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LINEARIZED STABILITY FOR SEMILINEAR NON-AUTONOMOUS EVOLUTION EQUATIONS WITH APPLICATIONS TO RETARDED DIFFERENTIAL EQUATIONS

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Abstract. We consider the semilinear non-autonomous evolution equation $\frac{d}{dt}u(t) = Au(t) + G(t, u(t)),\ t \geq s \geq 0$, where $(A, D(A))$ is a Hille-Yosida operator on a Banach space X and G is a continuous function on $\mathbb{R}_+ \times \overline{D(A)}$ with values in the extrapolated Favard class corresponding to A. In our main results we present principles of linearized stability and instability for a solution of such an equation. Our approach is based on the theory of extrapolation spaces. We apply the results to non-autonomous semilinear retarded differential equations.

1. Introduction. The purpose of this paper is to present a principle of linearized (in-)stability for the semilinear non-autonomous evolution equation

$$(SNP)_{s,x} \quad \begin{cases} \frac{d}{dt}u(t) & = Au(t) + G(t, u(t)), \quad t \geq s \geq 0, \\ u(s) & = x, \end{cases}$$

where $(A, D(A))$ is a Hille-Yosida operator on a Banach space X and G is a continuous function on $\mathbb{R}_+ \times \overline{D(A)}$ with values in the extrapolated Favard class F_0 corresponding to A satisfying certain differentiability conditions. This kind of equations naturally occurs when dealing with boundary conditions that change in time (see [14]) but also if one linearizes along nonstationary solutions (see [28]).

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Principles of linearized stability and instability for the semilinear equation (SNP) with A being the generator of a C_0-semigroup on X and $G : X \to X$ satisfying certain differentiability conditions are intensively studied e.g. in [33]. In a more general situation equation (SNP) has been investigated e.g. by Clément et al. [2], Desch, Schappacher and Zhang [5], Greiner [9] and Thieme [30]. In particular, they obtained existence and uniqueness of mild solutions and, in the *autonomous case*, principles of linearized (in-)stability for *stationary solutions*. Note, however, that none of the above authors investigated linearized (in-)stability when G also depends on t. In order to obtain principles of linearized (in-)stability for the non-autonomous equation (SNP), we deal with (linear and nonlinear) non-autonomous perturbations of the operator A. Our approach is based on the theory of extrapolation spaces developed by DaPrato and Grisvard [3] and Nagel and Sinestrari [17]. The basic facts on extrapolation spaces and Favard classes are collected in Section 2. The use of extrapolation spaces permits to consider perturbations G in (SNP) belonging to a rather general class of functions. In Section 3 we derive a formula for the solutions of (SNP) which is equivalent to the formulas used by the above authors. Section 4 contains the main results about linearized (in-)stability. Finally, in Section 5, we apply the previous results to non-autonomous semilinear retarded differential equations of the form

$$\begin{cases} \frac{d}{dt}v(t) &= Bv(t) + K(t, v_t), \quad t \geq s \geq 0, \\ \quad v_s &= g \in C([-1, 0], Y), \end{cases}$$

where $(B, D(B))$ is the generator of a C_0-semigroup on a Banach space Y and K is a continuous function on $\mathbb{R}_+ \times C([-1, 0], Y)$ with values in the extrapolated Favard class of B satisfying a certain differentiability condition.

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2. Extrapolation spaces and Favard classes. Throughout the paper, X denotes a Banach space and $(A, D(A))$ is a *Hille-Yosida operator* on X with constants $M \geq 1$ and $\omega \in \mathbb{R}$, i.e., A is linear, (ω, ∞) is contained in the resolvent set $\rho(A)$ of A and

$$M = \sup\{\|(\lambda - \omega)^n R(\lambda, A)^n\| : \lambda > \omega; n \in \mathbb{N}\} < \infty, \tag{2.1}$$

where $R(\lambda, A) := (\lambda - A)^{-1}$ is the *resolvent* of A at λ. The following result is well-known.

Lemma 2.1. ([12], Theorem 12.2.4). *The part* $(A_0, D(A_0))$ *of* A *in* $X_0 := \overline{D(A)}$ *given by*

$$D(A_0) := \{x \in D(A) : Ax \in X_0\}, \quad A_0 x := Ax \quad \text{for} \quad x \in D(A_0)$$

generates a C_0*-semigroup* $(T_0(t))_{t\geq 0}$ *on* X_0*, and* $\|T_0(t)\| \leq Me^{\omega t}$*,* $t \geq 0$*. Moreover,* $\rho(A) \subseteq \rho(A_0)$ *and* $R(\lambda, A_0) = R(\lambda, A)_{|X_0}$ *for* $\lambda \in \rho(A)$*.*

For fixed $\lambda_0 \in \rho(A)$ we introduce on X_0 a new norm by setting

$$\|x\|_{-1} := \|R(\lambda_0, A_0)x\|, \quad x \in X_0.$$

The completion X_{-1} of $(X_0, \|\cdot\|_{-1})$ is called the *extrapolation space* of X associated with A. Note that for $\lambda \in \rho(A)$, the norm on X_0 given by $\|R(\lambda, A)x\|$, $x \in X_0$, and $\|\cdot\|_{-1}$ are equivalent (see [19], Section 3.1). The operators $T_0(t)$ have a unique bounded linear extension $T_{-1}(t)$ to the Banach space X_{-1} and $(T_{-1}(t))_{t\geq 0}$ is a C_0-semigroup on X_{-1} (see [17] Proposition 1.3), the so-called *extrapolated semigroup* of $(T_0(t))_{t\geq 0}$. We denote by $(A_{-1}, D(A_{-1}))$ its generator. The subsequent lemmas can be found in [17].

Lemma 2.2. ([17] Proposition 1.3 and Theorem 1.4). *The following properties hold:*

(i) $\|T_{-1}(t)\|_{\mathcal{L}(X_{-1})} = \|T_0(t)\|_{\mathcal{L}(X_0)}$,

(ii) $D(A_{-1}) = X_0$,

(iii) $A_{-1} : X_0 \to X_{-1}$ *is the unique continuous extension of* $A_0 : D(A_0) \subseteq (X_0, \|\cdot\|) \to (X_{-1}, \|\cdot\|_{-1})$ *and* $\lambda_0 - A_{-1}$ *is an isometry from* $(X_0, \|\cdot\|)$ *onto* $(X_{-1}, \|\cdot\|_{-1})$,

(iv) *If* $\lambda \in \rho(A_0)$*, then* $\lambda - A_{-1}$ *is invertible and* $(\lambda - A_{-1})^{-1} \in \mathcal{L}(X_{-1})$*. In particular,* $\lambda \in \rho(A_{-1})$ *and* $R(\lambda, A_{-1})_{|X_0} = R(\lambda, A_0)$.

The following lemma shows that the space X lies between X_0 and X_{-1}. We write $Y \hookrightarrow Z$ if Y and Z are Banach spaces such that $Y \subseteq Z$ and $id : Y \hookrightarrow Z$ is continuous.

Lemma 2.3. ([17], Theorem 1.7). *The space* $X_0 := \overline{D(A)}$ *is dense in* $(X, \|\cdot\|_{-1})$*. Hence the extrapolation space* X_{-1} *is also the completion of* $(X, \|\cdot\|_{-1})$ *and* $X \hookrightarrow X_{-1}$*. Moreover, the operator* A_{-1} *is an extension of* A*. In particular, if* $\lambda \in \rho(A)$*, then* $R(\lambda, A_{-1})_{|X} = R(\lambda, A)$ *and* $R(\lambda, A_{-1})X = D(A)$*.*

Next we introduce the Favard class corresponding to a C_0-semigroup.

Definition 2.4. Let $(V(t))_{t\geq 0}$ be a C_0-semigroup with generator $(C, D(C))$ on the Banach space Z such that $\|V(t)\| \leq Ne^{\nu t}$ for constants $N \geq 1$ and $\nu \in \mathbb{R}$. The *Favard class* of $(V(t))_{t\geq 0}$ is the Banach space

$$F := \{x \in Z : \sup_{t>0} \frac{1}{t}\|e^{-\nu t}V(t)x - x\| < \infty\}$$

equipped with the norm $\|x\|_F := \|x\| + \sup_{t>0} \frac{1}{t}\|e^{-\nu t}V(t)x - x\|$.

It is easy to see that F is invariant for $V(t)$, $t \geq 0$, and that $D(C) \subseteq F$. Furthermore, if we denote by $\|\cdot\|_C$ the *graph-norm* of C, we have that $\|\cdot\|_F$ and $\|\cdot\|_C$ are equivalent norms on $D(C)$. For the rest of the paper we denote by $F_1 \subseteq X_0$ the Favard class of the C_0-semigroup $(T_0(t))_{t\geq 0}$ and by $F_0 \subseteq X_{-1}$ the Favard class of $(T_{-1}(t))_{t\geq 0}$.

