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UNIQUENESS OF SOLUTIONS TO THE INITIAL VALUE PROBLEM FOR AN INTEGRO-DIFFERENTIAL EQUATION

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Abstract. We establish uniqueness of a solution to the initial-value problem for the integro-differential equation

$$\frac{d}{dt}\int_0^1 J(x)\mu_t(dx) = \frac{1}{2}\int_0^1\int_0^1 \frac{J'(y)-J'(x)}{y-x}\cdot\frac{\mu_t(dx)\mu_t(dy)}{|y-x|^\gamma}, \qquad t>0$$

where the equality is required to hold for every smooth testing function J with $J'(0)=J'(1)=0$, and the solution $\mu_t=\mu_t(dx)$ is a finite measure on the unit interval $[0,1]$ for each t and γ a constant from the open interval $(-1,1)$. Stationary solutions are given explicitly and the convergence to them of general time-dependent solutions is proved.

1. Introduction. This paper concerns the non-linear integro-differential equation

$$\frac{d}{dt}\mu_t(J) = \frac{1}{2}\iint_Q \frac{J'(y)-J'(x)}{y-x}\cdot\frac{\mu_t(dx)\mu_t(dy)}{|y-x|^\gamma}, \qquad t>0 \tag{1.1}$$

with the initial condition

$$\lim_{t\downarrow 0}\mu_t = \mu_o. \tag{1.2}$$

Here $Q=[0,1]\times[0,1]$ (the unit square); γ is a constant such that $-1<\gamma<1$; (1.1) is understood to hold for every smooth testing function J with

$$J'(0)=J'(1)=0 \tag{1.3}$$

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(the condition of elastic boundary); $\mu_t = \mu_t(dx)$ is a finite measure on the unit interval $[0,1]$ for each t; $\mu_t(J) = \int_0^1 J(x)\mu_t(dx)$; and the convergence of measures is taken in the topology of weak convergence. (The way the sign of γ is chosen in (1.1) is due to our inclination of attention to the case of higher singularity $0 < \gamma < 1$.)

If the equation (1) is considered for the whole real line $\mathbf{R}$ instead of the unit interval and μ_t has a smooth density $\rho(x,t)$, it is reduced to the integro-differential equation

$$\frac{\partial}{\partial t}\rho(x,t) = \frac{\partial}{\partial x}\big(\rho(x,t)\int_{-\infty}^{\infty}\frac{\rho(y,t)dy}{(y-x)|y-x|^{\gamma}}\big). \tag{1.4}$$

Clearly the constants are stationary solutions of (1.4), while any constant multiple of the Lebesgue measure does not solve (1.1). (For the analogue to this form of the problem (1.1-2), we have only to reduce the range of integration to the unit interval.)

We do not assume the continuity of $(d/dt)\mu_t(J)$. Instead of it we impose that $\mu_t(J)$ is absolutely continuous relative to t and (1.1) holds a.e. $t \in [0,T]$, where T is a positive constant. This amounts to considering the integral equation

$$\mu_{t_2}(J) - \mu_{t_1}(J) = \frac{1}{2}\int_{t_1}^{t_2} dt \iint_Q \frac{J'(y) - J'(x)}{y-x}\cdot\frac{\mu_t(dx)\mu_t(dy)}{|y-x|^{\gamma}}, \tag{1.5}$$

which is to be valid for every pair $0 < t_1 < t_2 \le T$ as well as every smooth J satisfying (1.3). For the solution μ_t we also need to impose the condition

$$\int_0^T \mu_t(\{x\})dt = 0 \quad \text{for every } x \in [0,1] \quad \text{if} \quad \gamma < 0; \tag{1.6a}$$

$$\int_\delta^T dt \iint_Q \frac{\mu_t(dx)\mu_t(dy)}{|y-x|^{\gamma}} < \infty \quad \text{for every} \quad \delta \in (0,T) \quad \text{if} \quad \gamma > 0. \tag{1.6b}$$

(The function $|x|^{-p}$ $(p > 0)$ is considered to take the value $+\infty$ at $x = 0$, so that (1.6a) follows from (1.6b) if $\gamma > 0$.) The main result of this paper is now stated as follows.

Theorem 1. *Given $T > 0$ and a finite measure μ_o on the unit interval* $[0,1]$, *there exists one and only one solution of the non-linear problem* (1.5) *satisfying the initial condition* (1.2) *and the regularity condition* (1.6). *The solution* μ_t *necessarily satisfies*

$$\int_0^T dt \iint_Q \frac{\mu_t([x \wedge y, x \vee y])}{|y-x|} \frac{\mu_t(dx)\mu_t(dy)}{|y-x|^\gamma} < \infty, \tag{1.7}$$

where $x \wedge y = \max\{x, y\}$ *and* $x \vee y = \min\{x, y\}$.

As noticed above the Lebesgue measure is not a stationary solution of (1.1), which is therefore not trivial to identify. Fortunately, however, it turns out that it is given by a simple elementary function defined by

$$W_\gamma(x) = \frac{\Gamma(2\lambda+2)}{(\Gamma(\lambda+1))^2}[x(1-x)]^{-\lambda} \quad \lambda := \frac{1-\gamma}{2},$$

where the coefficient is chosen so that $\int_0^1 W_\gamma dx = 1$.

Theorem 2. *The measure* $\mu^W(dx) := W_\gamma(x)dx$ *is a unique stationary solution of* (1.5) *having unit total mass. Every solution* μ_t *of* (1.5) *and* (1.2) *that satisfies the regularity condition* (1.6) *weakly converges to* $M\mu^W$ *as* $t \to \infty$, *where* $M = \mu_o([0, 1])$.

The basic structure of our proof of uniqueness result of Theorem 1 bears some likeness to that of the corresponding uniqueness for the non-linear diffusion $u_t = \Delta f(u)$ as given in [3] and [9]. Because of nonlocality of the equation (1.1), however, we can not simply follow the proof of [3] or [9]. To overcome the difficulty it is crucial, in our approach, to consider the transform $\mu(dx) \longmapsto \hat{\mu}(x, y) := (y-x)^{-1} \int_x^y \mu(dr)$ $(x < y)$; actually large parts of this paper is devoted to deriving certain properties of this transform, which will allow us to rewrite some key formulae involving μ_t by means of $\hat{\mu}_t$ to conclude the desired uniqueness. The uniqueness of stationary solution stated in Theorem 2 is obtained as a byproduct of the proof of Theorem 1.

