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LOW REGULARITY SOLUTIONS FOR A GENERALIZED ZAKHAROV SYSTEM

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Abstract. Using the method of Bourgain (actually we use techniques developed in the paper of Ginibre, Tsutsumi, Velo (cf. [9])) we prove well-posedness of a generalized Zakharov system, describing the spontaneous generation of magnetic field in a cold plasma, in the framework of classical Sobolev spaces. Using the conservation laws of the system we extend these solutions globally in the energy space.

1. Introduction. In this paper we shall study the Cauchy problem for a generalized Zakharov system

$$\begin{cases} i\partial_t E = \alpha \text{ rot rot} E - \text{ grad div} E + nE - i(E \wedge B) \\ \partial_t^2 n = \Delta n + \Delta |E|^2 \\ \Delta B = i\eta \text{ rot rot}(E \wedge \bar{E}) + \beta B \end{cases} \tag{1}$$

with initial data

$$E(0,x) = E_0(x), \quad n(0,x) = n_0(x), \quad n_t(0,x) = n_1(x), \tag{2}$$

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where $x \in \mathbf{R}^d (d = 2, 3)$,$t \in [0, \infty)$, E is a vector valued function with values in $\mathbf{C}^d$, n is a real valued function, B is a vector function with values in $\mathbf{R}^d$, β is a nonpositive constant and $\alpha \geq 1$.

The system $(1) - (2)$ describes the spontaneous generation of a magnetic field in a cold plasma (cf. (4.15) in [10]). The derivation of the system (1) is similar to that for obtaining the Zakharov equations of strong Langmuir turbulence. E is the slowly varying complex amplitude of the high-frequency electric field, n is the fluctuation of the electron density from its equilibrium and B is the self generated magnetic field. If we omit the magnetic field B then we obtain the system describing Langmuir's turbulence (cf. [7], [14])

$$\begin{cases} i\partial_t E = \alpha \text{ rot rot} E - \text{ grad div} E + nE \\ \partial_t^2 n = \Delta n + \Delta |E|^2. \end{cases} \tag{3}$$

Using the conservation laws of the system (3) and compactness arguments (the method of Galerkin) C. Sulem and P.L. Sulem (cf. [13]) obtained global weak solutions in two and three space dimensions supposing the initial data are sufficiently small. In one spatial dimension global existence and uniqueness is obtained. Using a Brezis-Gallouet (cf. [5]) logarithmic Sobolev inequality H. Added and S. Added obtained global existence and uniqueness in two spatial dimensions. In both papers [1] and [13] the global existence is derived by obtaining well-posedness of the local problem in H^s for s sufficiently small. Using a method recently developed by Bourgain (cf. [2], [3]) J. Bourgain and J. Colliander proved (cf. [4]) that the system (3) is well-posed in H^s for s small enough to establish global existence using the conservation laws of the system. In the case of general space dimension J. Ginibre, Y. Tsutsumi and G. Velo (cf. [9]) defined a natural notion of criticality for the system (3) in the scale of the Sobolev spaces. Using the method of Bourgain they obtain sharp results in space dimensions higher than three. In two and three space dimensions they recover the result of Bourgain and Colliander, i.e., finite energy global solutions of (3) are obtained.

From the third equation of (1), using a Fourier transform, we obtain the representation of $B = B(E)$

$$B(E) = \mathcal{F}^{-1}\{\frac{i\eta}{|\xi|^2 - \beta}(\xi \wedge (\xi \wedge \mathcal{F}(E \wedge \bar{E})))\}, \quad \beta < 0.$$

Hence, the system (1) is transformed into a system of nonlinear wave and

Schrödinger equations. We shall show that (1) can be derived from a Lagrangian formalism. For that purpose we consider the Lagrangian

$$L = \int \mathcal{L},$$

with Lagrangian density $\mathcal{L}$ depending on the independent field variables $E, \bar{E}, u$

$$\begin{aligned}\mathcal{L}(E,\bar{E},u) = \frac{1}{2} i(\bar{E}\cdot\frac{\partial E}{\partial t} - \frac{\partial \bar{E}}{\partial t}\cdot E) - \alpha \ \mathrm{rot}E\cdot\mathrm{rot}\bar{E} - \mathrm{div}E\cdot\mathrm{div}\bar{E} \\ + \frac{1}{2}(\frac{\partial u}{\partial t} - E\cdot\bar{E})^2 + \frac{1}{2} i(-\Delta-\beta)^{-1/2} \ \mathrm{rot}(E\wedge\bar{E}) \\ \times (-\Delta-\beta)^{-1/2} \ \mathrm{rot}(\bar{E}\wedge E),\end{aligned}$$

where $\cdot$ denotes the scalar product in $\mathbf{C}^d$. The Euler-Lagrange equations corresponding to the Lagrangian density $\mathcal{L}$ have the form

$$\begin{aligned}& i\partial_t E = \alpha \ \mathrm{rot}\ \mathrm{rot}E - \mathrm{grad}\ \mathrm{div}E + (u_t - E\cdot\bar{E})E - i(E\wedge B(E)) \\ & \frac{\partial}{\partial t}(E\cdot\bar{E}) - \partial_t^2 u + \Delta u = 0.\end{aligned}$$

Setting $\partial_t u = n + |E|^2$ we arrive at the system (1).

Using that the Lagrangian is invariant by the group of time translations and Noether theorem we obtain that the total energy of the system is conserved

$$\begin{aligned}\alpha\| \ \mathrm{rot}\ E(t,.)\|_{L^2}^2 + \| \ \mathrm{div}\ E(t,.)\|_{L^2}^2 + \frac{1}{2}\|n(t,.)\|_{L^2}^2 \\ + \frac{1}{2}\|\partial_t^{-1} \ \mathrm{grad}\ (n+E^2)(t,.)\|_{L^2}^2 + \int_{\mathbf{R}^d}(n|E|^2)(t,x)dx \\ - \frac{\eta}{8}\|(-\Delta-\beta)^{-1/2} \ \mathrm{rot}(E\wedge\bar{E})(t,.)\|_{L^2}^2 = \ \text{constant}\ .\end{aligned} \tag{4}$$

Further we note that the Lagrangian is also invariant by the action of the group S^1 (the gauge transformations) $E \mapsto \exp(i\omega)E$, where $\omega \in \mathbf{R}$ and derive using Noether theorem that the L^2 norm of E is conserved

$$|E(t,.)|_{L^2} = \ \text{constant} \tag{5}$$

Using the conservation laws (4), (5) and compactness arguments C. Laurey (cf. [11]) obtained global weak solutions of (1) – (2) providing the initial data are sufficiently small. In [11] it is also shown that $(1)-(2)$ with initial data $(E_0, n_0, n_1) \in H^{s+1} \times H^s \times H^{s-1}$, where $s > d/2$ is well-posed in $H^{s+1} \times H^s \times H^s$, i.e., there exists (E, n, B) unique solution of $(1)-(2)$ defined on $[0,T) \times \mathbf{R}^d$ such that

$$(E, n, B) \in C([0,T); H^{s+1}(\mathbf{R}^d)) \times C([0,T); H^s(\mathbf{R}^d)) \times C([0,T); H^s(\mathbf{R}^d).$$

In the case when $d = 2, \alpha = 1$ and $s = 2$ the solutions are globally extended if the initial data are small enough.

