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A SHARPER DECAY ESTIMATE FOR THE QUASILINEAR WAVE EQUATION WITH VISCOSITY IN TWO SPACE DIMENSIONS

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1. Introduction. In this paper we are concerned with a decay property of solutions of the quasilinear wave equation with a strong dissipation:

$$u_{tt} - \operatorname{div}\{\sigma(|\nabla u|^2)\nabla u\} - \Delta u_t = 0 \quad \text{in } \Omega \times [0,\infty) \tag{1.1}$$

with the initial-boundary conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \ \text{ and } u|_{\partial\Omega} = 0, \tag{1.2}$$

where Ω is a bounded domain in R^2 with a C^2 class boundary $\partial\Omega$ and σ is a nonlinear function like $\sigma(v^2) = 1/\sqrt{1+v^2}$.

Let us consider the typical case $\sigma = 1/\sqrt{1+v^2}$. This equation was introduced by Greenberg [3] for the one dimensional case: $\Omega \subset R^1$, and the global existence and exponential decay of smooth solutions were proved by Greenberg [4], Greenberg, Mizel and MacCamy [5] and Yamada [13]. For N-dimensional case $\Omega \subset R^N$, the global existence and exponential decay of small amplitude solutions with small data were proved by Ebihara [2] and Kawashima and Shibata [7]. For large data in N-dimensional case, Kobayashi, Pecher and Shibata [8] proved the global existence of smooth solutions. In [8], however, no decay property of solutions is given for such solutions.

Recently in [10], the present author has proved that if the mean curvature $H(x)$ of $\partial\Omega$ at $x \in \partial\Omega$ is nonpositive, then for $(u_0, u_1) \in H_2 \cap H_1^0 \times H_1^0$, the

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problem (1.1)–(1.2) admits a unique solution $u(t) \in W^{2,2}([0,\infty); L^2(\Omega)) \cap W^{1,\infty}([0,\infty); H_1^0(\Omega)) \cap L^\infty([0,\infty); H_2(\Omega))$, satisfying the decay property

$$E(t) \leq \begin{cases} C_0(1+t)^{-(N+2)/(N-2)} & \text{if } N \geq 3 \\ C_q(1+t)^{-q} \ \text{ for any } q > 0 & \text{if } N = 2, \end{cases} \tag{1.3}$$

where

$$E(t) \equiv \frac{1}{2}\left(\|u_t(t)\|^2 + \int_\Omega \int_0^{|\nabla u(t)|^2} \sigma(\eta) d\eta dx\right)$$

and C_0, C_q are positive constants depending on $\|u_0\|_{H_2} + \|u_1\|_{H_1}$. (See also [6].)

The object of this paper is to give a sharper decay estimate of $E(t)$ for the case $N = 2$. That is, we shall prove for the case $N = 2$,

$$E(t) \leq C_0 e^{-\lambda t^{2/3}} \tag{1.4}$$

with some $\lambda > 0$.

For the proof we use a Trudinger type inequality, a sharper form of Gagliardo-Nirenberg inequality,

$$\|u\|_p \leq Cp^{1/2} \|u\|^{2/p} \|\nabla u\|^{1-2/p}, \quad u \in H_1^0(\Omega)$$

due to Cazenave [1] and Ogawa [12].

As a related problem we also consider the wave equation with a nonlinear weak dissipation

$$u_{tt} - \Delta u + \rho(u_t) = 0 \quad \text{in } \Omega \times [0,\infty) \tag{1.5}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \ \text{ and } u|_{\partial\Omega}, \tag{1.6}$$

where Ω is abounded domain in R^2, ρ is a nonlinear dissipation like $\rho = u_t/\sqrt{1+|u_t|^2}$.

