Differential and Integral Equations Volume 13 (4–6) April–June 2000, pp. 453–477

# EXISTENCE OF SOLUTIONS FOR ELLIPTIC SYSTEMS WITH HOLDER CONTINUOUS NONLINEARITIES

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**Abstract.** In this work we prove the existence of solutions for an elliptic system between lower and upper solutions when the nonlinearities are Hölder continuous functions without a Lipschitz condition. Specifically, under appropriate conditions of monotony on the nonlinear reaction terms we introduce two monotone sequences which converge to a minimal and a maximal solution respectively. Finally, we apply these results to a dynamical population problem with "slow" diffusion.

**1. Introduction.** In the recent years, reaction-diffusion systems have been intensively studied not only by its mathematical interest but also because they model many phenomena in biology, ecology, combustion theory, chemical reactions, population dynamics, etc. A typical form of these models is

$$\begin{cases} \mathcal{L}_1 u = f(x,u,v) & \text{in } \Omega, \\ \mathcal{L}_2 v = g(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with a smooth boundary $\partial\Omega$, $f, g : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ are two given functions and $\mathcal{L}_k$, $k = 1, 2$, two second order uniformly elliptic operators that will be detailed below.

One of the methods more often used for the study of the existence of solutions of (1) is the lower-upper solutions method and its associated monotone iterations, see [26]. This method was applied to systems by D.H. Sattinger

Accepted for publication August 1998.
AMS Subject Classifications: 35B50, 35J55, 47N60, 92D25.

in [29] motivated from the results obtained by H. Amann for the scalar case, see [1] and [2]. Observe that the definition itself of lower-upper solutions of (1) depends heavily on the monotony properties on $f$ and $g$. Thus, following C.V. Pao's notation in [26], we can classify (1) according to their relative monotony as follows:

- **Type I**. *Quasi-monotone* systems: $f$ is nondecreasing in $v$ and $g$ in $u$, or $f$ is nonincreasing in $v$ and $g$ in $u$.
- **Type II**. *Mixed quasi-monotone* systems: $f$ is nondecreasing in $v$ and $g$ is nonincreasing in $u$ or viceversa.
- **Type III**. *Nonquasi-monotone* systems: either $f$ or $g$ dose not have monotony properties in $v$ or $u$ respectively.

D.H. Sattinger generalized the lower-upper solutions definition to systems of Type I, see Definitions 5 and 6, and he obtained, as in the scalar case, the existence of a minimal and a maximal solution of (1). These solutions are the limits of two monotone sequences built as solutions of some linear elliptic problems. When the system (1) is of Type II or III the method of monotone iterations does not work. Actually, in [16] J. Hernández showed that the above definitions of lower-upper solutions are too weak to obtain solutions of systems of Type II, even when the operators are $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$. So, following some classical works by Müller, the author introduced a new version of the lower-upper solutions definition to systems (1), see Definition 4, which coincides with the above ones when the systems are of Type I. With this definition, the existence of a solution between lower and upper solutions was obtained when $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$, see [16] and [21]. On the other hand, the important loss of the existence of a minimal and a maximal solution can be replaced in part in systems of Type II through the construction of an alternating sequence of approximations to the solution, see [19].

We fix our attention now in the regularity of $f$ and $g$. In [29] and [16], the authors imposed on $f, g$ to be $C^1$; P.J. McKenna and W. Walter [21] obtained the same result when $f$ and $g$ are uniformly Hölder continuous in $x$ and Lipschitz continuous in $(u, v)$. However, it is noted in [25] that the Lipschitz condition on $f$ and $g$ can be replaced by the existence of a positive constant $M$ such that maps defined by

$$t \mapsto f(x, t, v) + Mt, \qquad s \mapsto g(x, u, s) + Ms, \tag{2}$$

are nondecreasing in $t \in [\underline{u}(x), \overline{u}(x)]$ and in $s \in [\underline{v}(x), \overline{v}(x)]$, respectively, a.a. $x \in \Omega$ and for all $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$, where $(\underline{u}, \overline{u}), (\underline{v}, \overline{v})$ is a pair of lower-upper solutions of (1). These results are also proved in [26], even with nonlinear boundary conditions.

More recently, A. Cañada and J.L. Gámez [7] have weakened condition (2), replacing it by the existence of two continuous and nondecreasing functions $h, j : \mathbb{R} \mapsto \mathbb{R}$ such that

$$t \mapsto f(x, t, v) + h(t), \qquad s \mapsto g(x, u, s) + j(s), \tag{3}$$

are nondecreasing in the same spaces above. Although our interest is the existence of classical solutions, we would like to emphasize that similar results are obtained for weak solutions of (1) for systems of Type III and I in [8] and [22].

The aim of this work is to obtain some results when $f$ and $g$ are only Hölder continuous functions without the Lipschitz condition (2) or (3). The existence theorem for this weakened condition on $f$ and $g$ is motivated by the study for porous medium analysis, population dynamics and certain "dead core" problems, cf. [4, 3, 15, 31, 27, 28, 18, 12].

We describe now the contents of this paper. In Section 2 we present some general results for the scalar case, which will be used in the next sections. In Section 3 we obtain, with the only hypothesis that $f$ and $g$ are Hölder continuous functions, the existence of classical solutions of (1) between lower and upper solutions. Here, we consider two different uniformly elliptic self-adjoint operators and the regularity imposed on lower-upper solutions is weaker than in [16] and [21]. In Section 4 we analyze systems of Type I with more general second order operators. In this case, we build two monotone sequences which converge to a minimal and a maximal classical solution of (1). The construction of these sequences is more technical than in the case that $f$ and $g$ satisfy (2) or (3). In fact, these sequences are solutions of certain nonlinear elliptic problems. Furthermore, it supposes an extension to systems of the results obtained by H. Amann in [1] for a single equation. In Section 5 we study systems of Type II and we build an alternating sequence which approaches the solution of (1), which can be used for numerical computations of the problem. Finally in the last Section we apply our results to a dynamical population problem. This model was proposed by M.E. Gurtin and R.C. MacCamy in [15], and it has been studied in [7, 12, 18, 20, 27, 30]. We analyze three typical interactions for two species:

competition, prey-predator and symbiosis.

**2. Preliminaries.** In this section we present general existence and uniqueness results for the scalar case which will be applied later. We will consider elliptic operators of the form

$$\mathcal{L}_k u := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}^k(x)\frac{\partial}{\partial x_j}u), \quad k=1,2, \tag{4}$$

or

$$\mathcal{L}_k u := -\sum_{i,j=1}^{N} a_{ij}^k(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i^k(x)\frac{\partial u}{\partial x_i} + a^k(x)u, \quad k=1,2, \tag{5}$$

with

$$a_{ij}^k \in C^{1,\alpha}(\overline{\Omega}),\ a_i^k,\ a^k \in C^{\alpha}(\overline{\Omega}) \quad a_{ij}^k = a_{ji}^k, \quad \text{with } 0<\alpha<1,$$

and uniformly elliptic in the sense that

$$\exists \theta > 0 \quad \text{such that} \quad \sum_{i,j=1}^{N} a_{ij}^k(x)\xi_i\xi_j \geq \theta|\xi|^2, \quad \forall \xi \in \mathbb{R}^N,\ \forall x \in \Omega.$$

We analyze the semilinear elliptic boundary value problem

$$\begin{cases} \mathcal{L}u &= h(x,u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, $\mathcal{L}$ is a operator as (4) or (5), and $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

Let $\mathcal{L}$ be as in (4). The following definitions have been previously used in [17], see also [9].

**Definition 1.** We say that $u \in H_0^1(\Omega)$ is a weak solution of (6) if $h(\cdot,u) \in L^{\frac{2N}{N+2}}(\Omega)$ and

$$\int_\Omega \sum_{i,j=1}^{N} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_j} = \int_\Omega h(x,u)\phi \quad \text{for all } \phi \in H_0^1(\Omega).$$

**Definition 2.** We say that $(\underline{u}, \overline{u})$ is a pair of weak lower-upper solutions of (6) if

1. $\underline{u}, \overline{u} \in H^1(\Omega) \bigcap L^\infty(\Omega)$ with $\underline{u} \leq \overline{u}$ in $\Omega$.
2. For all $\phi \in H_0^1(\Omega)$, $\phi \geq 0$ in $\Omega$, we have

$$\int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial \underline{u}}{\partial x_i} \frac{\partial \phi}{\partial x_j} \leq \int_\Omega h(x, \underline{u})\phi, \quad \int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial \overline{u}}{\partial x_i} \frac{\partial \phi}{\partial x_j} \geq \int_\Omega h(x, \overline{u})\phi.$$

3. $\underline{u} \leq 0 \leq \overline{u}$ on $\partial\Omega$.

