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GRADIENT BLOW-UP FOR MULTIDIMENSIONAL NONLINEAR PARABOLIC EQUATIONS WITH GENERAL BOUNDARY CONDITIONS

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Abstract. We consider nonlinear parabolic equations with gradient-dependent nonlinearities, of the form $u_t - \Delta u = F(u, \nabla u)$. These equations are studied on smoothly bounded domains of $\mathbb{R}^N$, $N \geq 1$, with arbitrary (continuous) Dirichlet boundary data. Under optimal assumptions of (superquadratic) growth of F with respect to ∇u, we show that gradient blow-up occurs for suitably large initial data; i.e., ∇u blows up in finite time while u remains uniformly bounded. Various extensions and additional results are given. We also consider some equations where the nonlinearity is nonlocal with respect to ∇u, and show that gradient blow-up usually does not occur in this case.

1. Introduction

Consider the following nonlinear parabolic problem:

$$u_t - \Delta u = F(u, \nabla u), \qquad t > 0, \quad x \in \Omega \tag{1.1}$$

$$u(t, x) = g(t, x), \qquad t > 0, \quad x \in \partial\Omega \tag{1.2}$$

$$u(0, x) = u_0(x), \qquad x \in \Omega. \tag{1.3}$$

In all the article, we assume that Ω is a smoothly bounded domain of $\mathbb{R}^N$, $F \in C^1(\mathbb{R} \times \mathbb{R}^N)$, $g \in C([0, T] \times \partial\Omega)$ for all $T > 0$, and $u_0 \in C^1(\overline{\Omega})$ satisfies the compatibility conditions $u_0(x) = g(0, x)$ on $\partial\Omega$. It is well-known [23, Theorem 10, p. 206] that (1.1)–(1.3) admits a unique, maximal-in-time,

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classical solution u, of maximal existence time $T^* \in (0,\infty]$. Moreover, if $T^* < \infty$, then u blows up in finite time in C^1 norm; i.e.,

$$\lim_{t\to T^*} \sup_{x\in\Omega} |u(t,x)| + |\nabla u(t,x)| = \infty. \tag{1.4}$$

It is said that *gradient blow-up* occurs if u stays uniformly bounded but does not exist globally in time. In view of (1.4), this means that

$$T^* < \infty, \qquad \sup_{[0,T^*)\times\Omega} |u| < \infty, \qquad \text{and} \qquad \lim_{t\to T^*} \sup_{x\in\Omega} |\nabla u(t,x)| = \infty.$$

If the growth of F is at most quadratic with respect to ∇u, then it is known that gradient blow-up cannot occur. Actually, a more precise condition on F is known to be

$$|F(u,\nabla u)| \le C(u)(1+|\nabla u|^2)h(|\nabla u|) \tag{1.5}$$

where h is positive nondecreasing and satisfies

$$\int^\infty \frac{ds}{sh(s)} = \infty, \tag{1.6}$$

and $C(u)$ is locally bounded (see [32, 33]; see also [14] for a different approach). On the contrary, if this condition is violated, namely, if

$$F(u,\nabla u) \ge |\nabla u|^2\, h(|\nabla u|), \qquad \int^\infty \frac{ds}{sh(s)} < \infty, \tag{1.7}$$

then examples of gradient blow-up were constructed in [33, Section 5] (see also [30, 20]), but the examples there involve specific, nonhomogeneous and time-dependent boundary data g.

Also, some results on gradient blow-up rates and profiles, and on continuation after blow-up, were obtained in [4, 6, 22, 4]. Moreover, gradient blow-up is known to occur for certain quasilinear parabolic problems [10, 26, 7]. On the other hand, the work [1] treats the equation

$$u_t - \Delta u = |\nabla u|^p, \qquad t > 0, \quad x \in \Omega, \tag{1.8}$$

with homogeneous Dirichlet boundary conditions from a different point of view. Namely, it addresses the question of existence of weak solutions for irregular data. Assuming that the initial data is given by $u_0 = \lambda h$, where $h \not\equiv 0$ is a positive, bounded measure, it is proved, among other things, that the problem (1.8), (1.2)–(1.3) for $p > 2$ and $g = 0$ cannot admit a global weak solution whenever λ is sufficiently large. However, the author of [1] does not relate his result to the general framework of gradient blow-up.

The main aim of the present article is to prove gradient blow-up for problems of type (1.1)–(1.3), under optimal growth conditions, for any boundary data g.

Our main results on gradient blow-up are stated in Section 2. We first focus on the case of the model equation (1.8) with arbitrary Dirichlet boundary data, and we prove that gradient blow-up occurs for suitably large initial data whenever $p > 2$. We then improve this result to show that the condition (1.5)–(1.6) is really optimal in general situations. Namely, for $F = F(\nabla u)$ satisfying (1.7), we show (under mild additional restrictions) that for arbitrary Dirichlet boundary data, gradient blow-up occurs when the initial data is large.

In Section 3, we extend these results in several directions and give additional information. For instance, we study equations of the form (1.1) such as

$$u_t - \Delta u = |\nabla u|^p + a(x)u^q, \quad \text{or} \quad u_t - \Delta u = \frac{|\nabla u|^p}{(1+u^2)^{m/2}}.$$

In Section 4, we turn to the case of nonlocal equations, for instance of the form

$$u_t - \Delta u = u^m(t,x)\Big(\int_\Omega |\nabla u(t,y)|^2\,dy\Big)^r,$$

and we show that the situation is quite different: gradient blow-up does *not* occur in this case, but L^∞ blow-up usually does.

Our proofs of gradient blow-up are different from those in previous gradient blow-up studies, which were based on the construction of suitable subsolutions [33, 20, 30, 22, 6]. They rely on the use of the first eigenfunction of the Laplacian and on differential inequalities, an idea first introduced in the celebrated paper [28] for equations of the type $u_t - \Delta u = u^p$. The method is related to that in [1]. All the proofs are collected in Section 5.

2. Main results on gradient blow-up

In what follows, we denote by $\lambda_1 > 0$ the lowest eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, and by φ_1 the associated eigenfunction, such that

$$\varphi_1 > 0 \text{ in } \Omega \quad \text{and} \quad \int_\Omega \varphi_1(x)\,dx = 1.$$

Theorem 2.1. *Assume $p > 2$, and consider the problem* (1.8), (1.2)–(1.3). *There exists $k_0 = k_0(\Omega, p, g) > 0$, such that if $\int_\Omega u_0(x)\varphi_1(x)\,dx > k_0$, then gradient blow-up occurs.*

Let us now consider nonlinearities $F = F(\nabla u)$ of class C^1, such that

$$F(\nabla u) \geq |\nabla u|^2 h(|\nabla u|),$$

where $h(s)$ is positive nondecreasing for $s > 0$, and is such that $s \mapsto s^2 h(s)$ is convex. Moreover, we assume that h satisfies

$$h(yz) \leq C(h(y) + h(z)) \quad \text{for all large } y,\ z > 0 \text{ and some } C > 0. \tag{2.1}$$

The condition (2.1) implies that h grows more slowly than any positive power. It is satisfied for instance if $h(s) = (\log s)^p$ or $h(s) = (\log s)^p (\log\log s)^q$ for large s, with $p,\ q > 0$.