The equation (1.1) is derived by Uchiyama [12] from a large system of classical particles that move on a one-dimensional segment subject to friction and interacting one another via repulsive pair-potential forces obeying the power law $|x|^{-1-\gamma}$ for large values of $|x|$. (As a more realistic physical model one may consider a system of three-dimensional particles moving in a narrow pipe with the elastic inner surface.) It in particular establishes the existence of a non-negative solution to the initial value problem for (1.1), of which we do not give any proof in this paper. In this derivation of the equation (1.1) its solutions arise as weak limits of empirical distributions of particles and it is important to establish a uniqueness result of a weak solution. Our result stated as Theorem 1 is strong enough at least for this purpose. (See (II) of Section 6 for more details.) According to the interpretation based on this model the equation (1.1) (and (1.4)) may be understood to be the equation of continuity for a flow driven by the force field $\int (y-x)^{-1}|y-x|^{-\gamma}\mu_t(dy)$ that is balanced with friction, which in turn is equated with the velocity field. From the derivation it is also recognized that the equation (1.1) may be regarded as a natural extension of the degenerate diffusion equation

$$\frac{\partial}{\partial t}\rho(\theta,t) = \frac{\partial^2}{\partial \theta^2}P(\rho(\theta,t)), \tag{1.8}$$

where $P(\rho)$ is an increasing function of $\rho \geq 0$ such that $P(\rho) = O(\rho^2)$ as $\rho \downarrow 0$ (the case $\gamma = 1$ corresponds to the case $P(\rho) = \rho^2$) [10,11]. One may then expect that solutions of (1.1) (especially if considered on $\mathbf{R}$) would have some fundamental features in common with those of a non-linear diffusion equation of this type. In fact the solutions possess, like those of the porous medium equation, such properties as a finite speed of propagation and the shifting comparison principle; we have an analogue to the Barenblatt solution and convergence of general solutions to it [13]. In the special case $\gamma = 0$, the equation (1.1) is particularly tractable and we can prove some detailed properties of solutions such as a local relaxation within their supports and also disprove the comparison principle.

When $\gamma = 0$ the uniqueness and existence of a weak solution to the initial value problem for (1.4) are virtually proved in [8], in which is studied the equation (1.4), modified with a centralizing drift term added to the right-hand side of it, in a relation to Wigner's semi-circle law for the eigenvalues of random matrices of large dimensions. Our method is quite different from theirs. It covers all the cases $-1 < \gamma < 1$, while the method of [8], using the special feature of the kernel $(y-x)^{-1}$, does not seem applicable to the case $\gamma \neq 0$ at all.

We shall give several preliminary lemmas concerning the transform $\mu \longmapsto \hat{\mu}$ in Sections 2 and 3. The proof of uniqueness part of Theorem 1 will be given in Section 4. Theorem 2 is proved in Section 5. In Section 6 we shall give some related results without proof: these are (I) Extension to a class of general kernels $K(y-x)$; (II) Derivation of the equation (1.1) by scaling limit from a system of interacting particles; and (III) Uniqueness for $u_t = \mathcal{L}f(u)$ with $\mathcal{L}$ the infinitesimal generator of a Markov semigroup of linear operators.

2. Auxiliary Lemmas I. Let $\mu(dx)$ be a finite, continuous measure on the unit interval (i.e., $\mu(\{x\}) = 0$ for every x and $\mu([0,1]) < \infty$) and γ a constant such that $-1 < \gamma < 1$. We put

$$\hat{\mu}(x,y) = \frac{1}{y-x}\int_x^y \mu(dr) \quad \text{for} \quad 0 < x < y < 1.$$

Lemma 2.1. *Let $p \geq 0$. Then*

$$\iint\limits_{x<y} \frac{\mu(dx)\mu(dy)\hat{\mu}^p(x,y)}{(y-x)^\gamma} = \frac{(p+\gamma)(p+1+\gamma)}{(p+2)(p+1)} \iint\limits_{x<y} \frac{\hat{\mu}^{p+2}(x,y)dxdy}{(y-x)^\gamma}$$

$$+ \frac{p+\gamma}{(p+2)(p+1)} \int_0^1 \big\{\hat{\mu}^{p+2}(0,x) + \hat{\mu}^{p+2}(1-x,1)\big\}x^{1-\gamma}dx + \frac{(\int_0^1 \mu(dr))^{p+2}}{(p+2)(p+1)},$$

where $\iint_{x<y}$ means that the integration ranges over the triangle $0 < x < y < 1$. (The two sides may be infinite simultaneously.)

Proof. Suppose that μ is absolutely continuous and has a continuous density f, say. Let δ be a positive constant. Using the formula

$$f(x)f(y)\Big(\int_x^y \mu(dr)\Big)^p = -\frac{1}{(p+2)(p+1)}\frac{\partial^2}{\partial x \partial y}\Big(\int_x^y \mu(dr)\Big)^{p+2}$$

we carry out integration by parts to see

$$\begin{aligned}
&\iint\limits_{x<y} \frac{f(x)f(y)dxdy}{(y-x+\delta)^{p+\gamma}}\Big(\int_x^y f(r)dr\Big)^p \\
&= \frac{1}{(p+2)(p+1)}\int_0^1 \frac{dy}{(y+\delta)^{p+\gamma}}\frac{\partial}{\partial y}\Big(\int_0^y \mu(dr)\Big)^{p+2} \\
&\quad + \frac{p+\gamma}{(p+2)(p+1)}\iint\limits_{x<y} \frac{dxdy}{(y-x+\delta)^{p+1+\gamma}}\frac{\partial}{\partial y}\Big(\int_x^y \mu(dr)\Big)^{p+2}.
\end{aligned}$$

Applying integration-by-parts once more to each integral on the right-hand side we see that it equals

$$\begin{aligned}
&\frac{(\int_0^1 \mu(dr))^{p+2}}{(p+2)(p+1)(1+\delta)^{p+\gamma}} \\
&+ \frac{p+\gamma}{(p+2)(p+1)}\int_0^1 \Big\{\frac{(\int_0^x \mu(dr))^{p+2}}{(x+\delta)^{p+1+\gamma}} + \frac{(\int_x^1 \mu(dr))^{p+2}}{(1-x+\delta)^{p+1+\gamma}}\Big\}dx \\
&+ \frac{(p+\gamma)(p+1+\gamma)}{(p+2)(p+1)}\iint\limits_{x<y}\Big(\int_x^y \mu(dr)\Big)^{p+2}\frac{dxdy}{(y-x+\delta)^{p+2+\gamma}}.
\end{aligned}$$

These equalities are valid if μ is a finite, continuous measure, since μ is then a weak limit of smooth measures μ_ϵ for which $(y-x+\delta)^{-1}\int_x^y \mu_\epsilon(dr)$ uniformly converges to $(y-x+\delta)^{-1}\int_x^y \mu(dr)$ as $\epsilon \downarrow 0$. Finally the equality of the lemma is obtained by letting $\delta \downarrow 0$ (if $p+\gamma \geq 0$, each term appearing in the equality is non-negative and monotone relative to δ; if $p+\gamma<0$, both of the first two terms on the right-hand side are negative and the other terms are convergent). □