Lemma 2.5. ([17], Proposition 3.2 and [16] Theorem 3.6). *For the Favard classes F_0 and F_1 the following holds:*

(i) $(\lambda_0 - A_{-1})F_1 = F_0$,

(ii) $\sup\{\|(\lambda - \omega)^n R(\lambda, A_{-1})^n\|_{\mathcal{L}(F_0)} : \lambda > \omega; n \in \mathbb{N}\} = M$,

(iii) $T_{-1}(t)F_0 \subseteq F_0$ *for* $t \geq 0$,

(iv) $D(A_0) \hookrightarrow D(A) \hookrightarrow F_1 \hookrightarrow X_0 \hookrightarrow X \hookrightarrow F_0 \hookrightarrow X_{-1}$, *where* $D(A)$ *is equipped with the graph norm* $\|x\|_A := \|x\| + \|Ax\|$, $x \in D(A)$.

The following lemma, due to Nagel and Sinestrari [17], is fundamental for our investigations.

Lemma 2.6. ([17], Proposition 3.3). *For* $f \in L^1_{loc}(\mathbb{R}_+, F_0)$ *define*

$$[T_{-1} \star f](t) := \int_0^t T_{-1}(t-s)f(s)\,ds, \quad t \geq 0.$$

Then

(i) $[T_{-1} \star f](t) \in X_0$,

(ii) $\|[T_{-1} \star f](t)\|_{X_0} \leq M_1 \int_0^t e^{\omega(t-s)}\|f(s)\|_{F_0}\,ds$ *for a constant* $M_1 > 0$, *independent of* f *and* t,

(iii) $\lim_{t\to 0} \|[T_{-1} \star f](t)\|_{X_0} = 0.$

Remark 2.7. From Lemma 2.6 it follows that $t \mapsto [T_{-1}\star f](t)$ is continuous from $\mathbb{R}_+$ into X_0.

In the following, we use the notations introduced above without further comment.

3. Existence and uniqueness of solutions of (SNP)**.** The main purpose of this paper is to discuss the semilinear non-autonomous Cauchy problem

$$(SNP)_{s,x} \quad \begin{cases} \frac{d}{dt}u(t) = Au(t) + G(t, u(t)), & t \geq s \geq 0, \\ u(s) = x \in X_0, \end{cases}$$

on the Banach space X, where $(A, D(A))$ is a Hille-Yosida operator on X with constants $M \geq 1$, $\omega \in \mathbb{R}$, and $G : \mathbb{R}_+ \times X_0 \to F_0$ is a function satisfying certain conditions to be specified below. We will discuss existence and uniqueness and, first of all, stability properties of solutions of (SNP). For our purposes the concept of a mild solution is most useful.

Definition 3.1. A function $u = u(\cdot; s; x) \in C([s, T], X_0)$ is called a *mild solution* of $(SNP)_{s,x}$ on the interval $[s, T] \subseteq \mathbb{R}_+$ if

$$u(t) = T_0(t-s)x + \int_s^t T_{-1}(t-\sigma)G(\sigma, u(\sigma))\,d\sigma \quad \text{for } t \in [s, T]. \tag{3.1}$$

Remark 3.2. Our definition of a mild solution coincides with that in [2], but also with the definition of an F-solution [4], a weak solution [5] and an integral solution [30]. It also includes mild solutions as defined in [9]. This is shown, for example, in [4], Proposition 12.3, [5], Section 4, [19], Proposition 3.2.6, [16], Theorem 4.5 and Section 4.3, in combination with [31], Section 6.

To prove existence and uniqueness of mild solutions of (SNP) we assume that G is locally Lipschitz continuous with respect to the second variable uniformly on compact intervals of $\mathbb{R}_+$, i.e. for every $c, t' \geq 0$ there is a constant $L(c, t')$ such that

$$\|G(t, x_1) - G(t, x_2)\|_{F_0} \leq L(c, t')\|x_1 - x_2\| \tag{3.2}$$

for all $x_1, x_2 \in X_0$ with $\|x_1\| \leq c$, $\|x_2\| \leq c$ and $t \in [0, t']$. By using Lemma 2.6 and a standard fixed point argument, one obtains existence and uniqueness of a mild solution of $(SNP)_{s,x}$ as in the case where A is the generator of a C_0-semigroup and $X_0 = X$ (c.f. [22], Theorem 6.1.2). We therefore omit the proof of the next proposition. Related results can be found e.g. in [2], Theorem 3.1, [5], Theorem 1, [9], Theorem 1.3, [11], Theorems 3.3.3 and 3.3.4, [15], Theorem 7.1.2 and Proposition 7.1.8, [16], Corollary 4.13, and [30], Theorem 1.3.

Proposition 3.3. *Let $G : \mathbb{R}_+ \times X_0 \to F_0$ be continuous and locally Lipschitz continuous with respect to the second variable uniformly on compact intervals of $\mathbb{R}_+$. Then for every $x \in X_0$ and every $s \in \mathbb{R}_+$ there exists a unique mild solution $u = u(\cdot; s; x)$ of $(SNP)_{s,x}$ defined on a maximal interval $[s, t_{max})$. Moreover, $t_{max} = \infty$ or $\|u(t; s; x)\| \to \infty$ as $t \to t_{max}$.*

Throughout the whole paper $v = v(\cdot; 0; x_0)$, $x_0 \in X_0$, denotes a fixed mild solution of $(SNP)_{0,x_0}$ defined on the half-line $[0, \infty)$.

In order to derive principles of linearized stability and instability for the semilinear non-autonomous equation (SNP) we have to overcome the difficulty that the nonlinearity G maps X_0 into F_0 and, at the same time is $t-$dependent. For that we impose the following assumptions on the nonlinearity G. Here and in the following we denote by B_Z the *open unit ball* of a Banach space Z.

(G1) The map $G : \mathbb{R}_+ \times X_0 \to F_0$ is continuous, locally Lipschitz continuous with respect to the second variable uniformly on compact intervals of $\mathbb{R}_+$, differentiable in the second variable along the solution v and for every $q > 0$ there exists $r > 0$ such that

$$\|G(t, v(t) + h) - G(t, v(t)) - DG(t, v(t))h\|_{F_0} \leq q\|h\|,$$

for $t \in \mathbb{R}_+$ and $h \in rB_{X_0}$.

(G2) The map $\mathbb{R}_+ \ni t \mapsto DG(t, v(t)) \in \mathcal{L}(X_0, F_0)$ is strongly continuous.

Assuming $(G1)$ and $(G2)$ we can pass to equation

$$(SNP')_{s,x} \begin{cases} \frac{d}{dt}u(t) &= Au(t) + B(t)u(t) + \rho(t, u(t)), \qquad t \geq s \in \mathbb{R}_+, \\ u(s) &= x \in X_0, \end{cases}$$

where $B(t) := DG(t, v(t))$ and

$$\rho(t, h) := G(t, v(t) + h) - G(t, v(t)) - DG(t, v(t))h$$

for $t \in \mathbb{R}_+$ and $h \in X_0$. The following proposition connects mild solutions of (SNP) with mild solutions of (SNP').

Proposition 3.4. *Let* $u : [s,T] \to X_0$. *Then* u *is a mild solution of* $(SNP)_{s,x}$ *if and only if* $u - v$ *is a mild solution of* $(SNP')_{s,x-v(s)}$ *on* $[s,T]$. *In particular, the mild solution* v *of* (SNP) *corresponds to the zero solution of* (SNP').

Proof. By (3.1)

$$v(t) = T_0(t-s)v(s) + \int_s^t T_{-1}(t-\sigma)G(\sigma, v(\sigma))\,d\sigma \quad \text{for } t \geq s.$$

Thus,

$$u(t;s;x) = T_0(t-s)x + \int_s^t T_{-1}(t-\sigma)G(\sigma, u(\sigma;s;x))\,d\sigma \quad \text{for } t \geq s$$

if and only if

$$\begin{aligned} u(t;s;x) - v(t) &= T_0(t-s)(x - v(s)) \\ &\quad + \int_s^t T_{-1}(t-\sigma)\Big(G(\sigma, u(\sigma;s;x)) - G(\sigma, v(\sigma))\Big)\,d\sigma \\ &= T_0(t-s)(x - v(s)) \\ &\quad + \int_s^t T_{-1}(t-\sigma)\Big(B(\sigma)(u(\sigma;s;x) - v(\sigma)) + \rho(\sigma, u(\sigma;s;x) - v(\sigma))\Big)\,d\sigma \end{aligned}$$

for $t \geq s$. □

Condition (G1) implies $\|\rho(t,h)\| = o(\|h\|), h \in X_0$, uniformly for $t \in \mathbb{R}_+$. So if we pass to the linearized non-autonomous equation

$$(LNP)_{s,x} \quad \begin{cases} \frac{d}{dt}u(t) &= Au(t) + B(t)u(t), \quad t \geq s \geq 0, \\ u(s) &= x \in X_0, \end{cases}$$

one may expect that (in-)stability of the zero solution of (LNP) leads to (in-)stability of the zero solution of (SNP') and hence to (in-)stability of v. This turns out to be the desired principle of linearized (in-)stability which will be proved in Section 4.

