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ON CONLEY INDEX THEORY FOR NON-SMOOTH DYNAMICAL SYSTEMS

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Abstract. We introduce Conley index theory for non-smooth dynamical systems, and by using this theory, we obtain a global bifurcation theorem for such systems.

1. Introduction. In many realistic mechanical problems, effects of dry friction have to be taken into account. As a simple example, imagine a pendulum moving up or down in a rigid tube, such that the pendulum has contact to the wall, where it may either slip or stick. The correct mathematical description of such problems requires the use of differential inclusions of the form

$$\ddot{x} + f(\dot{x}) + g(x) \in -\varphi(x, \dot{x})\,\mathrm{Sgn}(\dot{x}) \quad \text{a.e.}\,, \tag{1}$$

with suitable functions f, g, and φ, and the multi-valued sgn-function

$$\mathrm{Sgn}(x) = \begin{cases} \{-1\} & : \quad x < 0 \\ [-1, 1] & : \quad x = 0 \\ \{1\} & : \quad x > 0. \end{cases} \tag{2}$$

In addition, systems with an additional external forcing can be considered; cf., e.g., [6, 19, 12], also for additional references and other physically more relevant models.

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A systematic study of non-smooth dynamical systems is just at the beginning. In experiments and numerical simulations, many of the standard features of smooth dynamical systems have been observed, such as bifurcations of steady states, periodic, quasiperiodic, or chaotic motion. First analytical results have been obtained in [7, 10, 11, 13]. In addition, new bifurcation phenomena such as "grazing" have been detected; see e.g. [2, 12].

In the present paper, we want to develop a Conley index theory for non-smooth dynamical systems, having in mind that Conley index theory is a powerful tool for the analysis of the qualitative behavior of smooth systems. Here we aim at obtaining a global bifurcation theorem analogous to the one of Ward [23, Thm. 4] resp. [24, Thm. 2], which then can be applied to non-smooth systems like (1), in presence of a suitable bifurcation parameter.

With regard to bifurcation, systems like (1) have two main drawbacks: Firstly, they need not have a well-defined linearization at equilibrium points, and secondly, they need not define unique solutions, i.e., a global flow or even a semiflow. Since the approach of Ward does not rely on properties of the linearization, in this respect a generalization to non-smooth systems seems possible. The obstacle that a multi-valued system does not necessarily induce a flow (not even a "multi-valued flow"), can be overcome as follows: The general form of initial-value problem we are going to consider is

$$\dot{x} \in F(x) \quad \text{a.e.}, \qquad x(0) = x_0 \,, \tag{3}$$

with a multi-valued map $F : \mathbb{R}^n \to 2^{\mathbb{R}^n} \backslash \{\emptyset\}$ which is assumed to be ε-δ-upper semicontinuous and to have a certain boundedness property (cf. the basic assumptions (I) below). This implies that there are globally bounded C^1-functions $f : \mathbb{R}^n \to \mathbb{R}^n$ which are "almost" selections of F, i.e., $f(x) \in F(x)$ is "almost" true (up to some error which can be made arbitrarily small). Then $\dot{x} = f(x)$ defines a global flow, and it remains to show that, starting from an isolated invariant set of (3), there is a suitable approximation f such that $\dot{x} = f(x)$ has a related isolated invariant set. This program can be carried out, and therefore both above mentioned problems can be resolved.

Our construction of the Conley index for non-smooth systems is very similar to the definition of the Leray-Schauder mapping degree DEG for upper semicontinuous multi-valued maps, cf. [6, Ch. 11.4]. Roughly speaking, if f is an "almost" selection of such an F with $0 \notin (\mathrm{id} - F)(\partial\Omega)$, then f will also have $0 \notin (\mathrm{id} - f)(\partial\Omega)$, so one can reasonably set $DEG(\mathrm{id} - F, \Omega, 0) = deg(\mathrm{id} - f, \Omega, 0)$, with deg being the usual degree for single-valued functions.

This new DEG turns out to be well-defined, due to stability of deg w.r. to small perturbations (i.e., homotopy invariance). Analogously, we define the Conley index for non-smooth systems. If I is an isolated invariant set of (3) with isolating neighborhood $N = \overline{U}$, then we will define $H(I)$, the generalized Conley index, to be $h(\mathrm{inv}(N); f)$, where $h(\cdot; f)$ is the classical Conley index w.r. to the global flow generated by $\dot{x} = f(x)$, and $\mathrm{inv}(N)$ denotes the corresponding maximal invariant subset of N. Here it will be the stability of the classical Conley index under small perturbations (i.e., the continuation theorem), which ensures that H is well-defined, and moreover H inherits all the useful properties of h. Then the bifurcation theorem we are aiming at follows more or less analogously to the single-valued case.

To put our approach into more context w.r. to existing work, we note that in [5, Remark 1, p. 6] there is a remark on Conley index theory for single-valued equations with non-unique solutions. In [17] there was already introduced a Conley type index for multi-valued systems, but the underlying equation was supposed to define a multi-valued flow, cf. [17, Section 3]. Also our approach in this paper is more basic, since we are not going to use cohomology methods. Next, a Conley index for discrete multi-valued systems was developed in [9], cf. also [18], with the main goal of providing a theoretical framework for a computer-assisted proof of chaos in the Lorenz equation, cf. [15], [16]. Roughly speaking, when considering the discrete dynamics generated by a single-valued map f, a multi-valued $F(x)$ comes up in a natural way as the set of possible numerical values for the true value $f(x)$. Finally, as far as the bifurcation theorem is concerned, we heavily rely on some ideas of Ward in [23, 24].

The paper is organized as follows. In Section 2 we describe the basic definitions and concepts from both multi-valued differential equations and Conley index theory. In Section 3 we give the definition of the Conley index for non-smooth systems, and we prove its fundamental properties, like the addition property and two variants of the continuation theorem. In Section 4 we will apply these results to obtain bifurcation theorems, and we illustrate this by an example which is a special case of (1).

2. Preliminaries. We start by introducing some notation and facts concerning multi-valued differential equations and Conley index theory. To simplify the presentation and to avoid additional compactness conditions, we restrict our attention to finite-dimensional problems.

Throughout the paper we will assume that the multi-valued map F from

(3) satisfies the following technical assumptions.

Basic Assumptions (I).

(i) $F : \mathbb{R}^n \to 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ is ε-δ-upper semicontinuous (ε-δ-usc for short) with closed convex values $F(x) \subset \mathbb{R}^n$ for $x \in \mathbb{R}^n$.

(ii) $\|F(x)\| := \sup\{|y| : y \in F(x)\} \le \Phi(|x|)$ for some increasing function $\Phi : [0, \infty[\to [0, \infty[$.

Here F is called "ε-δ-usc", if for every $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $F(x) \subset F(x_0) + B_\varepsilon(0)$ for $x \in B_\delta(x_0)$. Since F has compact values (every $F(x) \subset \mathbb{R}^n$ is closed, and, by (ii), bounded), this is equivalent to usc, i.e., $\{x \in \mathbb{R}^n : F(x) \subset V\}$ is open whenever $V \subset \mathbb{R}^n$ is open, cf. [6]. See also [1] for more information on differential inclusions. Concerning our notation, here and in the following, $B_r(x_0)$ and $\overline{B}_r(x_0)$ are the open and closed balls with center x_0 and radius r, respectively, and $A + B = \{x + y : x \in A, y \in B\}$ for $A, B \subset \mathbb{R}^n$, as well as $\lambda A = \{\lambda x : x \in A\}$ for $\lambda \in \mathbb{R}$ and $A \subset \mathbb{R}^n$. For a multi-valued map F, $F(A) := \bigcup_{x \in A} F(x)$. Also, $\overline{A}$, A^0, resp. co$A$ will be the closure, interior, resp. convex hull of a set $A \subset \mathbb{R}^n$, and $\overline{\text{co}}A = \overline{\text{co}A}$. We will also need $\text{dist}(x, A) = \inf\{|x - a| : a \in A\}$, the distance from x to A, and

$$\mathrm{d_H}(A, B) = \max\left\{\sup_{x \in B} \text{dist}(x, A),\ \sup_{x \in A} \text{dist}(x, B)\right\},$$

the Hausdorff-distance between A and B. Continuity of a multi-valued function will always be understood as continuity w.r. to $\mathrm{d_H}$.

Remark 1. The above basic assumptions (I) imply in particular the following, cf. [6, Cor. 5.2] or [8, Thm. 1, p. 77] for (a) and (b).

(a) Every initial value problem (3) has at least one local solution $x = x(\cdot\,; x_0) :]\omega_x^-, \omega_x^+[\to \mathbb{R}^n$, with a maximal interval of existence (ω_x^-, ω_x^+) containing zero.

(b) If $x = x(\cdot\,; x_0) :]\omega_x^-, \omega_x^+[\to \mathbb{R}^n$ is any solution to (3) on a maximal interval of existence, and $\sup\{|x(t)| : t \in]\omega_x^-, \omega_x^+[\} < \infty$, then $\omega_x^\pm = \pm\infty$.

(c) For every bounded $D \subset \mathbb{R}^n$,

$$\sup_{x \in D} \|F(x)\| \le \sup_{x \in D} \Phi(|x|) =: \Phi(D) < \infty\,.$$

Here a solution x is understood to be absolutely continuous with derivative $\dot{x} \in L^1_{loc}(]\omega_x^-, \omega_x^+[)$.