Put

$$\Lambda(\mu) = \iint_Q \frac{\mu(dx)\mu(dy)\hat{\mu}(x,y)}{|y-x|^\gamma}.$$

Lemma 2.2. *Let ν be another finite, continuous measure on $[0,1]$. Then for $\delta > 0$*

$$\iint_{x<y} \frac{\mu(dx)\mu(dy)}{(y-x+\delta)^{1+\gamma}} \int_x^y \nu(dr) \tag{2.1}$$

$$= \frac{1+\gamma}{2} \iint_{x<y} \frac{(\int_x^y \mu(dr))^2}{(y-x+\delta)^{2+\gamma}} \Big\{ \frac{(2+\gamma)\int_x^y \nu(dr)dxdy}{y-x+a} - \nu(dx)dy - dx\nu(dy) \Big\}$$

$$+ \frac{1+\gamma}{2} \int_0^1 \Big\{ \frac{(\int_0^x \mu(dr))^2 \int_0^x \nu(dr)}{(x+\delta)^{2+\gamma}} + \frac{(\int_x^1 \mu(dr))^2 \int_x^1 \nu(dr)}{(1-x+\delta)^{2+\gamma}} \Big\} dx$$

$$- \frac{1}{2} \int_0^1 \Big\{ \frac{(\int_0^x \mu(dr))^2 \nu(dx)}{(x+\delta)^{1+\gamma}} + \frac{(\int_x^1 \mu(dr))^2 \nu(dx)}{(1-x+\delta)^{1+\gamma}} \Big\}$$

$$+ \frac{(\int_0^1 \mu(dr))^2 \int_0^1 \nu(dr)}{2(1+\delta)^{1+\gamma}}.$$

The equality is valid also for $\delta = 0$ with all the terms appearing therein finite, provided that one of the following conditions (a) *or* (b) *holds*

(a) $\iint_{x<y} \mu(dx)\mu(dy)(y-x)^{-\gamma} < \infty$ *and ν has a bounded density;*
(b) $\Lambda(\mu) < \infty$ *and* $\Lambda(\nu) < \infty$.

Proof. When $\delta > 0$ we may suppose that both μ and ν have continuous densities: otherwise we have only to employ the approximation method as in the proof of the previous lemma. Let f be the density of μ. Then, with the help of the formula

$$f(x)f(y) = -\frac{1}{2}(\partial^2/\partial x \partial y)\Big(\int_x^y f(r)dr\Big)^2,$$

an integration by parts turns the left-hand side of (2.1) into

$$
\begin{aligned}
&-\frac{1}{2}\int_0^1 \frac{(\partial/\partial x)(\int_x^1 f(r)dr)^2 \int_x^1 \nu(dr)}{(1-x+\delta)^{1+\gamma}}dx \\
&+\frac{1}{2}\iint\limits_{x<y} \frac{(\partial/\partial x)(\int_x^y f(r)dr)^2}{(y-x+\delta)^{1+\gamma}}\nu(dy)dx \\
&-\frac{1+\gamma}{2}\iint\limits_{x<y} \frac{(\partial/\partial x)(\int_x^y f(r)dr)^2 \int_x^y \nu(dr)}{(y-x+\delta)^{2+\gamma}}dxdy.
\end{aligned}
$$

Integrating by parts each term once more we get the right-hand side of (2.1).

Let $\Lambda(\mu) < \infty$ and $\Lambda(\nu) < \infty$. Then by Lemma 2.1 (with $p = 1$) we have

$$
\iint\limits_{x<y} \frac{\hat{\mu}^3(x,y)dxdy}{(y-x)^\gamma} < \infty
$$

and

$$
\int_0^1 \Big\{\frac{(\int_0^x \mu(dr))^3}{x^{2+\gamma}} + \frac{(\int_x^1 \mu(dr))^3}{(1-x)^{2+\gamma}}\Big\}dx < \infty,
$$

and similar bounds for ν. These together with Hölder's inequality show that all the positive terms on the right-hand side of (2.1) converge to finite limits as $\delta \downarrow 0$; hence also the negative terms and the left-hand side do as well. The case (i) is obvious. $\square$

Lemma 2.3. *Let μ and ν be as in Lemma 2.2. Then*

$$
\begin{aligned}
&\iint\limits_{x<y} \frac{\mu(dx)\nu(dy)}{(y-x)^{1+\gamma}} \int_x^y \nu(dr) \\
&\quad = \frac{1+\gamma}{2}\iint\limits_{x<y} \frac{\hat{\nu}^2(x,y)\mu(dx)dy}{(y-x)^\gamma} + \frac{1}{2}\int_0^1 \frac{(\int_x^1 \nu(dy))^2\mu(dx)}{(1-x)^{2+\gamma}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\iint\limits_{x<y} \frac{\nu(dx)\mu(dy)}{(y-x)^{1+\gamma}} \int_x^y \nu(dr) \\
&\quad = \frac{1+\gamma}{2}\iint\limits_{x<y} \frac{\hat{\nu}^2(x,y)dx\mu(dy)}{(y-x)^\gamma} + \frac{1}{2}\int_0^1 \frac{(\int_0^x \nu(dy))^2\mu(dx)}{x^{2+\gamma}}.
\end{aligned}
$$

If $\Lambda(\mu) < \infty$ and $\Lambda(\nu) < \infty$, then all the terms appearing above are finite.

Proof. The equalities are obtained as in Lemma 2.1 (by a much simpler computation). The second half follows from that of Lemma 2.2. □

3. Auxiliary Lemmas II. For the proof of Theorem 1 we prepare two more lemmas. Let $\mu(dx)$, $\hat{\mu}(x,y)$ and γ be as in the previous section. If μ is given in the form $f(x)dx$, we simply write $\hat{f}$ for $\hat{\mu}$:

$$\hat{f}(x,y) = \frac{1}{y-x}\int_x^y f(r)dr \qquad (0 < x < y < 1).$$

Let $p_t(x,y), 0 \le x,y \le 1, t > 0$, be the fundamental solution for the heat equation $(\partial/\partial dt)u = (\partial^2/\partial x^2)u$ on $[0,1]$ with the reflecting boundary condition, or explicitly,

$$p_\tau(x,y) = \sum_{n=-\infty}^{\infty}[g_\tau(x-y+2n) + g_\tau(x+y+2n)],$$

where

$$g_\tau(x) = \frac{1}{\sqrt{4\pi\tau}}\exp\bigl(-\frac{x^2}{4\tau}\bigr).$$

Let P_τ be the associated semi-group of operators:

$$P_\tau\mu(x) = \int_0^1 p_\tau(x,y)\mu(dy).$$

Lemma 3.1. *There exists a constant C such that for $0 \le a \le 1$*

$$\iint_Q \frac{[\widehat{P_a\mu}(x,y)]^3 dxdy}{|y-x|^\gamma} \le C\Lambda(\mu) \tag{i}$$

and

$$\int_0^1 [(\widehat{P_a\mu}(0,x))^3 + (\widehat{P_a\mu}(1-x,1))^3]x^{1-\gamma}dx \le C\Lambda(\mu). \tag{ii}$$