We point out that $(LNP)_{s,x}$ always has a unique mild solution defined on $[s,\infty)$. To make this more precise we introduce the following notion. A family $(U(t,s))_{t\geq s\geq 0} \subseteq \mathcal{L}(X_0)$ is called an *evolution family* on X_0 if $U(t,t) = Id$, $U(t,r)U(r,s) = U(t,s)$ for $t \geq r \geq s \geq 0$ and $(t,s) \mapsto U(t,s)x$ is continuous for $x \in X_0$, $t \geq s \geq 0$. By a result of Rhandi ([24], Theorem 2.3), there is a unique evolution family $(U_B(t,s))_{t\geq s\geq 0}$ on X_0 such that

$$U_B(t,s)x = T_0(t-s)x + \int_s^t T_{-1}(t-\sigma)B(\sigma)U_B(\sigma,s)x\,d\sigma \tag{3.3}$$
$$\|U_B(t,s)\| \leq Me^{(M_1-\omega)c(t,s)(t-s)}$$

for $t \geq s$ and $x \in X_0$, where $c(t,s) := \sup_{\sigma\in[s,t]} \|B(\sigma)\|_{\mathcal{L}(X_0,F_0)}$ and $M_1 > 0$ is the constant appearing in Lemma 2.6. In particular, $U_B(\cdot,s)x$, $s \geq 0$, $x \in X_0$, is a mild solution of $(LNP)_{s,x}$ in the sense of Definition 3.1. Uniqueness of the mild solutions of (LNP) follows from a standard argument based on Gronwall's inequality. If $u(\cdot;s;x)$ is a mild solution of $(SNP')_{s,x}$ on the maximal interval $[s,t_{max})$ we define

$$u_\lambda(t;s;x) = U_B(t,s)x + \int_s^t U_B(t,\sigma)\lambda R(\lambda,A_{-1})\rho(\sigma,u(\sigma;s;x))\,d\sigma \tag{3.4}$$

for $\lambda > \omega$ and $t \in [s,t_{max})$. We show that the mild solution $u(\cdot;s;x)$ is the limit of the $u_\lambda(\cdot;s;x)$ as $\lambda \to \infty$. For that we need the following lemma.

Lemma 3.5. *Let $x \in X_0$, $\lambda > \omega$ and $f \in C([s,T],F_0)$, where $T > s \geq 0$. If*

$$u_\lambda^f(t) = U_B(t,s)x + \int_s^t U_B(t,\sigma)\lambda R(\lambda,A_{-1})f(\sigma)\,d\sigma \text{ for } t \in [s,T], \tag{3.5}$$

then for $t \in [s,T]$

$$u_\lambda^f(t) = T_0(t-s)x + \int_s^t T_{-1}(t-\sigma)(B(\sigma)u_\lambda^f(\sigma) + \lambda R(\lambda,A_{-1})f(\sigma))\,d\sigma. \tag{3.6}$$

Proof. With formula (3.3) we obtain for $t \in [s,T]$

$$\begin{aligned} u_\lambda^f(t) = {}& T_0(t-s)x + \int_s^t T_{-1}(t-\sigma)B(\sigma)U_B(\sigma,s)x\,d\sigma \\ &+ \int_s^t T_0(t-\sigma)\lambda R(\lambda,A_{-1})f(\sigma)\,d\sigma \\ &+ \int_s^t \int_\sigma^t T_{-1}(t-\tau)B(\tau)U_B(\tau,\sigma)\lambda R(\lambda,A_{-1})f(\sigma)\,d\tau\,d\sigma \end{aligned}$$

$$
\begin{aligned}
&= T_0(t-s)x + \int_s^t T_{-1}(t-\sigma)B(\sigma)U_B(\sigma,s)x\,d\sigma \\
&\qquad + \int_s^t \int_s^\tau T_{-1}(t-\tau)B(\tau)U_B(\tau,\sigma)\lambda R(\lambda,A_{-1})f(\sigma)\,d\sigma\,d\tau \\
&\qquad + \int_s^t T_0(t-\sigma)\lambda R(\lambda,A_{-1})f(\sigma)\,d\sigma \\
&= T_0(t-s)x + \int_s^t T_{-1}(t-\tau)B(\tau)(U_B(\tau,s)x \\
&\quad + \int_s^\tau U_B(\tau,\sigma)\lambda R(\lambda,A_{-1})f(\sigma)\,d\sigma)\,d\tau + \int_s^t T_0(t-\sigma)\lambda R(\lambda,A_{-1})f(\sigma)\,d\sigma \\
&= T_0(t-s)x + \int_s^t T_{-1}(t-\tau)(B(\tau)u_\lambda^f(\tau) + \lambda R(\lambda,A_{-1})f(\tau))\,d\tau.
\end{aligned}
$$

Theorem 3.6. *Let $x \in X_0$, $s \in \mathbb{R}_+$ and $u(\cdot;s;x)$ be a mild solution of $(SNP')_{s,x}$ defined on the maximal interval $[s,t_{max})$. Then for $t \in [s,t_{max})$,*

$$
\begin{aligned}
u(t;s;x) &= \lim_{\lambda\to\infty} u_\lambda(t;s;x) \\
&= U_B(t,s)x + \lim_{\lambda\to\infty}\int_s^t U_B(t,\sigma)\lambda R(\lambda,A_{-1})\rho(\sigma,u(\sigma;s;x))\,d\sigma \qquad (3.7)
\end{aligned}
$$

Proof. Let $t \in [s,t_{max})$. By Lemma 3.5, we obtain for $\lambda > \omega$

$$
\begin{aligned}
u_\lambda(t;s;x) - u(t;s;x) &= \int_s^t T_{-1}(t-\sigma)B(\sigma)(u_\lambda(\sigma;s;x) - u(\sigma;s;x))\,d\sigma \\
&\quad + \int_s^t T_{-1}(t-\sigma)(\lambda R(\lambda,A_{-1}) - Id)\rho(\sigma,u(\sigma;s;x))\,d\sigma.
\end{aligned}
$$

Let $z_t := \int_s^t T_{-1}(t-\sigma)\rho(\sigma,u(\sigma;s;x))\,d\sigma$. Lemma 2.6 yields

$$
\begin{aligned}
\|u_\lambda(t;s;x) - u(t;s;x)\| &\le M_1 \int_s^t e^{\omega(t-\sigma)}\|B(\sigma)\|_{\mathcal{L}(X_0,F_0)} \\
&\qquad \times \|u_\lambda(\sigma;s;x) - u(\sigma;s;x)\|\,d\sigma + \|(\lambda R(\lambda,A_0) - Id)z_t\|
\end{aligned}
$$

for $\lambda > \omega$, where M_1 is the constant appearing in Lemma 2.6. An application of Gronwall's inequality leads to

$$
\begin{aligned}
&\|u_\lambda(t;s;x) - u(t;s;x)\| \le e^{\omega(t-s)}\|(\lambda R(\lambda,A_0) - Id)z_t\| + e^{\omega(t-s)}M_1 \\
&\times \int_s^t \|(\lambda R(\lambda,A_0) - Id)z_\sigma\| \sup_{s\le\tau\le t}\|B(\tau)\|_{\mathcal{L}(X_0,F_0)} e^{M_1 \sup_{s\le\tau\le t}\|B(\tau)\|(t-\sigma)}\,d\sigma
\end{aligned}
$$

for $\lambda > \omega$ and M_1 as above. Since $\lim_{\lambda\to\infty} \|(\lambda R(\lambda, A_0) - Id)z_t\| = 0$ and $\sup_{\lambda>\omega+1} \sup_{s\le\sigma\le t} \|(\lambda R(\lambda, A_0) - Id)z_\sigma\| < \infty$ the assertion follows with Lebesgue's Dominated Convergence Theorem.

4. Principles of linearized stability and instability. In this section we derive principles of linearized stability and instability for mild solutions of (SNP). Before we present the results we introduce the following notions.

Definition 4.1. Let $u(\cdot; s_0; x)$ be a mild solution of $(SNP)_{s_0,x}$ defined on $[s_0, \infty)$.

a) Then $u(\cdot; s_0; x)$ is called *stable* if for every $\epsilon > 0$ and every $s \geq s_0$ there exists $\delta = \delta(\epsilon, s) > 0$ such that for all $y \in X_0$ with $\|y - u(s; s_0; x)\| < \delta$ the mild solution $u(\cdot; s; y)$ exists on $[s, \infty)$ and $\|u(t; s; y) - u(t; s_0; x)\| < \epsilon$ for all $t \in [s, \infty)$.

b) The mild solution $u(\cdot; s_0; x)$ is called *exponentially stable* if there exist constants $N \geq 1$ and $\nu > 0$ and for every $s \geq s_0$ there exists $\delta = \delta(s) > 0$ such that for $y \in X_0$ with $\|y - u(s; s_0; x)\| < \delta$ the mild solution $u(\cdot; s; y)$ is defined on $[s, \infty)$ and $\|u(t; s_0; x) - u(t; s; y)\| \leq Ne^{-\nu(t-s)}$ for $t \in [s, \infty)$.

c) The mild solution $u(\cdot; s_0; x)$ is called *uniformly exponentially stable* if the constant δ in b) is independent of s.

d) The mild solution $u(\cdot; s_0; x)$ is *unstable* if it is not stable.