**Remark 1.** Given $u, v \in H^1(\Omega)$ we say that $u \leq v$ on $\partial\Omega$ when $\max\{0, u - v\} \in H_0^1(\Omega)$.

The following result is a particular case of the main theorem by J. Deuel and P. Hess in [11].

**Theorem 1.** *Let $\mathcal{L}$ be an operator of the form* (4) *and $\underline{u}, \overline{u}$ a pair of weak lower-upper solutions of* (6). *Assume also that there exists a function $k \in L^2(\Omega)$ such that*

$$|h(x,t)| \leq k(x), \quad \text{for a.a. } x \in \Omega \text{ and for all } t : \underline{u}(x) \leq t \leq \overline{u}(x).$$

*Then* (6) *possesses at least a weak solution $u$ such that*

$$\underline{u} \leq u \leq \overline{u}.$$

Concerning the uniqueness of weak solutions of (6), it is known the following lemma whose proof is included for the reader's convenience.

**Lemma 1.** *Let $\mathcal{L}$ be an operator of the form* (4) *and assume that for all $\xi, \eta \in \mathbb{R}$ such that $\xi \leq \eta$, we have*

$$h(x,\xi) \geq h(x,\eta), \quad a.a.\ x \in \Omega. \tag{7}$$

*Then there exists at most a weak solution of* (6).

**Proof.** Assume that $u_1, u_2 \in H_0^1(\Omega)$ are two weak solutions of (6). Then,

$$\theta \int_\Omega \nabla(u_1 - u_2) \cdot \nabla\phi \leq \int_\Omega [h(x,u_1) - h(x,u_2)]\phi \quad \text{for all } \phi \in H_0^1(\Omega).$$

Taking $\phi = (u_1 - u_2)^+ \in H_0^1(\Omega)$ as a nonnegative test function, we obtain

$$\theta \int_\Omega |\nabla(u_1 - u_2)^+|^2 \leq \int_\Omega [h(x,u_1) - h(x,u_2)](u_1 - u_2)^+ \leq 0$$

from (7). Hence

$$u_1(x) \leq u_2(x) \quad \text{a.a. } x \in \Omega.$$

Interchanging $u_1$ and $u_2$, the result follows. □

With respect to the existence and uniqueness of classical solution of (6), we present the following result due to H. Amann in [1].

**Definition 3.** We say that $(\underline{u}, \overline{u})$ is a pair of classical lower-upper solutions of (6) if

1. $\underline{u}, \overline{u} \in C^2(\Omega) \bigcap C^0(\overline{\Omega})$ and $\underline{u} \leq \overline{u}$ in $\Omega$.
2. $\mathcal{L}\underline{u} \leq h(x, \underline{u}), \quad \mathcal{L}\overline{u} \geq h(x, \overline{u})$.
3. $\underline{u} \leq 0 \leq \overline{u}$ on $\partial\Omega$.

**Theorem 2.** *Let $\mathcal{L}$ be an operator of the form* (5) *and $\underline{u}, \overline{u}$ a pair of classical lower-upper solutions of* (6). *Assume that*

$$h \in C^\alpha(\overline{\Omega} \times [\underline{u}, \overline{u}]). \tag{8}$$

*Then there exists a minimal $u_*$ and a maximal $u^*$ classical solutions of* (6), *such that, for every classical solution $u$ of* (6) *with $\underline{u} \leq u \leq \overline{u}$, we have*

$$\underline{u} \leq u_* \leq u \leq u^* \leq \overline{u}.$$

*Moreover, if $h$ is nonincreasing in $u$, then there exists a unique classical solution of* (6).

**Remark 2.** In [1] it is imposed that $a(x) \geq 0$ $x \in \Omega$. Observe that this is not necessary because we can consider the following problem equivalent to (6)

$$\begin{cases} (\mathcal{L} + M)u = \tilde{h}(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where $a(x) + M \geq 0$ and $\tilde{h}(x,u) = h(x,u) + Mu$ satisfies (8).

**3. Existence of solutions for systems.** In this section we consider the Dirichlet boundary value problem

$$\begin{cases} \mathcal{L}_1 u = f(x,u,v) & \text{in } \Omega, \\ \mathcal{L}_2 v = g(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

where $\Omega$ is as above, $\mathcal{L}_k$, $k = 1,2$ are uniformly elliptic operators of the form (4), and the nonlinear terms $f$ and $g$ satisfy

(H) $f, g \in C^\alpha(\overline{\Omega} \times \mathbb{R} \times \mathbb{R})$, for some $0 < \alpha < 1$.

**Definition 4.** We say that $(\underline{u}, \overline{u}), (\underline{v}, \overline{v})$ is a pair of weak lower-upper solutions of (10) if

1. $\underline{u}, \overline{u}, \underline{v}, \overline{v} \in H^1(\Omega) \bigcap L^\infty(\Omega)$, with $\underline{u} \le \overline{u}$, $\underline{v} \le \overline{v}$ in $\Omega$.
2. For all $\phi \in H_0^1(\Omega)$, $\phi \ge 0$ in $\Omega$, we have

$$\int_\Omega \sum_{i,j=1}^N a_{ij}^1 \frac{\partial \underline{u}}{\partial x_i} \frac{\partial \phi}{\partial x_j} \le \int_\Omega f(x, \underline{u}, v)\phi, \qquad \text{for } v \in [\underline{v}, \overline{v}],$$

$$\int_\Omega \sum_{i,j=1}^N a_{ij}^1 \frac{\partial \overline{u}}{\partial x_i} \frac{\partial \phi}{\partial x_j} \ge \int_\Omega f(x, \overline{u}, v)\phi, \qquad \text{for } v \in [\underline{v}, \overline{v}],$$

$$\int_\Omega \sum_{i,j=1}^N a_{ij}^2 \frac{\partial \underline{v}}{\partial x_i} \frac{\partial \phi}{\partial x_j} \le \int_\Omega g(x, u, \underline{v})\phi, \qquad \text{for } u \in [\underline{u}, \overline{u}],$$

$$\int_\Omega \sum_{i,j=1}^N a_{ij}^2 \frac{\partial \overline{v}}{\partial x_i} \frac{\partial \phi}{\partial x_j} \ge \int_\Omega g(x, u, \overline{v})\phi, \qquad \text{for } u \in [\underline{u}, \overline{u}].$$

3. $\underline{u} \le 0 \le \overline{u}$ and $\underline{v} \le 0 \le \overline{v}$ on $\partial\Omega$,

where for $z, w \in L^2(\Omega)$, we have defined

$$[z, w] := \{y \in L^2(\Omega) : z(x) \le y(x) \le w(x) \text{ a.a. in } \Omega\}.$$

The main result of this section is the following one.