Since $\lim_{v\to\pm\infty} |\rho(v)| < \infty$ in our case, the dissipation effect by $\rho(u_t)$ is much weaker compared with the usual one $\rho(u_t) = u_t$. So, we call such a dissipation as ' weak dissipation '. Recently in [10], we have investigated the decay property of the energy $E(t)$ of the problem (1.5)–(1.6), and in particular, for the case $N = 2$, we have proved that

$$E(t) \leq C_q(1+t)^{-q}$$

for any $q > 0$, where C_q again denotes a constant depending on $\|u_0\|_{H_2} + \|u_1\|_{H_1}$ and q. By use of a similar technique deriving (1.4) we shall prove a sharper estimate (1.4) also for the solutions of the problem (1.5)–(1.6).

2. Preliminaries and results.

The function spaces we use are all familiar and we omit the definition of them.

On the nonlinear term $\sigma(|\nabla u|^2)$ appearing in the equation (1.1), we make the following hypotheses.

Hypotheses A. $\sigma(\cdot)$ is a differentiable function on $R^+ \equiv [0, \infty)$, and satisfies the conditions:

(1) $\sigma(v^2) \geq k_0(1 + v^2)^{-\alpha}, \alpha > 0$,
(2) $\sigma(v^2) + 2\sigma' v^2 \geq 0$, and
(3) $k_0\sigma(v^2)v^2 \leq \int_0^{v^2} \sigma(\eta)d\eta \leq k_1\sigma(v^2)v^2$, where k_0, k_1 are positive constants.

Hypotheses B. The mean curvature $H(x)$ at $x \in \partial\Omega$ is nonpositive with respect to the outward normal.

Our result for (1.1)–(1.2) reads as follows.

Theorem 1. *Let* $(u_0, u_1) \in H_2(\Omega) \cap H_1^0(\Omega) \times H_1^0(\Omega)$. *Then, under the Hypotheses A and B, the problem* (1.1)–(1.2) *admits a unique solution* $u \in W^{2,2}([0, \infty); L^2(\Omega)) \cap W^{1,\infty}([0, \infty); H_1^0) \cap L^\infty([0, \infty); H_2)$, *satisfying the decay property*

$$E(t) \leq C_0 e^{-\lambda t^{1/(1+\alpha)}}, \tag{2.1}$$

where C_0 *and* λ *are positive constants depending on* $\|u_0\|_{H_2} + \|u_1\|_{H_1}$.

Concerning the nonlinearity $\rho(v)$ appearing in the equation (1.4) we make the following assumption:

Hypotheses $\tilde{A}$. $\rho(v)$ is a differentiable function on R, satisfying the conditions

(1) $k_1|v|^2 \geq \rho(v)v \geq k_0 v^2/(1 + v^2)^\alpha$, $\alpha > 0$, with some $k_0, k_1 > 0$, and
(2) $\rho'(v) \geq 0$.

Our result for (1.5)–(1.6) reads as follows.

Theorem 2. *Let* $(u_0, u_1) \in H_2 \cap H_1^0 \times H_1^0$. *Then, under the Hypotheses* $\tilde{A}$, *the problem* (1.4)–(1.5) *admits a unique solution* $u(t) \in W^{1,\infty}([0, \infty); H_1^0) \cap L^\infty([0, \infty); H_2 \cap H_1^0)$, *satisfying*

$$E(t) \leq C_0 e^{-\lambda t^{1/(1+\alpha)}}, \tag{2.2}$$

where C_0, λ are positive constants depending on $||u_0||_{H_2} + ||u_1||_{H_1}$.

To prove the theorems we use the following Lemma.

Lemma 1 ([7, 11]). *Let Ω is a domain in R^2. For $u \in H_1^0(\Omega)$, it holds that*

$$\|u\|_p \leq (4\pi)^{-(2-p)/2p}(\frac{p}{2})^{1/2}\|u\|^{2/p}\|\nabla u\|^{1-2/p} \tag{2.3}$$

for $p \geq 2$, where $\|\cdot\|$ denotes usual L^2 norm in Ω.