The main purpose of this paper is to prove that the Cauchy problem $(1)-(2)$ is locally well-posed in $H^k \times H^l \times H^l$, provided $1 \geq k - l \geq 0, \quad l \geq 0$ and $k \geq (l+1)/2$. Our arguments are similar to those of Ginibre, Tsutsumi and Velo (cf. [9]). If we omit the term $E \wedge B$ applying the result of [9] we obtain well-posedness of the system (3) in $H^k \times H^l$ for $l \geq 0$, $k \geq \frac{l+1}{2}$, $1 \geq k - l \geq 0$. In this paper we show that the term $E \wedge B$ may be estimated in a manner similar to nE as in [9] (cf. (16) below). From the representation of $B = B(E)$ we conclude that the nonlinearity $E \wedge B$ is cubic with respect to E. Since $\beta < 0$ we shall be able to control the derivatives of the term B in the integral representation of the nonlinear estimate. Using a result of T. Cazenave, F. Weissler (cf. [6]) one obtains that the nonlinear Schrödinger equation

$$i\partial_t u = \Delta u + |u|^2 u \tag{6}$$

is locally well-posed in H^k for $k \geq 1/2$. In fact well-posedness of (6) in the framework of Besov spaces is shown in [6]. The main ingredient of this paper is that we show that one is able to inject the result of [6], for cubic nonlinearity in a framework of Bourgain type spaces. More precisely we have the following Theorem.

Theorem 1. *Suppose that k and l satisfy $l + 1 \geq k > \frac{l+1}{2}$, $l \geq 0$. Then the Cauchy problem $(1)-(2)$ is locally well-posed for initial data*

$$(E_0, n_0, n_1) \in H^k(\mathbf{R}^d) \times H^l(\mathbf{R}^d) \times H^{l-1}(\mathbf{R}^d), \quad d = 2, 3.$$

In addition we have that

$$(E, n, n_t) \in C([0,T); H^k(\mathbf{R}^d) \times H^l(\mathbf{R}^d) \times H^{l-1}(\mathbf{R}^d)).$$

Using Theorem 1 for $k = 1$ and $l = 0$ and the conservation laws $(4)-(5)$ we will be able to extend the solutions globally.

Theorem 2. *The Cauchy problem* (1) − (2) *is globally well-posed for sufficiently small initial data* $(E_0, n_0, n_1) \in H^1(\mathbf{R}^d) \times L^2(\mathbf{R}^d) \times H^{-1}(\mathbf{R}^d)$.

2. Preliminaries. If we set $n_\pm = n \pm i\omega^{-1}\partial_t n$, where $\omega = (-\Delta)^{\frac{1}{2}}$, then the system (1) is transformed into the following one

$$i\partial_t u = \phi(D)u + f(u)$$

where $u = (E, n_+, n_-), D = -i\nabla_x$. The nonlinear term $f(u)$ has the form

$$f(u) = (\frac{1}{2}(n_+ + n_-)E - i(E \wedge B(E)), -\omega|E|^2, \omega|E|^2)$$

The matrix $\phi(D)$ is defined through its Fourier transform

$$\hat{\phi}(\xi) = \begin{pmatrix} M_d(\xi) & 0 & 0 \\ O_{d\times d} & |\xi| & 0 \\ O_{d\times d} & 0 & -|\xi| \end{pmatrix},$$

where $O_{d\times d}$ is the null $d \times d$ matrix and $M_d(\xi)$ is the following matrix

$$M_3(\xi) = (1-\alpha)\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2 \end{pmatrix} + \alpha I_{3\times 3},$$

in the case $d = 3$ and

$$M_2(\xi) = (1-\alpha)\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_2^2 \end{pmatrix} + \alpha I_{2\times 2},$$

in the case $d = 2$. Here $I_{d\times d}$ stands for the unit $d \times d$ matrix. We shall need the following estimate for $M_d(\xi)$.

Proposition 1. *There exists a constant* c *such that*

$$|\xi|^2 I_{d\times d} \le M_d(\xi) \le c|\xi|^2 I_{d\times d}.$$

Proof. We shall prove the Proposition for $d = 3$. The arguments in the case $d = 2$ are similar. We set $M_3(\xi) = (1-\alpha)\widetilde{M}_3(\xi) + \alpha I_{3\times 3}$. Using the elementary inequality $2ab \le a^2 + b^2$ several times we obtain

$$\begin{aligned} &< \widetilde{M}_3(\xi)E, E > \\ &= \xi_1^2E_1^2 + \xi_2^2E_2^2 + \xi_3^2E_3^2 + 2\xi_1\xi_2E_1E_2 + 2\xi_1\xi_3E_1E_3 + 2\xi_2\xi_3E_2E_3 \\ &\le \xi_1^2E_1^2 + \xi_2^2E_2^2 + \xi_3^2E_3^2 + \xi_1^2E_2^2 + \xi_2^2E_1^2 + \xi_1^2E_3^2 + \xi_3^2E_1^2 + \xi_2^2E_3^2 + \xi_3^2E_2^2 \\ &= |\xi|^2|E|^2, \end{aligned}$$

where $< .,. >$ denotes the scalar product in $\mathbf{R}^d$. Hence $\widetilde{M}_3(\xi) \leq |\xi|^2 I_{3\times 3}$, which is equivalent to the first inequality. To prove the second inequality we similarly use several times the inequality $\pm 2ab \leq a^2 + b^2$. This completes the proof of the Proposition.

Let $U(t) = \exp(-it\phi(D))$ be the unitary group which defines the free evolution of (1). Then (1) is reduced to the following integral equation

$$u(t) = U(t)u(0) - i\int_0^t U(t-t')f(u(t'))dt'. \tag{7}$$

Let ψ be a cut-off function such that $\psi \in C_0^\infty(\mathbf{R})$, supp $\psi \subset [-2,2]$, $\psi =$ 1 over the interval $[-1,1]$. We consider a cut-off version of (7)

$$u(t) = \psi(t)U(t)u(0) - i\psi(t/T)\int_0^t U(t-t')f(u(t'))dt'. \tag{8}$$

Note that (8) is equivalent to

$$u(t) = \psi(t)U(t)u(0) - i\psi(t/T)\int_0^t U(t-t')f(\psi(t'/2T)u(t'))dt'. \tag{9}$$

Hence, we can assume the nonlinearity to be truncated too. We shall solve globally in time the equation (9) (or what is the same (8)). To the solutions of (8) will correspond local solutions of (7) in time interval $[-T,T]$. The cut-off function will help us to derive positive powers of T in the estimates which will provide a small factor in the iteration scheme using a classical fixed point theorem for solving the equation (8). With $H^{b,s}$ we shall denote the usual Sobolev spaces $H^{b,s} = \{u : \|u; H^{b,s}\| < \infty\}$, where

$$\|u; H^{b,s}\| = \| < \tau >^b < \xi >^s \hat{u}(\tau,\xi); L^2_{(\tau,\xi)}\|.$$