Theorem 2.2. *Consider the problem* (1.1)–(1.3) *with F as described above, and assume that*

$$\int^{\infty} \frac{ds}{sh(s)} < \infty. \tag{2.2}$$

There exists $k_0 = k_0(\Omega, F, g) > 0$, such that if

$$\int_\Omega \big(u_0(x) - \tfrac{1}{2}\|u_0\|_\infty\big)\varphi_1(x)\,dx > k_0, \tag{2.3}$$

then gradient blow-up occurs.

3. Extensions and further results

Let us begin with some propositions which complement Theorem 2.1. The first result shows that a largeness hypothesis on the initial data cannot be avoided for the occurrence of gradient blow-up in equation (1.8).

Proposition 3.1. *Assume $p > 1$ and $g \equiv 0$, and let u be the solution of* (1.8), (1.2)–(1.3). *Then there exists $\epsilon > 0$ such that $|u_0|_{C^1(\overline{\Omega})} < \epsilon$ implies that u is global and bounded in C^1 norm for $t \geq 0$.*

However, there are some variants of equation (1.8) for which gradient blow-up occurs for *all* nontrivial nonnegative solutions. Namely, consider the equation

$$u_t - \Delta u = |\nabla u|^p + \lambda u, \qquad t > 0, \quad x \in \Omega, \tag{3.1}$$

with homogeneous Dirichlet conditions. We then have

Proposition 3.2. *Assume $p > 2$, $\lambda \geq \lambda_1$, and $g \equiv 0$, and let u be the solution of* (3.1), (1.2)–(1.3). *If $u_0 \geq 0$, $u_0 \not\equiv 0$, then gradient blow-up occurs.*

Remark 3.1. Let us note that the Cauchy problem for equation (1.8) was also investigated (see [34, 10, 5, 11, 12, 9, 35, 13, 27] and the references therein). It is known that no gradient blow-up occurs and all solutions exist globally. Moreover, precise results on the asymptotic behavior of solutions were obtained.

Let us now turn to different equations of type (1.1). For equations with zero-order nonlinearity, e.g. $u_t - \Delta u = u^q$, $q > 1$, it is well-known that L^∞ blow-up occurs for large (nonnegative) initial data. It is therefore natural to ask what happens if the nonlinearity involves both zero-order and first-order source terms, such as in

$$u_t - \Delta u = |\nabla u|^p + u^q. \tag{3.2}$$

Is it gradient blow-up or L^∞ blow-up which occurs? Conversely, if the zero-order term is an absorption term, such as in

$$u_t - \Delta u = |\nabla u|^p - u^q,$$

can it prevent gradient blow-up? More generally, one may consider the equation

$$u_t - \Delta u = |\nabla u|^p + a(x)u^q, \qquad t > 0, \quad x \in \Omega, \tag{3.3}$$

where $a \in C^\alpha(\overline{\Omega})$ $(0 < \alpha < 1)$. The following theorem provides partial answers to these questions.

Theorem 3.3. *Assume $p > 2$, $p > q \geq 1$, and let $u \geq 0$ be the solution of* (3.3), (1.2)–(1.3), *where $u_0 = \lambda\psi$, $\psi \geq 0$, $\psi \not\equiv 0$. Then there exists $\Lambda_0 = \Lambda_0(p, q, a, g, \Omega, \psi) > 0$, such that for all $\lambda > \Lambda_0$, gradient blow-up occurs.*

Remarks 3.2. (a) The conclusion of Theorem 3.3 remains valid if $q = p$ and $\|a\|_\infty$ is sufficiently small (depending on p, Ω, g and ψ).

(b) If $q > p$ and $a(x) \equiv C > 0$, we conjecture that L^∞ blow-up might occur. Of course, if $a(x) \leq 0$, L^∞ blow-up can never occur for $u \geq 0$, whatever the value of $q \geq 1$. We do not know if gradient blow-up may occur if $q > p$ and, e.g., $a(x) \equiv -C < 0$.

(c) Let us mention that for the related equation

$$u_t - \Delta u = -|\nabla u|^p + u^q, \qquad t > 0, \quad x \in \Omega, \tag{3.4}$$

with $g = 0$, no gradient blow-up can occur (see [37, 48]). On the other hand, global existence and L^∞ blow-up for equation (3.4) was extensively studied [17, 29, 19, 21, 36, 3, 37, 39, 40, 49, 46, 47, 48, 43, 38, 15, 45, 16]. See [44] for a recent survey. Also, for equation (3.2) with $q = 2$ and $p > 1$, detailed results on the form of L^∞ blow-up can be found in [31, 29, 24, 25].

Another extension of Theorem 2.1 for equations under general form (1.1), is given by the following theorem. It shows that the presence, in factor of a superquadratic gradient term, of a coefficient $C(u)$ which becomes very small for large values of u cannot generally prevent gradient blow-up.

Theorem 3.4. *Assume that F satisfies*

$$xF(x,0) \leq C_1 x^2, \quad |x| \geq A \tag{3.5}$$

and

$$F(x,y) \geq \begin{cases} C_2|x|^{-m}|y|^p, & \text{if } |x| \geq A \\ -C_3, & \text{if } |x| < A, \end{cases} \tag{3.6}$$

with C_1, C_2, C_3, $A > 0$, $p > 2$, and where $m \geq 0$ satisfies

$$\begin{cases} m < p-1, & \text{if } N = 1 \\ m \leq p/N, & \text{if } N \geq 2. \end{cases}$$

Let u be the solution of (1.1)–(1.3). *There exists $k_0 = k_0(F, \Omega, g) > 0$, such that if $\int_\Omega u_0(x)\varphi_1(x)\,dx > k_0$, then gradient blow-up occurs.*

Theorem 3.4 applies for instance for the equation

$$u_t - \Delta u = \frac{|\nabla u|^p}{(1+u^2)^{m/2}},$$

with p and m as above.

Remark 3.3. (a) As noted by several authors, if, e.g., F depends only on ∇u, it follows from the maximum principle that the maximal values of $|\nabla u|$ are attained on the parabolic boundary. Therefore, in case of gradient blow-up, we have

$$\limsup_{t \to T^*} \sup_{x \in \partial\Omega} |\nabla u(t,x)| = \infty.$$

For examples of interior gradient blow-up, see [6], and also [10, 7, 26] in the quasilinear case.

(b) As pointed out in [20], the nature of the boundary conditions determines the occurrence of gradient blow-up. For instance, in the case of Neumann boundary conditions in one space dimension, gradient blow-up never occurs for the equation (1.1) with $F \equiv F(u_x)$. To show this, first note that since $|u_x|$ is bounded on the parabolic boundary for all finite time, $|u_x|$ remains bounded in Ω locally in time (see Remark 3.3 (a)). Next, use the fact that the right-hand side of the equation is then bounded to deduce local boundedness of u.

