \eta^{-2-2\,\delta-\alpha}\,.\end{aligned}$$

Defining $s_\eta(v) = \frac{t_\eta(v)}{||t_\eta||_{L^1(\mathbf{T}^1)}}$, we get estimates (5.6)–(5.8).

Proof of Theorem A. We finally apply the estimates above to entropic solutions of eqs. (1.1.6)–(1.1.9). Let $0 \le f_0 \in L^\infty(\mathbf{T}^1 \times \mathbf{T}^1)$, and b satisfy (1.1.8). Then, we know thanks to Proposition 3 that there exists an entropic solution f to (1.1.6)–(1.1.9). Let g be defined by

$$g(t,x,v) = \begin{cases} \rho_f(t,x) f(t,x,v) & \text{if} \quad t \ge 0, \\ 0 & \text{if} \quad t \le 0. \end{cases}$$

Then, one can extend f on $\mathbf{R} \times \mathbf{T}^1 \times \mathbf{T}^1$ in such a way that (4.9) holds.

By the inequality (1.2.1) and the L^∞ bound on f, (4.8) also holds with $\alpha = \gamma - 1$. Applying Proposition 5, we see that the estimates of Proposition 6 hold with $\beta_1 = \frac{1}{2\,(1+\gamma+\epsilon)}$, $\beta_2 = \frac{1+2\,(\gamma-1-\epsilon)}{2\,(1+\gamma+\epsilon)}$, $\alpha = \gamma - 1$, and (by Proposition 5), $\delta = \frac{\gamma-1}{2} - \frac{\epsilon}{2}$ (or $\delta = \gamma - 1$ if $f_0 \ge R_0$ a.e. for some $R_0 > 0$).

This gives the following estimate (for all $\epsilon > 0$)

$$\int_{\mathbf{R}\times\mathbf{T}^1\times\mathbf{T}^1} 1_{\tau^2+\xi^2\ge 1}\, (\tau^2+\xi^2)^{\frac{\gamma-1}{2\,(\gamma+1)\,(\gamma+3)}-\epsilon} |\hat{f}(\tau,\xi,v)|^2 \, d\tau d\xi dv < +\infty \, .$$

In addition, Proposition 4 shows that $f \in L^2(\mathbf{R}_+ \times \mathbf{T}^1; H^{\frac{\gamma-1}{2}-\epsilon}(\mathbf{T}^1))$, which establishes the first part of Theorem A.

The corresponding estimates if $f_0 \ge R_0$ a.e. for some $R_0 > 0$ are

$$\int_{\mathbf{R}\times\mathbf{T}^1\times\mathbf{T}^1} 1_{\tau^2+\xi^2\ge 1} (\tau^2+\xi^2)^{\frac{\gamma-1}{2\,(\gamma+1)^2}-\epsilon} |\hat{f}(\tau,\xi,v)|^2 \, d\tau d\xi dv < +\infty$$

and $f \in L^2(\mathbf{R}_+ \times \mathbf{T}^1; H^{\gamma-1}(\mathbf{T}^1))$, and the second part of theorem A is proven.

6. The Landau-Fokker-Planck approximation. In this last section, we give a short proof of Theorem B.

Proof of Theorem B. The L^∞ estimate on f^ε shows that, up to extraction of a subsequence, f^ε converges to f in $L^\infty(\mathbf{R}_+^* \times \mathbf{T}^1 \times \mathbf{T}^1)$ weak-*.

Using the following variant of formulation (2.18),

$$\partial_t f^\varepsilon(t,x,v) + \cos(2\pi v)\partial_x f^\varepsilon(t,x,v) = \rho_f^\varepsilon(t,x)$$

$$\times\, \partial_v^2 \Big\{ \int_{-1/2}^{1/2} \int_{-1}^{1} 2\,(1-|u|)\, f^\varepsilon(v + \varepsilon\, u\, \theta)\, du\, |\theta|^2\, b(\theta)\, d\theta \Big\}$$

and observing that

$$\left| \int_{-1/2}^{1/2} \int_{-1}^{1} 2\,(1-|u|)\, f^{\varepsilon}(v+\varepsilon\, u\, \theta)\, du\, |\theta|^2\, b(\theta)\, d\theta \right|$$

$$\leq ||f^{\varepsilon}||_{L^{\infty}(\mathbf{R}_+^* \times \mathbf{T}^1 \times \mathbf{T}^1)} \int_{-1/2}^{1/2} \theta^2\, b(\theta)\, d\theta \leq ||f_0||_{L^{\infty}(\mathbf{T}^1 \times \mathbf{T}^1)}\, 2\, C_1\, (1/2)^{3-\gamma}\,,$$

we appeal to the standard averaging lemmas (see [12]) to show that the quantity $\rho_{f^{\varepsilon}}$ converges a.e. on $\mathbf{R}_+^* \times \mathbf{T}^1$ (again up to extraction of a subsequence) towards ρ_f.

In order to pass to the limit in the nonlinear collision term, we only need to show the following: for all smooth function φ of the variable v, the quantity

$$\rho_f^{\varepsilon}(t,x) \int_{\mathbf{T}^1} \int_{-\varepsilon/2}^{\varepsilon/2} f^{\varepsilon}(t,x,v) \left\{ \varphi(v+\theta) + \varphi(v-\theta) - 2\,\varphi(v) \right\} \varepsilon^{-3}\, b(\tfrac{\theta}{\epsilon})\, d\theta dv$$

$$= \rho_f^{\varepsilon}(t,x) \int_{\mathbf{T}^1} f^{\varepsilon}(t,x,v) \left\{ \int_{-1/2}^{1/2} \int_{-1}^{1} 2\,(1-|u|)\, \varphi''(v+\epsilon\, u\, \theta)\, du\, |\theta|^2\, b(\theta)\, d\theta \right\} dv$$

converges (in $L^{\infty}(\mathbf{R}_+^* \times \mathbf{T}^1)$ weak-*) towards

$$\rho_f(t,x) \int_{\mathbf{T}^1} f(t,x,v)\, \varphi''(v)\, dv \int_{-1/2}^{1/2} \theta^2\, b(\theta)\, d\theta\,.$$

But this follows at once from the convergence a.e. of $\rho_{f^{\varepsilon}}$.