The next lemma is an immediate consequence of Proposition 3.4.

Lemma 4.2. *Let $v = v(\cdot; 0; x_0)$, $x_0 \in X_0$, be a mild solution of $(SNP)_{0,x_0}$ defined on $[0, \infty)$. Then v is stable (exponentially stable, uniformly exponentially stable, unstable) if and only if the same holds for the zero solution of (SNP') as defined in Section 3.*

As our first main result we formulate the following principle of linearized stability. Related results can be found in [1] Theorem 7, [2], Theorem 4.2, [9] Theorem 2.4, [11] Theorem 5.1.1, [13] Theorem 2.1, [15] Theorem 9.1.2 and [30] Theorem 4.2.

Theorem 4.3. *Let $v = v(\cdot; 0; x_0)$, $x_0 \in X_0$, be a mild solution of $(SNP)_{0,x_0}$ defined on $[0, \infty)$. Assume that conditions $(G1)$ and $(G2)$ are satisfied and* *that the evolution family $(U_B(t,s))_{t\geq s\geq 0}$ defined by (3.3) is exponentially stable, i.e. there exist constants $N \geq 1$ and $\nu > 0$ such that $\|U_B(t,s)\| \leq Ne^{-\nu(t-s)}$ for $t \geq s \geq 0$. Then v is uniformly exponentially stable.*

Proof. By Lemma 4.2 it suffices to show that the zero solution of (SNP') is uniformly exponentially stable. Lemma 2.5 implies that

$$N_1 := \sup_{\lambda > \omega + 1} \|\lambda R(\lambda, A_{-1})\|_{\mathcal{L}(F_0)} < \infty.$$

Set $q := \frac{\nu}{2NN_1}$. By condition $(G1)$ there exists $r > 0$ such that $\|\rho(t,y)\| \leq q\|y\|$ for $y \in rB_{X_0}$. Let $x \in \frac{r}{N}B_{X_0}$. Take $T_0 \in (0,\infty)$ such that the mild solution $u(\cdot;s;x)$ of $(SNP')_{s,x}$ satisfies $\|u(t;s;x)\| < r$ for $t \in [s,T_0)$. Theorem 3.6 yields

$$\|u(t;s;x)\| \leq Ne^{-\nu(t-s)}\|x\| + N\int_s^t e^{-\nu(t-\sigma)}N_1 q\|u(\sigma;s;x)\|\,d\sigma \text{ for } t \in [s,T_0).$$

An application of Gronwall's inequality leads to

$$e^{\nu t}\|u(t;s;x)\| \leq Ne^{\nu s}\|x\|e^{NN_1 q(t-s)} \leq re^{\nu s}e^{\frac{\nu}{2}(t-s)} \qquad \text{for } t \in [s,T_0),$$

and hence

$$\|u(t;s;x)\| \leq re^{-\frac{\nu}{2}(t-s)} \qquad \text{for } t \in [s,T_0). \tag{4.1}$$

In order to prove that the solution $u(\cdot;s;x)$ exists on $[s,\infty)$ and satisfies (4.1) for $t \in [s,\infty)$ it suffices to show $\|u(t;s;x)\| < r$ for $t \in [s,\infty)$. If this is not the case we can find T_1 such that $\|u(T_1;s;x)\| = r$ and $\|u(t;s;x)\| < r$ for $t \in [s,T_1)$ which is a contradiction to (4.1). □

To show a principle of linearized instability the following concept plays a crucial role.

Definition 4.4. An evolution family $(U(t,s))_{t\geq s\geq 0}$ on the Banach space Z is said to have an *exponential dichotomy* (with constants $\beta > \alpha$) if there exists a family of projections $(P(t))_{t\geq 0} \subseteq \mathcal{L}(Z)$ and constants $N, L \geq 1$ such that

(i) $P(t)U(t,s) = U(t,s)P(s)$,
(ii) $\|U(t,s)z\| \leq Ne^{\alpha(t-s)}\|z\|$ for all $z \in P(s)Z$,
(iii) $\|U(t,s)z\| \geq Le^{\beta(t-s)}\|z\|$ for all $z \in (Id - P(s))Z$,
(iv) the map $U_|(t,s) : (Id - P(s))Z \to (Id - P(t))Z$: $z \mapsto U(t,s)z$ is invertible for every $t \geq s \geq 0$.

Note that in [20], Lemma 4.2, it is shown that the family $(P(t))_{t\geq 0}$ appearing in Definition 4.4 is automatically bounded and strongly continuous. Furthermore $(t,s)\mapsto [U_{|}(t,s)]^{-1}(Id-P(t))$ is strongly continuous for $t\geq s\geq 0$ (see [29], Lemma 5.15).

Now we can formulate the following principle of linearized instability. Related results can be found in [1], Theorem 8, [2], Theorem 4.3, [9], Theorem 2.5, [11], Theorem 5.1.3, [15], Theorem 9.1.3, and [30], Corollary 4.3.

Theorem 4.5. *Let $v=v(\cdot;0;x_0)$, $x_0\in X_0$, be a mild solution of $(SNP)_{0,x_0}$ defined on $[0,\infty)$. Assume that conditions $(G1)$ and $(G2)$ are satisfied and that the evolution family $(U_B(t,s))_{t\geq s\geq 0}$ defined by (3.3) has an exponential dichotomy with constants $\beta>\alpha$, $\beta>0$, and projections $(P(t))_{t\geq 0}$ such that* $\ker P(s_0)\neq 0$ *for some $s_0\in\mathbb{R}_+$. Then v is unstable.*

Proof. By Lemma 4.2 it suffices to show that the zero solution of (SNP') is unstable. We define $Q(t):=Id-P(t)$ for $t\geq 0$. There exist constants $L,N\geq 1$ such that $\|U_B(t,s)P(s)x\|\leq Ne^{\alpha(t-s)}\|P(s)x\|$, $\|U_B(t,s)Q(s)x\|\geq Le^{\beta(t-s)}\|Q(s)x\|$ and $\|[U_{B|}(t,s)]^{-1}Q(t)x\|\leq \frac{1}{L}e^{-\beta(t-s)}\|Q(t)x\|$ for $t\geq s\geq 0$ and $x\in X_0$. Choose $\gamma>0$ such that $\beta>\gamma>\alpha$ and define the rescaled evolution family $W(t,s):=e^{-\gamma(t-s)}U_B(t,s)$, $t\geq s\geq 0$. Note that $(W(t,s))_{t\geq s\geq 0}$ has an exponential dichotomy with constants $\beta-\gamma>0>\alpha-\gamma$. Let $C:=\sup\limits_{0\leq t<\infty}\{\|P(t)\|,\|Q(t)\|\}$ and $N_1:=\sup\limits_{\lambda>\omega+1}\|\lambda R(\lambda,A_{-1})\|_{\mathcal{L}(F_0)}$. Now take $0<q<\left(4CN_1(\frac{N}{\gamma-\alpha}+\frac{1}{L(\beta-\gamma)})\right)^{-1}$. By condition $(G1)$ there exists $r>0$ such that $\|\rho(t,y)\|\leq q\|y\|$ for $y\in rB_{X_0}$ and $t\in\mathbb{R}_+$. We prove the theorem by contradiction, i.e., we assume that the zero solution of (SNP') is stable. In particular, there is $\delta>0$ such that each mild solution $u(\cdot;s;x)$ of $(SNP')_{s,x}$, $x\in\delta B_{X_0}$, is defined on $[s,\infty)$ and satisfies $\|u(\cdot;s;x)\|\leq r$. If we choose $0\neq x\in\delta B_{X_0}$ such that $x\in Q(s_0)X_0$ Theorem 3.6 leads to

$$\begin{aligned}
&\|e^{-\gamma(\tau-s_0)}Q(\tau)u(\tau;s_0;x)\|\\
&\quad\geq \|e^{-\gamma(\tau-s_0)}u(\tau;s_0;x)\|-\|e^{-\gamma(\tau-s_0)}P(\tau)u(\tau;s_0;x)\|\\
&\geq \|e^{-\gamma(\tau-s_0)}u(\tau;s_0;x)\|\\
&\quad-\|\lim_{\lambda\to\infty}\int_{s_0}^{\tau}W(\tau,\sigma)P(\sigma)\lambda R(\lambda,A_{-1})e^{-\gamma(\sigma-s_0)}\rho(\sigma,u(\sigma;s_0;x))\,d\sigma\|\\
&\geq \|e^{-\gamma(\tau-s_0)}u(\tau;s_0;x)\|-\frac{NN_1Cq}{\gamma-\alpha}\sup_{\sigma\in[s_0,\infty)}\|e^{-\gamma(\sigma-s_0)}u(\sigma;s_0;x)\| \qquad (4.2)
\end{aligned}$$

for $\tau \in [s_0, \infty)$. Furthermore since $W_|(t,\tau) : Q(\tau)X_0 \to Q(t)X_0$ is invertible for $t \geq \tau \geq 0$ we have