Remark 3.4. The methods of the present paper can also be used to prove gradient blow-up for some quasilinear equations. This will be treated in a subsequent article.

4. Nonlocal equations

In this section, we show that the local character of the nonlinearity is essential in the previous gradient blow-up results. Indeed, we provide examples of nonlocal gradient terms with arbitrarily fast growth with respect to ∇u, such that gradient blow-up never occurs, while L^∞ blow-up does, either for Dirichlet or Neumann conditions. Namely, consider the equation

$$u_t - \Delta u = u^m(t,x)\Big(\int_\Omega |\nabla u(t,y)|^2\,dy\Big)^r, \qquad t>0, \quad x\in\Omega, \tag{4.1}$$

where $m \geq 1$ and $r > 0$, together with either

$$u(t,x) = 0, \qquad t>0, \quad x\in\partial\Omega, \tag{4.2}$$

or

$$\frac{\partial u}{\partial n}(t,x) = 0, \qquad t>0, \quad x\in\partial\Omega. \tag{4.3}$$

By standard arguments, one can show that for each nonnegative $u_0 \in C^1(\overline{\Omega})$ satisfying $u_0 \geq 0$ and $u_0(x) = 0$ (respectively $\partial u_0/\partial n = 0$) on $\partial\Omega$, the problem (4.1), (4.2), (1.3) (respectively (4.1), (4.3), (1.3)) admits a unique, maximal, classical solution $u \geq 0$ (see, e.g., [41, Theorem A4] for an existence proof in the case of Dirichlet conditions).

Theorem 4.1. *Assume $m \geq 1$ and $r > 0$, and let u be any nonnegative classical solution of* (4.1), (4.2), (1.3) or (4.1), (4.3), (1.3). *Then gradient blow-up cannot occur.*

We complement this theorem with the following L^∞ blow-up result.

Proposition 4.2. *Under the assumptions of Theorem 4.1, u blows up in finite time in the L^∞ norm, in each of the following situations:*

(i) *Dirichlet boundary conditions and $\int_\Omega u_0(x)\varphi_1(x)\,dx$ sufficiently large;*

(ii) *Neumann boundary conditions, $m = 1$, $u_0 \in C^2(\overline{\Omega})$, and $\int_\Omega |\Delta u_0|^2 - |\nabla u_0|^2 < 0$;*

(iii) *Neumann boundary conditions, $m \geq 1$, $N = 1$, $\Omega = (-a,a)$, with u_0 symmetric, and*

$$(u_0)_x \geq 0 \text{ on } (0,a), \quad u_0 \geq \delta > 0, \quad \textstyle\int_0^a (u_0)_x\,\varphi_1 \text{ large (depending on } \delta).$$

Remark 4.1. It was proved in in [20, Lemma 2] that ∇u blows up in finite time in cases (ii) of Proposition 4.2 for $N = 1$. We here complete the result of [20] by showing that u does also blow up.

Remark 4.2. One can be more precise about the form of L^∞ blow up for equation (4.1) in the case $m = 1$ (for either Dirichlet or Neumann conditions). Indeed, letting $G(t) = \int_0^t (\int_\Omega |\nabla u(s,y)|^2\,dy)^r ds$ and $v(t,x) = e^{-G(t)}u(t,x)$, we observe that $v_t - \Delta v = 0$ and $v(0) = u_0$. It follows that $u = e^{G(t)}e^{t\Delta}u_0$, where $e^{t\Delta}$ denotes the Dirichlet or the Neumann heat semigroup. Since $e^{t\Delta}u_0 > 0$ in Ω, we must have $G(T^*) = \infty$ if T^* is finite, and $u(t,x)$ will then blow-up at every point of Ω as t goes to T^*. Let us mention that related results of global blow-up for nonlocal reaction-diffusion equations can be found in [42] and in the references therein.

5. Proofs

For the proof of Theorem 2.1, we will need the following lemma on the first eigenfunction of the Laplacian. This lemma is stated without proof in [2, 1]. For completeness, we give a proof at the end of this section.

Lemma 5.1. *Assume that Ω is a bounded domain in $\mathbb{R}^N$ with smooth (say, C^2) boundary and let $\alpha \in (0,1)$. Then*

$$\int_\Omega \varphi_1^{-\alpha}(x)\,dx < \infty.$$

In the remainder of this section, C, C', ... will denote various constants depending only on the indicated arguments, and which may vary from line to line.

Proof of Theorem 2.1. First, by the maximum principle, it is clear that if $T^* < \infty$, then

$$\sup_{[0,T^*)\times\Omega} |u| \leq \max(\|u_0\|_\infty, \sup_{[0,T^*)\times\partial\Omega} |g|) < \infty.$$

Therefore, gradient blow-up has to occur if u is nonglobal.

Setting $z(t) = \int_\Omega u(t,x)\varphi_1(x)\,dx$, we obtain, after integration by parts,

$$z'(t)+\lambda_1 z(t) = \int_\Omega |\nabla u(t,x)|^p \varphi_1(x)\,dx + \int_{\partial\Omega} u(t,x)\frac{\partial\varphi_1}{\partial n}(x)\,d\sigma \qquad \text{in } (0,T^*).$$

Let $T_1 = \min(1,T^*)$, and denote $C_0(g) = \sup_{[0,1]\times\partial\Omega}|g|$. Taking $C(\Omega) = \int_{\partial\Omega}\left|\frac{\partial\varphi_1}{\partial n}(x)\right| d\sigma$ and using (1.2), it follows that

$$z'(t) + \lambda_1 z(t) \geq \int_\Omega |\nabla u(t,x)|^p \varphi_1(x)\,dx - C(\Omega)C_0(g) \qquad \text{in } (0,T_1). \quad (5.1)$$

On the other hand, by Hölder's inequality, we have

$$\int_\Omega |\nabla u(t,x)|\,dx = \int_\Omega \big(|\nabla u(t,x)|\varphi_1^{1/p}(x)\big)\varphi_1^{-1/p}(x)\,dx$$
$$\leq C(\Omega,p)\Big(\int_\Omega |\nabla u(t,x)|^p\varphi_1(x)\,dx\Big)^{1/p}, \tag{5.2}$$

where $C(\Omega,p) = \big(\int_\Omega \varphi_1^{-1/(p-1)}(x)\,dx\big)^{(p-1)/p}$ is finite by Lemma 5.1, since $p > 2$.