In fact, we use the above Lemma 1 in the following form:

Lemma 2. *Let Ω be a bounded domain in R^N with the C^1 class boundary $\partial\Omega$. Then, for $u \in H_1(\Omega)$ we have*

$$\|u\|_p \leq Cp^{1/2}\|u\|_2^{2/p}\|u\|_{H_1}^{1-2/p}, \quad p \geq 2, \tag{2.4}$$

for some $C > 0$ independent of p.

(2.4) easily follows from (2.3). Indeed, it is possible to extend u to a function $\tilde{u} \in H_1^0(\tilde{\Omega})$ with $\Omega \subset \tilde{\Omega}$ such that $\tilde{u} = u$ on Ω and $\|u\|_{L^2(\tilde{\Omega})} \leq C\|u\|_{L^2(\Omega)}, \|\tilde{u}\|_{H_1^o(\tilde{\Omega})} \leq C\|u\|_{H_1(\Omega)}$, which together with (2.3) implies (2.4).

3. A difference inequality. For the proof of decay of $E(t)$, we shall derive an inequality for the difference $E(t) - E(t+1)$, and we prepare the following proposition.

Proposition 1. *Let $\phi(t)$ be a continuous nonnegative nonincreasing function on $[0, \infty)$ satisfying*

$$\phi(t) \leq C_0\left(p^\alpha D(t)^2 + D(t)^{2-2k/p}\right) \tag{3.1}$$

for any $p \geq 2k$, where k, α are some positive numbers and we set $D(t)^2 = \phi(t) - \phi(t+1)$. Then, there exist positive constants C_1 and λ depending on $\phi(0)$ and C_0 such that

$$\phi(t) \leq C_1 e^{-\lambda t^{1/(1+\alpha)}}, \quad t \geq 0. \tag{3.2}$$

Remark. (1) When $\phi(t)$ satisfies the inequality $\phi(t) \leq c_0 p^\alpha D(t)^2$ we have $\phi(t) \leq C\phi(0)e^{-\lambda_p t}$ with a certain $\lambda_p > 0$ tending to 0 as $p \to \infty$, while if $\phi(t)$ satisfies $\phi(t) \leq C_0 D(t)^{2-k/p}$, we see

$$\phi(t) \leq C_p(1+t)^{-(2p-k)/k}.$$

(See [9]). Thus, our inequality (3.1) is very delicate as $p \to \infty$.

(2) If (3.1) holds for any $p \geq p_0$ with some $p_0 > 0$, then it holds for any $p \geq 2k$ with C_0 replaced by another constant if necessary.

Proof of Proposition 1. We take $K > 0$ so large that

$$\phi(t) \leq Ke^{-\lambda t^\theta}, 0 \leq t \leq \max\{2, T_0\}, \tag{3.3}$$

where we set $\theta = 1/(1+\alpha)$ (≤ 1) and T_0 is a positive constant to be fixed later.

If the estimate (3.2) with $C_1 = K$ was false for any $\lambda, 0 < \lambda \leq 1$, there would exist $T \geq 2$ such that

$$\phi(t) < Ke^{-\lambda t^\theta} \text{ for } 0 \leq t < T \tag{3.4}$$

and

$$\phi(T) = Ke^{-\lambda T^\theta}. \tag{3.5}$$

Then, taking $t = T-1$ in the inequality (3.1), we have

$$\begin{aligned} Ke^{-\lambda T^\theta} \leq \phi(T-1) \leq C_0\{&Kp^\alpha \left(e^{-\lambda(T-1)^\theta} - e^{-\lambda T^\theta}\right) \\ &+ K^{2-k/p}\left(e^{-\lambda(T-1)^\theta} - e^{-\lambda T^\theta}\right)^{1-k/p}\}. \end{aligned} \tag{3.6}$$

Here,

$$e^{-\lambda(T-1)^\theta} - e^{-\lambda T^\theta} = \theta\lambda e^{-\lambda \tilde{T}^\theta}\tilde{T}^{\theta-1} \leq \lambda\theta e^{-\lambda(T-1)^\theta}(T-1)^{\theta-1},$$