Proof. Choose a constant M such that $p_a(x,y) \le Mg_a(y-x)$ for $0 < a$, $x,y \le 1$ and put

$$F(x) = \mu([0,(x\vee 0)\wedge 1]),$$

where $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. F is continuous and non-decreasing. By Fubini's theorem and Hölder's inequality we obtain

$$\begin{aligned}\Big(\int_x^y P_a\mu(r)dr\Big)^3 &\le \Big(-M\int_x^y dr\int_{-\infty}^{\infty} g_a(s)d_sF(r-s)\Big)^3\\ &= \Big(M\int_{-\infty}^{\infty}\int_x^y d_rF(r-s)g_a(s)ds\Big)^3\\ &\le \int_{-\infty}^{\infty}\Big(M\int_x^y d_rF(r-s)\Big)^3 g_a(s)ds.\end{aligned}$$

(The equality above may be verified by approximating F with absolutely continuous functions.) By substituting this bound and introducing new variables u and λ of integration according to $x = u - \lambda$, $y = u + \lambda$, the double integral in (i) is dominated as follows:

$$\begin{aligned}&\iint_Q \frac{[\widehat{P_a\mu}(x,y)]^3dxdy}{|y-x|^\gamma}\\ &\le M^3 2^{1-\gamma}\int_{-1/2}^{1/2}\frac{d\lambda}{|\lambda|^\gamma}\int_{|\lambda|}^{1-|\lambda|}du\int_{-\infty}^{\infty}\Big(\frac{1}{2|\lambda|}\int_{-|\lambda|}^{|\lambda|}d_tF(u-s+t)\Big)^3 g_a(s)ds\\ &= C2^{1-\gamma}\int_{-1/2}^{1/2}\frac{d\lambda}{|\lambda|^\gamma}\int_{|\lambda|}^{1-|\lambda|}du\int_{-\infty}^{\infty}\Big(\frac{1}{2|\lambda|}\int_{-|\lambda|}^{|\lambda|}d_tF(y+t)\Big)^3 g_a(u-y)dy\\ &\le C2^{1-\gamma}\int_{-1/2}^{1/2}\frac{d\lambda}{|\lambda|^\gamma}\int_{-\infty}^{\infty}\Big(\frac{1}{2|\lambda|}\int_{-|\lambda|}^{|\lambda|}d_tF(y+t)\Big)^3 dy\\ &= C\iint_Q\frac{\hat\mu^3(x,y)dxdy}{|y-x|^\gamma} + C2^{2-\gamma}\int_0^{1/2}D(\lambda)\frac{d\lambda}{|\lambda|^\gamma},\end{aligned}$$

where

$$D(\lambda) = \int_{-\lambda}^{\lambda}\Big(\frac{1}{2\lambda}\int_{-\lambda}^{\lambda}d_tF(y+t)\Big)^3dy + \int_{1-\lambda}^{1+\lambda}\Big(\frac{1}{2\lambda}\int_{-\lambda}^{\lambda}d_tF(y+t)\Big)^3dy.$$

Now, observing that

$$D(\lambda) \le 2\lambda\hat\mu^3(0,2\lambda) + 2\lambda\hat\mu^3(1-2\lambda,1),$$

we have only to apply Lemma 2.1 (with $p = 1$) to finish the proof of (i).

The proof of (ii) is similar. As above we have for $0 < x < 1$

$$\Big(\int_0^x P_a\mu(r)dr\Big)^3 \le \int_{-1}^x \Big(M\int_0^x d_rF(r-s)\Big)^3 g_a(s)ds$$
$$\le 4\Big(M\int_0^{2x}\mu(dr)\Big)^3 + 4\int_x^1 \Big(M\int_s^{s+x}\mu(dr)\Big)^3 g_a(s)ds.$$

Since $xg_a(s) \le (e\sqrt{\pi})^{-1}$ if $s \ge x$, this yields

$$\int_0^{1/2} (\widehat{P_a\mu}(0,x))^3 x^{1-\gamma}dx$$
$$\le 2^{3+\gamma}M^3\int_0^1 (\hat\mu(0,x))^3x^{1-\gamma}dx + 4M^3\int_0^{1/2}dx\int_x^1(\hat\mu(s,s+x))^3x^{-\gamma}ds.$$

We have an analogous bound for $\int_0^{1/2}(\widehat{P_a\mu}(1-x,1))^3(1-x)^{-\gamma}dx$. In view of Lemma 2.1 these imply (ii). $\square$

Remark. The proof of Lemma 3.1 can be readily modified to show that if

$$\Lambda_\epsilon(\mu) := \iint_{|y-x|<\epsilon}\frac{\hat\mu^3(x,y)dxdy}{|y-x|^\gamma} + \int_0^\epsilon [\hat\mu^3(0,x)+\hat\mu^3(1-x,1)]x^{1-\gamma}dx,$$

then $\Lambda_\epsilon(P_a\mu) \le C\Lambda_\epsilon(\mu)$ for $0 \le \epsilon, a < 1$. This proves e.g. that $\Lambda(P_a\mu) \to \Lambda(\mu)$ as $a \downarrow 0$. Such results, though sometimes useful, are not used in this paper.

Put

$$\Gamma(\nu|\mu) = \iint_Q \frac{\mu(dx)\mu(dy)\hat\nu(x,y)}{|y-x|^\gamma},$$

where $\nu(dx)$ is another measure on $[0,1]$.

Lemma 3.2. *Let μ and ν be two finite continuous measures on $[0,1]$. Then for $0 \le a \le 1$*

$$\Gamma(P_a\nu|\mu) \le C(\Lambda(\mu)+\Lambda(\nu)) \tag{3.1}$$

for some constant C independent of μ, ν and a. Moreover if $\Lambda(\mu) < \infty$ and $\Lambda(\nu) < \infty$, then

$$\lim_{a\downarrow 0}\Gamma(P_a\nu|\mu) = \Gamma(\nu|\mu).$$

Proof. First we prove the second half of the lemma. By using Lemma 2.1, observe that the premise of the claim implies $\iint_Q \hat{\mu}^3(x,y)|y-x|^{-\gamma}dxdy < \infty$ and the corresponding relation for ν. Then, applying Hölder's inequality and Lemma 3.1 (i) in turn, we get

$$\lim_{\epsilon\downarrow 0} \sup_{0<a<1} \iint\limits_{|y-x|<\epsilon} \frac{\hat{\mu}^2(x,y)\widehat{P_a\nu}(x,y)}{|y-x|^\gamma} dxdy = 0. \tag{3.2}$$