Now, recall the following version of Poincaré's inequality: there exists a constant $C(\Omega) > 0$, such that for all $v \in C^1(\overline{\Omega})$,

$$\int_\Omega |v(x)|\,dx \leq C(\Omega)\Big(\sup_{\partial\Omega}|v| + \int_\Omega |\nabla v(x)|\,dx\Big). \tag{5.3}$$

Using (5.2), (5.3) and (1.2), it follows that

$$|z(t)|^p \leq \|\varphi_1\|_\infty^p\Big(\int_\Omega |u(t,x)|\,dx\Big)^p$$
$$\leq \|\varphi_1\|_\infty^p C(\Omega,p)C_0^p(g) + \int_\Omega |\nabla u(t,x)|^p\varphi_1(x)\,dx\Big) \qquad \text{in } (0,T_1).$$

This, combined with (5.1), yields

$$z'(t) + \lambda_1 z(t) \geq C(\Omega,p)|z(t)|^p - C'(\Omega,p)\big(C_0^p(g) + C_0(g)\big) \qquad \text{in } (0,T_1); \tag{5.4}$$

hence,

$$z'(t) \geq C(\Omega,p)|z(t)|^p - C'(\Omega,p)\big(1 + C_0^p(g)\big) \qquad \text{in } (0,T_1).$$

It then follows that

$$z'(t) \geq \frac{1}{2}C(\Omega,p)|z(t)|^p \qquad \text{in } (0,T_1), \tag{5.5}$$

whenever

$$z(0) \geq C'(\Omega,p)\big(1 + C_0^p(g)\big). \tag{5.6}$$

By taking a larger $C'(\Omega,p)$ if necessary, one easily deduces from (5.5), (5.6) that $T_1 < 1$, hence $T^* < 1$. Theorem 2.1 follows. □

Remark 5.1. Some comments concerning the proof of Theorem 2.1 are in order. A typical device of previous proofs of gradient blow up (see, e.g., [20, Theorem 1] or [22, Theorem 2.2]) is to construct a subsolution $\underline{u}$, which coincides with u at some point a of $\partial\Omega$ and at some time T, such that the normal derivative of $\underline{u}$ blows up at $t = T$ and $x = a$. By the comparison principle, the normal derivative of u is thus forced to blow-up at (T,a) if $T^* \geq$

T. In this respect, and on a formal level, those proofs of gradient blow-up can be considered as "direct" proofs (although u might cease to exist before $t = T$). On the contrary, our proof is highly indirect. Indeed, it relies on a superlinear differential inequality involving the quantity $\int_\Omega u(t,x)\varphi_1(x)\,dx$, by showing that this inequality cannot hold for t larger than some finite T_1. A careless reading of the proof could let one think that this quantity (hence also $\|u(t)\|_\infty$) should blow-up, while this is *not* the case, since uniform bounds on u are derived independently. Of course, the conclusion is that u will cease to exist before T_1. This is an additional illustration of a general observation of Ball [8, p. 473], concerning proofs of nonexistence theorems for evolution equations.

We now turn to the proof of Theorem 2.2. It is actually a consequence of the following slightly more general result.

Theorem 5.2. *Assume that $F = F(\nabla u) \geq G(|\nabla u|) - C$, where $C \geq 0$, $G(s)$ is convex increasing for $s \geq 0$, and $G(0) = 0$ without loss of generality. Let*

$$a(s) = \sup_{y>0} \frac{G^{-1}(ys)}{G^{-1}(y)}, \quad s \geq 0,$$

and denote by G^* the convex conjugate function of G. If

$$\int^\infty G^*(Ma(s))\frac{ds}{s^2} < \infty \tag{5.7}$$

for all $M > 0$, then there exists $k_0 = k_0(\Omega, F, g) > 0$ such that gradient blow-up occurs for problem (1.1)–(1.3) *whenever* (2.3) *is satisfied.*

Proof. From the definition of a, by letting $y = G(|\nabla u|)\varphi_1$, we see that

$$a(1/\varphi_1)G^{-1}(G(|\nabla u|)\varphi_1) \geq G^{-1}(y/\varphi_1) = |\nabla u|;$$

hence,

$$G(|\nabla u|)\varphi_1 \geq G\big(\frac{|\nabla u|}{a(1/\varphi_1)}\big) \tag{5.8}$$

(where the arguments t and x were omitted for brevity). Let $M > 0$ be fixed later. Since

$$M|\nabla u| = \frac{|\nabla u|}{a(1/\varphi_1)}\, Ma(1/\varphi_1) \leq G\big(\frac{|\nabla u|}{a(1/\varphi_1)}\big) + G^*\big(Ma(1/\varphi_1)\big),$$

it follows from (5.8) that

$$M\int_\Omega |\nabla u| \leq \int_\Omega G(|\nabla u|)\varphi_1 + \int_\Omega G^*\big(Ma(1/\varphi_1)\big). \tag{5.9}$$

Also, the assumption (5.7) can be recast as $\int_0^\varepsilon G^*(Ma(1/s))\,ds < \infty$, which implies that $\int_\Omega G^*\big(Ma(1/\varphi_1)\big) < \infty$, in a similar way as in Lemma 5.1.

Now assume that $T^* > 1$. Arguing as in the beginning of proof of Theorem 2.1 (cf. (5.1)), and using (5.9) and the Poincaré inequality (5.3), we see that for $t \in (0,1]$, $z(t) = \int_\Omega u(t,x)\varphi_1(x)\,dx$ satisfies

$$\begin{aligned} z'(t) + \lambda_1 z(t) &\geq \int_\Omega F(|\nabla u|)\varphi_1 - C(\Omega) \sup_{[0,1]\times\partial\Omega} |g| \geq \int_\Omega G(|\nabla u|)\varphi_1 - C(\Omega, F, g) \\ &\geq M \int_\Omega |\nabla u| - C(\Omega, F, M, g) \geq MC(\Omega) \int_\Omega |u| - C'(\Omega, F, M, g). \end{aligned}$$

Choosing $M = M(\Omega)$ sufficiently large, we deduce that $z' \geq z(t) - C(\Omega, F, g)$, $0 < t \leq 1$. Integrating this inequality, we obtain

$$z(t) \geq (z(0) - C(\Omega, F, g))e^t, \quad 0 < t \leq 1.$$

On the other hand, we know from the maximum principle that for $t \in [0,1]$,

$$\|u(t)\|_\infty \leq \max(\|u_0\|_\infty, \sup_{[0,1]\times\partial\Omega} |g|) \leq \|u_0\|_\infty + C(g).$$

By combining the last two inequalities for $t = 1$, it follows that

$$\int_\Omega u_0\varphi_1 \leq z(1)e^{-1} + C(\Omega, F, g) \leq \tfrac{1}{2}\|u_0\|_\infty + C'(\Omega, F, g);$$

hence,

$$\int_\Omega (u_0 - \tfrac{1}{2}\|u_0\|_\infty)\varphi_1 \leq C'(\Omega, F, g). \tag{5.10}$$

It is clearly possible to find u_0 such that (5.10) is violated. For such u_0, one then necessarily has $T^* \leq 1 < \infty$ and gradient blow-up occurs. □

To deduce Theorem 2.2 from Theorem 5.2, we then need the following elementary technical lemma.