As usual $< . >$ stays for $(1 + |.|^2)^{1/2}$. Further we define Bourgain type spaces associated to the group $U(t)$ in which we shall solve the equation (8) by a fixed point theorem $X^{b,s} = \{u : \|u; X^{b,s}\| < \infty\}$, where $\|u; X^{b,s}\| = \|U(-t)u; H^{b,s}\|$ A direct computation shows that in the case of equation (7) the spaces $X^{s,b}$ are the following:

$$\|E; X^{b,s}\| = \| < \tau I_{d\times d} + M_d(\xi) >^b < \xi >^s \hat{E}(\tau,\xi); L^2_{(\tau,\xi)}\|$$
$$\|n_\pm; X^{b,s}\| = \| < \tau \pm |\xi| >^b < \xi >^s \hat{n}_\pm(\tau,\xi); L^2_{(\tau,\xi)}\|.$$

Note that $\tau I_{d\times d} + M_d(\xi)$ is a symmetric matrix therefore we can define $< \tau I_{d\times d} + M_d(\xi) >^b$ for every $b \in \mathbf{R}$. Using Proposition 1 we obtain the following equivalent norm in $X^{b,s}$ for E

$$\|E; X^{b,s}\| = \| < \tau + |\xi|^2 >^b < \xi >^s \hat{E}(\tau,\xi); L^2_{(\tau,\xi)}\|$$

We can easily estimate the first term in the right hand-side of the integral equation (8)

$$\|\psi(t)U(t)u_0; X^{b,s}\| \le c\|u_0; H^s\|$$

For estimating the second term in the right hand-side of the integral equation (8) we use the next Proposition.

Proposition 2. (cf. [9] Lemma 2.1). *Let* $-1/2 < b' \le b \le 0 \le b' + 1$. *Then*

$$\|i\psi(t/T)\int_0^t U(t-t')f(u(t'))dt'; X^{b,s}\| \le cT^{1-b+b'}\|f; X^{b',s}\| \tag{10}$$

The estimate of Proposition 2 is in fact one dimensional (with respect to t) and can be applied for any equation of type (8). In most of the cases we shall be able to use Proposition 2 with $b = 1/2 - \epsilon$ and $b' = -1/2 + \epsilon$ for sufficiently small positive ϵ to estimate the second term of the right hand-side of (8). In these cases we have to estimate $X^{-1/2+\epsilon,s}$ norms of the nonlinear terms. But in a limit case we shall be obliged to take $b = 1/2$ and $b' = -1/2$. In this case an additional term appears in the right hand-side of (10). For that purpose we introduce the auxiliary spaces $Y^s = \{u : \|u; Y^s\| < \infty\}$, where $\|u; Y^s\| = \| < \tau >^{-1} < \xi >^s \mathcal{F}(U(-t)u); L^2_\xi(L^1_\tau)\|$. A direct computation shows that in the case of equation (7) the spaces Y^s are the following:

$$\|E; Y^s\| = \| < \tau I_{d\times d} + M_d(\xi) >^{-1} < \xi >^s \hat{E}(\tau,\xi); L^2_\xi(L^1_\tau)\|,$$
$$\|n_\pm; Y^s\| = \| < \tau \pm |\xi| >^{-1} < \xi >^s \hat{n}_\pm(\tau,\xi); L^2_\xi(L^1_\tau)\|.$$

Using once again Proposition 1, we obtain an equivalent norm for E in Y^s

$$\|E; Y^s\| = \| < \tau + |\xi|^2 >^{-1} < \xi >^s \hat{E}(\tau,\xi); L^2_\xi(L^1_\tau)\|.$$

In the limit case $b = 1/2$, $b' = -1/2$ we shall use the following estimate for the second term in the right hand-side of (8).

Proposition 3. (cf. [9] Lemma 2.1). *The following estimate holds*

$$\|i\psi(t/T)\int_0^t U(t-t')f(u(t'))dt'; X^{1/2,s}\| \le c(\|f; X^{-1/2,s}\| + \|f; Y^s\|). \quad (11)$$

One dimensional Sobolev embedding gives $X^{b,s} \subset C(\mathbf{R}; H^s)$ provided $b > 1/2$. But in our estimates we shall take $b = 1/2-\epsilon$. The next Proposition shows that if we can estimate Y^s norm of the nonlinearity then the solution belongs to $C(\mathbf{R}; H^s)$.

Proposition 4. (cf. [9] Lemma 2.2) *Let $f \in Y^s$. Then*

$$\int_0^t U(t-t')f(t')dt' \in C(\mathbf{R}; H^s).$$

We shall solve the integral equation (8) by a fixed point theorem with $E \in X^{1/2-\epsilon,k}$, $n_\pm \in X^{1/2-\epsilon,l}$, for ϵ sufficiently small. For that purpose we need the next estimates

$$\|n_\pm E; X^{-c,k}\| \le c\|E; X^{1/2-\epsilon,k}\| \cdot \|n_\pm; X^{1/2-\epsilon,l}\|, \quad (12)$$

$$\|E \wedge B(E); X^{-c,k}\| \le c\|E; X^{1/2-\epsilon,k}\|^3, \quad (13)$$

where $c \le 1/2+\epsilon$,

$$\|\omega|E|^2; X^{-b,l}\| \le c\|E; X^{1/2-\epsilon,k}\|^2, \quad (14)$$

where $b \le 1/2+\epsilon$. Hence, by the self-duality of L^2 one obtains that the estimates (12), (13), (14) are equivalent to the following three inequalities

$$\int\int |\hat{E}(\tau_1,\xi_1)\hat{n}_\pm(\tau-\tau_1,\xi-\xi_1)\hat{w}(\tau,\xi)|K_1(\tau,\xi,\tau_1,\xi_1)\,d\tau\,d\xi\,d\tau_1\,d\xi_1 \le c\|E\|_{L^2}\|n_\pm\|_{L^2}\|w\|_{L^2}, \quad (15)$$

where $K_1(\tau,\xi,\tau_1,\xi_1) = <\xi>^k <\xi_1>^{-k} <\xi-\xi_1>^{-l} <\tau+|\xi|^2>^{-c}$ $<\tau_1+|\xi_1|^2>^{-1/2+\epsilon} <\tau-\tau_1 \pm |\xi-\xi_1|>^{-1/2+\epsilon}$.

$$\int\int\int\big|\hat{E}_1(\tau-\tau_1,\xi-\xi_1)\hat{E}_2(\tau_1-\tau_2,\xi_1-\xi_2)\hat{E}_3(\tau_2,\xi_2)\hat{w}(\tau,\xi)\big| \times K_2(\tau,\xi,\tau_1,\xi_1)d\tau d\xi d\tau_1 d\xi_1 d\tau_2 d\xi_2 \le c\|E_1\|_{L^2}\|E_2\|_{L^2}\|E_3\|_{L^2}\|w\|_{L^2}, \quad (16)$$

where $K_2(\tau,\xi,\tau_1,\xi_1,\tau_2,\xi_2) =< \xi >^k< \xi_2 >^{-k}< \xi_1-\xi_2 >^{-k}< \xi-\xi_1 >^{-k}$ $< \tau+|\xi|^2 >^{-c}< \tau-\tau_1+|\xi-\xi_1|^2 >^{-1/2+\epsilon}< \tau_1-\tau_2+|\xi_1-\xi_2|^2 >^{-1/2+\epsilon}$ $< \tau_2+|\xi_2|^2 >^{-1/2+\epsilon}$.