We now apply Lemma 2.2 with $\delta = 0$ and with $P_a\nu(x)dx$ in place of $\nu(dx)$ to compute $\Gamma(P_a\nu|\mu)$. We examine the right-hand side of the equality in Lemma 2.2 term by term. By (3.2) and Fatou's lemma we get the following upper bound for the first term:

$$\begin{aligned}
&\limsup_{a\downarrow 0} \frac{1+\gamma}{2} \iint\limits_{x<y} \frac{\hat{\mu}^2(x,y)dxdy}{(y-x)^\gamma} \big\{(2+\gamma)\widehat{P_a\nu}(x,y) - P_a\nu(x) - P_a\nu(y)\big\} \\
&\le \frac{1+\gamma}{2} \iint\limits_{x<y} \frac{\hat{\mu}^2(x,y)}{(y-x)^\gamma} \big\{(2+\gamma)\hat{\nu}(x,y)dxdy - \nu(dx)dy - dx\nu(dy)\big\}.
\end{aligned}$$

We similarly prove, using Lemma 3.1 (ii) instead of (i), that the limit suprema of the other terms are bounded by the respective values of them at $a = 0$. Thus

$$\limsup_{a\downarrow 0} \Gamma(P_a\nu|\mu) \le \Gamma(\nu|\mu).$$

The inequality in the other direction is immediate from Fatou's lemma. The inequality (3.1) follows from Lemma 2.2 (the second half), Lemma 3.1 and Hölder's inequality. □

4. Proof of uniqueness. Let μ_t be a solution of the equation (1.5) satisfying (1.6). Then the total mass $\mu_t([0,1])$ is constant, $\mu_t(J)$ is absolutely continuous, and (1.1) holds a.e. Let p_τ and P_τ be as in Section 3. Let $0 < a < b$ and take with z fixed

$$J = \int_a^b p_\tau(\cdot, z)d\tau$$

as a testing function in (1.1). Then, integrating both sides of

$$(\partial/\partial\tau)p_\tau(u,z) = (\partial^2/\partial u^2)p_\tau(u,z)$$

on the square $a < \tau < b, x < u < y$ we obtain

$$J'(y) - J'(x) = \int_a^b [p'_\tau(y,z) - p'_\tau(x,z)]d\tau = \int_x^y [p_b(u,z) - p_a(u,z)]du,$$

where p'_τ denotes the partial derivative with respect to the first argument; hence for almost all $t \in [0,T]$

$$\begin{aligned} &\frac{\partial}{\partial t}\int_a^b P_\tau \mu_t(z)d\tau \\ &= \frac{1}{2}\iint_Q \frac{\mu_t(dx)\mu_t(dy)}{|y-x|^\gamma}\frac{1}{y-x}\int_x^y [p_b(u,z) - p_a(u,z)]du. \end{aligned} \tag{4.1}$$

Formal differentiation of $\int_a^b \mu_t(P_\tau \mu_t)d\tau$ and application of the identity (4.1) leads to

$$\int_a^b \big[\mu_T(P_\tau \mu_T) - \mu_\delta(P_\tau \mu_\delta)\big]d\tau = \int_\delta^T [\Gamma(P_b\mu_t|\mu_t) - \Gamma(P_a\mu_t|\mu_t)]dt, \tag{4.2}$$

where $0 < \delta < T$. ($\Gamma(\nu|\mu)$ is defined by (3.1).) Notice that

$$\sup_t \int_0^1 \mu_t(P_\tau \mu_t)d\tau < \infty.$$

Because of the hypothesis (1.6) the integral on the right-hand side is well defined and

$$\int_\delta^T \Gamma(P_b\mu_t|\mu_t)dt \le \|p_b\|_\infty \int_\delta^T dt \int |y-x|^{-\gamma}\mu_t(dx)\mu_t(dy) < \infty.$$

Taking (4.2) for granted we let $a \downarrow 0$ in it. Noticing that $\int_\delta^T \Gamma(P_a\mu_t|\mu_t)dt$ is bounded since the left-hand side is bounded, we apply Fatou's lemma to see that μ_t is a continuous measure for each t outside a Lebesgue null set even if $\gamma = 0$ and

$$\int_\delta^T \Lambda(\mu_t)dt \le A_1 + A_2\int_\delta^T dt \iint_Q \frac{\mu_t(dx)\mu_t(dy)}{|y-x|^\gamma},$$

where constants A_1 and A_2 are depending only on $\mu_0([0,1])$. ($\Lambda(\mu)$ is defined in Section 2.) If $\gamma > 0$, then the inner double integral on the right-hand side is bounded by $C_\gamma[\Lambda(\mu_t)]^{2/3}$ as is seen on using Lemma 2.1 with $p = 0$ and 1. Since δ may be arbitrarily small, we conclude the bound (1.7), namely

$$\int_0^T dt \iint_Q \frac{\mu_t(dx)\mu_t(dy)}{|y-x|^\gamma} < \infty \quad \text{and} \quad \int_0^T \Lambda(\mu_t)dt < \infty. \tag{4.3}$$

The identity (4.2) may be proved as follows. Let $\varphi(t)$ be a nonnegative smooth function such that $\varphi = 0$ outside $[-1,1]$ and $\int \varphi dt = 1$ and define $\varphi_\epsilon(s) = \varphi(s/\epsilon)/\epsilon$ and $\mu_t^{(\epsilon)}(dx) = \int_{-1}^1 \mu_{t-\epsilon s}(dx)\varphi(s)ds$ if $t \ge \delta > \epsilon$. Then we deduce from (4.1)

$$\begin{aligned}\frac{\partial}{\partial t}\int_a^b \mu_t^{(\epsilon)}(P_\tau \mu_t^{(\epsilon)})d\tau &= 2\int_{-\infty}^{\infty}\Big[\frac{\partial}{\partial s}\int_a^b \mu_t^{(\epsilon)}(P_\tau\mu_s)d\tau\Big]\varphi_\epsilon(t-s)ds \\ &= \int_{-\infty}^{\infty}\Gamma([P_b - P_a]\mu_t^{(\epsilon)}|\mu_s)\varphi_\epsilon(t-s)ds,\end{aligned}$$

where $[P_b - P_a]\mu = P_b\mu - P_a\mu$. Integrating both sides with respect to $t \in (\delta, T)$ and making a change of variables in the integration on the right-hand side we obtain

$$\begin{aligned}&\int_a^b \mu_t^{(\epsilon)}(P_\tau\mu_t^{(\epsilon)})d\tau\Big|_{t=\delta}^T \\ &= \int_{-1}^1 \Big[\int_{\delta-\epsilon s}^{T-\epsilon s} dt \int_{-1}^1 \Gamma([P_b - P_a]\mu_{t+\epsilon(s-r)}|\mu_t)\varphi(r)dr\Big]\varphi(s)ds.\end{aligned}$$

Clearly for $\tau = a, b$

$$\iint drds\frac{\varphi(r)\varphi(s)}{y-x}\int_x^y P_\tau\mu_{t+\epsilon(s-r)}(z)dz \longrightarrow \frac{1}{y-x}\int_x^y P_\tau\mu_t(z)dz \quad \text{as} \quad \epsilon \downarrow 0$$

with the left-hand side uniformly bounded in t, x, y $(x \ne y, t > 0)$. In view of the integrability condition (1.6) the bounded convergence theorem therefore yields (4.2).