$T - 1 \leq \tilde{T} \leq T$. Therefore, we have from (3.6) that

$$\begin{aligned} 1 &\leq C_0\{p^\alpha\lambda\theta(T-1)^{\theta-1} + K^{1-k/p}(\lambda\theta)^{1-k/p}e^{\lambda k(T-1)^\theta/p} \\ &\quad \times (T-1)^{(\theta-1)(1-k/p)}\}e^{\lambda(T^\theta-(T-1)^\theta} \\ &\leq C_0\{p^\alpha\lambda\theta(T-1)^{\theta-1} + K^{1-k/p}(\lambda\theta)^{1-k/p}e^{\lambda k(T-1)^\theta/p}\}e^{\lambda\theta}. \end{aligned} \tag{3.7}$$

Here, we fix $T_0 > 0$ such that

$$\frac{k}{(T_0-1)^{(1-\theta)/\alpha}} \leq \frac{1}{2}.$$

Since $p \geq 2k$ is arbitrary, we can take

$$p = (T-1)^{(1-\theta)/\alpha}.$$

Then, the inequality (3.7) implies

$$\begin{aligned} 1 &\leq C_0 \left(\lambda\theta + K^{1-k/p}(\lambda\theta)^{1-k/p} e^{\lambda k (T-1)^{\theta-(1-\theta)/\alpha}} \right) e^{\lambda\theta} \\ &\leq C_0 \left(\lambda\theta + (K + \sqrt{K})(\lambda\theta + \sqrt{\lambda\theta}) e^{\lambda k} \right) e^{\lambda\theta}, \end{aligned} \tag{3.8}$$

where we have used the facts that $0 < k/p \leq 1/2$ and $\theta - (1-\theta)/\alpha = 0$.

The inequality (3.8) is a contradiction if we choose a sufficiently small $\lambda > 0$, and we complete the proof of Proposition 1.

4. Proof of Theorem 1. The existence and uniqueness part is proved in [9] (see also [5]), and it suffices to prove the decay property (2.1). We shall derive the following difference inequality for $E(t)$, which implies (2.1) by Proposition 1.

Proposition 2. *Let $u(t)$ be a solution of* (1.1)–(1.2) *in the class stated in Theorem* 1. *Then, it holds that*

$$E(t) \leq C_0 \left(p^\alpha D(t)^2 + D(t)^{(2p+4\alpha)/(p+4\alpha)} \right) \tag{4.1}$$

for any $p \geq 1$, where C_0 is a constant depending on $\|u_0\|_{H_2} + \|u_1\|_{H_1}$, but, independent of p.

Proof. The proof is given by refining the argument in [9] with $N = 2$.

Multiplying the equation (1.1) by u_t and integrating we have

$$\int_t^{t+1} \|\nabla u_t(s)\|^2 ds = E(t) - E(t+1) \equiv D(t)^2 \tag{4.2}$$

where we recall

$$E(t) \equiv \frac{1}{2} \left(\|u_t(t)\|^2 + \int_\Omega \int_0^{|\nabla u(t)|^2} \sigma(\eta) d\eta dx \right).$$

($\|\cdot\| \equiv \|\cdot\|_2$.) From (4.2) we see

$$\int_t^{t+1} \|u_t(s)\|^2 ds \leq C D(t)^2 \tag{4.3}$$

and there exist $t_1 \in [t, t+1/4], t_2 \in [t+3/4, t+1]$ such that

$$\|u_t(t_i)\| \le 2CD(t), \quad i = 1, 2. \tag{4.4}$$

In what follows we denote by C any positive constant independent of (u_0, u_1) and p.

Next, multiplying the equation by u and integrating over $[t_1, t_2] \times \Omega$, we have

$$\begin{aligned} \int_{t_1}^{t_2} \int_\Omega \sigma |\nabla u|^2 dx\, ds &= -(u_t(t_2), u(t_2)) + (u_t(t_1), u(t_1)) \\ &+ \int_{t_1}^{t_2} \|u_t(s)\|^2 ds - \int_{t_1}^{t_2} \int_\Omega \nabla u_t \cdot \nabla u dx ds. \end{aligned} \tag{4.5}$$

To estimate the right hand side of (4.5) we first derive an estimate for $\|\nabla u(t)\|_{1+\delta}$, $0 < \delta < 1$.