$$\int\int |\hat{E}_1(\tau_1,\xi_1)\hat{E}_2(\tau-\tau_1,\xi-\xi_1)\hat{w}(\tau,\xi)|K_3(\tau,\xi,\tau_1,\xi_1) \qquad (17)$$
$$d\tau d\xi d\tau_1 d\xi_1 \le c\|E_1\|_{L^2}\|E_2\|_{L^2}\|w\|_{L^2},$$

where $K_3(\tau,\xi,\tau_1,\xi_1) = |\xi| < \xi >^l< \xi_1 >^{-k}< \xi-\xi_1 >^{-k}< \tau \pm |\xi| >^{-b}$ $< \tau_1+|\xi_1|^2 >^{-1/2+\epsilon}< \tau-\tau_1+|\xi-\xi_1|^2 >^{-1/2+\epsilon}$. In order to put the solution in the framework of the time continuous functions and to estimate the second term in the right hand-side of (11) we need to estimate some Y^s norms of the nonlinearities. More precisely we need the next estimates

$$\|n_\pm E; Y^k\| \le c\|E; X^{1/2-\epsilon,k}\| \cdot \|n_\pm; X^{1/2-\epsilon,l}\|, \qquad (18)$$
$$\|E \wedge B(E); Y^k\| \le c\|E; X^{1/2-\epsilon,k}\|^3, \qquad (19)$$
$$\|\omega|E|^2; Y^k\| \le c\|E; X^{1/2-\epsilon,k}\|^2. \qquad (20)$$

If we want to prove an estimate of type $|v|_{Y^s} \le c$, then by duality it is sufficient to prove that

$$\int < \tau+\phi(\xi) >^{-1}< \xi >^s |\hat{v}(\tau,\xi)||\hat{w}(\xi)| \le c|w|_{L^2},$$

with the same constant c. The real function ϕ is the phase function which the respective Bourgain space is associated to and w depends only on x variables. Hence using the above argument one obtains that the estimates (18), (19), (20) are equivalent to the following inequalities respectively

$$\int\int |\hat{E}(\tau_1,\xi_1)\hat{n}_\pm(\tau-\tau_1,\xi-\xi_1)\hat{w}(\xi)|K_4(\tau,\xi,\tau_1,\xi_1) \qquad (21)$$
$$d\tau d\xi d\tau_1 d\xi_1 \le c\|E\|_{L^2}\|n_\pm\|_{L^2}\|w\|_{L^2},$$

where $K_4(\tau,\xi,\tau_1,\xi_1) =< \xi >^k< \xi_1 >^{-k}< \xi-\xi_1 >^{-l}< \tau+|\xi|^2 >^{-1}< \tau_1+$ $|\xi_1|^2 >^{-1/2+\epsilon}< \tau-\tau_1 \pm |\xi-\xi_1| >^{-1/2+\epsilon}$.

$$\int\int\int |\hat{E}_1(\tau-\tau_1,\xi-\xi_1)\hat{E}_2(\tau_1-\tau_2,\xi_1-\xi_2)\hat{E}_3(\tau_2,\xi_2) \qquad (22)$$
$$\times\, \hat{w}(\xi)|K_5(\tau,\xi,\tau_1,\xi_1)d\tau d\xi d\tau_1 d\xi_1 d\tau_2 d\xi_2 \le c\|E_1\|_{L^2}\|E_2\|_{L^2}\|E_3\|_{L^2}\|w\|_{L^2},$$

where $K_5(\tau,\xi,\tau_1,\xi_1,\tau_2,\xi_2) =< \xi >^k< \xi_2 >^{-k}< \xi_1-\xi_2 >^{-k}< \xi-\xi_1 >^{-k}$ $< \tau+|\xi|^2 >^{-1} \{< \tau-\tau_1+|\xi-\xi_1|^2 >^{-1/2+\epsilon}< \tau_1-\tau_2+|\xi_1-\xi_2|^2 >^{-1/2+\epsilon}$ $< \tau_2+|\xi_2|^2 >\}^{-1/2+\epsilon}$.

$$\int\int |\hat{E}_1(\tau_1,\xi_1)\hat{E}_2(\tau-\tau_1,\xi-\xi_1)\hat{w}(\xi)|K_6(\tau,\xi,\tau_1,\xi_1)d\tau d\xi d\tau_1 d\xi_1 \qquad (23)$$
$$\leq c\|E_1\|_{L^2}\|E_2\|_{L^2}\|w\|_{L^2},$$

where $K_6(\tau,\xi,\tau_1,\xi_1) = |\xi| < \xi >^l< \xi_1 >^{-k}< \xi-\xi_1 >^{-k}< \tau\pm|\xi| >^{-1}$ $< \tau_1+|\xi_1|^2 >^{-1/2+\epsilon}< \tau-\tau_1+|\xi-\xi_1|^2 >^{-1/2+\epsilon}$. All integrals in the left hand sides of (15), (17), (21),(23) have the form

$$\int \hat{\phi}_1(\tau,\xi)(\hat{\phi}_2\star\hat{\phi}_3)(\tau,\xi)d\tau d\xi.$$

Using Plancherel identity and Hölder inequality one obtains

$$\int \hat{\phi}_1(\tau,\xi)(\hat{\phi}_2\star\hat{\phi}_3)(\tau,\xi)d\tau d\xi \leq \prod_{j=1}^{3}\|\phi_j;L_t^{q_j}(L_x^{r_j})\|,$$

where

$$\sum_{j=1}^{3}\frac{1}{r_j}=\sum_{j=1}^{3}\frac{1}{q_j}=1$$

The integrals in the left hand sides of (16) and (22) have the form

$$\int \hat{\phi}_1(\tau,\xi)(\hat{\phi}_2\star(\hat{\phi}_3\star\hat{\phi}_4))(\tau,\xi)d\tau d\xi.$$

Using Plancherel identity and Hölder inequality one obtains

$$\int \hat{\phi}_1(\tau,\xi)(\hat{\phi}_2\star(\hat{\phi}_3\star\hat{\phi}_4))(\tau,\xi)d\tau d\xi \leq \prod_{j=1}^{4}\|\phi_j;L_t^{q_j}(L_x^{r_j})\|,$$

where

$$\sum_{j=1}^{4}\frac{1}{r_j}=\sum_{j=1}^{4}\frac{1}{q_j}=1$$

Hence, in order to prove (15), (16), (17), (21), (22), (23) we need estimates of type $\|\mathcal{F}^{-1}(< \tau+|\xi|^2 >^{-1/2+\epsilon} |\hat{\phi}(\tau,\xi)|);L_t^q(L_x^r)\| \leq c\|\phi\|_{L^2}$, $\|\mathcal{F}^{-1}(< \tau\pm |\xi| >^{-1/2+\epsilon} |\hat{\phi}(\tau,\xi)|);L_t^q(L_x^r)\| \leq c\|\phi\|_{L^2}$. The above inequalities are versions of the Strichartz estimates for the linear Schrödinger and wave equations respectively. More precisely we have:

Proposition 5. (Strichartz inequalities, cf. [9], Lemma 3.1). *Let* $0 \leq \gamma \leq 1$, $0 < \eta \leq 1$, $\epsilon \geq 0$, $\epsilon_1 > 0$, *supp* $\phi \subset \{(t,x) : t \leq cT\}$. *Then for sufficiently small* ϵ *and* ϵ_1 *the following inequalities hold*

$$\|\mathcal{F}^{-1}(< \tau + |\xi|^2 >^{-1/2+\epsilon} |\hat{\phi}|)(\tau,\xi); L_t^q(L_x^r)\| \leq cT^{\gamma^\star/2}\|\phi\|_{L^2},$$

$$\|\mathcal{F}^{-1}(< \tau \pm |\xi| >^{-1/2+\epsilon} |\hat{\phi}|)(\tau,\xi); L_t^q(L_x^2)\| \leq cT^{\gamma^\star/2}\|\phi\|_{L^2},$$

where $\gamma^\star = \gamma$ *if* $\epsilon > 0$, $\gamma^\star = \gamma - 0$ *if* $\epsilon = 0$ *and provided* q *and* r *satisfy the relations*

$$\frac{2}{q} = 1 - \frac{\eta(1-\gamma)(1/2-\epsilon)}{1/2+\epsilon_1}.$$

$$\delta(r) := d(\frac{1}{2} - \frac{1}{r}) = \frac{(1-\eta)(1-\gamma)(1/2-\epsilon)}{1/2+\epsilon_1}.$$

We shall always be able to assume the support restriction of ϕ because (8) and (9) are equivalent and using Lemma 2.5 of [9]. In the cases when $\epsilon > 0$ we shall provide a small factor in the estimates from Proposition 2 and take $\gamma = 0$, but in a limit case ($k-l=1$) we will be forced to take $\epsilon = 0$ and then the small factor will come from the Strichartz inequality. In this limit case we shall be able to take $\gamma > 0$ very small. The above version of the Strichartz inequality for the wave equation is actually an interpolation between the obvious case $\eta = 1, \gamma = 1, \epsilon = 1/2$ and the energy inequality. In the case of the Schrödinger equation the estimate of Proposition 5 is an interpolation between the same trivial case and the well-known estimation for a mixed $L_t^q(L_x^r)$ norm of the free evolution of the Schrödinger equation (usually called Strichartz inequality). We refer to [8] and [9] for details.

3. Proof of Theorem 1 and Theorem 2. In order to apply a fixed point theorem it is sufficient to prove (15), (16), (17), (21), (22) and (23). We note by I_1, I_2, I_3, I_4, I_5 and I_6 the left hand sides of (15), (16), (17), (21), (22) and (23) respectively. The estimates (15), (17), (21) and (23) are essentially established in [9] but we shall prove them again for completeness. We take $b = c = 1/2 - \epsilon$, $\epsilon \geq 0$. When $k - l < 1$ we shall always be able to take $\epsilon > 0$ and zero the auxiliary parameter γ in Strichartz inequality.

Proof of (15). We have the decomposition of $I_1 = I_{11} + I_{12}$, where I_{11} is the restriction of I_1 over the region $\{(\tau,\xi,\tau_1,\xi_1) : |\xi| > 2|\xi_1|\}$ and I_{12} is the restriction of I_1 over the region $\{(\tau,\xi,\tau_1,\xi_1) : |\xi| \leq 2|\xi_1|\}$. Note that on the support of I_{12} we have that $|\xi| \leq 2|\xi - \xi_1|$. On the support of I_1, we have $K_1(\tau,\xi,\tau_1,\xi_1) \leq c < \xi - \xi_1 >^{-l} < \tau + |\xi|^2 >^{-1/2+\epsilon} < \tau_1 + |\xi_1|^2 >^{-1/2+\epsilon}$ $< \tau - \tau_1 \pm |\xi - \xi_1| >^{-1/2+\epsilon}$. Hence, using Hölder inequality we obtain

$$I_{11} \leq c\|\mathcal{F}^{-1}(< \tau + |\xi|^2 >^{-1/2+\epsilon} |\hat{w}(\tau,\xi)|); L_t^{q_1}(L_x^{r_1})\|$$
$$\|\mathcal{F}^{-1}(< \tau + |\xi|^2 >^{-1/2+\epsilon} |\hat{E}(\tau,\xi)|); L_t^{q_2}(L_x^{r_2})\|$$
$$\|\mathcal{F}^{-1}(< \xi >^{-l} < \tau \pm |\xi| >^{-1/2+\epsilon} |\hat{n}_\pm(\tau,\xi)|); L_t^{q_3}(L_x^{r_3})\|,$$

provided $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$ and $\delta(r_1) + \delta(r_2) + \delta(r_3) = \frac{d}{2}$. Using the Strichartz inequality with the same auxiliary parameters γ and η we obtain

$$\|\mathcal{F}^{-1}(< \tau + |\xi|^2 >^{-1/2+\epsilon} |\hat{w}(\tau,\xi)|); L_t^{q_1}(L_x^{r_1})\| \leq cT^{\gamma^\star/2}\|w\|_{L^2}$$
$$\|\mathcal{F}^{-1}(< \tau + |\xi|^2 >^{-1/2+\epsilon} |\hat{E}(\tau,\xi)|); L_t^{q_2}(L_x^{r_2})\| \leq cT^{\gamma^\star/2}\|E\|_{L^2}$$

provided

$$\frac{2}{q_j} = 1 - \frac{\eta(1-\gamma)(1/2-\epsilon)}{(1/2+\epsilon_1)}, \quad \delta(r_j) = \frac{(1-\eta)(1-\gamma)(1/2-\epsilon)}{(1/2+\epsilon_1)}, \quad j = 1, 2.$$

Next using Sobolev embedding and Strichartz estimate we obtain

$$\|\mathcal{F}^{-1}(< \xi >^{-l} < \tau \pm |\xi| >^{-1/2+\epsilon} |\hat{n}_\pm(\tau,\xi)|); L_t^{q_3}(L_x^{r_3})\| \leq$$
$$c\|\mathcal{F}^{-1}(< \tau \pm |\xi| >^{-1/2+\epsilon} |\hat{n}_\pm(\tau,\xi)|); L_t^{q_3}(L_x^2)\| \leq c\|n_\pm\|_{L^2},$$

provided $\frac{1}{2} - \frac{1}{r_3} \leq \frac{l}{d}$, i.e., $l \geq \delta(r_3)$ and $\frac{2}{q_3} = 1 - \frac{(1-\gamma)(1/2-\epsilon)}{(1/2+\epsilon_1)}$. Then the restrictions of the Hölder inequality become $(1-\gamma)(1+2\eta) = \frac{1+2\epsilon_1}{1-2\epsilon}$, $\delta(r_3) = d/2 - \delta(r_1) - \delta(r_2) = d/2 - \frac{2-2\eta}{1+2\eta}$.