Lemma 5.3. *Let F be as in Theorem 2.2, assume that (2.2) is satisfied, and let $G(s) = s^2 h(s)$. Then $G(s)G^*(s) \leq Cs^4$ for s large.*

Proof. Since h is nondecreasing, (2.2) implies that $h(s) \to \infty$ as $s \to \infty$. On the other hand, it is easy to see that (2.1) implies in particular $h(s) \leq \sqrt{s}$ for s large; hence $h(h(s)) \leq h(s/h(s))$. Again by (2.1), we have, for s large,

$$h(s) = h\Big(\frac{s}{h(s)}\, h(s)\Big) \leq Ch(s/h(s)) + Ch(h(s)) \leq 2Ch(s/h(s)).$$

Therefore, for all y, $z > 0$ large, if $y \geq z/h(z)$, then $h(z) \leq 2Ch(z/h(z)) \leq 2Ch(y)$; hence,

$$yz \leq y^2 h(y) + \frac{z^2}{h(y)} \leq y^2 h(y) + 2C\,\frac{z^2}{h(z)},$$

while, if $y < z/h(z)$, then $yz \leq \frac{z^2}{h(z)}$. In both cases, we get

$$yz \leq G(y) + C'\frac{z^2}{h(z)}$$

for all y, $z > 0$ large. It follows that $G^*(s) \leq C' s^2/h(s) = C' s^4/G(s)$ for s large, and the lemma is proved. □

Completion of the proof of Theorem 2.2. Set $G(s) = s^2 h(s)$. We have

$$\frac{G(yz)}{G(y)} = z^2 \frac{h(yz)}{h(y)} \geq z^2, \quad y > 0, \quad z \geq 1.$$

An easy computation then shows that

$$\frac{G^{-1}(yz)}{G^{-1}(y)} \leq \sqrt{z}, \quad y > 0, \quad z \geq 1;$$

hence, $a(s) \leq \sqrt{s}$ for $s \geq 1$, where $a(s)$ is defined in Theorem 5.2. Also, Lemma 5.3 implies that $G^*(s) \leq Cs^4/G(s) = Cs^2/h(s)$ for s large; hence,

$$G^*(Ma(s)) \leq G^*(M\sqrt{s}) \leq CM^2 s/h(M\sqrt{s}),$$

so that

$$\int^\infty G^*(Ma(s))\frac{ds}{s^2} \leq CM^2 \int^\infty \frac{1}{h(M\sqrt{s})}\,\frac{ds}{s} \leq C' \int^\infty \frac{d\tau}{\tau h(\tau)} < \infty.$$

Theorem 5.2 thus implies that gradient blow-up occurs under condition (2.3).

Proof of Proposition 3.1. Let χ_0 be the solution of

$$\begin{cases} -\Delta \chi_0 = 1 & \text{in } \Omega \\ \chi_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

As a consequence of Hopf's boundary lemma, we know that $\chi_0(x) \geq \epsilon_0 d(x)$ in Ω for some $\epsilon_0 > 0$, where $d(x) = \operatorname{dist}(x, \partial\Omega)$. Letting $M = \|\nabla\chi_0\|_\infty$ and $\chi = M^{-p/(p-1)}\chi_0$, we find that

$$-\Delta\chi \geq |\nabla\chi|^p \quad \text{in } \Omega. \tag{5.11}$$

Now, assume that $|u_0|_{C^1(\overline{\Omega})} < \epsilon = \epsilon_0 M^{-p/(p-1)}$ and $u_{0|\partial\Omega} = 0$. In particular, $u_0(x) \leq |u_0|_{C^1(\overline{\Omega})} d(x) \leq \chi(x)$ in Ω. It then follows from (5.11) and the comparison principle that $u \le \chi$ in $[0,T^*) \times \overline{\Omega}$. On the other hand, since $u_t - \Delta u \ge 0$, the maximum principle implies that $u \ge -|u_0|$ in $[0,T^*) \times \overline{\Omega}$. Therefore, using $u = u_0 = \chi = 0$ on $\partial\Omega$, we deduce that

$$-\Big|\frac{\partial u_0}{\partial n}\Big| \le -\frac{\partial u}{\partial n} \le -\frac{\partial \chi}{\partial n} \quad \text{ on } \partial\Omega.$$

Since the maximum values of $|\nabla u|$ are attained on the parabolic boundary (see Remark 3.3 (a)), it follows that $|\nabla u|$ is uniformly bounded in $[0,T^*)\times\overline{\Omega}$; hence $T^* = \infty$, and the result follows. □

Proof of Proposition 3.2. Proceeding exactly as in the proof of Theorem 2.1, with $g \equiv 0$, we arrive at

$$z'(t) \ge C(\Omega,p)|z(t)|^p + (\lambda - \lambda_1)z(t) \quad \text{ in } (0,T_1),$$

instead of (5.4), and the proposition follows. □

The proof of Theorem 3.3 proceeds in three steps. The first step uses precise comparison arguments to provide a uniform estimate of u up to a time of order λ^{1-q}, for large λ. In the second step, one combines this estimate with the eigenfunction argument of Theorem 2.1 to obtain an upper bound on T^*, of order λ^{1-p}. Finally, since $p > q$, this enables one to conclude that gradient blow-up has to take place first.

Proof of Theorem 3.3. We may restrict ourselves to the case $q > 1$, the case $q = 1$ being easier.

Step 1. Upper L^∞ estimate. Let $k = \|a\|_\infty$, and let $\overline{u}(t,x) = C(T_2 - t)^{-1/(q-1)}$, where $C = (k(q-1))^{-1/(q-1)}$ and $T_2 > 0$. A quick check reveals that $\overline{u}$ is a supersolution to (3.3), (1.2), (1.3) in $[0,T_2) \times \overline{\Omega}$ with

$$T_2 = \min\Big(1, C^{q-1} \sup_{[0,1]\times\partial\Omega} |g|^{1-q}, C^{q-1}\|\psi\|_\infty^{1-q}\lambda^{1-q}\Big).$$

Thus, there exists $\Lambda_1 = \Lambda_1(q,a,g,\psi) > 0$, such that for all $\lambda > \Lambda_1$,

$$T_2 = C(k,q)\|\psi\|_\infty^{1-q}\lambda^{1-q} < 1$$

and

$$u \le K = C'(k,q)\lambda\|\psi\|_\infty \quad \text{ in } [0,T_3) \times \overline{\Omega}, \tag{5.12}$$

with $T_3 = \min(T^*, \frac{T_2}{2}) < 1$.