**Theorem 3.** *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two operators of the form* (4)*. Assume* $(H)$ *and the existence of a pair* $(\underline{u},\overline{u})$, $(\underline{v},\overline{v})$ *of weak lower-upper solutions of* (10)*. Then there exists at least a classical solution* $(u,v)$ *of* (10) *such that*

$$\begin{aligned}\underline{u}(x)\le u(x)\le \overline{u}(x) \quad a.a.\ x\in\Omega,\\ \underline{v}(x)\le v(x)\le \overline{v}(x) \quad a.a.\ x\in\Omega.\end{aligned}$$

Before giving the proof of Theorem 3, we introduce the space

$$K\equiv[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]\subset E\equiv L^2(\Omega)\times L^2(\Omega).$$

$K$ is a bounded closed convex set in $E$. On the other hand, by $(H)$ given $(x_1,\xi_1,\eta_1),\ (x_2,\xi_2,\eta_2)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}$, there exists $\gamma>0$ such that

$$\begin{aligned}|f(x_1,\xi_1,\eta_1)-f(x_2,\xi_2,\eta_2)|\le\gamma(|x_1-x_2|^\alpha+|\xi_1-\xi_2|^\alpha+|\eta_1-\eta_2|^\alpha),\\ |g(x_1,\xi_1,\eta_1)-g(x_2,\xi_2,\eta_2)|\le\gamma(|x_1-x_2|^\alpha+|\xi_1-\xi_2|^\alpha+|\eta_1-\eta_2|^\alpha).\end{aligned}\tag{11}$$

Now, given $(u,v)\in K$, we define the functions $F_{(u,v)}:\Omega\times\mathbb{R}\to\mathbb{R}$; $G_{(u,v)}:\Omega\times\mathbb{R}\to\mathbb{R}$ a.a. $x\in\Omega$ by

$$F_{(u,v)}(x,\xi)=\begin{cases} f(x,u(x),v(x))-\gamma(\xi-u(x))^\alpha & \text{if } u(x)\le\xi,\\ f(x,u(x),v(x))+\gamma(u(x)-\xi)^\alpha & \text{if } \xi\le u(x),\end{cases}$$

$$G_{(u,v)}(x,\xi)=\begin{cases} g(x,u(x),v(x))-\gamma(\xi-v(x))^\alpha & \text{if } v(x)\le\xi,\\ g(x,u(x),v(x))+\gamma(v(x)-\xi)^\alpha & \text{if } \xi\le v(x).\end{cases}$$

From $(H)$ and the regularity of the weak lower-upper solutions, it is not hard to prove the following lemma:

**Lemma 2.** *Let $(u,v)\in K$. The following assertions are true:*

1. *For every $w\in[\underline{u},\overline{u}]$, $F_{(u,v)}(\cdot,w(\cdot))\in L^\infty(\Omega)$.*
2. *For every $z\in[\underline{v},\overline{v}]$, $G_{(u,v)}(\cdot,z(\cdot))\in L^\infty(\Omega)$.*

**Proof of Theorem 3.** We define the map $T:K\to E$ by $T(u,v)=(w,z)$ where $w$ and $z$ are the weak solutions, respectively, of problems

$$\begin{cases}\mathcal{L}_1 w = F_{(u,v)}(x,w) & \text{in } \Omega,\\ w=0 & \text{on } \partial\Omega,\end{cases}\tag{12}$$

$$\begin{cases}\mathcal{L}_2 z = G_{(u,v)}(x,z) & \text{in } \Omega,\\ z=0 & \text{on } \partial\Omega.\end{cases}\tag{13}$$

We claim:

1. $T$ is a well-defined operator.
2. $T(K) \subset K$.
3. $T$ is compact.

By using Schauder's fixed point theorem, we conclude that there exists $(u_0, v_0) \in K$ such that $T(u_0, v_0) = (u_0, v_0)$, then $(u_0, v_0)$ is a weak solution of (10), and a standard bootstrapping argument concludes the proof.

Now we prove the claim. First, we will see that $T$ is well defined and that $T(K) \subset K$ showing that $(\underline{u}, \overline{u})$ is a lower-upper solutions of (12). Indeed, let $\phi \in H_0^1(\Omega)$, $\phi \geq 0$, then by (11)

$$\int_\Omega \sum_{i,j=1}^N a_{ij}^1 \frac{\partial \underline{u}}{\partial x_i} \frac{\partial \phi}{\partial x_j} \leq \int_\Omega f(x, \underline{u}, v)\phi \leq \int_\Omega (f(x, u, v) + \gamma(u - \underline{u})^\alpha)\phi$$
$$= \int_\Omega F_{(u,v)}(x, \underline{u})\phi.$$

Similarly, it can be proved that $\overline{u}$ is an upper solution of (12). On the other hand, by Lemma 2 there exists $k \in L^2(\Omega)$ such that

$$|F_{(u,v)}(x,t)| \leq k(x) \text{ a.a. } x \in \Omega,\ t \in [\underline{u}(x), \overline{u}(x)].$$

Hence, by Theorem 1 and Lemma 1 there exists a unique weak solution $w$ of (12) with $\underline{u} \leq w \leq \overline{u}$. By a similar reasoning it follows the existence and uniqueness of a weak solution $z$ of (13) with $\underline{v} \leq z \leq \overline{v}$. Now, by the compact imbedding of $H_0^1(\Omega)$ in $L^2(\Omega)$, we only need to prove the continuity of $T$ from $E$ to $H_0^1(\Omega)$. Let $\{u_n, v_n\}_{n\geq 1}$ be two sequences such that $u_n \to u$ and $v_n \to v$ in $L^2(\Omega)$ and

$$(w_n, z_n) = T(u_n, v_n), \quad (w, z) = T(u, v),$$

where we know that $w_n, z_n, w, z \in H_0^1(\Omega) \cap K$. We will show that

$$w_n \to w \qquad \text{and} \qquad z_n \to z \quad \text{in } H_0^1(\Omega). \tag{14}$$

By definition, we have

$$\theta \int_\Omega |\nabla(w_n - w)|^2 \leq \int_\Omega [F_n(x, w_n) - F(x, w)](w_n - w) \tag{15}$$

where we have written $F_n := F_{(u_n,v_n)}$ and $F := F_{(u,v)}$. Fix $n \geq 1$ and define the sets

$$A := \{x \in \Omega : \; w_n(x) \geq w(x)\} \quad \text{and} \quad B := \{x \in \Omega : \; w_n(x) < w(x)\}.$$

It is clear that $\Omega = A \cup B$ and $A \cap B = \emptyset$ and, from the monotony properties of $F_n$ and $F$, we obtain

$$\begin{aligned} (F_n(x, w_n) - F(x, w))(w_n - w) &\leq (F_n(x, w) - F(x, w))(w_n - w) \quad \text{in } A, \\ (F_n(x, w_n) - F(x, w))(w_n - w) &\leq (F_n(x, w_n) - F(x, w_n))(w_n - w) \quad \text{in } B. \end{aligned} \tag{16}$$

Therefore, from (15) and (16), it follows

$$\theta \int_\Omega |\nabla(w_n - w)|^2 \\ \leq (\|F_n(x, w) - F(x, w)\|_{L^2} + \|F_n(x, w_n) - F(x, w_n)\|_{L^2})\|w_n - w\|_{L^2},$$

and so, from the Poincaré's inequality ,

$$\|w_n - w\|_{H_0^1(\Omega)} \leq C(\|F_n(x, w) - F(x, w)\|_{L^2} + \|F_n(x, w_n) - F(x, w_n)\|_{L^2}), \tag{17}$$

where $C$ is a positive constant. Assume that for $y = w(x)$ or $y = w_n(x)$ we have

$$|F_n(x, y) - F(x, y)| \leq C(|u_n(x) - u(x)|^\alpha + |v_n(x) - v(x)|^\alpha). \quad \text{a.a. } x \in \Omega, \tag{18}$$

We will deduce the continuity. Indeed, if (18) is true, then

$$\begin{aligned} \|F_n(x, y) - F(x, y)\|_{L^2} &\leq C(\|u_n - u\|^\alpha_{L^{2\alpha}} + \|v_n - v\|^\alpha_{L^{2\alpha}}) \\ &\leq C(\|u_n - u\|^\alpha_{L^2} + \|v_n - v\|^\alpha_{L^2}) \end{aligned} \tag{19}$$

this last inequality follows because of $0 < \alpha < 1$ and an imbedding result for $L^p$-spaces even with $p < 1$, see Proposition 6.12 in [13]. Now, from (17) and (19) it follows the continuity of $T$. We must prove (18). This depends on the relative positions of $y, u_n(x)$ and $u(x)$. For example, assume that $u(x) \leq u_n(x) \leq y$, then

$$\begin{aligned} &|F_n(x, y) - F(x, y)| \\ &\leq |f(x, u_n(x), v_n(x)) - f(x, u(x), v(x))| + \gamma[(y - u(x))^\alpha - (y - u_n(x))^\alpha] \\ &\leq \gamma[|u_n(x) - u(x)|^\alpha + |v_n(x) - v(x)|^\alpha + (y - u(x))^\alpha - (y - u_n(x))^\alpha] \\ &\leq C(|u_n(x) - u(x)|^\alpha + |v_n(x) - v(x)|^\alpha), \end{aligned}$$

this last inequality is obtained by taking $t_1 = u_n(x) - u(x)$ and $t_2 = y - u_n(x)$ in Lemma 3, whose enunciation is postponed up to the end of this proof. The other possibilities can be proved by a similar method. This proves (18) and completes the proof. □