$$\begin{aligned}
&\|e^{-\gamma(t-s_0)}Q(t)u(t;s_0;x)\| = \|W(t,\tau)Q(\tau)e^{-\gamma(\tau-s_0)}u(\tau;s_0;x) \\
&\quad + \lim_{\lambda\to\infty}\int_\tau^t W(t,\sigma)Q(\sigma)\lambda R(\lambda,A_{-1})e^{-\gamma(\sigma-s_0)}\rho(\sigma,u(\sigma;s_0;x))\,d\sigma\| \\
&\geq Le^{(\beta-\gamma)(t-\tau)}(\|e^{-\gamma(\tau-s_0)}Q(\tau)u(\tau;s_0;x) \\
&\quad + [W_|(t,\tau)]^{-1}Q(t) \\
&\quad \times \lim_{\lambda\to\infty}\int_\tau^t W(t,\sigma)Q(\sigma)\lambda R(\lambda,A_{-1})e^{-\gamma(\sigma-s_0)}\rho(\sigma,u(\sigma;s_0;x))\,d\sigma\|) \\
&= Le^{(\beta-\gamma)(t-\tau)}(\|e^{-\gamma(\tau-s_0)}Q(\tau)u(\tau;s_0;x) \\
&\quad + \lim_{\lambda\to\infty}\int_\tau^t [W_|(\sigma,\tau)]^{-1}Q(\sigma)\lambda R(\lambda,A_{-1})e^{-\gamma(\sigma-s_0)}\rho(\sigma,u(\sigma;s_0;x))\,d\sigma\|) \\
&\geq Le^{(\beta-\gamma)(t-\tau)}(\|e^{-\gamma(\tau-s_0)}Q(\tau)u(\tau;s_0;x)\| \\
&\quad - \frac{qN_1C}{(\beta-\gamma)L}\sup_{\sigma\in[s,\infty)}\|e^{-\gamma(\sigma-s_0)}u(\sigma;s_0;x)\,d\sigma\|)
\end{aligned}$$

for $t \geq \tau \geq s_0$. Using estimate (4.2) we obtain

$$\begin{aligned}
\|e^{-\gamma(t-s_0)}Q(t)u(t;s_0;x)\| \geq & Le^{(\beta-\gamma)(t-\tau)}(\|e^{-\gamma(\tau-s_0)}u(\tau;s_0;x)\| \\
& - \frac{1}{4}\sup_{\sigma\in[s_0,\infty)}\|e^{-\gamma(\sigma-s_0)}u(\sigma;s_0;x)\|)
\end{aligned} \tag{4.3}$$

for $t \geq \tau \geq s_0$. Now choose $\tau \geq s_0$ such that

$$\|e^{-\gamma(\tau-s_0)}u(\tau;s_0;x)\| \geq \frac{1}{2}\sup_{\sigma\in[s,\infty)}\|e^{-\gamma(\sigma-s_0)}u(\sigma;s_0;x)\|.$$

By (4.3),

$$\|e^{-\gamma(t-s_0)}Q(t)u(t;s_0;x)\| \geq \frac{L}{4}e^{(\beta-\gamma)(t-\tau)}\sup_{\sigma\in[s_0,\infty)}\|e^{-\gamma(\sigma-s_0)}u(\sigma;s_0;x)\|$$

for $t \geq \tau \geq s_0$, which is a contradiction to the boundedness of $u(\cdot;s_0;x)$.

5. Non-autonomous retarded differential equations. In this section we apply the results obtained for (SNP) to retarded differential equations. Throughout the whole section Y is a fixed Banach space and $E :=$

$C([-1,0],Y)$. For $0 \le s \le T$, a function $w \in C([s-1,T],Y)$ and $t \in [s,T]$ we define $w_t \in E$ by $w_t(r) := w(t+r)$, $r \in [-1,0]$. We consider the non-autonomous semilinear retarded differential equation

$$(RSNP)_{s,g} \quad \begin{cases} \frac{d}{dt}w(t) &= Bw(t) + K(t,w_t), \qquad t \ge s \ge 0, \\ \quad w_s &= g \in E. \end{cases}$$

We assume that $(B, D(B))$ is the generator of a C_0-semigroup $(S(t))_{t\ge 0}$ on Y. Denote by $(S_{-1}(t))_{t\ge 0}$ the extrapolated C_0-semigroup of $(S(t))_{t\ge 0}$, by B_{-1} its generator on the extrapolation space Y_{-1} and by F_1^B and F_0^B the Favard class of $(S(t))_{t\ge 0}$ and $(S_{-1}(t))_{t\ge 0}$ respectively. Furthermore we assume $K : \mathbb{R}_+ \times E \to F_0^B$ to be a continuous nonlinear mapping such that

(K1) $K(t,0) = 0$ for every $t \in \mathbb{R}_+$,

(K2) K is Fréchet-differentiable at $0 \in E$ with Fréchet-derivative $DK(t,0)$, locally Lipschitz continuous with respect to the second variable uniformly on compact intervals of $\mathbb{R}_+$ and for every $q > 0$ there exists $r > 0$ such that $\|K(t,h) - K(t,0) - DK(t,0)h\|_{F_0^B} \le q\|h\|_E$ for $t \in \mathbb{R}_+$ and $h \in rB_E$,

(K3) the map $\mathbb{R}_+ \ni t \mapsto DK(t,0)$ is strongly continuous.

A comprehensive discussion of retarded differential equations can be found in [10]. For results connected with existence, uniqueness and stability of solutions of $(RSNP)$ with K having values in Y we refer to [21], [26], [27], [32] in the autonomous case and to [1], [7], [8], [23], [25] in the non-autonomous situation. However all of the above authors considered equation $(RSNP)$ when K takes values in Y. We now give the definition of a mild solution of $(RSNP)$.

Definition 5.1. A function $w(\cdot;s;g) \in C([s-1,T],Y)$, $T > s \ge 0$, is called a *mild solution* of $(RSNP)_{s,g}$ on the interval $[s-1,T]$ if

$$w(t;s;g) = \begin{cases} S(t-s)g(0) + \int_s^t S_{-1}(t-\sigma)K(\sigma, w_\sigma(\cdot;s;g))\,d\sigma, & t \in [s,T], \\ g(t-s), \qquad t \in [s-1,s]. \end{cases}$$

Remark 5.2. a) If K admits only values in the Banach space Y the above definition of a mild solution coincides with the definition of a mild solution in [1], [7], [8], [21], [23] and [32] .

b) If K is differentiable from $\mathbb{R}_+ \times E$ into F_0^B, $g \in C^1([-1,0],Y)$, $g(0) \in F_1^B$ and $B_{-1}g(0) + K(0,g) \in Y$, then a mild solution $w = w(\cdot;s;g)$ of $(RSNP)_{s,g}$ on $[s,T]$, $T > s$, satisfies $w \in C^1([s-1,T],Y) \cap C([s-1,T],F_1^B)$ and

$$\begin{cases} \frac{d}{dt}w(t) = & B_{-1}w(t) + K(t,w_t), \quad t \in [s,T], \\ \quad w_s = g. & \end{cases}$$

(see [17], Corollary 3.5).

c) A function $w \in C([s-1,T],Y)$, $T > s \geq 0$, is a mild solution of $(RSNP)_{s,g}$ if and only if

$$w(t;s;g) = \begin{cases} S(t-s)g(0) + \lim_{\lambda\to\infty} \int_s^t S(t-\sigma)\lambda R(\lambda,B_{-1})K(\sigma,w_\sigma(\cdot;s;g))d\sigma, \\ \hfill t \in [s,T], \\ g(t-s),\ t \in [s-1,s]. \end{cases}$$

We now transform $(RSNP)$ into another equation so that we can apply the results of the previous sections. For that we set $X := F_0^B \times E$ and consider the equation

$$(SNP)_{s,g} \quad \begin{cases} \frac{d}{dt}u(t) & = Au(t) + G(t,u(t)), \qquad t \geq s \geq 0, \\ \quad u(s) & = (0,g) \in \{0\} \times E, \end{cases}$$

where $A : D(A) \to X$ is the linear operator on X given by

$$A\begin{pmatrix} 0 \\ f \end{pmatrix} = \begin{pmatrix} -f'(0) + B_{-1}f(0) \\ f' \end{pmatrix},$$

$$D(A) := \{\begin{pmatrix} 0 \\ f \end{pmatrix} \in \{0\} \times E : f \in C^1([-1,0],Y), f(0) \in F_1^B\},$$

and $G : \mathbb{R}_+ \times \overline{D(A)} \to F_0^B \times E : (t, \binom{0}{f}) \mapsto \binom{K(t,f)}{0}$. It is known that on the Banach space E the operator A_0 defined by $A_0 f := f', D(A_0) := \{f \in C^1([-1,0],Y) : f(0) \in D(B) \text{ and } f'(0) = Bf(0)\}$, is the generator of a C_0-semigroup $(T_0(t))_{t\geq 0}$ given by

$$(T_0(t)f)(r) = \begin{cases} f(t+r) & \text{if } t+r \leq 0, \\ S(t+r)f(0) & \text{if } t+r > 0, \end{cases}$$