Step 2. Upper bound on T^*. Denote again $z(t) = \int_\Omega u(t,x)\varphi_1(x)\,dx$. By a slight modification in the proof of Theorem 2.1, one finds that

$$z'(t) \ge \frac{1}{2}C(\Omega,p)z^p(t) - k\int_\Omega u^q(t,x)\varphi_1(x)\,dx \quad \text{ in } (0,T_3)$$

whenever (5.6) holds, which in turn is satisfied if λ is larger than some $\Lambda_0(q,a,g,\psi) > \Lambda_1$. Applying estimate (5.12), it follows that

$$z'(t) \geq \big(C(\Omega,p)z^{p-1}(t) - C(k,q)\lambda^{q-1}\|\psi\|_\infty^{q-1}\big)z(t) \quad \text{in } (0,T_3);$$

hence,

$$z'(t) \geq C(\Omega,p)z^p(t) \quad \text{in } (0,T_3) \tag{5.13}$$

whenever

$$\lambda^{p-1}\Big(\int_\Omega \psi\varphi_1\Big)^{p-1} \geq C(\Omega,p,k,q)\lambda^{q-1}\|\psi\|_\infty^{q-1}.$$

Since $p > q$, the latter inequality is satisfied for all $\lambda > \Lambda_0$ (possibly larger). By integrating (5.13), it then follows that

$$T_3 = \min(T^*, \frac{T_2}{2}) \leq C(\Omega,p)\|\psi\|_\infty^{1-p}\lambda^{1-p} \equiv T_4.$$

Using $p > q$ again, one infers that $T_4 < \frac{T_2}{2}$, for all $\lambda > \Lambda_0$ (still larger), which means that

$$T^* < \frac{T_2}{2}. \tag{5.14}$$

Step 3. Conclusion. By combining (5.12) and (5.14), one obtains that u is uniformly bounded in $[0,T^*) \times \overline{\Omega}$, with $T^* < \infty$; that is, gradient blow-up occurs. □

Proof of Theorem 3.4. The constants $C, C' > 0$ below may vary from line to line, and depend only on Ω, F and g. Assumption (3.5), together with the maximum principle imply that L^∞ blow-up cannot occur. The same argument as in the proof of Theorem 2.1 yields

$$z'(t) + \lambda_1 z(t) \geq \int_\Omega F(u,\nabla u)\varphi_1\,dx - C \sup_{[0,1]\times\partial\Omega} |g| \quad \text{in } (0,T_1).$$

Using assumption (3.6), and arguing as before, we obtain

$$\begin{aligned}\int_\Omega F(u,\nabla u)\varphi_1\,dx &\geq C\int_{|u(t)|\geq A} |\nabla u|^p\,|u|^{-m}\varphi_1\,dx - C' \\ &\geq C\Big(\int_{|u(t)|\geq A} |\nabla u|\,|u|^{-m/p}\,dx\Big)^p - C' \quad \text{in } (0,T_1).\end{aligned}$$

Let now

$$H(s) = \begin{cases} s^{1-(m/p)} - B^{1-(m/p)}, & s \geq B \\ 0, & 0 \leq s < B,\end{cases}$$

with $B \geq A$ to be fixed. We have

$$|\nabla H(u)| = \frac{p-m}{p} |\nabla u|\, |u|^{-m/p} \mathbf{1}_{\{|u(t)| \geq B\}}$$

(where $p - m > 0$ by assumption). We deduce that

$$\int_\Omega F(u, \nabla u)\varphi_1 \, dx \geq C\Big(\int_\Omega |\nabla H(u)|\, dx\Big)^p - C' \quad \text{ in } (0, T_1).$$

Now select $B = \max\big(A, \sup_{[0,1]\times\partial\Omega} |g|\big)$, so that $H(u(t,\cdot)) = 0$ on $\partial\Omega$. Applying the Sobolev inequality, it follows that

$$\int_\Omega F(u, \nabla u)\varphi_1 \, dx \geq C\|H(u(t,\cdot))\|^p_{L^r(\Omega)} - C',$$

where $r = \infty$ if $N = 1$ and $r = N/(N-1)$ if $N \geq 2$. Note that $H(u(t,\cdot)) \geq C|u|^{1-(m/p)} \mathbf{1}_{\{|u(t)|\geq 2B\}}$. In the case $N \geq 2$, since $\frac{p-m}{p}\frac{N}{N-1} = \alpha \geq 1$, we then have

$$\begin{aligned} \int_\Omega F(u, \nabla u)\varphi_1 \, dx &\geq C\|u(t)\, \mathbf{1}_{\{|u(t)|\geq 2B\}}\|^{p-m}_{L^\alpha(\Omega)} - C' \\ &\geq C\|u(t)\|^{p-m}_{L^1(\Omega)} - C' \geq C|z(t)|^{p-m} - C' \quad \text{ in } (0, T_1). \end{aligned} \tag{5.15}$$

In the case $N = 1$, we get

$$\int_\Omega F(u, \nabla u)\varphi_1 \, dx \geq C\|u(t)\, \mathbf{1}_{\{|u(t)|\geq 2B\}}\|^{p-m}_{L^\infty(\Omega)} - C',$$

which yields (5.15) again. From (5.15), since $p - m > 1$, we then conclude similarly to the proof of Theorem 2.1. □

Proof of Theorem 4.1. We proceed by contradiction and assume that gradient blow-up occurs; i.e., $T^* < \infty$ and $|u| \leq M$ in $[0, T^*) \times \overline{\Omega}$. First, it is clear that $\lim_{n\to\infty} \int_\Omega |\nabla u(t_n, y)|^2 \, dy = \infty$ for some sequence $t_n \to T^*$, since otherwise, by standard parabolic estimates, u would be extendable after T^*. Next, multiplying (4.1) by $-\Delta u$ and integrating by parts, with either Dirichlet or Neumann homogeneous boundary conditions, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u(t,y)|^2 \, dy \\ &= -\int_\Omega |\Delta u(t,y)|^2 dy + m\Big(\int_\Omega u^{m-1}|\nabla u(t,y)|^2 \, dy\Big)\Big(\int_\Omega |\nabla u(t,y)|^2 \, dy\Big)^r; \end{aligned}$$

hence,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u(t,y)|^2 \, dy \leq mM^{m-1}\Big(\int_\Omega |\nabla u(t,y)|^2 \, dy\Big)^{r+1}. \tag{5.16}$$

(Differentiating inside the integral is justified, since $\partial_t \nabla u$ belongs to $C^\alpha((0, T^*) \times \overline{\Omega})$, by parabolic regularity.) Integrating (5.16) between t and s, with $0 < t < s < T^*$, yields

$$\Big(\int_\Omega |\nabla u(t,y)|^2\,dy\Big)^{-r} - \Big(\int_\Omega |\nabla u(s,y)|^2\,dy\Big)^{-r} \le C(s-t).$$

Taking $s = t_n$ and letting $n \to \infty$, we obtain

$$\int_\Omega |\nabla u(t,y)|^2\,dy \ge C(T^* - t)^{-1/r} \quad \text{in } (0,T^*)$$

for some $C > 0$; hence,

$$u_t - \Delta u \ge \frac{Cu^m}{T^* - t} \quad \text{in } (0,T^*) \times \overline{\Omega}.$$

Multiplying by φ_1 and integrating by parts (with either Dirichlet or Neumann homogeneous boundary conditions), it follows that

$$\frac{d}{dt}\int_\Omega u\varphi_1\,dx + \lambda_1 \int_\Omega u\varphi_1\,dx \ge \frac{C\int_\Omega u^m \varphi_1\,dx}{T^* - t}. \tag{5.17}$$