**Lemma 3.** *Let* $\alpha \in (0,1)$ *and* $f(t) = t^\alpha$. *If* $0 \le t_1 < t_2$, *then*

$$f(t_1 + t_2) \le f(t_1) + f(t_2) \le 2f(t_1 + t_2).$$

**Remark 3.** Notice that the hypothesis $(H)$ can be weakened, because it is sufficient that $f$ and $g$ are Hölder continuous in

$$\overline{\Omega} \times [\operatorname{ess}\inf_{x\in\Omega} \underline{u}(x), \operatorname{ess}\sup_{x\in\Omega} \overline{u}(x)] \times [\operatorname{ess}\inf_{x\in\Omega} \underline{v}(x), \operatorname{ess}\sup_{x\in\Omega} \overline{v}(x)].$$

**4. Minimal and maximal solutions for systems of Type I.** In this section we will prove the existence of a maximal and a minimal classical solution of systems of Type I. These solutions are built as limits of some monotone sequences which are solutions of some nonlinear elliptic problems. This is a great difference with respect to the problem with $f$ and $g$ satisfying a Lipschitz condition and it requires a more technical proof.

We consider the problem (10), $\mathcal{L}_k$ are now operators in the form (5), and $f$ and $g$ satisfy $(H)$ with

(H1) $f$ is nondecreasing in $v$ and $g$ is nondecreasing in $u$.

Moreover, in this section we consider the lower-upper solutions in a classical sense; i.e.,

**Definition 5.** We say that $(\underline{u}, \overline{u})$, $(\underline{v}, \overline{v})$ is a pair of classical lower-upper solutions of (10) if

1. $\underline{u}, \overline{u}, \underline{v}, \overline{v} \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $\underline{u} \le \overline{u}$, $\underline{v} \le \overline{v}$ in $\Omega$.
2. $\mathcal{L}_1 \underline{u} \le f(x, \underline{u}, \underline{v})$, $\mathcal{L}_1 \overline{u} \ge f(x, \overline{u}, \overline{v})$,
   $\mathcal{L}_2 \underline{v} \le g(x, \underline{u}, \underline{v})$, $\mathcal{L}_2 \overline{v} \ge g(x, \overline{u}, \overline{v})$.
3. $\underline{u} \le 0 \le \overline{u}$ and $\underline{v} \le 0 \le \overline{v}$ on $\partial\Omega$.

The main result in this section is:

**Theorem 4.** *Let* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *be two operators of the form* (5). *Assume* $(H) - (H1)$ *and the existence of a pair* $(\underline{u}, \overline{u})$, $(\underline{v}, \overline{v})$ *of classical lower-upper* *solutions of* (10) *in the sense of Definition* 5. *Then there exists a minimal* $(u_*, v_*)$ *and a maximal* $(u^*, v^*)$ *classical solution of* (10) *such that for every classical solution* $(u, v)$ *of* (10) *with* $(\underline{u}, \underline{v}) \leq (u, v) \leq (\overline{u}, \overline{v})$, *we have*

$$(\underline{u}, \underline{v}) \leq (u_*, v_*) \leq (u, v) \leq (u^*, v^*) \leq (\overline{u}, \overline{v}).$$

We need some preliminaries for the proof of this result. Fix $\underline{w}, \overline{w}, \underline{z}, \overline{z} \in C^{2,\alpha}(\overline{\Omega})$ such that $\underline{u} \leq \underline{w} \leq \overline{w} \leq \overline{u}$ and $\underline{v} \leq \underline{z} \leq \overline{z} \leq \overline{v}$ in $\Omega$. Now, we consider the problems

$$\begin{cases} \mathcal{L}_1 u = f(x, \underline{w}, \underline{z}) - \gamma(u - \underline{w})^\alpha \equiv \underline{f}(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{20}$$

$$\begin{cases} \mathcal{L}_1 u = f(x, \overline{w}, \overline{z}) + \gamma(\overline{w} - u)^\alpha \equiv \overline{f}(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{21}$$

$$\begin{cases} \mathcal{L}_2 v = g(x, \underline{w}, \underline{z}) - \gamma(v - \underline{z})^\alpha \equiv \underline{g}(x, v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{22}$$

$$\begin{cases} \mathcal{L}_2 v = g(x, \overline{w}, \overline{z}) + \gamma(\overline{z} - v)^\alpha \equiv \overline{g}(x, v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{23}$$

We have the

**Lemma 4.** *Assume that* $(\underline{w}, \overline{w}), (\underline{z}, \overline{z})$ *is a pair of classical lower-upper solutions of* (10). *Then there exists a unique classical solution* $u_*, u^*, v_*, v^*$ *of* (20), (21), (22) *and* (23), *respectively, such that*

$$\underline{u} \leq \underline{w} \leq u_* \leq u^* \leq \overline{w} \leq \overline{u},$$
$$\underline{v} \leq \underline{z} \leq v_* \leq v^* \leq \overline{z} \leq \overline{v}.$$

**Proof.** From $(H)$, it is easy to prove that $\underline{f}, \overline{f}, \underline{g}$ and $\overline{g}$ are Hölder continuous functions. We will show that $(\underline{w}, \overline{w})$ is a classical lower-upper solutions of (20). Indeed,

$$\mathcal{L}_1 \underline{w} \leq f(x, \underline{w}, \underline{z}) = \underline{f}(x, \underline{w}),$$

and, on the other hand, by $(H1)$ and (11),

$$\mathcal{L}_1 \overline{w} \geq f(x, \overline{w}, \overline{z}) \geq f(x, \overline{w}, \underline{z}) \geq f(x, \underline{w}, \underline{z}) - \gamma(\overline{w} - \underline{w})^\alpha = \underline{f}(x, \overline{w}).$$

Moreover, from the monotony property of $\underline{f}$ and Theorem 2 it follows the existence and uniqueness of $u_*$. Similarly, we can prove the existence and uniqueness of a solution to (21), which we call $u^*$. We have to prove that $u_* \le u^*$. First, we will show that

$$\underline{f}(x,\xi) \le \overline{f}(x,\xi), \quad \text{for all } \xi \in [\underline{w}(x), \overline{w}(x)]. \tag{24}$$

Using $(H1)$, (11) and Lemma 3, we have

$$\begin{aligned} f(x,\underline{w}(x),\underline{z}(x)) - f(x,\overline{w}(x),\overline{z}(x)) &\le f(x,\underline{w}(x),\overline{z}(x)) - f(x,\overline{w}(x),\overline{z}(x)) \\ &\le \gamma(\overline{w}(x) - \underline{w}(x))^\alpha \le \gamma((\overline{w}(x)-\xi)^\alpha + (\xi - \underline{w}(x))^\alpha) \end{aligned}$$

and so (24) is proved. Now, let $M$ be a positive constant such that $M + a^1(x) \ge 0$. Then from (24) it follows that

$$\begin{aligned} (\mathcal{L}_1 + M)(u^* - u_*) &= M(u^* - u_*) + \overline{f}(x,u^*) - \underline{f}(x,u_*) \\ &\ge M(u^* - u_*) + \overline{f}(x,u^*) - \overline{f}(x,u_*), \end{aligned}$$

and by a standard application of the maximum principle, it follows that

$$u_* \le u^* \tag{25}$$

We can reason similarly with problems (22) and (23). This completes the proof. □

Motivated by a result in [1], we obtain:

**Lemma 5.** *Let $\{h_n^k\}_{n\ge 1}$, $k = 1,2$, be two sequences of bounded functions in $C^\alpha(\overline{\Omega} \times [\underline{u},\overline{u}]^2 \times [\underline{v},\overline{v}])$ for $k=1$ and $C^\alpha(\overline{\Omega} \times [\underline{u},\overline{u}] \times [\underline{v},\overline{v}]^2)$ for $k=2$ and such that $h_n^k \to h^k$ pointwise. Moreover, let $\{u_n\}, \{v_n\}_{n\ge 0}$ be two sequences in $C^{2,\alpha}(\overline{\Omega})$ satisfying $(\underline{u},\underline{v}) \le (u_n,v_n) \le (\overline{u},\overline{v})$ which converge to $u$ and $v$ pointwise, respectively. Suppose that we have for $n \ge 1$*

$$\begin{cases} \mathcal{L}_1 u_n = h_n^1(x,u_n,u_{n-1};v_{n-1}) & \text{in } \Omega, \\ \mathcal{L}_2 v_n = h_n^2(x,u_{n-1};v_n,v_{n-1}) & \text{in } \Omega, \\ u_n = v_n = 0 & \text{on } \partial\Omega. \end{cases}$$

*Then $\{u_n\}, \{v_n\}$ converge to $u$ and $v$ in $C^2(\overline{\Omega})$ respectively, $u, v \in C^{2,\alpha}(\overline{\Omega})$ and satisfy*

$$\begin{cases} \mathcal{L}_1 u = h^1(x,u,u;v) & \text{in } \Omega, \\ \mathcal{L}_2 v = h^2(x,u;v,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

**Proof.** Since $(\underline{u},\underline{v}) \le (u_n,v_n) \le (\overline{u},\overline{v})$, then

$$h_n^1(x,u_n,u_{n-1};v_{n-1}),\ h_n^2(x,u_{n-1};v_n,v_{n-1}) \in L^p(\Omega), \quad \text{for all } p \ge 1.$$

Therefore, by elliptic regularity, see [14], and a bootstrapping argument it follows that $\{u_n\}$ and $\{v_n\}$ are bounded in $C^{2,\alpha}(\overline{\Omega})$. By Ascoli-Arzela's Theorem, there exist two subsequences, again labeled by $n$, such that $u_n \to u$ and $v_n \to v$ in $C^2(\overline{\Omega})$, with $u,v \in C^{2,\alpha}(\overline{\Omega})$. Thanks to the properties of $h_n^k$, we get

$$\begin{aligned} h_n^1(x,u_n,u_{n-1};v_{n-1}) &\to h^1(x,u,u;v),\\ h_n^2(x,u_{n-1};v_n,v_{n-1}) &\to h^2(x,u;v,v). \end{aligned}$$

The proof is complete. □

Finally, we define $\underline{f},\overline{f} : \overline{\Omega} \times [\underline{u},\overline{u}]^2 \times [\underline{v},\overline{v}] \to \mathbb{R}$ and $\underline{g},\overline{g} : \overline{\Omega} \times [\underline{u},\overline{u}] \times [\underline{v},\overline{v}]^2) \to \mathbb{R}$ by

$$\underline{f}(x,\xi,\xi';\eta) = \begin{cases} f(x,\xi',\eta) & \underline{u}(x) \le \xi \le \xi' \le \overline{u}(x),\\ & \eta \in [\underline{v}(x),\overline{v}(x)]\\ f(x,\xi',\eta) - \gamma(\xi-\xi')^\alpha & \underline{u}(x) \le \xi' \le \xi \le \overline{u}(x),\\ & \eta \in [\underline{v}(x),\overline{v}(x)] \end{cases}$$

$$\overline{f}(x,\xi,\xi';\eta) = \begin{cases} f(x,\xi',\eta) + \gamma(\xi'-\xi)^\alpha & \underline{u}(x) \le \xi \le \xi' \le \overline{u}(x),\\ & \eta \in [\underline{v}(x),\overline{v}(x)]\\ f(x,\xi',\eta) & \underline{u}(x) \le \xi' \le \xi \le \overline{u}(x),\\ & \eta \in [\underline{v}(x),\overline{v}(x)] \end{cases}$$

$$\underline{g}(x,\xi;\eta,\eta') = \begin{cases} g(x,\xi,\eta') & \underline{v}(x) \le \eta \le \eta' \le \overline{v}(x),\\ & \xi \in [\underline{u}(x),\overline{u}(x)]\\ g(x,\xi,\eta') - \gamma(\eta-\eta')^\alpha & \underline{v}(x) \le \eta' \le \eta \le \overline{v}(x),\\ & \xi \in [\underline{u}(x),\overline{u}(x)] \end{cases}$$

$$\overline{g}(x,\xi;\eta,\eta') = \begin{cases} g(x,\xi,\eta') + \gamma(\eta'-\eta)^\alpha & \underline{v}(x) \le \eta \le \eta' \le \overline{v}(x),\\ & \xi \in [\underline{u}(x),\overline{u}(x)]\\ g(x,\xi,\eta') & \underline{v}(x) \le \eta' \le \eta \le \overline{v}(x),\\ & \xi \in [\underline{u}(x),\overline{u}(x)] \end{cases}$$

Now we summarize the main properties of the previous functions.

**Lemma 6.** *The following assertions hold:*

1. $\underline{f},\overline{f} \in C^\alpha(\overline{\Omega} \times [\underline{u},\overline{u}]^2 \times [\underline{v},\overline{v}])$ *and* $\underline{g},\overline{g} \in C^\alpha(\overline{\Omega} \times [\underline{u},\overline{u}] \times [\underline{v},\overline{v}]^2)$.

2. $\underline{f}, \overline{f}$ *are nonincreasing in* $\xi$ and $\underline{g}, \overline{g}$ *are nonincreasing in* $\eta$.

3. *For all* $x \in \overline{\Omega}; \xi, \xi' \in [\underline{u}(x), \overline{u}(x)],$ *and* $\eta \in [\underline{v}(x), \overline{v}(x)]$,

$$\underline{f}(x, \xi, \xi'; \eta) \leq \overline{f}(x, \xi, \xi'; \eta).$$

*Similarly, for all* $x \in \overline{\Omega}; \xi \in [\underline{u}(x), \overline{u}(x)]$ *and* $\eta, \eta' \in [\underline{v}(x), \overline{v}(x)]$,

$$\underline{g}(x, \xi; \eta, \eta') \leq \overline{g}(x, \xi; \eta, \eta').$$

This result is not hard to prove and we omit the proof here. We only would like to note that paragraph 3 can be proved as (24).

**Proof of Theorem 4.** We take $u_0 = \underline{u}$, $v_0 = \underline{v}$ and define the sequences $\{u_n\}, \{v_n\}_{n \geq 1}$ by:

$$\begin{cases} \mathcal{L}_1 u_n = f(x, u_{n-1}, v_{n-1}) - \gamma(u_n - u_{n-1})^\alpha & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{26}$$

$$\begin{cases} \mathcal{L}_2 v_n = g(x, u_{n-1}, v_{n-1}) - \gamma(v_n - v_{n-1})^\alpha & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{27}$$

Analogously, we take $u^0 = \overline{u}$, $v^0 = \overline{v}$ and define the sequences $\{u^n\}, \{v^n\}_{n \geq 1}$ by:

$$\begin{cases} \mathcal{L}_1 u^n = f(x, u^{n-1}, v^{n-1}) + \gamma(u^{n-1} - u^n)^\alpha & \text{in } \Omega, \\ u^n = 0 & \text{on } \partial\Omega, \end{cases} \tag{28}$$

$$\begin{cases} \mathcal{L}_2 v^n = g(x, u^{n-1}, v^{n-1}) + \gamma(v^{n-1} - v^n)^\alpha & \text{in } \Omega, \\ v^n = 0 & \text{on } \partial\Omega. \end{cases} \tag{29}$$

We apply Lemma 4 to problems (26), (27), (28) and (29) with $(\underline{w}, \overline{w}) = (u_0, u^0)$ and $(\underline{z}, \overline{z}) = (v_0, v^0)$, and we conclude the existence and uniqueness of $u_1, u^1, v_1$ and $v^1$ such that

$$u_0 \leq u_1 \leq u^1 \leq u^0, \qquad v_0 \leq v_1 \leq v^1 \leq v^0.$$

Now, suppose that

$$\begin{array}{l} \underline{u} \leq u_1 \leq \ldots \leq u_{k-1} \leq u_k \leq u^k \leq u^{k-1} \leq \ldots \leq u^1 \leq \overline{u}, \\ \underline{v} \leq v_1 \leq \ldots \leq v_{k-1} \leq v_k \leq v^k \leq v^{k-1} \leq \ldots \leq v^1 \leq \overline{v}. \end{array} \tag{30}$$