(see [18] B.IV.3.1). Obviously, the operator A_0 can be identified with the part of A on $X_0 := \overline{D(A)} = \{0\} \times E$. Therefore we do not distinguish between $(T_0(t))_{t\geq 0}$ and the C_0-semigroup generated by the part of A on X_0. As in [24] it can be shown that $(A, D(A))$ is a Hille-Yosida operator on X and for $\lambda \in \rho(B_{-1}) \cap \rho(A_0)$ the resolvent $R(\lambda, A)$ is given by

$$R(\lambda, A)\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ e_\lambda \otimes R(\lambda, B_{-1})x + R(\lambda, A_0)f \end{pmatrix} \text{ for } \begin{pmatrix} x \\ f \end{pmatrix} \in F_0^B \times E, \quad (5.1)$$

where $e_\lambda \otimes y : [-1, 0] \to F_0^B : s \mapsto e^{\lambda s}y$ for $y \in F_0^B$. From Theorem 3.3 we know that for every $s \geq 0$ and $g \in E$ there exists $t_{max} \in [s, \infty)$ such that $(SNP)_{s,g}$ has a unique mild solution on $[s, t_{max})$. Note that $t \mapsto \binom{0}{u(t)}$ with $u \in C([s, T], E)$, $T > s$, is a mild solution of $(SNP)_{s,g}$ if

$$\begin{pmatrix} 0 \\ u(t) \end{pmatrix} = T_0(t-s)\begin{pmatrix} 0 \\ g \end{pmatrix} + \int_s^t T_{-1}(t-\sigma)G(\sigma, \begin{pmatrix} 0 \\ u(\sigma) \end{pmatrix})\, d\sigma, \quad t \in [s, T]. \quad (5.2)$$

We show how mild solutions of (SNP) are connected with mild solutions of $(RSNP)$.

Theorem 5.3. *If $u \in C([s, T], E)$ satisfies equation (5.2) for all $t \in [s, T]$, then*

$$t \mapsto \begin{cases} u(t)(0) & \text{for } t \in [s, T], \\ g(t-s) & \text{for } t \in [s-1, s], \end{cases}$$

is a mild solution of $(RSNP)_{s,g}$. Conversely, if $t \mapsto w(t)$ is a mild solution of $(RSNP)_{s,g}$ on $[s-1, T]$, then $t \mapsto w_t$ satisfies equation (5.2) on $[s, T]$.

Proof. At first we look at the expression

$$\int_0^t T_{-1}(\sigma)\begin{pmatrix} x \\ 0 \end{pmatrix} d\sigma \in \{0\} \times E \text{ for } x \in F_0^B \text{ and } t \geq 0.$$

We have

$$\begin{aligned} \int_0^t T_{-1}(\sigma)\begin{pmatrix} x \\ 0 \end{pmatrix} d\sigma &= \lim_{\lambda\to\infty} \lambda R(\lambda, A_0)\int_0^t T_{-1}(\sigma)\begin{pmatrix} x \\ 0 \end{pmatrix} d\sigma \\ &= \lim_{\lambda\to\infty} \int_0^t T_0(\sigma)\lambda R(\lambda, A_{-1})\begin{pmatrix} x \\ 0 \end{pmatrix} d\sigma \end{aligned}$$

for $x \in F_0^B$ and $t \geq 0$. If $\tau \in [-1, 0]$ and pr_E is the coordinate projection from X onto E, we obtain from (5.1)

$$\begin{aligned} &pr_E(T_0(\sigma)\lambda R(\lambda, A_{-1})\begin{pmatrix} x \\ 0 \end{pmatrix})(\tau) \\ &= (T_0(\sigma)\lambda e_\lambda \otimes R(\lambda, B_{-1})x)(\tau) = \begin{cases} \lambda e^{\lambda(\tau+\sigma)}R(\lambda, B_{-1})x & \text{if } \tau + \sigma \leq 0, \\ S(\tau+\sigma)\lambda R(\lambda, B_{-1})x & \text{if } \tau + \sigma > 0. \end{cases} \end{aligned}$$

Thus

$$pr_E(\int_0^t T_0(\sigma)\lambda R(\lambda, A_{-1})\begin{pmatrix} x \\ 0 \end{pmatrix} d\sigma)(\tau)$$
$$= \begin{cases} R(\lambda, B_{-1})(e^{\lambda(t+\tau)} - e^{\lambda\tau})x & \text{if } t+\tau \leq 0, \\ R(\lambda, B_{-1})(1 - e^{\lambda\tau})x + \int_0^{t+\tau} S(\sigma)\lambda R(\lambda, B_{-1})x \, d\sigma & \text{if } t+\tau > 0. \end{cases}$$

Passing to the limit $\lambda \to \infty$ we obtain

$$pr_E(\int_0^t T_{-1}(\sigma)\begin{pmatrix} x \\ 0 \end{pmatrix} d\sigma)(\tau) = \begin{cases} 0 & \text{if } t+\tau \leq 0, \\ \int_0^{t+\tau} S_{-1}(\sigma)x \, d\sigma & \text{if } t+\tau > 0. \end{cases}$$

Now let $u \in C([s,T], E)$ such that (5.2) holds. Then for $t \in [s,T]$ and $\tau \in [-1, 0]$, we have

$$(u(t))(\tau) = (T_0(t-s)g)(\tau) + pr_E(\frac{d}{dt}\int_s^t\int_s^r T_{-1}(r-\sigma)G(\sigma, \begin{pmatrix} 0 \\ u(\sigma) \end{pmatrix})d\sigma dr)(\tau)$$
$$= (T_0(t-s)g)(\tau) + pr_E(\frac{d}{dt}\int_s^t \int_\sigma^t T_{-1}(r-\sigma)G(\sigma, \begin{pmatrix} 0 \\ u(\sigma) \end{pmatrix}) \, dr \, d\sigma)(\tau)$$
$$= (T_0(t-s)g)(\tau) + pr_E(\frac{d}{dt}\int_s^t \int_0^{t-\sigma} T_{-1}(r)G(\sigma, \begin{pmatrix} 0 \\ u(\sigma) \end{pmatrix}) \, dr \, d\sigma)(\tau) \qquad (5.3)$$
$$= \begin{cases} (T_0(t-s)g)(\tau) & \text{if } t+\tau \leq s, \\ (T_0(t-s)g)(\tau) + \frac{d}{dt}\int_s^{t+\tau}\int_0^{t-\sigma+\tau} S_{-1}(r)K(\sigma, u(\sigma)) \, dr \, d\sigma & \text{if } t+\tau > s. \end{cases}$$

From (5.3) we obtain for $\tau = 0$ and $t \in [s,T]$

$$\begin{aligned} [u(t)](0) &= S(t-s)g(0) + \frac{d}{dt}\int_s^t \int_0^{t-\sigma} S_{-1}(r)K(\sigma, u(\sigma)) \, dr \, d\sigma \\ &= S(t-s)g(0) + \frac{d}{dt}\int_s^t \int_\sigma^t S_{-1}(r-\sigma)K(\sigma, u(\sigma)) \, dr \, d\sigma \\ &= S(t-s)g(0) + \int_s^t S_{-1}(t-\sigma)K(\sigma, u(\sigma)) \, d\sigma. \end{aligned}$$

We define

$$w(t) = \begin{cases} u(t)(0) & \text{for } t \in [s,T], \\ g(t-s) & \text{for } t \in [s-1, s]. \end{cases}$$

With (5.3) we obtain $w_\sigma(\tau) = w(\sigma+\tau) = u(\sigma)(\tau)$ for $\tau \in [-1, 0]$ and $\sigma \in [s,T]$. Thus, $w_\sigma(\cdot) = u(\sigma)(\cdot)$ for $\sigma \in [s,T]$, and hence w is a mild solution of $(RSNP)$. The second statement is proved in the same way.

Proposition 5.4. *If $(U(t,s))_{t\geq s\geq 0}$ is the evolution family on E determined by the formula*

$$\begin{pmatrix} 0 \\ U(t,s)g \end{pmatrix} = T_0(t-s)\begin{pmatrix} 0 \\ g \end{pmatrix} + \int_s^t T_{-1}(t-\sigma)DG(\sigma,0)\begin{pmatrix} 0 \\ U(\sigma,s)g \end{pmatrix} d\sigma \quad (5.4)$$

for $t \geq s \geq 0$, $g \in E$ and $DG(\sigma,0)\binom{0}{h} = \binom{DK(\sigma,0)h}{0}$, $h \in E$, then the map

$$t \mapsto \begin{cases} [U(t,s)g](0) & \text{for } t \geq s, \\ g(t-s) & \text{for } s \in [s-1,s], \end{cases}$$

is the unique mild solution of the linear problem

$$(RLNP)_{s,g} \quad \begin{cases} \frac{d}{dt}w(t) = Bw(t) + DK(t,0)w_t, & t \geq s \geq 0, \\ w(s) = g \in C([-1,0],Y). \end{cases}$$

On the other hand, if $t \mapsto w(t)$ is a mild solution of $(RLNP)_{s,g}$, then $t \mapsto w_t$ satisfies equation (5.4) on $[s,\infty)$.