The Conley index will be defined for isolated invariant sets of (3), which we are going to introduce now, together with some other notions and results from Conley index theory. Readers who are not familiar with these concepts may consult [3], [22, Ch. 22-23], [4], [21], or [14]. Facts from classical Conley index theory which will be used in the sequel without reference may be found in these sources.

For a multi-valued F as above and $N \subset \mathbb{R}^d$, let

$$\begin{aligned} \operatorname{inv}(N; F) = \big\{ x_0 \in N :\ & \text{there exists a solution } x = x(\cdot\,; x_0) :]\omega_x^-, \omega_x^+[\to \mathbb{R}^n \\ & \text{of (3) such that } x(t) \in N\,,\ t \in]\omega_x^-, \omega_x^+[\, \big\}\,. \end{aligned} \tag{4}$$

Note that (3) may have several solutions, so we have a "weak" concept of a maximal invariant set, i.e., we only require that there is *one* solution through x_0 which remains in N, instead of imposing this condition for *all* solutions, cf. also [17, Section 4]. If N is bounded, then $\omega_x^\pm = \pm\infty$ for the solutions appearing in the definition of $\operatorname{inv}(N; F)$, by Remark 1(b). Then $\operatorname{inv}(N; F)$ is "weakly invariant", meaning that for $x_0 \in \operatorname{inv}(N; F)$ and $t_1 \in \mathbb{R}$ there exists a solution $x(\cdot\,; x_0)$ such that $x_1 = x(t_1; x_0) \in \operatorname{inv}(N; F)$, as is a consequence of the following lemma.

Lemma 1. *Assume $I = \operatorname{inv}(\overline{U}; F) \subset U$ for some open bounded $U \subset \mathbb{R}^n$. If $x_0 \in I$ and $x = x(\cdot\,; x_0) : \mathbb{R} \to \overline{U}$ is a solution of* (3)*, then $x(\mathbb{R}) \subset I \subset U$.*

Proof. Fix $t_1 \in \mathbb{R}$ and define $x_1 = x(t_1) \in \overline{U}$ as well as $y(t) = x(t_1 + t)$, $t \in \mathbb{R}$. Then $y : \mathbb{R} \to \overline{U}$ is a solution of $\dot{y} \in F(y)$ with $y(0) = x_1$, hence $x_1 \in \operatorname{inv}(\overline{U}; F) = I$, by definition in (4). □

We also fix a consequence of this for later use.

Lemma 2. *If $I = \operatorname{inv}(\overline{U}; F) \subset U \cap B_r(0)$ for $r > 0$ and some open bounded $U \subset \mathbb{R}^n$, then $I = \operatorname{inv}(\overline{U \cap B_r(0)}; F) \subset U \cap B_r(0)$.*

Proof. On the one hand, $\operatorname{inv}(\overline{U \cap B_r(0)}; F) \subset \operatorname{inv}(\overline{U}; F) = I$. Conversely, if $x_0 \in I$ and $x = x(\cdot\,; x_0) : \mathbb{R} \to \overline{U}$ is a solution of (3), then, by Lemma 1, $x(\mathbb{R}) \subset I \subset U \cap B_r(0)$, and hence $x_0 \in \operatorname{inv}(\overline{U \cap B_r(0)}; F)$. □

Compactness of N implies compactness of $\operatorname{inv}(N; F)$, as we are going to prove next.

Lemma 3. *If $N \subset \mathbb{R}^n$ is compact, then also* $\operatorname{inv}(N; F)$ *is compact.*

Proof. To show closedness, let $x_0^k \in I = \operatorname{inv}(N; F)$ with $x_0^k \to x_0 \in \mathbb{R}^n$ as $k \to \infty$. By Remark 1(b) we find global solutions $x_k = x_k(\cdot\,; x_0^k) : \mathbb{R} \to N$ of $\dot{x} \in F(x)$ a.e. with $x_k(0) = x_0^k$. In particular, by Remark 1(c), $|\dot{x}_k(t)| \le \|F(x_k(t))\| \le \Phi(N)$ for $k \in \mathbb{N}$ and a.e. $t \in \mathbb{R}$. Hence the Arzelà-Ascoli theorem and a diagonal argument implies that we find a subsequence (w.l.o.g. the whole sequence) and a continuous $x : \mathbb{R} \to N$ such that $x_k \to x$ uniformly on every compact subinterval of $\mathbb{R}$. We also may assume that $\dot{x}_k \to \dot{x}$ weakly in $L^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n)$ as $k \to \infty$, and we have $x(0) = x_0$. Since F is ε-δ-usc and has closed convex values, a standard argument for multi-valued differential equations shows that $\dot{x} \in F(x)$ a.e. in $\mathbb{R}$, i.e., x is a solution to (3); cf. [6, Lemma 5.1]. Therefore $x_0 \in I$, by definition. The proof that I is relatively compact is analogous, since for $(x_0^k) \subset I \subset N$ we may first choose a subsequence converging to some $x_0 \in \mathbb{R}^n$, and then repeat the argument. □

Now we can introduce isolated invariant sets and isolating neighborhoods of (3).

Definition 1. A set $I \subset \mathbb{R}^n$ is called an isolated invariant set of (3), if $I = \operatorname{inv}(\overline{U}; F) \subset U$ for some open bounded $U \subset \mathbb{R}^n$. In this case, the compact $N = \overline{U}$ is called an isolating neighborhood for F.

We also need the analogous classical concepts for single-valued equations. To this purpose, let $f \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ be given such that $\dot{x} = f(x)$ induces a global flow $\pi_f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. For $N \subset \mathbb{R}^n$ define

$$\operatorname{inv}(N; f) = \{x_0 \in N : \pi_f(\mathbb{R}; x_0) \subset N\}\,.$$

In addition, we let $\operatorname{inv}(N; \pi) = \{x_0 \in N : \pi(\mathbb{R}; x_0) \subset N\}$, for a flow $\pi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Note that these sets are invariant w.r. to the corresponding flows, and compact, in case that N is compact.

In a later reduction step we will apply the following simple result, which is in the same spirit as Lemma 2.

Lemma 4. *Let $f \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ induce a global flow π_f on $\mathbb{R}^n$, and assume* $\operatorname{inv}(\overline{V}; f) \subset U$ *for open bounded $V, U \subset \mathbb{R}^n$ such that $V \supset U$. Then*

$$\operatorname{inv}(\overline{V}; f) = \operatorname{inv}(\overline{U}; f) \subset U\,.$$

Proof. Clearly, $\text{inv}(\overline{V}; f) \supset \text{inv}(\overline{U}; f)$ by definition. Conversely, if $x_0 \in \text{inv}(\overline{V}; f)$, then also $\pi_f(\mathbb{R}; x_0) \subset \text{inv}(\overline{V}; f)$, as $\text{inv}(\dots)$ is invariant. Hence $\pi_f(\mathbb{R}; x_0) \subset U \subset \overline{U}$. □

Next we recall the concepts of isolated invariant sets and isolating neighborhoods in the "classical" setting.

Definition 2. A set $I \subset \mathbb{R}^n$ is called an isolated invariant set of f, if $I = \text{inv}(\overline{U}; f) \subset U$ for some open bounded $U \subset \mathbb{R}^n$. Again $N = \overline{U}$ is termed an isolating neighborhood for f (or π_f).

In the situation of Definition 2, the Conley index $h(I) = h(I; \pi_f) = h(I; f)$ of I is defined and equals $h(I) = [B/b^+]$ for some isolating block B for I, with b^+ being the exit set on ∂B, cf. [22, Ch. 22]. Then B/b^+ is a pointed space, and $[B/b^+]$ denotes the equivalence class of B/b^+ modulo homotopy equivalence in the category of pointed spaces. Important examples of such homotopy classes are $\overline{0}$, the equivalence class of the pointed one-point space, and Σ^k, the equivalence class of the pointed k-sphere. Moreover, we have $h(\emptyset) = \overline{0}$. As usual, for pointed spaces (X, x_0) and (Y, y_0), we denote by $(X, x_0) \vee (Y, y_0)$ the sum (or wedge) of (X, x_0) and (Y, y_0), and this definition carries over in a natural way to the corresponding homotopy equivalence classes, cf. [22, Lemma 22.26].

In fact, below we do not need the specific definition of $h(I)$, but instead we will build on the nice properties of this index. In particular, the continuation theorem will be helpful.

Definition 3. A family (π_λ), $\lambda \in [a, b] \subset \mathbb{R}$, of flows on $\mathbb{R}^n$ will be called continuous in λ, if $\lambda_k \to \lambda$ and $(t_k, x_k) \to (t, x)$ implies $\pi_{\lambda_k}(t_k, x_k) \to \pi_\lambda(t, x)$.

Theorem 1 (Continuation Theorem). *For each $\lambda \in [a, b]$ let π_λ be a flow on $\mathbb{R}^n$ such that (π_λ) is continuous in λ and such that $I_\lambda = \text{inv}(\overline{U}; \pi_\lambda) \subset U$ for some open bounded $U \subset \mathbb{R}^n$, i.e., $\overline{U}$ is an isolating neighborhood for every π_λ, $\lambda \in [a, b]$. Then $h(I_\lambda; \pi_\lambda)$ is independent of $\lambda \in [a, b]$.*

Proof. Cf. [21, Thm. 12.2, p. 65] or [23, Thm. 2] for much more general results. □

We shall introduce the Conley index for differential inclusions by approximation of the multi-valued map through single-valued maps. For this, the following result will be needed.