By Hypotheses A,(1),

$$\begin{aligned} \int_\Omega |\nabla u|^{1+\delta} dx &= \int_\Omega \left(\sigma |\nabla u|^2\right)^{(1+\delta)/2} \sigma^{-(1+\delta)/2} dx \\ &\le C \left(\int_\Omega \sigma |\nabla u|^2 dx\right)^{(1+\delta)/2} \left(\int_\Omega (1 + |\nabla u|)^{2\alpha(1+\delta)/(1-\delta)} dx\right)^{(1-\delta)/2} \\ &\le C F(\nabla u)^{(1+\delta)/2} (1 + \|\nabla u\|_{2\alpha(1+\delta)/(1-\delta)}^{\alpha(1+\delta)}), \end{aligned} \tag{4.6}$$

where we set

$$F(\nabla u) \equiv \int_\Omega \sigma |\nabla u|^2 dx.$$

We note that by assumption Hypotheses A,(3), $F(\nabla u)$ is equivalent to

$$\int_\Omega \int_0^{|\nabla u|^2} \sigma(\eta) d\eta dx.$$

Taking $1 > \delta \ge (1-\alpha)/(1+\alpha)$ and setting $p = 2\alpha(1+\delta)/(1-\delta) \ge 2$, we have by Lemma 2 that

$$\|\nabla u\|_p \le C p^{1/2} \|\nabla u\|^{2/p} \|\Delta u\|^{1-2/p}$$

and hence, from (4.6),

$$\|\nabla u\|_{1+\delta}^{1+\delta} \le CF(\nabla u)^{(1+\delta)/2}(1 + p^{\alpha(1+\delta)/2}\|\nabla u\|^{2\alpha(1+\delta)/p}\|\Delta u\|^{\alpha(1+\delta)(1-2/p)}). \tag{4.7}$$

Further, we use Gagliardo-Nirenberg inequality (cf. Friedmann [4])

$$\|\nabla u\| \le C\|\nabla u\|_{1+\delta}^{(1+\delta)/2}\|\Delta u\|^{(1-\delta)/2}$$

to get from (4.7) that

$$\begin{aligned}\|\nabla u\| &\le CF(\nabla u)^{(1+\delta)/4}(1 + p^{\alpha(1+\delta)/4}\|\nabla u\|^{\alpha(1+\delta)/p}\|\Delta u\|^{\alpha(1+\delta)(p-2)/2p}) \\ &\qquad \times \|\Delta u\|^{(1-\delta)/2} \qquad\qquad (4.8)\\ &\le CF(\nabla u)^{p/2(p+2\alpha)}(1+ p^{p\alpha/2(p+2\alpha)}\|\nabla u\|^{2\alpha/(p+2\alpha)}\|\Delta u\|^{(p-2)\alpha/(p+2\alpha)}) \\ &\qquad \times \|\Delta u\|^{2\alpha/(p+2\alpha)},\end{aligned}$$

where we note that $\delta = (p-2\alpha)/(p+2\alpha)$. To estimate $\|\Delta u(t)\|$ we multiply the equation (1.1) by $-\Delta u(t)$ to get (see [10])

$$\begin{aligned}\frac{d}{dt}\big((u_t, -\Delta u) + \frac{1}{2}\|\Delta u(t)\|^2\big) + \int_\Omega \big(\sigma|D^2u|^2 + 2\sigma' \sum_j |\nabla u\cdot\nabla u_j|^2\big)dx \\ + \int_{\partial\Omega} \sigma|\frac{\partial u}{\partial n}|^2 H(x)dx = \|\nabla u_t\|^2,\end{aligned} \tag{4.9}$$