Proof. By (3.3) the operator family $(U(t,s))_{t\geq s\geq 0}$ defined by (5.4) is an evolution family. The assertion follows by applying Theorem 5.3 to $(RLNP)$ instead of $(RSNP)$. □

We are now in a position to prove the following principle of linearized stability. It should be clear how uniform exponential stability of a mild solution of $(RSNP)$ is defined.

Theorem 5.5. *If the zero solution of (RLNP) is uniformly exponentially stable, then the zero solution of $(RSNP)$ is uniformly exponentially stable.*

Proof. Let the zero solution of $(RLNP)$ be uniformly exponentially stable. Then we can find $\delta > 0$, $N \geq 1$ and $\nu > 0$ such that for every $g \in \delta B_{C([-1,0],Y)}$ and every $s \geq 0$ the mild solution $w(\cdot;s;g)$ of $(RLNP)_{s,g}$ is defined on $[s-1,\infty)$ and

$$\|w(t+r;s;g)\|_Y \leq Ne^{-\nu(t+r-s)} \quad \text{for } t \geq s \text{ and } r \in [-1,0].$$

Together with Proposition 5.4 this implies

$$\|U(t,s)g\|_E = \|w_t(\cdot;s;g)\|_E \leq Ne^{-\nu(t-s)} \quad \text{for } t \geq s \text{ and a constant } N \geq 1,$$

where $(U(t,s))_{t\geq s\geq 0}$ is the evolution family given by formula (5.4). Combining Theorem 4.3 and Theorem 5.3 we obtain the desired result. □

As a concrete example we discuss the retarded differential equation

$$(E)\begin{cases}\frac{\partial}{\partial t}w(x,t) = -\frac{\partial}{\partial x}(g(x)w(x,t)) + k(t)m(x)f(w(x,t-1)), \\ \qquad\qquad 0 \le x \le 1,\ t \ge s \ge 0, \\ w(x,t) = \varphi(x,t-s), \quad 0 \le x \le 1, \quad s-1 \le t \le s,\end{cases}$$

with initial value $\varphi \in C([0,1]\times[-1,0])$. We assume that $g \in C^1([0,1])$, $g(0) = 0$, $g(x) > 0$ for $0 < x \le 1$, $\int_0^1 \frac{1}{g(s)}\,ds = \infty$, $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, differentiable in 0 and $f(0) = 0$. The function $k : \mathbb{R}_+ \to \mathbb{R}$ is continuous and $m(\cdot) \in L^\infty(0,1)$. For $k \equiv c_0 \in \mathbb{R}$ and $m \equiv c_1 \in \mathbb{R}$ a similar equation is studied in [6]. There it is shown that on the Banach space $Y = C([0,1])$ the operator $(B, D(B))$ defined by

$$(B\psi)(x) = \begin{cases} -\frac{\partial}{\partial x}(g(x)\psi(x)) & \text{if } 0 < x \le 1, \\ -g'(0)\psi(0) & \text{if } x = 0,\end{cases}$$

$D(B) = \{\psi \in C[0,1] : \psi$ is differentiable on $(0,1]$, $\psi' \in C(0,1]$, $\lim_{x\to 0} g(x)\psi'(x) = 0\}$, is the generator of a C_0-semigroup $(S(t))_{t\ge 0}$ given by

$$(S(t)\psi)(x) = \exp(-\int_0^t g'(h^{-1}(h(x)e^{-s}))\,ds)\psi(h^{-1}(h(x)e^{-t}))$$

for $\psi \in C[0,1]$, where $h(x) = \exp(\int_1^x \frac{1}{g(s)}\,ds)$. In particular, if $g'(x) > 0$ on $[0,1]$ and $I := \inf_{0\le x\le 1} g'(x)$, then $\|S(t)\| \le e^{-It}$, $t \ge 0$. The operator $(B, D(B))$ can be written as the sum $(C + D, D(C))$, where

$$(C\psi)(x) = \begin{cases} -g(x)\frac{\partial}{\partial x}\psi(x), & \text{for } 0 < x \le 1, \\ 0, & \text{for } x = 0,\end{cases} \qquad D(C) = D(B),$$

is the generator of the C_0-semigroup $(R(t))_{t\ge 0}$ on $C([0,1])$ with $[R(t)\psi](x) = \psi(h^{-1}(h(x)e^{-t}))$ for $x \in [0,1]$, $t \ge 0$, and D is the bounded multiplication operator on $C([0,1])$ given by $(D\psi)(x) = -g'(x)\psi(x)$.

We write $(S_{-1}(t))_{t\ge 0}$ and $(R_{-1}(t))_{t\ge 0}$ for the extrapolated C_0-semigroups of $(S(t))_{t\ge 0}$ and $(R(t))_{t\ge 0}$ respectively. We denote by F_1^C the Favard class of $(R(t))_{t\ge 0}$ and by F_0^C the Favard class of $(R_{-1}(t))_{t\ge 0}$.

Lemma 5.6. *If $\psi \in F_1^C$, then ψ is differentiable a.e. on $[0,1]$ and $g(\cdot)\frac{\partial}{\partial x}\psi \in L^\infty(0,1)$.*

Proof. Let $\delta \in (0,1)$. We prove that $\psi \in F_1^C$ is Lipschitz continuous on the interval $[\delta, 1]$. Take $x, y \in [\delta, 1]$ and assume that $y < x$. Then there exists $t_y = \ln \frac{h(x)}{h(y)} \in \mathbb{R}_+$ such that $y = h^{-1}(h(x)e^{-t_y})$. If we set $L_\psi = \sup_{t>0} \left\| \frac{R(t)\psi - \psi}{t} \right\|_\infty$, this implies

$$|\psi(x) - \psi(y)| = |\psi(x) - \psi(h^{-1}(h(x)e^{-t_y}))| \le L_\psi \ln \frac{h(x)}{h(y)}$$
$$= L_\psi(\ln h(x) - \ln h(y)) \le L_\psi L_\delta |x - y|,$$

where L_δ is a Lipschitz constant of the map $x \mapsto \ln h(x)$ on $[\delta, 1]$. We can therefore conclude that ψ is differentiable a.e. on $[0,1]$. Since

$$\sup_{t>0} \left\| \frac{R(t)\psi - \psi}{t} \right\|_\infty < \infty \quad \text{and} \quad \lim_{t \to 0} \left| \frac{[R(t)\psi](x) - \psi(x)}{t} \right| = g(x) \frac{\partial}{\partial x} \psi(x)$$

for a.e. $x \in [0,1]$ the assertion is proved.

Lemma 5.7. $L^\infty(0,1) = F_0^C$.

Proof. On $L^\infty(0,1)$ we define the operator $(\tilde{C}, D(\tilde{C}))$ by

$$(\tilde{C}\psi)(x) = \begin{cases} -g(x)\frac{\partial}{\partial x}\psi \ \text{ a.e.,} & \text{for } x \in (0,1], \\ 0, & \text{for } x = 0. \end{cases} \qquad D(\tilde{C}) = F_1^C.$$

Because of Lemma 5.6 this operator is well defined. The operator $(\tilde{C}, D(\tilde{C}))$ is a Hille-Yosida operator on $L^\infty(0,1)$. To show this we define for $\lambda > 0$ the operator $R(\lambda) : L^\infty(0,1) \to C([0,1])$ by

$$[R(\lambda)\psi](x) = \frac{1}{h_\lambda(x)} \int_0^x \frac{\psi(t)}{\lambda g(t)} h_\lambda(t)\, dt, \quad 0 < x \le 1, \quad [R(\lambda)\psi](0) = \psi(0),$$

where $h_\lambda(x) = \exp(\int_1^x \frac{ds}{\lambda g(s)})$, $h_\lambda(0) = 0$. It is obvious that $\|R(\lambda)\psi\|_\infty \le \|\psi\|_\infty$ for $\psi \in L^\infty(0,1)$. In fact, $R(\lambda) \in \mathcal{L}(L^\infty(0,1), F_1^C)$ since for $x, y \in (0,1]$ and $t > 0$

$$\frac{1}{t}|[R(\lambda)\psi](x) - [R(\lambda)\psi](h^{-1}(h(x)e^{-t}))| = \frac{1}{t}\Big| \frac{1}{h_\lambda(x)} \int_0^x \frac{\psi(\tau)}{\lambda g(\tau)} h_\lambda(\tau)\, d\tau$$
$$- \frac{1}{h_\lambda(h^{-1}(h(x)e^{-t}))} \int_0^{h^{-1}(h(x)e^{-t})} \frac{\psi(\tau)}{\lambda g(\tau)} h_\lambda(\tau)\, d\tau \Big|$$

$$= \frac{1}{t}\Big| \int_0^x \frac{\psi(\tau)}{\lambda g(\tau)} h_\lambda(\tau) \Big[\frac{1}{h_\lambda(s)}\Big]_{h^{-1}(h(x)e^{-t})}^{x} d\tau$$

$$+ \int_{h^{-1}(h(x)e^{-t})}^{x} \frac{\psi(\tau)}{\lambda g(\tau)} h_\lambda(\tau) \Big[\frac{1}{h_\lambda(s)}\Big]_{\tau}^{h^{-1}(h(x)e^{-t})} d\tau + \int_{h^{-1}(h(x)e^{-t})}^{x} \frac{\psi(\tau)}{\lambda g(\tau)}\, d\tau \Big|$$