Now let μ_t and ν_t be two solutions starting at the same initial measure. We put $\omega_t = \mu_t - \nu_t$. Then the same argument as leading to (4.2) shows that if $0 < \delta < s \le T$

$$\int_a^b \omega_s(P_\tau\omega_s)d\tau - \int_a^b \omega_\delta(P_\tau\omega_\delta)d\tau = \int_\delta^s I_b(t)dt - \int_\delta^s I_a(t)dt, \tag{4.4}$$

where we put

$$I_\tau(t) = \iint\limits_{y>x} \frac{\mu_t(dx)\mu_t(dy) - \nu_t(dx)\nu_t(dy)}{|y-x|^\gamma} \cdot \frac{1}{y-x}\int_x^y P_\tau\omega_t(z)dz.$$

By the semigroup property $P_\tau = P_{\tau/2}P_{\tau/2}$ together with the symmetry of p_τ the quadratic form $\omega(P_\tau\omega)$ is non-negative. Since $\int_a^b p_\tau(x,y)d\tau$ is continuous in $(x,y) \in Q$ for each $b > a > 0$, the initial condition therefore implies

$$\int_a^b \omega_\delta(P_\tau\omega_\delta)d\tau \to 0 \quad \text{as} \quad \delta \downarrow 0.$$

We let $\delta \downarrow 0$, $b \to \infty$ and $a \downarrow 0$ in (4.4) in this order. Then the term $\int_\delta^s I_b(t)dt$ disappears from the right-hand side of (4.4) as verified by using the first relation of (4.3) and the fact that $P_\tau\omega_t(z)$ is bounded uniformly for $\tau > 1$ and for each z and t

$$P_\tau\omega_t(z) \to \mu_t([0,1]) - \nu_t([0,1]) = 0 \quad \text{as} \quad \tau \to \infty. \tag{4.5}$$

In view of Lemma 3.2 and (4.3) the dominated convergence theorem verifies that $\int_0^s I_a(t)dt$ converges to $\int_0^s I_0(t)dt$ as $a \downarrow 0$ (P_0 is understood to be the identity operator). Combining these facts we now deduce from (4.4) the following inequality

$$2\int_0^\infty d\tau \int_0^1 (P_\tau\omega_s(x))^2 dx \le -\int_0^s I_0(t)dt. \tag{4.6}$$

We write I_0 in the form

$$I_0(t) = \iint\limits_{y>x} \frac{\mu_t(dx)\omega_t(dy) + \omega_t(dx)\nu_t(dy)}{|y-x|^\gamma} \cdot \frac{1}{y-x}\int_x^y \omega_t(dz).$$

We further rewrite it by applying Lemma 2.3 with ω_t in place of ν. That ω_t is not necessarily non-negative causes no problem since $\Lambda(\mu_t + \nu_t) < \infty$ a.e. Observe that all the terms that arise thereupon are non-negative; hence $I_0(t) \ge 0$. By (4.6) we therefore conclude that $P_\tau\omega_s(x)$ vanishes a.e. $(\tau, x) \in [0,\infty)\times[0,1]$. Thus μ_s is identical to ν_s. The proof of Theorem 1 is complete.

5. The stationary solution and the convergence to it. In this section we are concerned with stationary solutions of (1.1). Let μ be a finite continuous measure on $[0,1]$ and suppose that $\iint_Q |x-y|^{-\gamma} mu(dx)\mu(dy) < \infty$ if $\gamma > 0$. Put

$$\mathcal{A}(J;\mu) = \frac{1}{2}\iint_Q \frac{J(y)-J(x)}{y-x} \cdot \frac{\mu(dx)\mu(dy)}{|y-x|^\gamma},$$

so that the equation (1.1) may be written as $\frac{d}{dt}\mu_t(J) = \mathcal{A}(J';\mu_t)$. Clearly μ is then a stationary solution of (1.1) if and only if

$$\int_0^1 \frac{\mu(dy)}{|y-x|^\gamma(y-x)} = 0 \quad \text{for} \quad x \in \mathrm{supp}(\mu) \tag{5.1}$$

in the sense that

$$\mathcal{A}(J;\mu) = 0 \quad \text{for all } J \in C^\infty([0,1]) \text{ satisfying } \ J(0) = J(1) = 0. \tag{5.2}$$

Recall that $W_\gamma(x) = C_\gamma[x(1-x)]^{-\lambda}$ where $\lambda := \frac{1}{2}(1-\gamma)$ and the constant C_γ is chosen so that $\int_0^1 W_\gamma dx = 1$. Put $\mu^W(dx) := W_\gamma(x)dx$. The first half of Theorem 2 may read as follows.

Theorem 2a. *The measure μ^W is a solution of* (5.2), *which is unique among continuous measures of unit total mass, satisfying*

$$\iint_Q |x-y|^{-\gamma}\mu(dx)\mu(dy) < \infty \quad \textit{if} \ \ \gamma > 0.$$

Proof. According to Erdélyi [6]

$$\int_0^{1/2} [\tfrac{1}{4} - x^2]^{-\lambda} \cos tx\, dx = \tfrac{1}{2}\sqrt{\pi}\,\Gamma(\tfrac{1}{2}(1+\gamma))t^{-\gamma/2}J_{\gamma/2}(t/2)$$

($J_{\gamma/2}$ is the Bessel function of the first kind of order $\gamma/2$) and

$$\int_0^\infty t^{\gamma/2}J_{\gamma/2}(t/2)\sin yt\, dt = \begin{cases} 0 & \text{if} \quad |y| < 1/2 \\ \sqrt{\pi}[\Gamma(\lambda)]^{-1}(y^2-\frac{1}{4})^{-(\gamma+1)/2} & \text{if} \quad |y| > 1/2, \end{cases}$$

from which we deduce that μ^W satisfies (5.1) for all $x \in (0,1)$ if the integral on the right-hand side of (5.1) is regarded as the Cauchy principal value, in particular it is a solution of (5.2). (cf. [7; pages 296, 325, 342] for a different approach.)