Now we take $(\underline{w},\overline{w}) = (u_k, u^k)$ and $(\underline{z},\overline{z}) = (v_k, v^k)$ and apply Lemma 4 again. For this, we show that $(\underline{w},\overline{w}),(\underline{z},\overline{z})$ is a pair of classical lower-upper solutions of (10). Indeed, from $(H1)$, (11) and (30), it follows

$$\mathcal{L}_1 u_k = f(x, u_{k-1}, v_{k-1}) - \gamma(u_k - u_{k-1})^\alpha \leq f(x, u_k, v_{k-1}) \leq f(x, u_k, v_k)$$

and we can prove similarly the other inequalities. Then we get

$$u_k \leq u_{k+1} \leq u^{k+1} \leq u^k, \qquad v_k \leq v_{k+1} \leq v^{k+1} \leq v^k.$$

Therefore, there exist functions $u_*, u^*, v_*$ and $v^*$ such that $u_n \to u_*, u^n \to u^*, v_n \to v_*$ and $v^n \to v^*$ pointwise. Now, we define

$$h_n^1(x,\xi,\xi';\eta) \equiv \underline{f}(x,\xi,\xi';\eta) \qquad \text{and} \qquad h_n^2(x,\xi;\eta,\eta') \equiv \underline{g}(x,\xi;\eta,\eta').$$

Lemma 6 allows to apply Lemma 5 and gives the result. The minimality and maximality of $(u_*, v_*)$ and $(u^*, v^*)$ follow by the same reasoning used to prove (25). This completes the proof. □

Instead of (H1), now we assume

(H2) $f$ is nonincreasing in $v$ and $g$ is nonincreasing in $u$.

We define

**Definition 6.** We say that $(\underline{u},\overline{u})$, $(\underline{v},\overline{v})$ is a pair of classical lower-upper solutions of (10) if

1. $\underline{u},\overline{u},\underline{v},\overline{v} \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $\underline{u} \leq \overline{u}$, $\underline{v} \leq \overline{v}$ in $\Omega$.
2. $\mathcal{L}_1\underline{u} \leq f(x,\underline{u},\overline{v})$, $\quad \mathcal{L}_1\overline{u} \geq f(x,\overline{u},\underline{v})$,
   $\mathcal{L}_2\underline{v} \leq g(x,\overline{u},\underline{v})$, $\quad \mathcal{L}_2\overline{v} \geq g(x,\underline{u},\overline{v})$.
3. $\underline{u} \leq 0 \leq \overline{u}$ and $\underline{v} \leq 0 \leq \overline{v}$ on $\partial\Omega$.

In this case, the main result is

**Theorem 5.** *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two operators of the form* (5). *Assume $(H) - (H2)$ and the existence of a pair $(\underline{u},\overline{u})$, $(\underline{v},\overline{v})$ of classical lower-upper solutions of* (10) *in the sense of Definition* 6. *Then there exists a min-max $(u_*, v^*)$ and a max-min $(u^*, v_*)$ classical solution of* (10) *such that for every classical solution $(u,v)$ of* (10) *with $(\underline{u},\underline{v}) \leq (u,v) \leq (\overline{u},\overline{v})$, we have*

$$(\underline{u},\underline{v}) \leq (u_*, v_*) \leq (u,v) \leq (u^*, v^*) \leq (\overline{u},\overline{v}).$$

**Proof.** The proof is rather similar to the proof of Theorem 4, so that we only sketch it. In fact, we only show the sequences built in this case. We take $u_0 = \underline{u}$, $v^0 = \overline{v}$ and define the sequences $\{u_n\}_{n\geq 1}, \{v^n\}_{n\geq 1}$ as the solutions of the following problems :

$$\left\{ \begin{array}{rcll} \mathcal{L}_1 u_n &=& f(x, u_{n-1}, v^{n-1}) - \gamma(u_n - u_{n-1})^\alpha & \text{in } \Omega, \\ u_n &=& 0 & \text{on } \partial\Omega, \end{array} \right. \tag{31}$$

$$\left\{ \begin{array}{rcll} \mathcal{L}_2 v^n &=& g(x, u_{n-1}, v^{n-1}) + \gamma(v^{n-1} - v^n)^\alpha & \text{in } \Omega, \\ v^n &=& 0 & \text{on } \partial\Omega, \end{array} \right. \tag{32}$$

and taking $v_0 = \underline{v}$ and $u^0 = \overline{u}$, we define $\{v_n\}_{n\geq 1}, \{u^n\}_{n\geq 1}$ by

$$\left\{ \begin{array}{rcll} \mathcal{L}_2 v_n &=& g(x, u^{n-1}, v_{n-1}) - \gamma(v_n - v_{n-1})^\alpha & \text{in } \Omega, \\ v_n &=& 0 & \text{on } \partial\Omega, \end{array} \right. \tag{33}$$

$$\left\{ \begin{array}{rcll} \mathcal{L}_1 u^n &=& f(x, u^{n-1}, v_{n-1}) + \gamma(u^{n-1} - u^n)^\alpha & \text{in } \Omega, \\ u^n &=& 0 & \text{on } \partial\Omega. \end{array} \right. \tag{34}$$

**5. Alternating schemes to systems of Type II.** As we have mentioned in the Introduction when the system is not of Type I, the method of monotone iterations does not work. However, if the system is of Type II we can build two alternating sequences which approach the solution. This can be of great value for numerical procedures for the computation of solutions and error estimates. Again, we consider (10) and, for example, we assume

(H3) $f$ is nondecreasing in $v$ and $g$ is nonincreasing in $u$.

In this particular case, the Definition 4 is equivalent to

**Definition 7.** We say that $(\underline{u}, \overline{u}), (\underline{v}, \overline{v})$ is a pair of weak lower-upper solutions of (10) if

1. $\underline{u}, \overline{u}, \underline{v}, \overline{v} \in H^1(\Omega) \bigcap L^\infty(\Omega)$, with $\underline{u} \leq \overline{u}$, $\underline{v} \leq \overline{v}$ in $\Omega$.
2. For all $\phi \in H_0^1(\Omega)$, $\phi \geq 0$ in $\Omega$, we have

$$\int_\Omega \sum_{i,j=1}^N a_{ij}^1 \frac{\partial \underline{u}}{\partial x_i} \frac{\partial \phi}{\partial x_j} \leq \int_\Omega f(x, \underline{u}, \underline{v})\phi, \quad \int_\Omega \sum_{i,j=1}^N a_{ij}^1 \frac{\partial \overline{u}}{\partial x_i} \frac{\partial \phi}{\partial x_j} \geq \int_\Omega f(x, \overline{u}, \overline{v})\phi,$$

$$\int_\Omega \sum_{i,j=1}^N a_{ij}^2 \frac{\partial \underline{v}}{\partial x_i} \frac{\partial \phi}{\partial x_j} \leq \int_\Omega g(x, \overline{u}, \underline{v})\phi, \quad \int_\Omega \sum_{i,j=1}^N a_{ij}^2 \frac{\partial \overline{v}}{\partial x_i} \frac{\partial \phi}{\partial x_j} \geq \int_\Omega g(x, \underline{u}, \overline{v})\phi.$$

3. $\underline{u} \le 0 \le \overline{u}$ and $\underline{v} \le 0 \le \overline{v}$ on $\partial\Omega$.