• If $k - l < 1$, then we take $\gamma = 0, \epsilon_1 = 2\epsilon, \epsilon > 0$ and we have that $\eta = \frac{3\epsilon}{1-2\epsilon}$. Therefore, $0 < \eta \leq 1$, for ϵ sufficiently small. The restriction $l \geq \delta(r_3)$ becomes $l \geq d/2 - 2\frac{1-5\epsilon}{1+4\epsilon}$, which is fulfilled if $l \geq 0$ and ϵ sufficiently small.

• If $k - l = 1$, then we take $\gamma = \epsilon_2, \epsilon_1 = 2\epsilon_2$, $\epsilon = 0$ and we have that $\eta = \frac{5\epsilon_2}{1-\epsilon_2}$. The restriction $l \geq \delta(r_3)$ becomes $l \geq d/2 - \frac{2-7\epsilon_2}{1+4\epsilon_2}$, which is fulfilled if $l \geq 0$ and ϵ_2 sufficiently small. Note that for the estimate of I_{11} we are not obliged to take $\epsilon = 0$. This completes the estimate of I_{11}.

To estimate I_{12} we note that on the support of I_2 the following holds

$$K_1(\tau,\xi,\tau_1,\xi_1) \leq c < \xi >^{k-l} < \xi_1 >^{-k} < \tau + |\xi|^2 >^{-1/2+\epsilon}$$
$$< \tau_1 + |\xi_1|^2 >^{-1/2+\epsilon} < \tau - \tau_1 \pm |\xi - \xi_1| >^{-1/2+\epsilon} .$$

Now we need the following proposition.

Proposition 6. *If* $|\xi| > 2|\xi_1|$, *then there exists a constant* c *such that*

$$<\xi>^2 \leq c(<\tau+|\xi|^2> + <\tau_1+|\xi_1|^2> + <\tau-\tau_1 \pm |\xi-\xi_1|>).$$

Proof. If $|\xi| \leq 4$, then the estimate is obvious. Let $|\xi| > 4$. Then we have

$$|\tau+|\xi|^2| + |\tau_1+|\xi_1|^2| + |\tau-\tau_1 \pm |\xi-\xi_1|| \geq ||\xi|^2 - ||\xi_1|^2 \mp |\xi-\xi_1||.$$

Moreover,

$$|\xi|^2 - ||\xi_1|^2 \mp |\xi-\xi_1| \geq \frac{3}{4}(\frac{|\xi|}{2}(|\xi|-4) + |\xi|^2) \geq \frac{3}{8}|\xi|^2.$$

Hence,

$$\frac{3}{8}|\xi|^2 \leq |\tau+|\xi|^2| + |\tau_1+|\xi_1|^2| + |\tau-\tau_1 \pm |\xi-\xi_1||,$$

which completes the proof of Proposition 6.

Using Proposition 6 we obtain the following estimate for K_1 on the support of I_2

$$\begin{aligned} K_1(\tau,\xi,\tau_1,\xi_1) \leq c(&<\xi_1>^{-k}<\tau+|\xi|^2>^{-1/2+\epsilon+(k-l)/2} \\ &<\tau_1+|\xi_1|^2>^{-1/2+\epsilon}<\tau-\tau_1 \pm |\xi-\xi_1|>^{-1/2+\epsilon} + \\ &<\xi_1>^{-k}<\tau+|\xi|^2>^{-1/2+\epsilon} \\ &<\tau_1+|\xi_1|^2>^{-1/2+\epsilon+(k-l)/2}<\tau-\tau_1 \pm |\xi-\xi_1|>^{-1/2+\epsilon} + \\ &<\xi_1>^{-k}<\tau+|\xi|^2>^{-1/2+\epsilon} \\ &<\tau_1+|\xi_1|^2>^{-1/2+\epsilon}<\tau-\tau_1 \pm |\xi-\xi_1|>^{-1/2+\epsilon+(k-l)/2}). \end{aligned}$$

Hence, we deduce $I_{12} \leq c(I_{121}+I_{122}+I_{123})$. Using the Hölder inequality we obtain

$$\begin{aligned} I_{121} \leq c\|&\mathcal{F}^{-1}(<\xi>^{-k}<\tau+|\xi|^2>^{-1/2+\epsilon}|\hat{E}(\tau,\xi)|); L_t^{q_1}(L_x^{r_1})\| \\ &\|\mathcal{F}^{-1}(<\tau+|\xi|^2>^{-1/2+\epsilon+(k-l)/2}|\hat{w}(\tau,\xi)|); L_t^{q_2}(L_x^{r_2})\| \\ &\|\mathcal{F}^{-1}(<\tau \pm |\xi|>^{-1/2+\epsilon}|\hat{n}_\pm(\tau,\xi)|); L_t^{q_3}(L_x^2)\|, \end{aligned}$$

provided $\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3} = 1$ and $\delta(r_1)+\delta(r_2) = \frac{d}{2}$. Using Sobolev and Strichartz inequalities we obtain

$$\begin{aligned} &\|\mathcal{F}^{-1}(<\xi>^{-k}<\tau+|\xi|^2>^{-1/2+\epsilon}|\hat{E}(\tau,\xi)|); L_t^{q_1}(L_x^{r_1})\| \\ \leq c&\|\mathcal{F}^{-1}(<\tau+|\xi|^2>^{-1/2+\epsilon}|\hat{E}(\tau,\xi)|); L_t^{q_1}(L_x^{r_1'})\| \leq cT^{\gamma^\star/2}\|E\|_{L^2}, \end{aligned}$$

provided $\frac{1}{r_1} - \frac{1}{r_1'} \leq \frac{k}{d}$, i.e., $k \geq \delta(r_1) - \delta(r_1')$ and $\frac{2}{q_1} = 1 - \frac{\eta(1-\gamma)(1/2-\epsilon)}{(1/2+\epsilon_1)}$, $\delta(r_1') = \frac{(1-\eta)(1-\gamma)(1/2-\epsilon)}{(1/2+\epsilon_1)}$. Using Strichartz inequality we obtain

$$\|\mathcal{F}^{-1}(<\tau+|\xi|^2>^{-1/2+\epsilon+(k-l)/2}|\hat{w}(\tau,\xi)|);L_t^{q_2}(L_x^{r_2})\| \leq cT^{\gamma^\star/2}\|w\|_{L^2},$$

provided $\frac{2}{q_2} = 1 - \frac{\eta(1-\gamma)(1/2-\epsilon-(k-l)/2)}{(1/2+\epsilon_1)}$, $\delta(r_2) = \frac{(1-\eta)(1-\gamma)(1/2-\epsilon-(k-l)/2)}{(1/2+\epsilon_1)}$ and

$$\|\mathcal{F}^{-1}(<\tau\pm|\xi|>^{-1/2+\epsilon}|\hat{n}_\pm(\tau,\xi)|);L_t^{q_3}(L_x^{2})\| \leq cT^{\gamma^\star/2}\|n_\pm\|_{L^2},$$

if $\frac{2}{q_3} = 1 - \frac{(1-\gamma)(1/2-\epsilon)}{(1/2+\epsilon_1)}$. Then the first restriction of the Hölder inequality becomes

$$(2\eta+1)\frac{(1-\gamma)(1-2\epsilon)}{(1+2\epsilon_1)} - \frac{\eta(1-\gamma)(k-l)}{1+2\epsilon_1} = 1$$

The restriction $k \geq \delta(r_1) - \delta(r_1')$ becomes

$$k \geq d/2 - \delta(r_1') - \delta(r_2) = d/2 + 1 - 3\frac{(1-\gamma)(1-2\epsilon)}{1+2\epsilon_1} + \frac{(1-\gamma)(k-l)}{1+2\epsilon_1}.$$

Now we consider two cases for (k,l).