For the proof of the uniqueness of the solution let μ be a continuous measure of unit mass satisfying (5.2) and

$$\iint_Q |x-y|^{-\gamma}\mu(dx)\mu(dy) < \infty,$$

and put $\omega = \mu - \mu^W$. Since both μ and μ^W are stationary solutions of (1.1) we have the relation (4.4) with $\omega_t = \omega$. Proceeding as before and applying Lemma 2.3 we infer that

$$\int_0^1 x^{-(\gamma+2)}\Big(\int_0^x \omega(dy)\Big)^2(\mu+\mu^W)(dx) = 0,$$

showing $\omega = 0$ as required. □

Theorem 2b. *Every solution* μ_t *of* (1.5) *and* (1.2) *that satisfies the uniqueness condition* (1.6) *weakly converges to* $M\mu^W$ *as* $t \to \infty$, *where* $M = \mu_o([0,1])$.

Proof. We give only an outline of the proof since the essential ingredients are found in [13], where the equation is considered on the whole real line. Put

$$V(x) = \begin{cases} \dfrac{1}{\gamma|x|^\gamma} & \text{if} \quad \gamma \neq 0 \\ -\log|x| & \text{if} \quad \gamma = 0, \end{cases}$$

and for a finite, continuous measure μ on $[0,1]$ define

$$\mathcal{E}(\mu) = \iint_Q V(y-x)\mu(dx)\mu(dy)$$

and

$$h_\mu(x) = \int_0^1 \frac{\mu(dy)}{|y-x|^\gamma(y-x)}.$$

For the latter we shall suppose that for some constant $C = C_\mu$, $|\mathcal{A}(J;\mu)| \le C\sqrt{\mu(J^2)}$ for all $J \in L^2(\mu)$, so that h_μ is well defined as an element of $L^2(\mu)$ such that $\mu([h_\mu]^2) \le C$. By employing the construction of solutions of (1.1) from large dynamical systems as given by (6.II-4) in the next section we can show that for each constant $\delta > 0$ there exists a constant $C = C_\delta$ such that

$$\mathcal{E}(\mu_t) < C \quad \text{and} \quad |\mathcal{A}(J;\mu_t)| \le C\sqrt{\mu_t(J^2)} \quad (J \in L^2(\mu_t)) \quad \text{for} \quad t > \delta,$$

and that $\mathcal{E}(\mu_t)$ is a non-increasing function of t and

$$\mathcal{E}(\mu_s) - \mathcal{E}(\mu_t) \ge \int_s^t \mu_\tau([h_{\mu_\tau}]^2)d\tau \quad \text{for} \quad 0 < s < t \tag{5.3}$$

(see [13]). Now let μ_∞ be a limit point of μ_t as $t \to \infty$. Since the solution of (1.1) changes continuously as a functional of the initial measure (cf. [12, 13]), $\mathcal{E}(\cdot)$ must be constant along the solution starting from μ_∞. But in view of (5.3) and Theorem 2a this implies that μ_∞ coincides with $M\mu^W$. $\square$

6. Appendices. In this section we give miscellaneous results without proof.

(I) Extension to a class of general kernels $K(y-x)$. Let $K(x)$ be an even, non-negative, smooth function on $\mathbf{R} \setminus \{0\}$ satisfying

$$\int_{-1}^{1} K(x)dx < \infty;$$

$$\frac{d}{dx}\frac{K(x)}{x} < 0 \quad \text{for} \quad x > 0; \quad \text{and} \tag{6.I-1}$$

$$x\frac{d^2}{dx^2}\frac{K(x)}{x} > \frac{1}{C}|K''(x)| \quad \text{for} \quad 0 < x < r \tag{6.I-2}$$

with some positive numbers C and r. Then our proof of uniqueness works for the equation (1.5) with the kernel $K(y-x)$ replacing $1/|y-x|^\gamma$, i.e., for the equation

$$\mu_{t_2}(J) - \mu_{t_1}(J) = \frac{1}{2}\int_{t_1}^{t_2} dt \iint_Q \frac{J'(y) - J'(x)}{y-x}\, K(y-x)\mu_t(dx)\mu_t(dy).$$

If the condition in (6.I-2) is strengthened by removing the restriction $x < r$ therein the same arguments as given in Section 4 prove with minor changes the corresponding uniqueness theorem, e.g. in places of the integrals

$$\iint_{y>x} |y-x|^{-\gamma}\hat{\mu}^p(x,y)\,dx\,dy \quad (p = 2,3)$$

appearing in Lemmas 2.1 and 2.2 there arise the following expressions

$$\iint_{y>x} K_0(y-x)\hat{\mu}^2(x,y)dxdy \quad (\text{if} \quad p=0), \quad \text{where} \quad K_0(x)=x^2K''(x)$$

$$\iint_{y>x} K_1(y-x)\hat{\mu}^3(x,y)dxdy \quad (\text{if} \quad p=1), \quad \text{where} \quad K_1(x)=x^3\frac{d^2}{dx^2}\frac{K(x)}{x}$$

and similarly for integrals in Lemmas 2.2 and 2.3

$$\iint_{y>x} K_2(y-x)\hat{\mu}^2(x,y)\nu(dx)dy, \quad \text{where} \quad K_2(x)=x^2\frac{d}{dx}\frac{K(x)}{x}$$

and analogous integrals for boundary terms. We need the condition (6.I-2) in order to estimate

$$\iint_{y>x} K_0(y-x)\hat{\mu}^2(x,y)\,dx\,dy$$

by means of

$$\iint_{y>x} K_1(y-x)\hat{\mu}^3(x,y)\,dx\,dy.$$

With the restriction $x<r$ in (6.I-2) this does not necessarily follow straightforwardly. But the singularity is only along the diagonal $x=y$, so that the contribution of an off-diagonal part is uniformly bounded by a constant, which fact is enough for the relevant part of our proof of uniqueness. In the present approach the condition (6.I-1) seems difficult to weaken (except for the strict inequality) since it is owing to it that there arise only non-negative terms in the equalities analogous to those of Lemma 2.3, which fact is applied to prove $I_0(t)\geq 0$ in the final step of the proof given in Section 4.

(II) Scaling limit of a system of interacting particles. The integro-differential equation (1.1) can be derived from a system of N particles moving on a segment $(0,N)$ according to the classical law of motion by passing to the limit as the particle number N goes to infinity. In the system each particle is subject to the resistance equal to its velocity, interacts with the other particles by pair potential forces and is repelled with potential forces exerted by the two 'walls' at 0 and N. The equation of motion for the system is written as follows: for $i = 1, ..., N$,

$$\ddot{q}_i(t) = -\dot{q}_i(t) - \sum_{j \neq i} U'(q_i(t) - q_j(t)) - W'(q_i(t)) - W'(q_i(t) - N), \quad (6.\text{II-1})$$

where $q_i(t)$ denotes the location of the i-th particle at time $t \geq 0$, $\dot{q}, \ddot{q}$ the time derivatives and U', W' the derivatives of functions U, W. The pair potential function U, defined on $\mathbf{R} \setminus \{0\}$, is supposed to be continuously differentiable and even ($U(x) = U(-x)$) and to satisfy that $-xU'(x) \geq 0$ (the pair interaction works repulsively) and $-xU'(x)$ is positive and non-increasing on a small interval $(0, \delta)$; W is a wall potential and set so that particles are kept away from the two end points 0 and N.