Lemma 5. *Let $\varepsilon \in]0,1]$ and $r > 0$. Then there exists a C^∞-function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $|f|_\infty \le \Phi(\overline{B}_r(0)) + 1$ and*

$$f(x) \in F(B_\varepsilon(x)) + B_\varepsilon(0)\,, \quad x \in \overline{B}_r(0)\,.$$

Proof. Since F is ε-δ-usc with convex values and $D = \overline{B}_r(0)$ is compact, we find a continuous $g : D \to \mathbb{R}^n$ such that $g(x) \in F(B_\varepsilon(x) \cap D) + B_{\varepsilon/2}(0)$ for $x \in D$, cf. [6, Prop. 1.1(d)], and hence in particular $|g(x)| \le \sup_{y\in D} \|F(y)\| + \varepsilon/2 \le \Phi(D) + 1$ for $x \in D$, with $\Phi(D)$ from Remark 1(c). Since D is compact, we may extend g to a continuous function $\tilde{g} : \mathbb{R}^n \to \mathbb{R}^n$ such that $|\tilde{g}(x)| \le \Phi(D) + 1$ for $x \in \mathbb{R}^n$. We also have $\tilde{g}(x) \in F(B_\varepsilon(x)) + B_{\varepsilon/2}(0)$ for $x \in D$. This $\tilde{g}$ in turn may be approximated uniformly on the compact D by C^∞-functions $\tilde{g}_j = \tilde{g} * \varphi_j$, φ_j being the standard mollifiers. Thus

$$|\tilde{g}_j(x)| \le \int_{\mathbb{R}^n} |\tilde{g}(y)|\varphi_j(x-y)\,dy \le [\Phi(D)+1] \int_{\mathbb{R}^n} \varphi_j(x-y)\,dy = \Phi(D) + 1\,,$$

$j \in \mathbb{N}$, $x \in \mathbb{R}^n$. Consequently, for j large, $f = \tilde{g}_j$ will have the desired properties. □

3. Definition of the Conley index and basic properties. In this section we will define the Conley index for isolated invariant sets of (3), under the basic assumptions (I) on the right-hand side F. First we describe the approximations of F we are going to use for the definition of the index.

Definition 4. Let $\varepsilon, r > 0$. A (single-valued) function f is called an (ε, r)-approximation of F, if $f \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, $|f|_{\infty, r} := \sup\{|f(x)| : x \in \overline{B}_r(0)\} \le \Phi(\overline{B}_r(0)) + 1$, f generates a global flow on $\mathbb{R}^n$ via the differential equation $\dot{x} = f(x)$, and if

$$f(x) \in \overline{\mathrm{co}}\, F(B_\varepsilon(x)) + B_\varepsilon(0)\,, \quad x \in \overline{B}_r(0)\,.$$

Remark 2. For every $\varepsilon \in]0,1]$ and $r > 0$ there exists an (ε, r)-approximation of F, by Lemma 5, since even $|f|_\infty \le \Phi(\overline{B}_r(0)) + 1$ in this lemma, and hence f generates a global flow.

Lemma 6. *Let $r > 0$, and assume I is an isolated invariant set of (3) with two representations $I = \mathrm{inv}(N_0; F) \subset U_0$ and $I = \mathrm{inv}(N_1; F) \subset U_1$, where $N_i = \overline{U_i}$, $i = 0, 1$, and $U_0, U_1 \subset B_r(0)$ are open. Then there is an $\varepsilon_* > 0$*

such that for all $\varepsilon_0, \varepsilon_1 \in]0, \varepsilon_*]$ *the following holds. If* f_i, $i = 0, 1$, *are* (ε_i, r)-*approximations of* F, *then*

$$\operatorname{inv}(N_i\,; f_\lambda) = \operatorname{inv}(\overline{U_i}\,; f_\lambda) \subset U\,, \quad i = 0, 1\,, \quad \lambda \in [0, 1]\,,$$

where $f_\lambda = (1-\lambda) f_0 + \lambda f_1$ *and* $U = U_0 \cap U_1$. *In particular, if we take* $U_0 = U_1 = U$ *and* $f_0 = f_1 = f$, *then* $\operatorname{inv}(\overline{U}; f) \subset U$.

Proof. If the first claim were false, we would find $\varepsilon_0^k \to 0$ and $\varepsilon_1^k \to 0$ as $k \to \infty$, and sequences (f_i^k) of (ε_i^k, r)-approximations of F, $i = 0, 1$, such that, w.l.o.g. for all $k \in \mathbb{N}$ we have $\operatorname{inv}(N_0; f_k) \not\subset U$, where $f_k = (1-\lambda_k) f_0^k + \lambda_k f_1^k$ for some sequence $(\lambda_k) \subset [0, 1]$. Hence there is some $x_0^k \in \operatorname{inv}(N_0; f_k) \cap (\mathbb{R}^n \setminus U)$ for every $k \in \mathbb{N}$, and therefore in particular $x_k(\mathbb{R}) \subset N_0 \subset \overline{B}_r(0)$, with $x_k = x_k(\cdot\,; x_0^k)$ being the solution of $\dot{x} = f_k(x)$, $x(0) = x_0^k$. As N_0 is compact, w.l.o.g. $x_0^k \to x_0$ for some $x_0 \in N_0 \cap (\overline{\mathbb{R}^n \setminus U})$.

We have $|f_i^k|_{\infty, r} \le \Phi(\overline{B}_r(0)) + 1 =: r_0$ for $i = 0, 1$ and $k \in \mathbb{N}$ by definition of an (ε_i^k, r)-approximation. Therefore may may proceed as in the proof to Lemma 3. We have

$$|x_k(t) - x_k(s)| = \left| \int_s^t f_k(x_k(\tau))\, d\tau \right| \le r_0\, |t - s|\,, \quad k \in \mathbb{N}\,, \quad t, s \in \mathbb{R}\,,$$

and hence, again as a consequence of the Arzelà-Ascoli theorem and a diagonal sequence argument, there is a function $x : \mathbb{R} \to \mathbb{R}^n$ being Lipschitz with constant r_0 and a subsequence (w.l.o.g. the whole sequence) such that $x_k \to x$ uniformly on every compact subinterval of $\mathbb{R}$; whence $x(0) = x_0$ and $x(\mathbb{R}) \subset N_0$. In addition, we may once more suppose that $\dot{x}_k \to \dot{x}$ weakly in $L^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n)$. Next, $\varepsilon_k = \max\{\varepsilon_0^k, \varepsilon_1^k\} \to 0$ as $k \to \infty$. By definition of an (ε_i^k, r)-approximation we obtain for $i = 0, 1$

$$f_i^k(x) \in \overline{\operatorname{co}}\, F(B_{\varepsilon_k}(x)) + B_{\varepsilon_k}(0)\,, \quad x \in \overline{B}_r(0)\,,$$

and since every $\overline{\operatorname{co}}\, F(B_{\varepsilon_k}(x)) + B_{\varepsilon_k}(0)$ on the right-hand side is convex and $x_k(\mathbb{R}) \subset \overline{B}_r(0)$, we conclude that

$$\dot{x}_k(t) = f_k(x_k(t)) \in \overline{\operatorname{co}}\, F(B_{\varepsilon_k}(x_k(t))) + B_{\varepsilon_k}(0)\,, \quad k \in \mathbb{N}\,,\ t \in \mathbb{R}\,.$$

Therefore, as $k \to \infty$ it follows that $\dot{x}(t) \in F(x(t))$ for a.e. $t \in \mathbb{R}$, cf. again [6, Lemma 5.1]. Consequently, x is a solution to (3) with $x(0) = x_0$, and thus

$x_0 \in \text{inv}(N_0; F) = I \subset U_0 \cap U_1 = U$. On the other hand, $x_0 \in \overline{\mathbb{R}^n \setminus U} = \mathbb{R} \setminus U$, a contradiction. □

Now we can proceed to define the Conley index for ε-δ-usc multi-valued maps as above.

Definition 5. Let F be a multi-valued map satisfying the basic assumptions (I), and let I be an isolated invariant set of (3). Hence $I \subset B_r(0)$ for some $r > 0$. First we choose an isolating neighborhood $N = \overline{U}$ for F such that $U \subset B_r(0)$; note that this is possible according to Lemma 2. Next we choose an (ε, r)-approximation f of F, with $\varepsilon > 0$ being sufficiently small, cf. Remark 2 and Lemma 6. Then we define $H(I) = H(I; F) = h(\text{inv}(N; f); f)$ to be the Conley index of I.