$$= \frac{1}{t}\Big| \int_0^x \frac{\psi(\tau)}{\lambda g(\tau)} h_\lambda(\tau) \Big(\int_{h^{-1}(h(x)e^{-t})}^{x} \frac{-1}{\lambda h_\lambda(s) g(s)}\, ds \Big)\, d\tau$$

$$+ \int_{h^{-1}(h(x)e^{-t})}^{x} \frac{\psi(\tau)}{\lambda g(\tau)} h_\lambda(\tau) \Big(\int_{\tau}^{h^{-1}(h(x)e^{-t})} \frac{-1}{\lambda h_\lambda(s) g(s)}\, ds \Big)\, d\tau$$

$$+ \frac{1}{\lambda} \int_{h^{-1}(h(x)e^{-t})}^{x} \frac{\psi(\tau)}{\lambda g(\tau)}\, d\tau \Big|$$

$$\leq \frac{1}{t}\Big| \int_{h^{-1}(h(x)e^{-t})}^{x} \frac{-1}{\lambda h_\lambda(s) g(s)} \Big(\int_0^x \frac{\psi(\tau)}{\lambda g(\tau)} h_\lambda(\tau)\, d\tau \Big)\, ds$$

$$+ \int_{h^{-1}(h(x)e^{-t})}^{x} \frac{-1}{\lambda h_\lambda(s) g(s)} \Big(\int_x^s \frac{\psi(\tau)}{\lambda g(\tau)} h_\lambda(\tau)\, d\tau \Big)\, ds + \int_{h^{-1}(h(x)e^{-t})}^{x} \frac{\psi(\tau)}{\lambda g(\tau)}\, d\tau \Big|$$

$$= \frac{1}{t}\Big| \int_{h^{-1}(h(x)e^{-t})}^{x} \frac{-1}{\lambda g(s)} [R_\lambda \psi](s)\, ds + \frac{1}{\lambda} \int_{h^{-1}(h(x)e^{-t})}^{x} \frac{\psi(\tau)}{\lambda g(\tau)}\, d\tau \Big|$$

$$\leq \frac{2\|\psi\|_\infty}{\lambda t} \int_{h^{-1}(h(x)e^{-t})}^{x} \frac{1}{g(s)}\, ds \leq L \frac{2\|\psi\|_\infty}{\lambda} = \frac{2\|\psi\|_\infty}{\lambda},$$

where $L = \sup_{\xi \in [0,\infty)} \sup_{x \in [0,1]} \big| \big(\frac{\partial}{\partial t} \int_{h^{-1}(h(x)e^{-t})}^{x} \frac{1}{g(s)}\, ds\big)_{|t=\xi} \big| = 1$. It is easily verified that $(Id - \lambda \tilde{C}) R(\lambda) = Id_{L^\infty(0,1)}$ and $R(\lambda)(Id - \lambda \tilde{C}) = Id_{F_1^C}$ which implies that $\frac{1}{\lambda} R(\frac{1}{\lambda}) = R(\lambda, \tilde{C})$ for $\lambda > 0$. Since $\|\frac{1}{\lambda} R(\frac{1}{\lambda})\|_{\mathcal{L}(L^\infty(0,1))} < \frac{1}{\lambda}$ we proved that $\tilde{C}$ is a Hille-Yosida operator on $L^\infty(0,1)$. As C is the part of $\tilde{C}$ on $C([0,1])$, we obtain that $L^\infty(0,1) = F_0^C$.

Lemma 5.8. *The Favard class of $(S_{-1}(t))_{t\geq 0}$ and $(R_{-1}(t))_{t\geq 0}$ coincide.*

Proof. The extrapolation space Y_{-1} of Y associated with B and the extrapolation space of Y associated with C coincide (see for example [19] Theorem

4.1.3). Furthermore, it is shown in [19], Theorem 4.1.9, that for $\psi \in Y_{-1}$

$$S_{-1}(t)\psi = R_{-1}(t)\psi + \lim_{\lambda\to\infty} \int_0^t S_{-1}(t-\sigma)D\lambda R(\lambda, C_{-1})R_{-1}(\sigma)\psi \, d\sigma$$
$$= R_{-1}(t)\psi + \lim_{\lambda\to\infty} \int_0^t R_{-1}(t-\sigma)D\lambda R(\lambda, C_{-1})S_{-1}(\sigma)\psi \, d\sigma$$

for $t \geq 0$. If we assume that $\|R(t)\|$, $\|S(t)\| \leq Ne^{\nu t}$ for $t \geq 0$ and constants $N \geq 1$, $\nu \in \mathbb{R}$, we obtain for $\psi \in F_0^C$

$$\sup_{t>0} \left\| \frac{e^{-\nu t}S_{-1}(t)\psi - \psi}{t} \right\|_{-1} \leq \sup_{t>0} \left\| \frac{e^{-\nu t}R_{-1}(t)\psi - \psi}{t} \right\|_{-1}$$
$$+ \sup_{t>0} \left\| \frac{\lim_{\lambda\to\infty} \int_0^t e^{-\nu(t-\sigma)} S_{-1}(t-\sigma)D\lambda R(\lambda, C_{-1})e^{-\nu\sigma}R_{-1}(\sigma)\psi \, d\sigma}{t} \right\|_{-1}$$
$$\leq \sup_{t>0} \left\| \frac{e^{-\nu t}R_{-1}(t)\psi - \psi}{t} \right\|_{-1} + cN\|D\|_{\mathcal{L}(Y,Y_{-1})} N_1(N\|\psi\|_{F_0^C} + N\|\psi\|_{-1})$$

where $N_1 = \sup_{\lambda>\nu+1} \|\lambda R(\lambda, C)\|_{\mathcal{L}(F_0^C)}$ and $c > 0$ is a constant such that $\|y\|_\infty \leq c\|y\|_{F_0^C}$ on Y. This implies $F_0^C \subseteq F_0^B$. The other inclusion can be shown in the same way. □

We now consider on $E = C([-1, 0], Y)$ the function $K : \mathbb{R}_+ \times E \to L^\infty(0, 1) : (t, \varphi) \mapsto [x \mapsto k(t)m(x)f(\varphi(x)(-1))]$. With Lemma 5.7 and Lemma 5.8 we can conclude that K satisfies conditions (K1) and (K2) and

$$(DK(t, 0)\varphi)(x) = k(t)m(x)f'(0)\varphi(x, -1)$$

for $t \geq 0$, $x \in [0, 1]$ and $\varphi \in C[-1, 0]$. Therefore with Theorem 3.3, Theorem 5.3 and Remark 5.2 c) we obtain that there is a unique function $w \in C([0, 1]\times \mathbb{R}_+)$ satisfying the following integrated version of (E)

$$w(x,t) = \varphi(h^{-1}(h(x)e^{-(t-s)}), 0) \exp(-\int_0^{t-s} g'(h^{-1}(h(x)e^{-\sigma})) \, d\sigma)$$
$$+ \lim_{\lambda\to\infty} \int_s^t \Big[\frac{1}{h_\lambda(h^{-1}(h(x)e^{-(t-\sigma)})}$$
$$\times \int_0^{h^{-1}(h(x)e^{-(t-\sigma)})} \frac{k(\sigma)m(\tau)f(w(\tau, \sigma - 1))}{\lambda g(\tau)} h_\lambda(\tau) \, d\tau\Big]$$
$$\times \exp(-\int_0^{t-\sigma} g'(h^{-1}(h(x)e^{-\xi})) \, d\xi) \, d\sigma$$

for $0 \leq x \leq 1$, $t > s \geq 0$, and $w(x,t) = \varphi(x, t-s)$ for $0 \leq x \leq 1$, $-1+s \leq t \leq s$. Now assume $|k(t)m(x)f'(0)| < I$ for $t \geq 0$ and $x \in [0,1]$. Then, with Gronwalls's lemma we obtain that the zero solution

$$\begin{aligned} w(x,t) &= \varphi(h^{-1}(h(x)e^{-(t-s)}),0)\exp(-\int_0^{t-s} g'(h^{-1}(h(x)e^{-\sigma}))\,d\sigma) \\ &\quad + \lim_{\lambda\to\infty}\int_s^t \Big[\frac{1}{h_\lambda(h^{-1}(h(x)e^{-(t-\sigma)})} \\ &\quad \times \int_0^{h^{-1}(h(x)e^{-(t-\sigma)})} \frac{k(\sigma)m(\tau)f'(0)w(\tau,\sigma-1)}{\lambda g(\tau)} h_\lambda(\tau)\,d\tau\Big] \\ &\qquad \times \exp(-\int_0^{t-\sigma} g'(h^{-1}(h(x)e^{-\xi}))\,d\xi)\,d\sigma \end{aligned}$$

for $0 \leq x \leq 1$, $t > s \geq 0$, and $w(x,t) = \varphi(x,t-s)$ for $0 \leq x \leq 1$, $-1+s \leq t \leq s$, of the linearized equation is uniformly exponentially stable. Thus, by Theorem 5.5, the zero solution of (E) is uniformly exponentially stable.

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