The integro-differential equation (1.1) arises in the case when the potential has a long tail:

$$-xU'(x) \sim \frac{1}{|x|^\gamma} \quad \text{as} \quad |x| \to \infty \qquad (6.\text{II-2})$$

where γ is a constant such that $-1 < \gamma < 1$ and $\sim$ means that the ratio of two sides of it approaches to unity. Introducing the macroscopic position variables

$$x_i(t) = N^{-1} q_i(\lambda_N t) \quad \text{where} \quad \lambda_N = N^{1+\gamma} \quad (-1 < \gamma < 1), \qquad (6.\text{II-3})$$

one defines the normalized counting measure

$$\alpha_t^N(d\theta) = N^{-1} \sum_{i=1}^{N} \delta_{x_i(t)}(d\theta), \quad \theta \in (0, 1).$$

It is demonstrated by Uchiyama [12] that if (6.II-2) holds with $0 < \gamma < 1$ (the extension to the case $-1 < \gamma \leq 0$ is easy), then under a certain mild condition on the initial phases $(q_i(0), \dot{q}_i(0))_{i=1}^N$ every (weak) limit point of α_t^N as $N \to \infty$ satisfies (1.5); this combined with our uniqueness theorem implies that if $\alpha_0^N \to \mu_o$ in addition, then α_t^N converges to the solution of (1.5) satisfying the initial condition (1.2).

If U has a short range in the sense that $\int_1^\infty |xU'(x)| dx < \infty$ the numbers λ_N is taken to be N^2 (diffusion scaling) and if U is further supposed strictly convex then the limit equation is shown to be the non-linear diffusion equation (1.8), where $P(u) = -\sum_{k=1}^\infty kU'(k/u)$ for $u > 0$ and $P(0) = 0$ (see

[10]). In [12] it is also proved that if $\gamma = 1$ in (6.II-2) and $\lambda_N = N^2/\log N$ in (6.II-3) the measure α_t^N converges to a solution of the non-linear diffusion equation $\frac{\partial}{\partial t}\rho(\theta, t) = \frac{\partial^2}{\partial \theta^2}\rho^2(\theta, t)$ that satisfies the boundary condition $(\partial/\partial\theta)\rho(0, t) = (\partial/\partial)\rho(1, t) = 0$.

The analogous convergence result can be obtained for the system of first order differential equations

$$\dot{q}_i(t) = -\sum_{j \neq i} U'(q_i(t) - q_j(t)) - W'(q_i(t)) - W'(q_i(t) - N). \qquad (6.\ \text{II-4})$$

It is emphasized that the macroscopic equation (1.1) is determined only by the tail behavior of the pair potential as in (6.II-2), not depending on the details of it.

(III) Uniqueness for $u_t = \mathcal{L}f(u)$. Apart from the idea of introducing the transform $\mu(dx) \mapsto \hat{\mu}(x, y)$ (and accordingly the lemmas (except Lemma 3.2) in Sections 2 and 3) our uniqueness proof is similar to that given in [9] for the equation $u_t = \Delta f(u)$. The method is easily adapted for the equation of the form

$$\frac{\partial}{\partial t}u = \mathcal{L}f(u) \qquad (6.\text{III-1})$$

where $\mathcal{L}$ is the infinitesimal generator of a Markov semi-group of linear operators subject to certain mild conditions as described below. Uniqueness results for such equations are useful for obtaining scaling limits of various models (e.g. particle systems on fractals). Let $(X, \mathcal{F})$ be a measurable space and $m = m(dx)$ a $\sigma-$finite measure on $(X, \mathcal{F})$. Let $P_\tau, \tau \geq 0$ be a strongly continuous semigroup on $L^1(m)$ and $\mathcal{L}$ its infinitesimal generator in the sense of Hille-Yosida. We denote by $\mathcal{D}(\mathcal{L})$ the domain of $\mathcal{L}$. Suppose that there exists a non-negative kernel $p_\tau(x, y)$ which is measurable relative to $(\tau, x, y) \in (0, \infty) \times X \times X$ and satisfies the following conditions

(1) For each $\tau > 0$, $p_\tau(x, y)$ is bounded and symmetric in (x, y), and for $\varphi \in L^1(m)$

$$P_\tau\varphi(x) = \int_X p_\tau(x, y)\varphi(y)m(dy) \quad \text{and} \quad \int_X p_\tau(x, y)m(dy) \leq 1.$$

(2) Either

$$\lim_{\tau \to \infty} P_\tau\varphi(x) = 0$$

for each $\varphi \in L^1(m)$ and $x \in X$ or

$$\lim_{\tau \to \infty} P_\tau \varphi(x) = m(\varphi)$$

for each such φ and x. Here $m(\varphi) = \int_X \varphi dm$.

In the second case of (2) it follows that P_τ is conservative ($P_\tau 1 \equiv 1$) and $m(X) < \infty$.

Theorem. *Suppose that f is non-increasing and $f(0) = 0$. If μ is a non-negative finite measure on $(X, \mathcal{F})$ satisfying*

$$\iint_{X \times X} \mu(dx)\mu(dy) \int_0^1 d\tau p_\tau(x,y) < \infty,$$

then there is at most one non-negative solution $u(x,t)$ to the non-linear problem

$$m(u(t)J) - \mu(J) = \int_0^t m(f(u(s))\mathcal{L}J)ds \qquad 0 < t \leq T$$
$$\textit{valid for} \quad J \in \{\textstyle\int_a^b d\tau P_\tau \varphi : \ \varphi \in L^1(m), 0 < a < b < \infty\}$$

such that

$$\sup_{0 \leq t \leq T} m(u(t)) < \infty$$

and

$$\int_0^T m(f(u(t)))dt < \infty.$$

When $X = \mathbf{R}^N$ and $\mathcal{L} = \Delta$, the equation (6.III-1) has been extensively studied (see [2], [14] for surveys and further references to the literature). Among others it is proved that a weak non-negative solution to (6.III-1) in a domain $G \subset R^{N+1}$ has always a continuous modification ([1], [5]), provided that $1 + C^{-1} < uf'(u)/f(u) < C$ (a.e. $u > 0$) for some positive constant C and that such continuity property of a solution implies the uniqueness of a solution to the corresponding initial value problem ([4]); especially the uniqueness in the class of functions that are not necessarily integrable has been established if $f(u) = \text{const}\, u^\kappa$ with $\kappa > 1$. Unfortunately our method is not (at least directly) applicable to solutions $u(t)$ with $m(u(t)) = \infty$.

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