We first deduce from (5.17) that $\int_\Omega u\varphi_1\,dx \ge Ce^{-\lambda_1 t} \ge C > 0$ in $(0,T^*)$. Plugging this estimate into the right-hand side of (5.17) after applying Jensen's inequality, we then infer that

$$\frac{d}{dt}\int_\Omega u\varphi_1\,dx + \lambda_1 \int_\Omega u\varphi_1\,dx \ge \frac{C\big(\int_\Omega u\varphi_1\,dx\big)^m}{T^* - t} \ge \frac{C}{T^* - t} \quad \text{in } (0,T^*).$$

A further integration yields $\lim_{t\to T^*} \int_\Omega u\varphi_1\,dx = \infty$, a contradiction to the assumption u bounded. □

Proof of Proposition 4.2. (i) Multiplying (4.1) by φ_1, integrating by parts, and using Hölder's and Poincaré's inequalities, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega u\varphi_1\,dx + \lambda_1 \int_\Omega u\varphi_1\,dx &= \Big(\int_\Omega u^m \varphi_1\,dx\Big)\Big(\int_\Omega |\nabla u(t,y)|^2\,dy\Big)^r \\ &\ge C\Big(\int_\Omega u\varphi_1\,dx\Big)^m \Big(\int_\Omega u^2(t,y)\,dy\Big)^r \ge C\Big(\int_\Omega u\varphi_1\,dx\Big)^{m+2r} \quad \text{in } (0,T^*).\end{aligned}$$

Since $m + 2r > 1$, it follows that $T^* < \infty$ whenever $\int_\Omega u\varphi_1\,dx$ is sufficiently large.

(ii) The proof is a straightforward modification of the energy argument of [20, Lemma 2] (given there for $N = 1$).

(iii) From the assumptions, we deduce that $u(t,\cdot)$ is symmetric for all t. Also, $u \geq \delta$ since $\underline{u}(t,x) = \delta$ is a subsolution. Let $v = u_x$. Differentiating the equation (4.1), we see that v solves

$$v_t - v_{xx} = mu^{m-1}v\Big(\int_{-a}^{a} |u_x(t,y)|^2\,dy\Big)^r \quad \text{in } (0,T^*)\times(0,a),$$

with the boundary conditions $v(t,0) = v(t,a) = 0$. By the maximum principle, the assumption on $(u_0)_x$ implies that $v \geq 0$. Multiplying the equation for v by φ_1 and integrating by parts yields

$$\begin{aligned}\frac{d}{dt}\int_0^a v\varphi_1\,dx + \lambda_1\int_0^a v\varphi_1\,dx &\geq 2^r m\delta^{m-1}\Big(\int_0^a v\varphi_1\,dx\Big)\Big(\int_0^a v^2\,dx\Big)^r \\ &\geq C\delta^{m-1}\Big(\int_0^a v\varphi_1\,dx\Big)^{1+2r} \quad \text{in } (0,T^*),\end{aligned}$$

where $C = C(a,m,r) > 0$, and the conclusion follows. □

Proof of Lemma 5.1. It is well-known that $\varphi_1(x) \geq c\,\mathrm{dist}(x,\partial\Omega)$ for all $x \in \Omega$ and some constant $c > 0$. Moreover, it follows from the assumption that $\partial\Omega$ can be locally represented as the graph of a C^2 function (C^1 would do) in some orthonormal system of coordinates.

Using a partition of unity, we are therefore reduced to proving that the integral $I = \int_\omega \mathrm{dist}^{-\alpha}(x,\Gamma)\,dx$ is finite, where ω and Γ are defined in the following way: $x = (x'_N, x_N)$, $\quad x'_N \in \mathbb{R}^{N-1}$, $x_N \in \mathbb{R}$,

$U = \{x \in \mathbb{R}^N : |x'_N| \leq \epsilon,\ |x_N| < \epsilon\}$, $U_0 = \{x'_N \in \mathbb{R}^{N-1} : |x'_N| \leq \epsilon\}$, $\quad \epsilon > 0$,

$f : U_0 \to (-\epsilon,\epsilon)$ is a function of class C^1, with $f(0) = 0$,

$$\omega = \{x \in U : x_N < f(x'_N)\}, \quad \Gamma = \{x \in U : x_N = f(x'_N)\}.$$

Now it is easily seen that for all $x \in \omega$, $\mathrm{dist}(x,\Gamma) \geq K(f(x'_N) - x_N)$, where $K = (1 + \|\nabla f\|_\infty^2)^{-1/2}$. Using Fubini's theorem, we then have

$$\begin{aligned} I &= \int_{U_0} dx'_N \int_{-\epsilon}^{f(x'_N)} \mathrm{dist}^{-\alpha}(x,\Gamma)\,dx_N \\ &\leq K^{-\alpha}\int_{U_0} dx'_N \int_{-\epsilon}^{f(x'_N)} (f(x'_n) - x_N)^{-\alpha}\,dx_N,\end{aligned}$$

which is finite since $\alpha < 1$. The lemma is proved.

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References

[1] N. Alaa, *Solutions faibles d'équations paraboliques quasi-linéaires avec données initiales mesures,* Ann. Math. Blaise-Pascal, 3 (1996), 1–15.

[2] N. Alaa and M. Pierre, *Weak solutions of some quasilinear elliptic equations with data measures,* SIAM J. Math. Anal., 24 (1993), 23–35.

[3] L. Alfonsi and F.B. Weissler, *Blow-up in* $\mathbb{R}^N$ *for a parabolic equation with a damping nonlinear gradient term,* Progress in Nonlinear Differential Equations, N.G. Lloyd et al. editors, Birkhäuser, 1992.

[4] N. Alikakos, P. Bates, and C. Grant, *Blow up for a diffusion-advection equation,* Proc. Roy. Soc. Edinburgh, A113 (1989), 181–190.

[5] L. Amour and M. Ben-Artzi, *Global existence and decay for viscous Hamilton-Jacobi equations,* Nonlinear Anal., TMA, 31 (1998), 621–628.

[6] S. Angenent and M. Fila, *Interior gradient blow-up in a semilinear parabolic equation,* Diff. Integral Equations, 9 (1996), 865–877.

[7] K. Asai and N. Ishimura, *On the interior derivative blow-up for the curvature evolution of capillary surfaces,* Proc. Amer. Math. Soc., 126 (1998), 835–840.

[8] J. Ball, *Remarks on blow-up and nonexistence theorems for nonlinear evolution equations,* Quart. J. Math. Oxford, 28 (1977), 473–486.

[9] S. Benachour and P. Laurençot, *Global solutions to viscous Hamilton–Jacobi equation with irregular data,* Commun. partial Differ. Eq., 24 (1999), 1999–2021.

[10] M. Ben-Artzi, *Global existence and decay for a nonlinear parabolic equation,* Nonlinear Anal., TMA, 19 (1992), 763–768.

[11] M. Ben-Artzi, J. Goodman, and A. Levy, *Remarks on a nonlinear parabolic equation,* Trans. Amer. Math. Soc., 352 (2000), 731–751.