• If $k-l<1$, then we can take $\epsilon>0, \epsilon_1 = 2\epsilon$ and $\gamma=0$. In this case we obtain a small factor in the iteration scheme using Proposition 2 (note that $1-b+b' = 2\epsilon > 0$ in this case). We have that $\eta = \frac{6\epsilon}{2-(k-l)-4\epsilon}$. Therefore, $0<\eta\leq 1$, for sufficiently small ϵ. The Sobolev embedding condition $k \geq \delta(r_1) - \delta(r_1')$ becomes

$$k \geq d/2 + 1 - 3\frac{1-2\epsilon}{1+4\epsilon} + \frac{k-l}{1+4\epsilon}, \tag{24}$$

which is fulfilled for $l > d/2 - 2$ (the limit of (24) when $\epsilon \to 0$), provided ϵ sufficiently small.

• If $k-l=1$, then we can not take $\epsilon>0$ in order to ensure the non-negativeness of the number $1/2-\epsilon-(k-l)/2$. Hence, we are forced to take $\epsilon = 0$. In this case we have to derive a small factor in the iteration schema from Strichartz inequalities. For that purpose we have to choose γ to be a positive number. If $k-l=1$ then we set $\gamma = \epsilon_2, \epsilon_1 = 2\epsilon_2$. Hence $\eta = \frac{5\epsilon_2}{2-(1-\epsilon_2)(k-l)-2\epsilon_2}$. Therefore, $0<\eta\leq 1$, for sufficiently small ϵ_2. The Sobolev embedding condition $k \geq \delta(r_1) - \delta(r_1')$ becomes

$$k \geq d/2 + 1 - 3\frac{1-\epsilon_2}{1+4\epsilon_2} + \frac{(1-\epsilon_2)(k-l)}{1+4\epsilon_2},$$

which is fulfilled for $l > d/2-2$, provided ϵ sufficiently small. This completes the estimation of I_{121}. The integrals I_{122} and I_{223} can be estimated in the same way. This completes the proof of (15)

Proof of (16). To prove (16) we use the next elementary inequality

$$< \xi >^k \leq c(< \xi - \xi_1 >^k + < \xi_1 - \xi_2 >^k + < \xi_2 >^k).$$

Therefore, we can estimate $K_2(\tau, \xi, \tau_1, \xi_1, \tau_2, \xi_2)$

$$\begin{gathered} K_2(\tau, \xi, \tau_1, \xi_1, \tau_2, \xi_2) \leq c(< \xi_2 >^{-k} < \xi_1 - \xi_2 >^{-k} + \\ < \xi_2 >^{-k} < \xi - \xi_1 >^{-k} + < \xi_1 - \xi_2 >^{-k} < \xi - \xi_1 >^{-k}) \\ < \tau - \tau_1 + |\xi - \xi_1|^2 >^{-1/2+\epsilon} < \tau + |\xi|^2 >^{-1/2+\epsilon} \\ < \tau_1 - \tau_2 + |\xi_1 - \xi_2|^2 >^{-1/2+\epsilon} < \tau_2 + |\xi_2|^2 >^{-1/2+\epsilon} . \end{gathered}$$

Hence, $I_2 \leq c(I_{21} + I_{22} + I_{23})$. Setting $E_4 := w$ and using Hölder inequality we obtain

$$\begin{gathered} I_{21} \leq c \prod_{j=1}^{2} \|\mathcal{F}^{-1}(< \tau + |\xi|^2 >^{-1/2+\epsilon} |\hat{E}_j(\tau, \xi)|); L_t^{q_j}(L_x^{r_j})\| \times \\ \prod_{j=3}^{4} \|\mathcal{F}^{-1}(< \xi >^{-k} < \tau + |\xi|^2 >^{-1/2+\epsilon} |\hat{E}_j(\tau, \xi)|); L_t^{q_j}(L_x^{r_j})\| \end{gathered}$$

provided $\sum_{j=1}^4 \frac{1}{q_j} = 1$ and $\sum_{j=1}^4 \delta(r_j) = d$. Using the Strichartz inequality with the same auxiliary parameters γ and η we obtain

$$\|\mathcal{F}^{-1}(< \tau + |\xi|^2 >^{-1/2+\epsilon} |\hat{E}_j(\tau, \xi)|); L_t^{q_j}(L_x^{r_j})\| \leq cT^{\gamma^\star/2}\|E_j\|_{L^2},$$

if $\frac{2}{q_j} = 1 - \frac{\eta(1-\gamma)(1/2-\epsilon)}{(1/2+\epsilon_1)}$, $\delta(r_j) = \frac{(1-\eta)(1-\gamma)(1/2-\epsilon)}{(1/2+\epsilon_1)}$, $j = 1, 2$. Using Sobolev and Strichartz inequalities we obtain

$$\begin{gathered} \|\mathcal{F}^{-1}(< \xi >^{-k} < \tau + |\xi|^2 >^{-1/2+\epsilon} |\hat{E}_j(\tau, \xi)|); L_t^{q_j}(L_x^{r_j})\| \leq \\ c\|\mathcal{F}^{-1}(< \tau + |\xi|^2 >^{-1/2+\epsilon} |\hat{E}(\tau, \xi)|); L_t^{q_j}(L_x^{r_j'})\| \leq cT^{\gamma^\star/2}\|E\|_{L^2}, \end{gathered}$$

provided $\frac{1}{r_j} - \frac{1}{r_j'} \leq \frac{k}{d}$, i.e., $k \geq \delta(r_j) - \delta(r_j')$, $j = 3, 4$ and

$$\frac{2}{q_j} = 1 - \frac{\eta(1-\gamma)(1/2-\epsilon)}{(1/2+\epsilon_1)}, \quad \delta(r_j') = \frac{(1-\eta)(1-\gamma)(1/2-\epsilon)}{(1/2+\epsilon_1)}.$$

The first restriction of the Hölder inequality becomes $\frac{2\eta(1-\gamma)(1-2\epsilon)}{1+2\epsilon_1}=1$. Using the above identity and the fact $\delta(r_3')=\delta(r_4')$ we obtain that the restriction $k\geq \delta(r_j)-\delta(r_j'), j=3,4$ reads

$$2k\geq d-(\delta(r_1)+\delta(r_2)+\delta(r_3')+\delta(r_4'))=d+2-4\frac{(1-\gamma)(1-2\epsilon)}{1+2\epsilon_1}.$$

• If $k-l<1$, then we take $\gamma=0$, $\epsilon_1=2\epsilon$ and we have that $\eta=\frac{1+4\epsilon}{2-4\epsilon}$, i.e., $0<\eta\leq 1$ is fulfilled for sufficiently small ϵ. The Sobolev embedding condition $k\geq\delta(r_j)-\delta(r_j')$, $j=3,4$ becomes $k\geq d/2+1-2\frac{1-2\epsilon}{1+4\epsilon}$, which is fulfilled for $k>d/2-1(\geq 1/2)$, provided ϵ sufficiently small. In the estimate of I_{12} when $k-l=1$ we were forced to take $\epsilon=0$. Hence we are obliged to do the same in all other estimates.