Since on first sight this definition depends on U and f, we additionally need to prove

Lemma 7. *$H(I)$ is well-defined.*

Proof. Suppose we have two isolating neighborhoods $N_i = \overline{U_i}$ with $U_i \subset B_r(0)$, $i = 1, 2$. Note that $r > 0$ is determined solely through I. Additionally, let f_i, $i = 1, 2$, be (ε_i, r)-approximations for some $\varepsilon_i \in]0, \varepsilon_*]$, $i = 1, 2$, with $\varepsilon_* > 0$ from Lemma 6. Hence we conclude that $\text{inv}(N_i; f_\lambda) \subset U_0 \cap U_1 = U$ for $\lambda \in [0, 1]$ and $i = 0, 1$, f_λ being the path from f_0 to f_1; whence in particular $\text{inv}(N_i; f_\lambda) = \text{inv}(\overline{U}; f_\lambda) \subset U$ for $\lambda \in [0, 1]$ and $i = 0, 1$, by Lemma 4. Therefore, we obtain

$$h(\text{inv}(N_0; f_0); f_0) = h(\text{inv}(\overline{U}; f_0); f_0) = h(\text{inv}(\overline{U}; f_1); f_1) = h(\text{inv}(N_1; f_1); f_1),$$

where the middle equality follows from the continuation theorem, cf. Theorem 1. □

We start with some elementary properties of H.

Remark 3. The Conley index H is a generalization of the classical Conley index h for single-valued functions $f \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ which generate global flows on $\mathbb{R}^n$.

Proof. If $F(x) = \{f(x)\}$ and $I = \text{inv}(\overline{U}; F) = \text{inv}(\overline{U}; f) \subset U$ is an isolated invariant set, then $H(I; F) = h(I; f)$, since f itself is an admissible (ε, r)-approximation of F for all $\varepsilon, r > 0$. Note that here $\Phi(\rho) = \sup\{|f(x)| : x \in \overline{B}_\rho(0)\}$ in (ii) of the basic assumptions (I). □

Theorem 2. *If $H(I) \neq \overline{0}$, then $I \neq \emptyset$, i.e., if $\overline{U}$ is any isolating neighborhood for F which isolates I, then U contains a whole orbit (a "full solution").*

Proof. If $I = \emptyset$, then $I = \mathrm{inv}(\overline{U}; F) \subset U$ and also $I = \mathrm{inv}(\overline{U_1}; F) \subset U_1$ with $U_1 = \emptyset$. Since $H(I)$ is independent of the isolating neighborhood, we obtain with some (ε, r)-approximation f of F, for $\varepsilon > 0$ sufficiently small, $H(I) = h(\mathrm{inv}(\overline{U_1}; f)) = h(\emptyset) = \overline{0}$, a contradiction. Hence U contains a full solution, cf. Lemma 1. □

Again the generalized Conley index H enjoys the addition property. For sets $I_1, I_2 \subset \mathbb{R}^n$ we let $I_1 \dot{\cup} I_2$ denote their disjoint union.

Theorem 3. *If I_1, I_2 are disjoint isolated invariant sets of* (3), *then $I = I_1 \dot{\cup} I_2$ is an isolated invariant set of* (3), *and*

$$H(I_1 \dot{\cup} I_2) = H(I_1) \vee H(I_2)\,. \tag{5}$$

Proof. Let $I_i = \mathrm{inv}(\overline{U}_i; F) \subset U_i$, $i = 1, 2$. Because I_1 and I_2 are compact and disjoint, we may assume w.l.o.g. that also U_1 and U_2 are disjoint with positive distance, since otherwise we may replace U_i through $U_i \cap (I_i + B_\delta(0))$ for small δ, cf. the argument of Lemma 2. Then $I = \mathrm{inv}(\overline{U}; F) \subset U$ with $U = U_1 \dot{\cup} U_2$. Indeed, we have $\mathrm{inv}(\overline{U}; F) \supset \mathrm{inv}(\overline{U}_i; F) = I_i$ for $i = 1, 2$, hence $\mathrm{inv}(\overline{U}; F) \supset I$. On the other hand, if $x_0 \in \mathrm{inv}(\overline{U}; F)$ and $x = x(\cdot\,; x_0) : \mathbb{R} \to \overline{U} = \overline{U}_1 \dot{\cup} \overline{U}_2$ is a solution of (3) with $x(0) = x_0$, then either $x(\mathbb{R}) \subset \overline{U}_1$ or $x(\mathbb{R}) \subset \overline{U}_2$, in both cases of which we obtain $x_0 \in I_1 \dot{\cup} I_2 = S$.

To prove (5), we choose $r > 0$ such that $I \subset B_r(0)$ and $U \subset B_r(0)$. Next we choose an (ε, r)-approximation f of F with $\varepsilon > 0$ small, so that we have

$$H(I) = h(\mathrm{inv}(\overline{U}; f)) \tag{6}$$

by definition. Since also $I_i \subset B_r(0)$ and $U_i \subset B_r(0)$, $i = 1, 2$, Definition 5 shows that additionally

$$H(I_i) = h(\mathrm{inv}(\overline{U_i}; f))\,, \quad i = 1, 2\,. \tag{7}$$

Moreover, $\mathrm{inv}(\overline{U}; f) = \mathrm{inv}(\overline{U_1}; f) \dot{\cup}\, \mathrm{inv}(\overline{U_2}; f)$, as follows from $\overline{U_1} \cap \overline{U_2} = \emptyset$ and the fact that f generates a global flow, cf. Definition 5. Because $\mathrm{inv}(\overline{U}; f) \subset U$ by Lemma 6, we also obtain from $\overline{U_1} \cap \overline{U_2} = \emptyset$ that $\mathrm{inv}(\overline{U_i}; f) \subset U_i$, $i = 1, 2$. Therefore (6), (7), and the addition property of the classical Conley index imply

$$\begin{aligned} H(I) &= h(\mathrm{inv}(\overline{U}; f)) = h\Big(\mathrm{inv}(\overline{U_1}; f) \,\dot\cup\, \mathrm{inv}(\overline{U_2}; f)\Big) \\ &= h(\mathrm{inv}(\overline{U_1}; f)) \vee h(\mathrm{inv}(\overline{U_2}; f)) = H(I_1) \vee H(I_2)\,, \end{aligned}$$

as was to be shown. □

This addition property can be used to obtain a generalization of a classical theorem of Conley. For that, we recall $x_1 \in \mathbb{R}^n$ is an equilibrium of (3), if $0 \in F(x_1)$.

Theorem 4. *Let U be open bounded such $x_1, x_2 \in U$ for exactly two equilibria $x_1 \neq x_2$ of* (3). *Assume that $I = \mathrm{inv}(\overline{U}; F) \subset U$ is an isolated invariant set of* (3) *and that $\{x_i\}$, $i = 1, 2$, are isolated invariant sets of (3). If $H(I) \neq H(\{x_1, x_2\})$, then U contains a full nonconstant solution different from x_1 and x_2. In particular, this holds if $H(I) = \overline{0}$, but $H(\{x_i\}) \neq \overline{0}$ for at least one i.*

Proof. By Theorem 3, $I_* = \{x_1, x_2\}$ is an isolated invariant set of (3), and we have $I \supset I_*$. As $H(I) \neq H(I_*)$, $I = I_*$ is impossible. Thus we find $x_0 \in I$ with $x_0 \neq x_1, x_2$, and a solution $x = x(\cdot\,; x_0) : \mathbb{R} \to \overline{U}$, which exists by definition of $\mathrm{inv}(\dots)$ and provides us with a full nonconstant solution remaining in U, cf. Lemma 1. For the second claim we note that

$$H(I_*) = H(\{x_1\}) \vee H(\{x_2\})$$

by Theorem 3. Thus, if $H(I) = \overline{0}$, then $H(I) \neq H(I_*)$, since otherwise $\overline{0} = H(\{x_1\}) \vee H(\{x_2\})$ would imply $H(\{x_1\}) = \overline{0} = H(\{x_2\})$. □

Our next objective is to prove a basic continuation theorem for the generalized Conley index H. For this we first need to introduce parameterized families of differential inclusions

$$\dot{x} \in F(\mu, x) = F_\mu(x) \ \text{ a.e.}\,, \quad x(0) = x_0\,.$$

In the following, caligraphic letters like $\mathcal{U}$, etc., will be reserved for subsets of $[a, b] \times \mathbb{R}^n$, whereas we continue to denote by U, etc., subsets of $\mathbb{R}^n$. In particular, we let $\mathcal{B}_r(\mu_0, x_0) = \{(\mu, x) \in \mathbb{R} \times \mathbb{R}^n : |\mu - \mu_0| + |x - x_0| < r\}$.

Basic Assumptions (II).

(i) $F : [a,b] \times \mathbb{R}^n \to 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ has closed convex values and is such that $F(\mu,\cdot)$ is ε-δ-usc for every $\mu \in [a,b]$, $F(\cdot,x)$ is $\mathrm{d_H}$-continuous for every $x \in \mathbb{R}^n$, and $F(\cdot,x)$ is ε-δ-usc, uniformly for x in bounded subsets of $\mathbb{R}^n$.

(ii) $\|F(\mu,x)\| \le \Phi(|x|)$ for all $(\mu,x) \in [a,b] \times \mathbb{R}^n$, with some increasing function $\Phi : [0,\infty[\to [0,\infty[$.