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REFERENCES

[1] V.I. Agoshkov, *Spaces of functions with differential-difference characteristics and the smoothness of solutions of the transport equation*, Dokl. Akad. Nauk SSSR., 276 (1984), 1289–1293.

[2] A.A. Arsen'ev and O.E. Buryak, *On the connection between a solution of the Boltzmann equation and a solution of the Landau-Fokker-Planck equation*, Math. USSR Sbornik, 69 (1991), 465–478.

[3] F. Bouchut and L. Desvillettes, *A proof of the smoothing properties of the positive part of Boltzmann's kernel*, Rev. Mat. Iberoam., to appear.

[4] N. Sibgatullin and H. Cabannes, *Homogeneous solutions for a semicontinuous model of the Boltzmann equation*, Math. Models Methods Appl. Sci., 5 (1995), 1129–1138.

[5] C. Cercignani, "Mathematical Methods in Kinetic Theory," Second edition, Plenum Press, New York, (1990).

[6] P. Degond and B. Lucquin-Desreux, *The Fokker-Planck asymptotics of the Boltzmann collision operator in the Coulomb case*, Math. Models Methods Appl. Sci., 2 (1992), 167–182.

[7] L. Desvillettes, *About the regularizing properties of the non cutoff Kac equation*, Comm. Math. Phys., 168 (1995), 417–440.

[8] L. Desvillettes, *On asymptotics of the Boltzmann equation when the collisions become grazing*, Transport Theory Statist. Phys., 21 (1992), 259–276.

[9] L. Desvillettes, *Regularization for the non cutoff 2D radially symmetric Boltzmann equation with a velocity dependent cross section*, Proceedings of the Second International Workshop on Nonlinear Kinetic Theories and Mathematical Aspects of Hyperbolic Systems (Sanremo, 1994), Transport Theory Statist. Phys., 25 (1996), 383–394.

[10] L. Desvillettes, *Regularization properties of the 2-dimensional non radially symmetric non cutoff spatially homogeneous Boltzmann equation for Maxwellian molecules*, Transport Theory Statist. Phys., 26 (1997), 341–357.

[11] L. Desvillettes and F. Golse, *On the smoothing properties of a model Boltzmann equation without Grad's cutoff assumption*, 21st Symposium on Rarefied Gas Dynamics, Marseilles (1998).

[12] R.J. DiPerna and P.-L. Lions, *Global weak solutions of Vlasov-Maxwell systems*, Comm. Pure Appl. Math., 42 (1989), 729–757.

[13] P. Gérard and F. Golse, *Averaging regularity results for PDEs under transversality assumptions*, Comm. Pure Appl. Math., 45 (1992), 1–26.

[14] F. Golse, P.-L. Lions, B. Perthame, and R. Sentis, *Regularity of the moments of the solution of a transport equation*, J. Funct. Anal., 76 (1988), 110–125.

[15] F. Golse, B. Perthame, and R. Sentis, *Un résultat de compacité pour les équations de transport et application au calcul de la limite de la valeur propre principale d'un opérateur de transport*, C.RȦcad. Sci. Paris Sér. I Math., 301 (1985), 341–344.

[16] F. Golse and F. Poupaud, *Limite fluide des équations de Boltzmann des semiconducteurs pour une statistique de Fermi-Dirac*, Asymptotic Anal., 6 (1992), 135–160.

[17] T. Goudon, *On Boltzmann equations and Fokker-Planck asymptotics: influence of grazing collisions*, J. Statist. Phys., 89 (1997), 751–776.

[18] H. Grad, *Principles of the kinetic theory of gases*, Flügge's Handbuch der Physik, 12 (1958), 205–294.

[19] L. Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis*, Grundlehren der Mathematischen Wissenschaften, 256, Springer-Verlag, Berlin-New York, (1983).

[20] M. Kac, *Foundations of Kinetic Theory*, Proceedings of the 3rd Berkeley Symposium on Mathematical Statistics and Probability 1954–55, vol. III, U. of California Press, Berkeley and Los Angeles (1956), 171–197.

[21] P.-L. Lions, *Compactness in Boltzmann's equation via Fourier integral operators and applications, I, II*, J. Math. Kyoto Univ., 34 (1994), 391–427, 429–461. III, J. Math. Kyoto Univ., 34 (1994), 539–584.

[22] P.-L. Lions, *On Boltzmann and Landau equations*, Philos. Trans. Roy. Soc. London Ser. A, 346 (1994), 191–204.

[23] P.-L. Lions, *Régularité et compacité pour des noyaux de collision de Boltzmann sans troncature angulaire*, C.R. Acad. Sci. Paris Sér. I Math., 326 (1998), 37–42.

[24] E. Lifchitz and L. Pitayevski, *Physique théorique ("Landau-Lifchitz"), Tome 10, Cinétique physique*, Editions Mir, Moscow, (1990).

[25] A. Proutière, *New results of regularization for weak solutions of Boltzmann equation*, Preprint.

[26] C. Villani, *On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations*, Preprint.

[27] C. Villani, *Conservative forms of Boltzmann's collision operator: Landau revisited*, Preprint.

[28] B. Wennberg, *Regularity in the Boltzmann equation and the Radon transform*, Comm. Part. Diff. Eq., 19 (1994), 2057–2074.