[12] M. Ben-Artzi and H. Koch, *Decay of mass for a semilinear parabolic equation,* Commun. Partial Differ. Eq., 24 (1999), 869–881.

[13] M. Ben-Artzi, Ph. Souplet, and F. B. Weissler, *Sur la non-existence et la non-unicité des solutions du problème de Cauchy pour une équation parabolique semi-linéaire,* C. R. Acad. Sc. Paris, 329, Série I (1999), 371–376.

[14] L. Boccardo, F. Murat, and J.-P. Puel, *Existence results for some quasilinear parabolic equations,* Nonlinear Anal., 13 (1989), 373–392.

[15] M. Chlebik and M. Fila, *From critical exponents to blow-up rates for parabolic problems,* Rend. Math., Ser. VII, 19 (1999), 449–470.

[16] M. Chlebik, M. Fila, and P. Quittner, *Blow-up of positive solutions of a semilinear parabolic equation with a gradient term,* Dyn. Contin. Discrete Impulsive Syst., to appear.

[17] M. Chipot and F.B. Weissler, *Some blow up results for a nonlinear parabolic problem with a gradient term,* SIAM J. Math. Anal., 20 (1989), 886–907.

[18] G. Conner and C. Grant, *Asymptotics of blowup for a convection-diffusion equation with conservation,* Diff. Integral Equations, 9 (1996), 719–728.

[19] K. Deng, *Stabilization of solutions of a nonlinear parabolic equation with a gradient term,* Math. Z., 216 (1994), 147–155.

[20] T. Dlotko, *Examples of parabolic problems with blowing-up derivatives,* J. Math. Anal. Appl., 154 (1991), 226–237.

[21] M. Fila, *Remarks on blow up for a nonlinear parabolic equation with a gradient term,* Proc. Amer. Math. Soc., 111 (1991), 795–801.

[22] M. Fila and G. Lieberman, *Derivative blow-up and beyond for quasilinear parabolic equations,* Diff. Integral Equations, 7 (1994), 811–821.

[23] A. Friedman, "Partial Differential Equations of Parabolic Type," Prentice-Hall, 1964.

[24] V.A. Galaktionov and J.L. Vazquez, *Regional blow-up in a semilinear heat equation with convergence to a Hamilton-Jacobi equation,* SIAM J. Math. Anal., 24 (1993), 1254–1276.

[25] V. A. Galaktionov and J. L. Vazquez, *Blow-up for quasilinear heat equations described by means of nonlinear Hamilton–Jacobi equations,* J. Diff. Equations, 127 (1996), 1–40.

[26] Y. Giga, *Interior derivative blow-up for quasilinear parabolic equations,* Discrete Conti. Dyn. Syst., 1 (1995), 449–461.

[27] B. Gilding, M. Guedda, and R. Kersner, *The Cauchy problem for the KPZ equation,* preprint, Université de Picardie, 1998.

[28] S. Kaplan, *On the growth of solutions of quasilinear parabolic equations,* Comm. Pure Appl. Math., 16 (1963), 327–343.

[29] B. Kawohl and L.A. Peletier, *Observations on blow up and dead cores for nonlinear parabolic equations,* Math. Z., 202 (1989), 207–217.

[30] N. Kutev, *Gradient blow-ups and solvability after the blow-up time for nonlinear parabolic equations,* Proc. of the Third International Conference on Evolution Equations, Control Theory and Biomathematics (Ph. Clement and G. Lumer, eds.), Marcel Dekker, 1994, 301–306.

[31] A.A. Lacey, *Global blow-up of a nonlinear heat equation,* Proc. Royal Soc. Edinburgh, 104 (1986), 161–167.

[32] O. Ladyzenskaya, V.A. Solonnikov, and N.N. Uralceva, "Linear and Quasilinear Equations of Parabolic Type," Amer. Math. Soc. Translations, Providence, RI, 1967.

[33] G. Lieberman, *The first initial-boundary value problem for quasilinear second order parabolic equations,* Ann. Scuola Norm. Sup. Pisa, 13 (1986), 347–387.

[34] O. Oleinik and S. Kruzkov, *Quasi-linear parabolic second order equations of many independent variables,* Uspekhi Mat. Nauk, 16 (1961), 115–155.

[35] R.G. Pinsky, *Decay of mass for the equation* $u_t = \Delta u - a(x)|\nabla u|^q$, J. Differ. Equations, 165 (2000), 1–23.

[36] P. Quittner, *Blow-up for semilinear parabolic equations with a gradient term,* Math. Meth. Appl. Sc., 14 (1991), 413–417.

[37] P. Quittner, *On global existence and stationary solutions for two classes of semilinear parabolic equations,* Comment. Math. Univ. Carolinae, 34 (1993), 105–124.

[38] S. Snoussi, S. Tayachi, and F.B. Weissler, *Asymptotically self-similar global solutions of a semilinear parabolic equation with a nonlinear gradient terms,* Proc. Royal Soc. Edinburgh, A129 (1999), 1291–1307.

[39] Ph. Souplet, *Résultats d'explosion en temps fini pour une équation de la chaleur non linéaire,* C. R. Acad. Sc. Paris, 321, Série I (1995), 721–726.

[40] Ph. Souplet, *Finite time blow up for a nonlinear parabolic equation with a gradient term and applications,* Math. Methods. Appl. Sciences, 19 (1996), 1317–1333.

[41] Ph. Souplet, *Blow-up in nonlocal reaction-diffusion equations,* SIAM J. Math. Anal., 29 (1998), 1301–1334.

[42] Ph. Souplet, *Uniform blow-up profiles and boundary behavior for diffusion equations with nonlocal nonlinear source,* J. Differ. Equations, 153 (1999), 374–406.

[43] Ph. Souplet, *Geometry of unbounded domains, Poincaré inequalities and stability in semilinear parabolic equations,* Commun. Partial Differ. Eq., 24 (1999), 951–973.

[44] Ph. Souplet, *Recent results and open problems on parabolic equations with gradient nonlinearities,* Ellectronic J. Differ. Eq.,No. 20 (2001), 1–19.

[45] Ph. Souplet and S. Tayachi, *Blow-up rates for heat equations with gradient terms and for parabolic inequalities,* Colloquium Math., 88 (2001), 135–154.

[46] Ph. Souplet, S. Tayachi, and F.B. Weissler, *Exact self-similar blow-up of solutions of a semilinear parabolic equation with a nonlinear gradient term,* Indiana Univ. Math. J., 48 (1996), 655–682.

[47] Ph. Souplet and F.B. Weissler, *Self-similar sub-solutions and blow-up for nonlinear parabolic equations,* J. Math. Anal. Appl., 212 (1997), 60–74.

[48] Ph. Souplet and F. B. Weissler, *Poincaré's inequality and global solutions of a nonlinear parabolic equation,* Ann. Inst. H. Poincaré, Analyse non linéaire, 16 (1999), 337–373.

[49] S. Tayachi, *Forward self-similar solutions of a semilinear parabolic equation with a nonlinear gradient term,* Diff. Integral Equations, 9 (1996), 1107–1117.