Here the uniformity condition in (i) means that given $D \subset \mathbb{R}^n$ bounded, $\mu_0 \in [a,b]$, and $\varepsilon > 0$, there is $\delta > 0$ such that $F(\mu,x) \subset F(\mu_0,x) + B_\varepsilon(0)$ for $\mu \in [a,b]$ with $|\mu - \mu_0| < \delta$ and $x \in D$. Thus in particular all assumptions on $F(\cdot,x)$ hold, if $F(\cdot,x)$ is $\mathrm{d_H}$-continuous, uniformly for x in bounded sets.

Lemma 8. *Assumption* (i) *in particular implies the following.*

(a) $F : [a,b] \times \mathbb{R}^n \to 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ *is jointly* ε-δ-*usc in* (μ,x).

(b) *For every* $\varepsilon > 0$ *and* $r > 0$ *there exists* $\delta > 0$ *such that*

$$F\Big(([\mu-\delta,\mu+\delta] \cap [a,b]) \times B_\delta(x)\Big) \subset F(\mu, B_\varepsilon(x)) + B_\varepsilon(0)\,, \tag{8}$$

$(\mu,x) \in [a,b] \times \overline{B}_r(0)\,.$

Proof. We only prove (b), since (a) follows from the uniformity condition in (i).

If (8) were wrong, then we would find $\varepsilon > 0$ and $r > 0$, as well as sequences $\delta_k \to 0^+$, $(\mu_k) \subset [a,b]$ and $(x_k) \subset \overline{B}_r(0)$, and also $\theta_k \in [\mu_k - \delta_k, \mu_k + \delta_k] \cap [a,b]$, $z_k \in B_{\delta_k}(x_k)$, and $y_k \in F(\theta_k, z_k)$ such that $\mathrm{dist}(y_k, F(\mu_k, B_\varepsilon(x_k))) \ge \varepsilon$. With no loss of generality we may assume $\mu_k \to \mu_0 \in [a,b]$, $x_k \to x_0 \in \overline{B}_r(0)$, $\theta_k \to \mu_0$, and $z_k \to x_0$. By (a), we have $y_k \in F(\theta_k, z_k) \subset F(\mu_0, x_0) + B_{\varepsilon/3}(0)$ for k large, and hence $\mathrm{dist}(\bar{y}_k, F(\mu_k, B_\varepsilon(x_k))) \ge 2\varepsilon/3$ for k large and some sequence $(\bar{y}_k) \subset F(\mu_0, x_0)$. Moreover, $x_0 \in B_\varepsilon(x_k)$ for k large, and consequently $F(\mu_k, B_\varepsilon(x_k))) \supset F(\mu_k, x_0)$, what in turn implies $\mathrm{dist}(\bar{y}_k, F(\mu_k, x_0)) \ge \mathrm{dist}(\bar{y}_k, F(\mu_k, B_\varepsilon(x_k))) \ge 2\varepsilon/3$ for k large. Because $F(\cdot, x_0)$ is $\mathrm{d_H}$-continuous, we have $F(\mu_0, x_0) \subset F(\mu_k, x_0) + B_{\varepsilon/3}(0)$ for k large, and thus $\mathrm{dist}(\bar{y}_k, F(\mu_0, x_0)) \ge \varepsilon/3$ for k large, a contradiction to $\bar{y}_k \in F(\mu_0, x_0)$. □

The following approximation lemma is a parametric version of Lemma 5, but nevertheless we state it separately to keep things clearer.

Lemma 9. *Let $\varepsilon \in]0,1]$, $r > 0$, and suppose that the multi-valued function $F = F(\mu, x) : [a,b] \times \mathbb{R}^n \to 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ has convex values and is jointly ε-δ-usc in (μ, x). If (ii) from the basic assumptions (II) holds, then there exists a continuous function $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\mu, \cdot) \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ and $|f(\mu; \cdot)|_\infty \leq \Phi(\overline{B}_r(0)) + 1$ for $\mu \in [a,b]$, and also*

$$f(\mu, x) \in F\Big(([\mu - \varepsilon, \mu + \varepsilon] \cap [a,b]) \times B_\varepsilon(x)\Big) + B_\varepsilon(0)\,, \tag{9}$$

$(\mu, x) \in [a,b] \times \overline{B}_r(0)$. In addition, f can be chosen to satisfy $|f'(\mu, x)| \leq C$, $(\mu, x) \in [a,b] \times \overline{B}_r(0)$, with a constant C depending on f; here f' denotes the Jakobian of f w.r. to x.

Proof. Analogously to the proof to Lemma 5 we find a continuous $g : \mathcal{D} \to \mathbb{R}^n$ such that $g(\mu, x) \in F(\mathcal{B}_\varepsilon(\mu, x) \cap \mathcal{D}) + B_{\varepsilon/2}(0)$ for $(\mu, x) \in \mathcal{D} = [a,b] \times \overline{B}_r(0)$, once more by application of [6, Prop. 1.1(d)], which is also valid if there Ω is an open subset of a general complete metric space. Since $\|F(\mu, x)\| \leq \Phi(\overline{B}_r(0))$ for $(\mu, x) \in \mathcal{D}$, we obtain $|g(\mu, x)| \leq \Phi(\overline{B}_r(0)) + 1$ for $(\mu, x) \in \mathcal{D}$, and again this estimate transfers to a continuous extension $\tilde{g} : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ of g, even for all $(\mu, x) \in [a,b] \times \mathbb{R}^n$. We can choose $f(\mu, x) = \int_{\mathbb{R}^n} \tilde{g}(\mu, y) \varphi_j(x - y)\, dy$, $(\mu, x) \in [a,b] \times \mathbb{R}^n$, for some large j, with standard mollifiers φ_j, to obtain a suitable approximating function f which also satisfies (9), since $\mathcal{B}_\varepsilon(\mu, x) \cap \mathcal{D} \subset ([\mu - \varepsilon, \mu + \varepsilon] \cap [a,b]) \times B_\varepsilon(x)$ for $(\mu, x) \in \mathcal{D}$. The estimate for $|f'(\mu, x)|$ is obtained from the defining formula. $\Box$

Next we need to introduce parameterized (ε, r)-approximations.

Definition 6. Let $\varepsilon, r > 0$. A function $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ is called an (ε, r)-approximating family of $F = F(\mu, x)$, if

(a) $f \in C([a,b] \times \mathbb{R}^n; \mathbb{R}^n)$;

(b) for every $\mu \in [a,b]$ we have $f(\mu, \cdot) \in C^1(\mathbb{R}^n; \mathbb{R}^n)$,

$$|f(\mu; \cdot)|_{\infty, r} \leq \Phi(\overline{B}_r(0)) + 1,$$

and $f(\mu, \cdot)$ generates a global flow on $\mathbb{R}^n$;

(c) there is a constant $C \geq 0$ such that $|f'(\mu, x)| \leq C$, $(\mu, x) \in [a,b] \times \overline{B}_r(0)$;

(d) we have

$$f(\mu, x) \in \overline{\text{co}}\, F(\mu, B_\varepsilon(x)) + B_\varepsilon(0)\,, \quad (\mu, x) \in [a, b] \times \overline{B}_r(0)\,.$$

Note that in this case in particular every $f_\mu = f(\mu, \cdot)$ is an (ε, r)-approximation of $F_\mu = F(\mu, \cdot)$ in the sense of Definition 4, and the family $(\pi_\mu) = (\pi_{f_\mu})$ of generated global flows is continuous in the sense of Definition 3.

From now on we suppose the basic assumptions (II) from above to hold for $F = F(\mu, x)$. In particular this implies that every F_μ satisfies the basic assumptions (I).

Remark 4. For every $\varepsilon \in]0, 1]$ and $r > 0$ there is an (ε, r)-approximating family of F.

Proof. For $\varepsilon, r > 0$ given, we choose the $\delta > 0$ from Lemma 8(b), and we can assume that $\delta \le \min\{\varepsilon, 1\}$. Hence, by Lemma 8(a), application of Lemma 9 with δ and r is possible, and this yields a function $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ as is described there. This f is an $(2\varepsilon, r)$-approximating family of F, since (a), (b), and (c) from Definition 6 hold, and also, by (9) and (8),

$$\begin{aligned} f(\mu, x) &\in F\Big(([\mu - \delta, \mu + \delta] \cap [a, b]) \times B_\delta(x)\Big) + B_\delta(0) \\ &\subset F(\mu, B_\varepsilon(x)) + B_\varepsilon(0) + B_\delta(0) \subset F(\mu, B_{2\varepsilon}(x)) + B_{2\varepsilon}(0) \end{aligned}$$

for $(\mu, x) \in [a, b] \times \overline{B}_r(0)$. As we can start with $\varepsilon/2$ instead of ε, the claim follows. $\square$

Since ε_* from Lemma 6 in general depends on I, U_0, and U_1, we also need a "parameterized" version of this result.

Lemma 10. *Let $r > 0$ and assume that $I_\mu = \text{inv}(\overline{U}; F_\mu) \subset U$ for $\mu \in [a, b]$, where $U \subset B_r(0)$ is open. Then there exists an $\varepsilon_* > 0$ such that for every (ε, r)-approximating family f of F we have*

$$\text{inv}(\overline{U}; f(\mu; \cdot)) \subset U\,, \quad \mu \in [a, b]\,.$$

Remark 5. In fact, this statement holds in the same generality as Lemma 6, i.e., instead of U one can allow two isolating neighborhoods U_0, U_1, both of which isolate all I_μ, and it is also possible to treat simultaneously two different (ε, r)-approximating families f_0, f_1 of F, defining then $f_\lambda(\mu, x) = (1 - \lambda) f_0(\mu, x) + \lambda f_1(\mu, x)$. We refrained from this generalization in order as to simplify the presentation.