**Theorem 6.** *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two operators of the form* (4). *Assume* $(H)-(H3)$ *and let* $(\underline{u},\overline{u}),(\underline{v},\overline{v})$ *be a weak lower-upper solutions of* (10) *in the sense of Definition* 7. *We define $u_{-1}=\overline{u}$, $u_0=\underline{u}$, $v_{-1}=\overline{v}$, $v_0=\underline{v}$ and the sequences $\{u_n\}$ and $\{v_n\}$ as the solutions of*

$$\begin{cases} \mathcal{L}_1 u_n &= F_{(u_{n-2},v_{n-2})}(x,u_n) \quad \text{in } \Omega,\\ \quad u_n &= 0 \qquad\qquad\qquad\quad\ \text{on } \partial\Omega, \end{cases} \tag{35}$$

$$\begin{cases} \mathcal{L}_2 v_n &= G_{(u_{n-1},v_{n-2})}(x,v_n) \quad \text{in } \Omega,\\ \quad v_n &= 0 \qquad\qquad\qquad\quad\ \text{on } \partial\Omega. \end{cases} \tag{36}$$

*Then* (10) *possesses at least a classical solution* $(u,v)$ *verifying*

$$\begin{array}{c} \underline{u}=u_0\le u_2\le\cdots\le u_{2r}\le\cdots\le u\le\cdots\le u_{2r-1}\le\cdots\le u_1\le u_{-1}=\overline{u},\\ \underline{v}=v_0\le v_2\le\cdots\le v_{2r}\le\cdots\le v\le\cdots\le v_{2r-1}\le\cdots\le v_1\le v_{-1}=\overline{v}. \end{array} \tag{37}$$

**Proof.** Theorem 3 provides us the existence of $(u,v)$. We have to prove (37). Given $u_{n-1}$, $u_{n-2}$ and $v_{n-2}$, the uniqueness of solution of (35) and (36) follows like existence and uniqueness proof of problem (12) in Theorem 3. We will show that

$$u_2 \le u_1. \tag{38}$$

Indeed, by definition of solution of (35), we have, for all $\phi \in H_0^1(\Omega)$,

$$\int_\Omega \sum_{i,j=1}^N a_{ij}^1 \frac{\partial(u_2-u_1)}{\partial x_i}\frac{\partial\phi}{\partial x_j} = \int_\Omega [F_{(\underline{u},\underline{v})}(x,u_2) - F_{(\overline{u},\overline{v})}(x,u_1)]\phi.$$

Now we take $\phi=(u_2-u_1)^+$ and, using $(H3)$ and (11), we obtain

$$\begin{aligned} \theta\int_\Omega |\nabla(u_2-u_1)^+|^2 &\le \int_\Omega [F_{(\underline{u},\underline{v})}(x,u_2) - F_{(\overline{u},\overline{v})}(x,u_1)](u_2-u_1)^+\\ &= \int_\Omega [f(x,\underline{u},\underline{v}) - \gamma(u_2-\underline{u})^\alpha - f(x,\overline{u},\overline{v}) - \gamma(\overline{u}-u_1)^\alpha](u_2-u_1)^+\\ &\le \gamma\int_\Omega [(\overline{u}-\underline{u})^\alpha - (u_2-\underline{u})^\alpha - (\overline{u}-u_1)^\alpha](u_2-u_1)^+\\ &\le \gamma\int_\Omega [(\overline{u}-u_2)^\alpha - (\overline{u}-u_1)^\alpha](u_2-u_1)^+ \le 0, \end{aligned}$$

and so we deduce (38). Similarly, it can be shown that $\underline{v} \le v_2 \le v_1 \le \overline{v}$. Now we reason by induction.

1. Assume that $k$ is odd. Suppose that

$$\begin{aligned} \underline{u} = u_0 \le u_2 \le \cdots \le u_{k-2} \le u_{k-1} \le \cdots \le u_1 \le u_{-1} = \overline{u}, \\ \underline{v} = v_0 \le v_2 \le \cdots \le v_{k-2} \le v_{k-1} \le \cdots \le v_1 \le v_{-1} = \overline{v}, \end{aligned}$$

and we will show that

$$u_{k-2} \le u_k \le u_{k-1}, \quad v_{k-2} \le v_k \le v_{k-1}. \tag{39}$$

By definition of $u_k$ and $u_{k-1}$ and taking $\phi = (u_k - u_{k-1})^+$, we obtain

$$\begin{aligned} \theta \int_\Omega |\nabla (u_k - u_{k-1})^+|^2 &\le \int_\Omega \Big[ f(x, u_{k-2}, v_{k-2}) \\ &- \gamma (u_k - u_{k-2})^\alpha - f(x, u_{k-3}, v_{k-3}) - \gamma(u_{k-3} - u_{k-1})^\alpha \Big] (u_k - u_{k-1})^+ \\ &\le \gamma \int_\Omega [(u_{k-3} - u_{k-2})^\alpha - (u_{k-3} - u_{k-1})^\alpha - (u_k - u_{k-2})^\alpha](u_k - u_{k-1})^+ \\ &\le \gamma \int_\Omega [(u_{k-1} - u_{k-2})^\alpha - (u_k - u_{k-2})^\alpha](u_k - u_{k-1})^+ \le 0, \end{aligned}$$

and then $u_k \le u_{k-1}$. Analogously, we can prove the other inequalities in (39).

2. Assume that $k$ is even. Suppose that

$$\begin{aligned} \underline{u} = u_0 \le u_2 \le \cdots \le u_{k-1} \le u_{k-2} \le \cdots \le u_1 \le u_{-1} = \overline{u}, \\ \underline{v} = v_0 \le v_2 \le \cdots \le v_{k-1} \le v_{k-2} \le \cdots \le v_1 \le v_{-1} = \overline{v}. \end{aligned}$$

From a similar reasoning, we can prove

$$u_{k-1} \le u_k \le u_{k-2}, \qquad v_{k-1} \le v_k \le v_{k-2}.$$

The ordering of $u$ and $v$ in (37) is proved in the same way as above. This completes the proof. □

**Remark 4.** Theorems 3, 4, 5 and 6 are also valid if we consider systems with an arbitrary number of equations. In this case, we should generalize the corresponding definition of lower-upper solutions, see [21], [19] and [8].

**6. Applications to ecological models.** We apply the previous results to the systems

$$\left\{\begin{array}{ll} -d_1\Delta U^m = U(A - BU^p \pm CV^q) & \text{in } \Omega, \\ -d_2\Delta V^n = V(D - EV^r \pm FU^s) & \text{in } \Omega, \\ U = V = 0 & \text{on } \partial\Omega, \end{array}\right. \tag{40}$$

where $m, n, p, q, r, s, d_1, d_2, B, C, E, F$ are positive constants with $m, n > 1$ and $A, D \in \mathbb{R}$. This type of problem was introduced in [15], see also [24] and [23], for modeling the steady-state behavior of two species, where $U(x)$ and $V(x)$ are the densities of each of them. In this case, the diffusion, the rate the moving of these species from high density regions to low density ones, is slow ($m, n > 1$). Moreover, in (40) the interactions of the species are more general that the classical Lotka-Volterra's model, which seems give rise to "more realistic " models than the last one, see [15] and [27]. In (40) we are assuming that $\Omega$, the inhabiting region, is fully surrounded by inhospitable areas, because both population densities are subject to homogeneous Dirichlet boundary conditions. Here, $d_1$ and $d_2$ are the diffusion rates of each species, $B$ and $E$ account for self-regulation in each population, $C$ and $F$ are the interaction rates between the species and, finally, $A$ and $D$ are the growth rates of the species, positive on favorable regions and negative on unfavorable ones.

Thanks to the meaning of (40), we only are interested in nonnegative solutions of (40). In fact, we only look for positive solutions of (40), that we call *coexistence states*. Observe that, in general, the strong maximum principle does not work in this case, and so there exist nonnegative solutions that are not positive; i.e., there exist "dead cores" of the solutions.

If we make change of variables $U = \alpha w$ and $V = \beta z$ with $\alpha$ and $\beta$ such that $\alpha^{m-p-1} = \frac{B}{d_1}$ and $\beta^{n-r-1} = \frac{E}{d_2}$, we obtain

$$\left\{\begin{array}{ll} -\Delta w^m = w(\lambda - w^p \pm bz^q) & \text{in } \Omega, \\ -\Delta z^n = z(\mu - z^r \pm cw^s) & \text{in } \Omega, \\ w = z = 0 & \text{on } \partial\Omega, \end{array}\right.$$

where $\lambda,\ \mu \in \mathbb{R}$ and $b, c > 0$. Lastly, we write $w^m = u,\ z^n = v$, and obtain

$$\left\{\begin{array}{ll} -\Delta u = \lambda u^{1/m} - u^{(p+1)/m} \pm bu^{1/m}v^{q/n} & \text{in } \Omega, \\ -\Delta v = \mu v^{1/n} - v^{(r+1)/n} \pm cv^{1/n}u^{s/m} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{41}$$

To study these systems, we need first consider the following equations

$$\begin{cases} -\Delta w = \sigma w^a & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{42}$$

$$\begin{cases} -\Delta w = \sigma w^a - dw^e & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{43}$$

with

$$0 < a < 1, \quad e > a, \quad d > 0. \tag{44}$$

Problem (42) has been studied for porous medium analysis, so we only state the following result.