where $D^2u = \big(\frac{\partial^2 u}{\partial x_i \partial x_j}\big)$ and $u_j = \frac{\partial u}{\partial x_j}$. Thus, by the use of Hypotheses A,(2), Hypotheses B and the fact

$$\int_0^\infty \|\nabla u_t(s)\|^2 ds \le E(0) < \infty$$

(see (4.2)), we have

$$\|\Delta u(t)\|^2 \le C(\|u_0\|_{H_2} + \|u_1\|) < \infty. \tag{4.10}$$

Hereafter, we denote by C_0 any constants depending on $\|u_0\|_{H_2} + \|u_1\|$. It follows from (4.8) and (4.10) that

$$\|\nabla u(t)\| \le C_0E(t)^{p/2(p+2\alpha)}\left(1 + p^{\alpha/2}\|\nabla u\|^{2\alpha/(p+2\alpha)}\right)$$

and, by Young's inequality,

$$\begin{aligned}\|\nabla u(t)\| &\le C_0\left(E(t)^{p/2(p+2\alpha)} + p^{(p+2\alpha)\alpha/2p}\sqrt{E(t)}\right)\\ &\le C_0\left(E(t)^{p/2(p+2\alpha)} + p^{\alpha/2}\sqrt{E(t)}\right). \qquad (4.11)\end{aligned}$$

Let us return to (4.5). Then, by(4.11) just obtained, we see

$$\begin{aligned}|\int_{t_1}^{t_2}\int_\Omega(\nabla u_t\cdot\nabla u)dxds| &\le \left(\int_t^{t+1}\|\nabla u_t\|^2 ds\right)^{1/2}\sup_{t\le s\le t+1}\|\nabla u(s)\|\\ &\le C_0 D(t)(E(t)^{p/2(p+2\alpha)} + p^{\alpha/2}\sqrt{E(t)}) \equiv A(t)^2. \qquad (4.12)\end{aligned}$$

Similarly,

$$|\,(u_t(t_i), u(t_i))\,| \le C\sup_{t\le s\le t+1}\|u_t(s)\|\|\nabla u(s)\| \le A(t)^2.$$

Thus, we obtain from (4.5) that

$$\int_{t_1}^{t_2}\int_\Omega \sigma|\nabla u|^2 dx\,ds \le C(D(t)^2 + A(t)^2). \qquad (4.13)$$

It follows from (4.2) and (4.13) that

$$E(t+1) \le \int_t^{t+1} E(s)ds \le C(D(t)^2 + A(t)^2)$$

and consequently,

$$E(t) \le C(D(t)^2 + A(t)^2). \qquad (4.14)$$

Recalling (4.12), the definition of $A(t)^2$, we can easily derive from (4.14) the desired inequality

$$E(t) \le C_0\left(p^\alpha D(t)^2 + D(t)^{(2p+4\alpha)/(p+4\alpha)}\right).$$

5. Proof of Theorem 2. For the proof of Theorem 2 it suffices to derive the decay estimate (2.2). Here, the geometrical condition on Ω is not required. The following proposition together with Proposition 3.1 will give the desired result.

Proposition 3. *Let $u(t)$ be a solution in the class stated in Theorem* 2. *Then we have*

$$E(t) \le C_0 \left(p^\alpha D(t)^2 + D(t)^{2p/(p+2\alpha)}\right).$$

Proof. The proof is given by a similar and simpler way than that of Proposition 4.1, and we sketch it briefly.