Proof of Lemma 10. The method of proof is the same as for Lemma 6, and we omit the details. It is used that F is jointly ε-δ-usc in (μ, x), cf. Lemma 8(a), in order to conclude from $\varepsilon_k \to 0^+$, $x_k(0) = x_0^k \to x_0 \in \partial U$,

$$\dot{x}_k(t) = f(\mu_k, x_k(t)) \in \overline{\mathrm{co}}\, F(\mu_k, B_{\varepsilon_k}(x_k(t))) + B_{\varepsilon_k}(0)\,,$$

$x_k \to x$ uniformly on every compact subinterval of $\mathbb{R}$, and $\dot{x_k} \to \dot{x}$ weakly in $L^2_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$, that x is a solution of (3). □

With this preparation we can prove the following basic continuation theorem.

Theorem 5. *Let the basic assumptions (II) be satisfied for $F = F(\mu, x)$. Assume that there is a bounded open $U \subset \mathbb{R}^n$ such that $I_\mu = \mathrm{inv}(\overline{U}; F_\mu) \subset U$ for every $\mu \in [a, b]$. Then $H(I_\mu; F_\mu)$ is independent of $\mu \in [a, b]$.*

Proof. Choose $r > 0$ such that $U \subset B_r(0)$, hence also $I_\mu \subset B_r(0)$ for $\mu \in [a, b]$. By Definition 5, Lemma 10, and Remark 5, we have

$$H(I_\mu; F_\mu) = h(\mathrm{inv}(\overline{U}; f(\mu; \cdot)))$$

and $\mathrm{inv}(\overline{U}; f(\mu; \cdot)) \subset U$, $\mu \in [a, b]$, for some (ε, r)-approximating family f of F with $\varepsilon > 0$ sufficiently small, and this is well-defined. Since the family $(\pi_\mu) = (\pi_{f_\mu})$ of generated flows is continuous in the sense of Definition 3, the claim follows from Theorem 1. □

In Section 4 we will give a version of the global bifurcation theorem of Ward [23, Thm. 4] (cf. [24, Thm. 2] for a slightly corrected version) for multi-valued equations. As in [23], [24], this heavily relies on an extension of the continuation theorem, which we shall derive here first. The difference to Theorem 5 is that it will not be enough to have parameterized isolated invariant sets with a *single* isolating neighborhood, i.e., $I_\mu = \mathrm{inv}(\overline{U}; F_\mu) \subset U$ with a fixed U, but instead we also have to allow the isolating neighborhood U to vary with μ. More precisely, the situation will be as follows, cf. Ward [23, Thm. 2]. We are given a relatively open and bounded $\mathcal{U} \subset [a, b] \times \mathbb{R}^n$, and we define for $\mu \in [a, b]$ the sections $\overline{\mathcal{U}}_\mu = \{x \in \mathbb{R}^n : (\mu, x) \in \overline{\mathcal{U}}\}$. To avoid confusion, we want to state explicitly that $\overline{\mathcal{U}}_\mu$ is the section at parameter value μ of the closure of $\mathcal{U}$, but not the closure of the section $\mathcal{U}_\mu = \{x \in \mathbb{R}^n : (\mu, x) \in \mathcal{U}\}$. In general, $\overline{\mathcal{U}_\mu} \subset \overline{\mathcal{U}}_\mu$, but $\overline{\mathcal{U}_\mu} \neq \overline{\mathcal{U}}_\mu$. Let $I_\mu = \mathrm{inv}(\overline{\mathcal{U}}_\mu; F_\mu)$.

Theorem 6. *In the setting described above, assume the basic assumptions (II) to hold for* $F = F(\mu, x)$*, and*

$$\mu \in [a, b]\,, \; x \in I_\mu \quad \Longrightarrow \quad (\mu, x) \notin \partial\mathcal{U}\,, \tag{10}$$

with the boundary being taken w.r. to $[a, b] \times \mathbb{R}^n$*. Then* $H(I_\mu; F_\mu)$ *is defined,* $\mu \in [a, b]$*, and independent of* μ*.*

Proof. In principle, the proof is analogous to the one of WARD [23, Thm. 2], and we avoid most details. To begin with, (10) in particular implies that $\overline{\mathcal{U}}_\mu$ is an isolating neighborhood for I_μ, and thus every $H(I_\mu; F_\mu)$ is defined, $\mu \in [a, b]$. The idea is now to reduce Theorem 6 to Theorem 5 by showing that for every fixed $\mu_0 \in [a, b]$ there is an V_{μ_0} such that V_{μ_0} may serve as an isolating neighborhood for I_μ with μ in a whole neighborhood of μ_0. As $H(I_\mu; F_\mu)$ is independent of the special choice of isolating neighborhood, Theorem 5 can be applied to yield that $\mu \mapsto H(I_\mu; F_\mu)$ is locally constant around μ_0.

So let $\mu_0 \in [a, b]$ be fixed and construct the open V_{μ_0} as it was done in the proof of WARD [23, Thm. 2]; whence I_{μ_0} is compactly contained in V_{μ_0}, i.e., there exists $\varepsilon_0 > 0$ with $\text{dist}(x_0, \mathbb{R}^n \setminus V_{\mu_0}) \geq \varepsilon_0$ for all $x_0 \in I_{\mu_0}$. In order to complete the proof it is enough to show $I_\mu \subset V_{\mu_0}$ for μ sufficiently close to μ_0. To see the latter, assume the contrary. Then we would find $\mu_k \to \mu_0$ and $x_0^k \in I_{\mu_k} \cap (\mathbb{R}^n \setminus V_{\mu_0})$, and thus corresponding solutions $x_k : \mathbb{R} \to \overline{\mathcal{U}}_{\mu_k}$ of $\dot{x} \in F(\mu_k, x)$ a.e. with $x_k(0) = x_0^k$. Hence by definition $\{\mu_k\} \times x_k(\mathbb{R}) \subset \overline{\mathcal{U}}$. Since all $\overline{\mathcal{U}}_\mu$ are uniformly bounded, w.l.o.g. we may assume $x_0^k \to x_0$ as $k \to \infty$, and (II)(ii) implies $|\dot{x}_k(t)| \leq \|F(\mu_k, x_k(t))\| \leq \Phi(|x_k(t)|) \leq C$ for some constant $C > 0$, all $k \in \mathbb{N}$, and a.e. $t \in \mathbb{R}$. As in the proof of Lemma 6, cf. also the proof of Lemma 10, we thus find a subsequence (relabeled as x_k) and a Lipschitz continuous solution $x : \mathbb{R} \to \mathbb{R}^n$ of $\dot{x} \in F(\mu_0, x)$ a.e., $x(0) = x_0$, such that $x_k \to x$ uniformly on bounded intervals in $\mathbb{R}$. This implies $\{\mu_0\} \times x(\mathbb{R}) \subset \overline{\mathcal{U}}$, and therefore $x(\mathbb{R}) \subset \overline{\mathcal{U}}_{\mu_0}$, i.e., $x_0 \in I_{\mu_0}$. Consequently, $|x_0 - x_0^k| \geq \varepsilon_0$ for all k, a contradiction. □

4. Application to bifurcation results. In this section we first state a simple (but useful) bifurcation theorem which is in the same spirit as [23, Thm. 4(c1)], see [24, Thm. 2] for a slightly corrected version. As with degree theory, cf. [25, Ch. 15.1], one can conclude from a jump of the index (here: the Conley index) in some parameter interval that there has to be a bifurcation point. We remark that we are concerned with bifurcations of non-equilibrium solutions. Afterwards we will apply this bifurcation theorem to a typical non-smooth example problem. Then we prove, along the lines of [23, Thm. 4] or [23, Thm. 2], a result concerning global bifurcations for non-smooth systems; in principle, the assertion is analogous to the classical global bifurcation theorem of Rabinowitz [20]. Our basic setup is again

$$\dot{x} \in F(\mu, x) = F_\mu(x) \ \text{ a.e.}\,, \quad \mu \in [a, b]\,, \tag{11}$$

with an $F : [a, b] \times \mathbb{R}^n \to 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ satisfying the basic assumptions (II). In addition, we suppose

$$0 \in F(\mu, 0)\,, \quad \mu \in [\mu_1, \mu_2]\,, \tag{E}$$

where $a \leq \mu_1 < \mu_2 \leq b$, i.e., $x = 0$ is an equilibrium solution of (11) for $\mu \in [\mu_1, \mu_2]$.

Definition 7. The point $(\mu_0, 0) \in [\mu_1, \mu_2] \times \mathbb{R}^n$ is called a bifurcation point of (11), if for every $\varepsilon > 0$ there exists $(\mu, x_0) \in [\mu_1, \mu_2] \times (\mathbb{R}^n \setminus \{0\})$ and a global solution $x : \mathbb{R} \to \mathbb{R}^n$ of (11) with $x(0) = x_0$ such that $|x(t)| + |\mu - \mu_0| < \varepsilon$ for all $t \in \mathbb{R}$.

The following theorem implies that a jump of the Conley index causes bifurcation.