**Theorem 7.** *Let $\sigma > 0$. Then there exists a unique classical positive solution $w$ of* (42)*. If $\sigma \leq 0$,* (42) *does not possess any positive solution.*

**Remark 5.** We denote by $\phi_{[\sigma,a]} > 0$ the unique positive solution of (42). The proof of Theorem 7 can be found in many papers, see for example [4], [5] and [17].

Concerning problem (43), we obtain:

**Theorem 8.** *Let $\sigma > 0$. Then there exists a unique classical positive solution of* (43)*. Moreover, if $\sigma \leq 0$ there is no positive solution of* (43)*.*

**Proof.** We use Theorem 2. We take $\underline{w} = \epsilon\phi_{[\sigma,a]}$, with $\epsilon > 0$ to choose. We can prove that $\underline{w}$ is a lower solution of (43) if we have

$$\epsilon^a(\phi_{[\sigma,a]})^a(\sigma(1-\epsilon^{1-a}) - \epsilon^{e-a}(\phi_{[\sigma,a]})^{e-a}) \geq 0,$$

which holds if $\epsilon$ is sufficiently small. As an upper solution we can pick a sufficiently large positive constant. In fact, it can be proved that for every solution $w$ of (43)

$$\|w\|_{L^\infty} \leq (\frac{\sigma}{d})^{\frac{1}{e-a}}. \tag{45}$$

For the uniqueness we use [6]. Observe that the function

$$t \mapsto \frac{\sigma t^a - dt^e}{t} = t^{a-1}(\sigma - dt^{e-a})$$

is nonincreasing from (45). If $\sigma \leq 0$, the maximum principle ends the proof.

**Remark 6.** We will denote by $\theta_{[\sigma,d;a,e]} > 0$ the unique positive solution of (43) if $\sigma > 0$. Problem (43) has been studied in [17], [28] and [10].

**6.1. Competition.** Consider the boundary value problem

$$\begin{cases} -\Delta u = \lambda u^{1/m} - u^{(p+1)/m} - bu^{1/m}v^{q/n} & \text{in } \Omega, \\ -\Delta v = \mu v^{1/n} - v^{(r+1)/n} - cv^{1/n}u^{s/m} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{46}$$

It is easy to prove that $(H)$ and $(H2)$ hold for (46), so we can apply Theorem 5. First, we observe that $\lambda > 0$ and $\mu > 0$ is a necessary condition for the existence of nonnegative solutions of (46).

**Theorem 9.** *Assume that*

$$\lambda > b\mu^{q/r} \quad \textit{and} \quad \mu > c\lambda^{s/p}. \tag{47}$$

*Then* (46) *possesses at least a coexistence state.*

**Proof.** We pick

$$(\underline{u}, \underline{v}) = (\rho\phi_{[\lambda,1/m]}, \epsilon\phi_{[\mu,1/n]}),$$

$$(\overline{u}, \overline{v}) = (\theta_{[\lambda,1;1/m,(p+1)/m]}, \theta_{[\mu,1;1/n,(r+1)/n]}) \equiv (\theta_\lambda, \theta_\mu),$$

with $\rho$ and $\epsilon$ positive constants to be chosen later. It is not hard to prove that $(\underline{u}, \overline{u}), (\underline{v}, \overline{v})$ is a pair of lower-upper solutions of (46) if

$$\rho^{1/m}\phi^{1/m}_{[\lambda,1/m]}(\rho^{(m-1)/m}\lambda - \lambda + \rho^{p/m}\phi^{p/m}_{[\lambda,1/m]} + b\theta^{q/n}_\mu) \leq 0,$$

$$\epsilon^{1/n}\phi^{1/n}_{[\mu,1/n]}(\epsilon^{(n-1)/n}\mu - \mu + \epsilon^{r/n}\phi^{r/n}_{[\mu,1/n]} + c\theta^{s/m}_\lambda) \leq 0.$$

This holds if $b\theta^{q/n}_\mu < \lambda$ and $c\theta^{s/m}_\lambda < \mu$, which follows from (45) and (47). This completes the proof.

**6.2. Predator-prey.** In the following problem $u(x)$ and $v(x)$ denote the predator and prey populations respectively,

$$\begin{cases} -\Delta u = \lambda u^{1/m} - u^{(p+1)/m} + bu^{1/m}v^{q/n} & \text{in } \Omega, \\ -\Delta v = \mu v^{1/n} - v^{(r+1)/n} - cv^{1/n}u^{s/m} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{48}$$

In this case, condition $\mu > 0$ is necessary for the existence of coexistence states. Moreover, $(H)$ and $(H3)$ hold for (48).

**Theorem 10.** *Assume that*

$$\lambda > 0 \quad \textit{and} \quad \mu > c(\lambda + b\mu^{q/r})^{s/p}. \tag{49}$$

*Then* (48) *possesses at least a coexistence state.*

**Proof.** We show that

$$(\underline{u}, \underline{v}) = (\rho\phi_{[\lambda,1/m]}, \epsilon\phi_{[\mu,1/n]}), \qquad (\overline{u}, \overline{v}) = (M, \theta_\mu),$$

with $\rho$, $\epsilon$ and $M$ positive constants to choose later, is a pair of lower-upper solutions of (48). Indeed, we need to check the following conditions:

$$\rho^{1/m}\phi^{1/m}_{[\lambda,1/m]}(\rho^{(m-1)/m}\lambda - \lambda + \rho^{p/m}\phi^{p/m}_{[\lambda,1/m]} - b\phi^{q/n}_{[\mu,1/n]}\epsilon^{q/n}) \le 0,$$

$$\epsilon^{1/n}\phi^{1/n}_{[\mu,1/n]}(\epsilon^{(n-1)/n}\mu - \mu + \epsilon^{r/n}\phi^{r/n}_{[\mu,1/n]} + cM^{s/m}) \le 0,$$

$$M^{p/m} \ge \lambda + b\theta^{q/n}_\mu.$$

From (45), we can choose $\lambda > 0$, $M = (\lambda + b\mu^{q/r})^{m/p}$ with $\rho$ and $\epsilon$ sufficiently small. Theorem 6 provides us a positive solution and approximations to it.

**6.3. Symbiosis.** Consider the problem

$$\begin{cases} -\Delta u = \lambda u^{1/m} - u^{(p+1)/m} + bu^{1/m}v^{q/n} & \text{in } \Omega, \\ -\Delta v = \mu v^{1/n} - v^{(r+1)/n} + cv^{1/n}u^{s/m} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{50}$$

**Theorem 11.** *Assume that $\lambda > 0$, $\mu > 0$ and $qs < rp$. Then* (50) *possesses at least a coexistence state.*

**Proof.** Let $(\underline{u}, \underline{v}) = (\theta_\lambda, \theta_\mu)$, $(\overline{u}, \overline{v}) = (M, N)$, with $M$ and $N$ positive constants to be chosen later. This is a pair of lower-upper solutions of (50) if $\lambda - M^{p/m} + bN^{q/n} \le 0$, and $\mu - N^{r/n} + cM^{s/m} \le 0$. It is sufficient to show that the function $f(x) = \mu - x^{r/n} + c(\lambda + bx^{q/n})^{s/p}$ has a zero, which it is true by assumption.

**Remark 7.** 1. Problem (40) has been also studied in [7], [20] and [27]. In these works, the authors looked for nonnegative solutions of (40). In a forthcoming paper, [10], we will study an interesting particular case of (40), where some results on "dead cores", existence and uniqueness will be given.

2. Conditions (47) and (49) define two regions in $(\lambda - \mu)$-space. It is not hard to give some sufficient conditions showing that these regions are not empty.

3. In order to give other sufficient conditions in Theorem 11, we can try other ways to get $x > 0$ such that $f(x) = 0$. If, for example, $qs = rp$, a sufficient condition is $bc < 1$.

**Acknowledgments.** We are grateful to Professor J. Hernández for helpful comments. The authors thank to DGICYT of Spain for research support under grant DGICYT PB95-1242.

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