• In the case of I_{21} if $k-l=1$ then we take $\epsilon=0,\gamma=\epsilon_2,\epsilon_1=2\epsilon_2$ and we have that $\eta=\frac{1+4\epsilon_2}{2-2\epsilon_2}$, i.e., $0<\eta\leq 1$ is fulfilled for sufficiently small ϵ_2. The Sobolev embedding condition $k\geq\delta(r_j)-\delta(r_j')$, $j=3,4$ becomes $k\geq d/2+1-2\frac{1-\epsilon_2}{1+2\epsilon_2}$, which is fulfilled for $k>d/2-1(\geq 1/2)$, provided ϵ_2 sufficiently small. This completes the estimate of I_{21}. The terms I_{22} and I_{23} can be estimated in the same way. This completes the proof of (16).

Proof of (17). We have the next decomposition of I_3, $I_3=I_{31}+I_{32}$, where I_{31} is the restriction of I_3 over the region $\{(\tau,\xi,\tau_1,\xi_1):|\xi|>2|\xi_1|\}$ and I_{32} is the restriction of I_3 over the region $\{(\tau,\xi,\tau_1,\xi_1):|\xi|\leq 2|\xi_1|\}$. Note that on the support of I_{32} we have that $|\xi|\leq 2|\xi-\xi_1|$. On the support of I_{31}, we have $K_3(\tau,\xi,\tau_1,\xi_1)\leq c<\xi_1>^{k-(2k-l-1)}<\xi_1>^{-k}<\tau\pm|\xi|>^{-1/2+\epsilon}$ $<\tau_1+|\xi_1|^2>^{-1/2+\epsilon}<\tau-\tau_1+|\xi-\xi_1|^2>^{-1/2+\epsilon}$. On the support of I_{32}, we have $K_3(\tau,\xi,\tau_1,\xi_1)\leq c<\xi-\xi_1>^{k-(2k-l-1)}<\xi-\xi_1>^{-k}<\tau\pm|\xi|>^{-1/2+\epsilon}$ $<\tau_1+|\xi_1|^2>^{-1/2+\epsilon}<\tau-\tau_1+|\xi-\xi_1|^2>^{-1/2+\epsilon}$. Hence, using an analogue of Proposition 6 we reduce the estimation of I_{31} and I_{32} to estimates similar to these we obtained for I_{121}. The condition $l\geq 0$ is transformed into $2k-l-1\geq 0$. These completes the proof of (17).

Proof of (21), (22) and (23). We have that $K_4=<\tau+|\xi|^2>^{-1/2-\epsilon}K_1$, $K_5=<\tau+|\xi|^2>^{-1/2-\epsilon}K_2$, $K_6=<\tau\pm|\xi|>^{-1/2-\epsilon}K_3$. For $\epsilon>0$ the following inequalities hold

$$\|<\tau+|\xi|^2>^{-1/2-\epsilon}\hat{w}(\xi)\|_{L^2_{(\tau,\xi)}}\leq\|w\|_{L^2}, \tag{25}$$

$$\|<\tau\pm|\xi|>^{-1/2-\epsilon}\hat{w}(\xi)\|_{L^2_{(\tau,\xi)}}\leq\|w\|_{L^2}. \tag{26}$$

Hence, when $\epsilon > 0$ (i.e., $k - l < 1$) we can reduce the estimations of I_4, I_5, I_6 to these of I_1, I_2, I_3 respectively. The functions $< \tau + |\xi|^2 >^{-1/2-\epsilon} \hat{w}(\xi)$ or $< \tau \pm |\xi| >^{-1/2-\epsilon} \hat{w}(\xi)$ play the same role as $\hat{w}(\tau, \xi)$ in I_1, I_2, I_3. In the case when $\epsilon = 0$ (i.e., $k - l = 1$) logarithmic singularities appear in the right hand sides of (25) and (26). In the case $|\xi| \geq 2|\xi_1|$ we need an analogue of Proposition 6 with cut-off functions in order to cancel the logarithmic singularities. For this we refer to [9], Lemma 3.3. In the case $|\xi| \leq 2|\xi_1|$ we reduce I_4, I_5, I_6 to integrals of type I_{11} and estimate them in the same way.

It remains to use standard fixed point arguments to prove that there exists $(E, n) \in X^{1/2-\epsilon,k} \times X^{1/2-\epsilon,l}$, solution of the cut-off integral equation (8). Using (21), (22) and (23) we inject the solutions in the framework of time continuous functions using Proposition 4. Up to now we proved well-posedness of the cut-off equation (8). For the proof of the independence of the cut-off function ψ we refer to [9]. This completes the proof of Theorem 1.

In order to prove Theorem 2, we obtain using the conservation of energy (4)

$$\|\nabla E(t,.)\|_{L^2}^2 + \frac{1}{2}\|n(t,.)\|_{L^2}^2 \leq c_1 + \int_{\mathbf{R}^d} (n|E|^2)(t,x)dx$$
$$+\frac{\eta}{4}\|(-\Delta - \beta)^{-1/2} \text{ rot}(E \wedge \bar{E})(t,.)\|_{L^2}^2,$$

where we used that $\|\nabla E\|_{L^2}^2 = \|\text{rot} E\|_{L^2}^2 + \|\text{div} E\|_{L^2}^2$. The constant c_1 depends only on the initial data. In fact c_1 is the absolute value of the total energy of the system (1). Furthermore we have

$$\|\nabla E(t,.)\|_{L^2}^2 + \frac{1}{2}\|n(t,.)\|_{L^2}^2 \leq c_1 + \frac{1}{4}\|n(t,.)\|_{L^2}^2 + c_2\|\nabla E(t,.)\|_{L^2}^d,$$

where we used the classical inequality $\|f\|_{L^4} \leq c\|f\|_{L^2}^{1-d/4}\|\nabla f\|_{L^2}^{d/4}$ and that $(\Delta - \beta)^{-1/2}\text{rot}$ is a pseudo differential operator of order zero and therefore L^2 continuous. The constant c_2 depends on the initial data. Hence under some smallness assumptions on the initial data (cf. [11], Lemma 3.3 for details) we obtain that the quantity $\|\nabla E(t,.)\|_{L^2} + \|n(t,.)\|_{L^2}$ remains bounded as a function of t. Using the conservation of the L^2-norm of E we derive the boundedness of $\|E(t,.)\|_{H^1} + \|n(t,.)\|_{L^2}$. It remains to apply Theorem 1 for $k = 1, l = 0$ and the continuation principle (cf. [12]) to obtain global solutions of (1) – (2) in the energy space. This completes the proof of Theorem 2.

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