Theorem 7. *Let the basic assumptions (II) and* (E) *hold for* F. *If* $\{0\}$ *is an isolated invariant set for both* F_{μ_1} *and* F_{μ_2}, *and if* $H(\{0\}; F_{\mu_1}) \neq H(\{0\}; F_{\mu_2})$, *then there exists a* $\mu_0 \in [\mu_1, \mu_1]$ *such that* $(\mu_0, 0)$ *is a bifurcation point of* (11).

Proof. Assume on the contrary that the claim is false. Then for every $\mu_0 \in [\mu_1, \mu_2]$ we find $\varepsilon_0 = \varepsilon_0(\mu_0) > 0$ such that for all $(\mu, x_0) \in [\mu_1, \mu_2] \times \mathbb{R}^n$, $x_0 \neq 0$, and global solutions x of (11) with initial value x_0 there is a $t \in \mathbb{R}$ such that $|x(t)| + |\mu - \mu_0| \geq 3\,\varepsilon_0$. This implies that given $\mu_0 \in [\mu_1, \mu_2]$, $H(\{0\}; F_\mu)$ is defined and constant for $\mu \in [\mu_1, \mu_2]$ with $|\mu - \mu_0| \leq \varepsilon_0$. Indeed, Definition 4 of $\text{inv}(\dots)$, the assumption, and (E) show that $\text{inv}(\overline{B}_{\varepsilon_0}(0); F_\mu) = \{0\}$ for such μ, i.e., $\{0\}$ is an isolated invariant set with uniform isolating neighborhood $\overline{B}_{\varepsilon_0}(0)$. Hence the local constancy of $\mu \mapsto H(\{0\}; F_\mu)$ follows from Theorem 5. As $[\mu_1, \mu_2]$ is connected, this yields (cf. [21, p. 66]) that $\mu \mapsto H(\{0\}; F_\mu)$ is constant on $[\mu_1, \mu_2]$, a contradiction. □

Example 1. For illustration, we return to a special case of (1) from the introduction and consider the differential inclusion

$$\ddot{x} + \mu\dot{x} + x^{2k+1} \in -\varphi(x,\dot{x})\,\mathrm{Sgn}(\dot{x}) \quad \text{a.e.}\,, \tag{12}$$

with $k \in \mathbb{N}_0$. With $(x,v) = (x,\dot{x})$, (12) becomes a first-order system in $\mathbb{R}^2$, namely

$$(x,v)' \in F(\mu,(x,v)) := \Big(v,\, -\mu v - x^{2k+1} - \varphi(x,v)\,\mathrm{Sgn}(v)\Big)\,, \tag{13}$$

where $' = d/dt$. If $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is continuous, then $F = F(\mu,(x,v))$ may be seen to satisfy the basic assumptions (II), since Sgn is ε-δ-usc, cf. (2) for the definition of Sgn. In addition, (E) holds for arbitrary $\mu_1 < \mu_2$.

The following theorem says that a bifurcation occurs at $\mu_0 = 0$, if φ is sufficiently small near $v = 0$.

Theorem 8. *Let φ be continuous, and assume there exist $C,\alpha > 0$ such that $|\varphi(x,v)| \le C|v|^{1+\alpha}$ for (x,v) close to zero in $\mathbb{R}^2$. Then $(\mu_0,(0,0)) = (0,(0,0))$ is a bifurcation point of* (13) *resp.* (12).

Proof. By Theorem 7 it is enough to show that $\{(0,0)\}$ is an isolated invariant set for F_μ with $\mu \neq 0$ and that $H(\{(0,0)\};F_\mu) = \Sigma^0$ for $\mu > 0$, but $H(\{(0,0)\};F_\mu) = \Sigma^2$ for $\mu < 0$.

We first consider the case $\mu > 0$. Define the Lyapunov function $V(x,v) = \frac{1}{2}v^2 + \frac{1}{2k+2}\,x^{2k+2}$, $(x,v) \in \mathbb{R}^2$, and $U_r = \{(x,v) \in \mathbb{R}^2 : V(x,v) < r\}$. Then $2C|v|^\alpha \le \mu$ and $|\varphi(x,v)| \le C|v|^{1+\alpha}$ for $(x,v) \in \overline{U_r}$, if $r = r(\mu) > 0$ is small enough. We claim that $\mathrm{inv}(\overline{U_r};F_\mu) = \{(0,0)\}$, i.e., $\{(0,0)\}$ is an isolated invariant set for F_μ with isolating neighborhood $\overline{U_r}$. To see this, assume that there is a solution $X = (x,\dot{x}) : \mathbb{R} \to \overline{U_r}$ of (13) with $X(0) = (x_0,v_0) \neq (0,0)$. Then there exists a measurable $w : \mathbb{R} \to [-1,1]$ satisfying $w(t) \in \mathrm{Sgn}(\dot{x}(t))$ a.e. in $\mathbb{R}$ as well as

$$\ddot{x} + \mu\dot{x} + x^{2k+1} + \varphi(x,\dot{x})\,w(t) = 0 \quad \text{a.e. in } \mathbb{R}\,. \tag{14}$$

By definition of Sgn and choice of r it follows that

$$\begin{aligned} \frac{d}{dt}V(X(t)) &= \dot{x}\,[\ddot{x} + x^{2k+1}] = \dot{x}\,[-\mu\dot{x} - \varphi(x,\dot{x})\,w(t)] = -\mu\,|\dot{x}|^2 - \varphi(x,\dot{x})\,|\dot{x}| \\ &\le -\mu\,|\dot{x}|^2 + C\,|\dot{x}|^{2+\alpha} \le -(\mu/2)\,|\dot{x}|^2\,. \end{aligned} \tag{15}$$

This enforces $(x_0, v_0) = (0,0)$, a contradiction; cf. Lemma 11 below for the technical part. Hence $\{(0,0)\}$ is an isolated invariant set for F_μ when $\mu > 0$. On the other hand, in case that $\mu < 0$, a similar argument may be used, since then, with analogous notation,

$$\frac{d}{dt} V(X(t)) = -\mu\,|\dot{x}|^2 - \varphi(x,\dot{x})\,|\dot{x}| \geq -\mu\,|\dot{x}|^2 - C\,|\dot{x}|^{2+\alpha} \geq -(\mu/2)\,|\dot{x}|^2 .$$

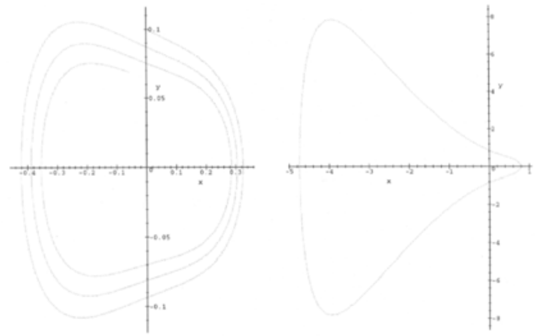
Next we will calculate $H(\{(0,0)\}; F_\mu) = \Sigma^0$ in the case $\mu > 0$. For that, we can use Definition 5. Let $w_\delta(v) = (2/\pi)\arctan(v/\delta)$, $v \in \mathbb{R}$, be the standard approximations of $\mathrm{Sgn}(v)$. It may be seen that given $\varepsilon > 0$, $w_\delta(v) \in \mathrm{Sgn}([v-\varepsilon, v+\varepsilon]) + [-\varepsilon, \varepsilon]$, $v \in \mathbb{R}$, for δ sufficiently small; whence $f_\delta(x,v) = (v,\, -\mu v - x^{2k+1} - \varphi(x,v)\, w_\delta(v))$ serves as a suitable (ε, R)-approximation of F_μ, for every $R > 0$ such that $U_r \subset B_R(0)$, with $r = r(\mu)$ from above. Thus $H(\{(0,0)\}; F_\mu) = h(\mathrm{inv}(\overline{U_r}; f_\delta); f_\delta)$ by definition. Since $|w_\delta(v)| \leq 1$, $v \in \mathbb{R}$, V is also a Lyapunov function for the system $(x,v)' = f_\delta(x,v)$. As above we obtain $\mathrm{inv}(\overline{U_r}; f_\delta) = \{(0,0)\}$, and $B = \overline{U_r}$ is a corresponding isolating block with exit set $b^+ = \emptyset$. Hence $H(\{(0,0)\}; F_\mu) = [B/b^+] = \Sigma^0$. In case that $\mu < 0$, we may proceed similarly, because then $b^+ = \partial B$, and therefore $H(\{(0,0)\}; F_\mu) = [B/b^+] = [\overline{B}_1(0)/S^1] = \Sigma^2$. □

We add some further remarks.

Remark 6. The method of proof for Theorem 8 shows that similar results may be derived under various different assumptions on φ, which can be allowed to depend on μ, and also with x^{2k+1} replaced by more general functions.

Remark 7. Note that the isolating neighborhood $N = \overline{U_r}$ from the proof of Theorem 8 in general shrinks with $\mu > 0$ approaching $\mu = 0$, since we have to ensure $2C|v|^\alpha \leq \mu$ for $(x,v) \in N$. Thus N in fact depends on μ, and this shows that the argument of [23, Thm. 5] cannot be applied to yield nontrivial periodic solutions which encircle the origin in $\mathbb{R}^2$, since for this an isolating neighborhood being independent of μ would be needed ($\mu > 0$ in the situation of Theorem 8 corresponds to $\mu < 0$ in the cited paper and vice versa). Nevertheless, for specific examples it may be easily shown that periodic solutions in fact do bifurcate for $\mu < 0$. The phase portraits below have been obtained numerically, with $\varphi(x,\dot{x}) = \dot{x}^2$.