Multiplying the equation (1.4) by u_t, we have

$$\int_t^{t+1}\int_\Omega \rho(u_t)u_t dx\, ds = E(t) - E(t+1) \equiv D(t)^2. \tag{5.1}$$

By Hypotheses $\tilde{A}$, we see, for $0 < \delta < 1$,

$$\begin{aligned}\int_1^{t+1} \|u_t(s)\|_{1+\delta}^{1+\delta} &= \int_t^{t+1}\int_\Omega \Big(\frac{|u_t|^2}{(1+|u_t|^2)^\alpha}\Big)^{(1+\delta)/2}(1+|u_t|^2)^{-(1+\delta)\alpha/2} dx\, ds \\ &\le C\Big(\int_t^{t+1}\int_\Omega \rho(u_t)u_t dx\, ds\Big)^{(1+\delta)/2}\Big(1+\int_t^{t+1}\|u_t(s)\|_{2\alpha(1+\delta)/(1-\delta)}^{\alpha(1+\delta)} ds\Big) \\ &\le CD(t)^{(1+\delta)}\left(1+\sup_{t\le s\le t+1}\|u_t(s)\|_{2\alpha(1+\delta)/(1-\delta)}^{\alpha(1+\delta)}\right).\end{aligned} \tag{5.2}$$

Here, setting $p = 2\alpha(1+\delta)/(1-\delta)$, we have

$$\|u_t\|_p \le Cp^{1/2}\|u_t\|^{2/p}\|\nabla u_t\|^{1-2/p} \le C_0 p^{1/2}E(t)^{1/p}, \tag{5.3}$$

where we have used the fact that

$$\|u_{tt}(t)\| + \|\nabla u_t(t)\| \le C_0 = C_0(\|u_0\|_{H_2}, \|u_1\|_{H_1}) < \infty,$$

which follows by multiplying the equation by $-\Delta u_t$. Hence,

$$\int_t^{t+1}\|u_t(s)\|_{1+\delta}^{1+\delta} ds \le CD(t)^{1+\delta}\left(1 + C_0 p^{\alpha(1+\delta)/2}E(t)^{\alpha(1+\delta)/p}\right) \equiv A(t)^2. \tag{5.4}$$

Then, by Gagliardo-Nirenberg inequality, we have

$$\int_t^{t+1}\|u_t(s)\|^2 \le C\int_t^{t+1}\|u_t(s)\|_{1+\delta}^{1+\delta}\|\nabla u_t(s)\|^{1-\delta} ds \le C_0 A(t)^2. \tag{5.5}$$

There exist $t_1 \in [t, t+1/4]$, $t_2 \in [t+3/4, t+1]$ such that

$$\|u_t(t_i)\| \le 2C_0 A(t), \quad i = 1, 2,$$

and, multiplying the equation by u, we have

$$\begin{aligned}
\int_{t_1}^{t_2} \|\nabla u(s)\|^2 ds &= -(u_t(t_2), u(t_2)) + (u_t(t_1), u(t_1)) \\
&\quad + \int_{t_1}^{t_2} \|u_t(s)\|^2 ds - \int_{t_1}^{t_2} \int_\Omega \rho(u_t) u dx ds \\
&\le C_0 A(t)\sqrt{E(T)} + C_0 A(t)^2 + \Big(\int_t^{t+1} \int_\Omega \rho(u_t) u_t ds\Big)^{1/2} \Big(\int_t^{t+1} \|u\|^2 ds\Big)^{1/2} \\
&\le C_0 A(t)\sqrt{E(t)} + C_0 A(t)^2 + CD(t)\sqrt{E(t)}, \qquad (5.6)
\end{aligned}$$

where we have used the inequality $|\rho(u_t)|^2 \le k_1 \rho(u_t) u_t$.

It follows from (5.5) and (5.6) that

$$E(t+1) \le C_0 A(T)\sqrt{E(T)} + C_0 A(T)^2 + CD(t)\sqrt{E(t)}$$

and

$$E(t) \le C_0 A(t)\sqrt{E(t)} + C_0 A(t)^2 + CD(t)\sqrt{E(t)} + D(t)^2. \qquad (5.7)$$

This gives

$$E(t) \le C_0 A(t)^2 + CD(T)^2$$

and hence,

$$E(t) \le C_0 \left(p^\alpha D(t)^2 + C_0 D(t)^{2p/(p+2\alpha)}\right).$$

The proof of Proposition 3 is complete.

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