The following technical lemma was needed above.



$\mu = 0.01$, $x(0) = 0$, $\dot{x}(0) = 0.1$ $\qquad$ $\mu = -0.01$, $x(0) = 0$, $\dot{x}(0) \approx 0.83$

Figure 1: Bifurcation of periodic solutions

Lemma 11. *In the setting of the proof to Theorem* 8, (15) *implies* $(x_0, v_0) = (0, 0)$.

Proof. Since $X(\mathbb{R}) \subset \overline{U_r}$ and $t \mapsto V(X(t))$ is non-negative and decreasing, it is enough to show that $V_- := \lim_{t\to-\infty} V(X(t)) = 0$, since then $V(X(t)) = 0$ for $t \in \mathbb{R}$, hence in particular $V(x_0, v_0) = 0$. We have $\dot{x}(t) \to 0$ as $t \to -\infty$. Indeed, suppose $|\dot{x}(t_k)| \geq 2\delta_0$ for some sequence $t_k \downarrow -\infty$ with $t_{k+1} \leq t_k - 1$ and some $\delta_0 > 0$. Because X is globally bounded, (14) yields $|\ddot{x}|_{L^\infty(\mathbb{R})} < \infty$, and thus the existence of $\eta \in]0, 1]$ with $|\dot{x}(t)| \geq \delta_0$ for $t \in [t_k, t_k + \eta]$ and all $k \in \mathbb{N}$. Hence we obtain from (15)

$$V(X(t_{k-1})) \leq V(X(t_k + \eta)) \leq V(X(t_k)) - (\mu/2)\,\delta_0^2\,\eta\,, \quad k \in \mathbb{N}\,.$$

Iteration of this gives $V(X(t_k)) \geq V(X(t_1)) + (\mu/2)\,\delta_0^2\,\eta\,(k-1)$. Contradiction. Therefore in fact $\dot{x}(t) \to 0$ as $t \to -\infty$, and by definition of V, and since $V(X(t)) \to V_-$, also $x(t) \to x_1$ as $t \to -\infty$ for some $x_1 \in \mathbb{R}$. It follows that $|\int_t^{t+1} x(s)^{2k+1}\,ds - x_1^{2k+1}| \to 0$ as $t \to -\infty$, and thus from integrating (14) for sufficiently negative t,

$$|x_1^{2k+1}| \leq \Big| \int_t^{t+1} x(s)^{2k+1}\,ds - x_1^{2k+1} \Big| + |\dot{x}(t+1) - \dot{x}(t)| + \mu |x(t+1) - x(t)| \\ + C \int_t^{t+1} |\dot{x}(s)|^{1+\alpha}\,ds \to 0\,, \quad t \to -\infty\,.$$

Hence $x_1 = 0$ yields $V_- = \lim_{t\to-\infty} V(x(t), \dot{x}(t)) = V(0,0) = 0$ as claimed. □

Now we turn to the investigation of global bifurcation. We introduce

$$\begin{aligned}\mathcal{S}_0 &= \{(\mu, x_0) \in [a,b] \times (\mathbb{R}^n \setminus \{0\}) : \text{ there exists a global bounded} \\ &\qquad \text{solution } x \text{ of (11) with } x(0) = x_0\}, \quad \text{and} \\ \mathcal{S} &= \overline{\mathcal{S}_0}^{\,[a,b]\times\mathbb{R}^n}.\end{aligned}$$

Note that we can restrict ourselves to $x(0) = x_0$, since (11) is an autonomous differential inclusion. As a last result, we obtain the following global bifurcation theorem, cf. [23, Thm. 4] and [24, Thm. 2].

Theorem 9. *Let the basic assumptions (II) and* (E) (*with* $\mu_1 = a$ *and* $\mu_2 = b$) *hold for* F. *Assume* $\mu_0 \in [a,b]$, $(\mu, 0)$ *is not a bifurcation point for* $\mu \in [a,b]$, $\mu \neq \mu_0$, *and* $H(\{0\}; F_a) \neq H(\{0\}; F_b)$. *Then* $(\mu_0, 0)$ *is a bifurcation point of* (11). *Moreover, if* $\mathcal{C}$ *denotes the component of* $\mathcal{S}$ *containing* $(\mu_0, 0)$, *then the following alternative holds: Either* $\mathcal{C} \subset [a,b] \times \mathbb{R}^n$ *is unbounded, or* $\mathcal{C}$ *has nonempty intersection with* $\{a,b\} \times \mathbb{R}^n$.

Proof. We do not give the details here, since we have at hand a continuation theorem that allows a varying isolating neighborhood, cf. Theorem 6, and thus the proof from [23, Thm. 4] resp. [24, Thm. 2] carries over. Note in particular that $\{0\}$ is an isolated invariant set for every F_μ, $\mu \in [a,b] \setminus \{\mu_0\}$, cf. the proof of Theorem 7. □

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REFERENCES

[1] J.P. Aubin and A. Cellina, "Differential Inclusions," Springer, Berlin-Heidelberg-New York 1984.

[2] C. Budd, *Grazing in impact oscillators,* lecture presented at the NATO advanced study institute on Real and Complex Dynamical Systems, Hillerod, Denmark 1993, in "University of Bristol Research Reports," AM-93-09 (1993).

[3] C. Conley, "Isolated Invariant Sets and the Morse Index," CBMS, 38, American Mathematical Society, Providence/RI 1978.

[4] C. Conley and E. Zehnder, *Morse type index theory for flows and periodic solutions for Hamiltonian equations,* Comm. Pure Appl. Math., 37 (1984), 207–254.

[5] E.N. Dancer, *Degenerate critical points, homotopy indices and Morse inequalities,* J. Reine Angew. Math., 350 (1984), 1–22.

[6] K. Deimling, "Multivalued Differential Equations," de Gruyter, Berlin-New York 1992.

[7] M. Fečkan, *Chaotic solutions in differential inclusions: chaos in dry friction problems,* to appear in Trans. AMS.

[8] A.F. Filippov, "Differential Equations with Discontinuous Right-Hand Sides," Kluwer, Dordrecht 1988.

[9] T. Kaczynski and M. Mrozek, *Conley index for discrete multi-valued dynamical systems,* Topology Appl., 65 (1995), 83–96.

[10] M. Kunze, *On Lyapunov exponents for non-smooth dynamical systems with an application to a pendulum with dry friction,* to appear in J. Dynamics Differential Equations.

[11] M. Kunze, "Non-Smooth Dynamical Systems," Habilitationsschrift (1998).

[12] M. Kunze and T. Küpper, *Qualitative bifurcation analysis of a non-smooth friction-oscillator model,* Z. Angew. Math. Phys., 48 (1997), 87–101.

[13] M. Kunze, T. Küpper, and J. You, *On the application of KAM theory to discontinuous dynamical systems,* J. Differential Equations, 139 (1997), 1–21.

[14] K. Mischaikow, *Conley index theory,* in "Dynamical Systems," Montecatini Terme 1994, Ed. Johnson R., LNM 1609, Springer, Berlin-Heidelberg-New York-Tokyo 1995, pp. 119–207.

[15] K. Mischaikow and M. Mrozek, *Chaos in the Lorenz equation: a computer-assisted proof,* Bull. AMS, 32 (1995), 66–72.

[16] K. Mischaikow and M. Mrozek, *Chaos in the Lorenz equation: a computer-assisted proof II. Details,* to appear in Math. Comp.

[17] M. Mrozek, *A cohomological index of Conley type for multi-valued admissible flows,* J. Differential Equations, 84 (1990), 15–51.

[18] M. Mrozek *From the theorem of Ważewski to computer assisted proofs in dynamics,* in "Panoramas of Mathematics," Banach Center Publications, 34, Polish Acad. Science, Warsaw 1995, pp. 105–120.

[19] K. Popp, N. Hinrichs, and M. Oestreich, *Dynamical behaviour of a friction oscillator with simultaneous self and external excitation,* Sādhanā, 20 (1995), 627–654.

[20] P. Rabinowitz, *Some global results for nonlinear eigenvalue problems,* J. Functional Anal., 7 (1971), 487–513.

[21] K.P. Rybakowski, "The Homotopy Index and Partial Differential Equations," Springer, Berlin-Heidelberg-New York 1987.

[22] J. Smoller, "Shock Waves and Reaction-Diffusion Equations," 2nd edition, Springer, Berlin-Heidelberg-New York 1994.

[23] J.R. Ward, Jr., *Bifurcating continua in infinite dimensional dynamical systems and applications to differential equations,* J. Differential Equations, 125 (1996), 117–132.

[24] J.R. Ward, Jr., *Global bifurcation of periodic solutions to ordinary differential equations,* to appear in J. Differential Equations.

[25] E. Zeidler, "Nonlinear Functional Analysis and its Applications I: Fixed-Point Theorems," Springer, Berlin-Heidelberg-New York-Tokyo 1986.