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ON AN ESTIMATE FOR THE WAVE EQUATION AND APPLICATIONS TO NONLINEAR PROBLEMS

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Abstract. We prove estimates for solutions of the Cauchy problem for the inhomogeneous wave equation on $\mathbb{R}^{1+n}$ in a class of Banach spaces whose norms depend only on the size of the space–time Fourier transform. The estimates are local in time, and this allows one, essentially, to replace the symbol of the wave operator, which vanishes on the light cone in Fourier space, with an inhomogeneous symbol, which can be inverted. Our result improves earlier estimates of this type proved by Klainerman–Machedon [4, 5]. As a corollary, one obtains a rather general result concerning local well-posedness of nonlinear wave equations, which was used extensively in the recent article [8].

1. Introduction

Consider the Cauchy problem for the wave equation on $\mathbb{R}^{1+n}$,

$$\Box u = F, \quad u\big|_{t=0} = f, \quad \partial_t u\big|_{t=0} = g, \tag{1.1}$$

where $\Box = -\partial_t^2 + \Delta$ is the wave operator.

The purpose of this note is to prove estimates for u in a certain class of Banach spaces whose norms depend only on the size of the space-time Fourier transform. The estimates are local with respect to time t, and this allows one, essentially, to replace the symbol of the wave operator, which vanishes on the light cone in Fourier space, with an inhomogeneous symbol, which can be inverted. This idea originates in the work of Bourgain [1] on the Schrödinger and KdV equations, and was later simplified by Kenig–Ponce–Vega [3] in their work on KdV. Following this, Klainerman–Machedon [4, Lemma 4.3], [5, Lemma 1.3] proved estimates of this type for the wave equation; see also Klainerman–Tataru [9].

The improvement in our result compared to [3, 4, 5] lies mainly in showing, as suggested in [9, Remark 1.8], that for sufficiently small $\varepsilon > 0$, the norm of

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the inhomogeneous part of the solution, restricted to the time slab $[0,T]\times\mathbb{R}^n$, is $O(T^\varepsilon)$ as $T \to 0$, at the expense of a loss of essentially ε derivatives. As shown in the recent article [8], this allows one to remove the assumption of small-norm data in the well-posedness results proved in [4, 5, 6, 7, 9].

As an application of our estimate, we also prove a simple but useful result concerning local well-posedness of nonlinear wave equations, which is used extensively in [8].

2. The main estimate

We are interested in finding complete subspaces $\mathcal{X}^s$ of

$$C_b(\mathbb{R}, H^s) \cap C_b^1(\mathbb{R}, H^{s-1}) \tag{2.1}$$

such that solutions of (1.1) satisfy estimates of the type

$$\|u\|_{\mathcal{X}_T^s} \le C \, \|(f,g)\|_{(s)} + C_{T,\varepsilon} \left\| \widetilde{\square}^{-1} F^\varepsilon \right\|_{\mathcal{X}^s} \tag{2.2}$$

for all $0 < T < 1$ and $\varepsilon \ge 0$. We also want

$$\lim_{T\to 0} C_{T,\varepsilon} = 0 \tag{2.3}$$

when ε is strictly positive.

Precise definitions will be supplied presently. For the moment suffice it to say that $\mathcal{X}_T^s$ stands for the restriction to the time slab $[0,T]\times\mathbb{R}^n$, $\widetilde{\square}$ may be thought of as an inhomogeneous and invertible version of the wave operator $\square$, F^ε is F with a certain operator of order ε applied to it, and

$$\|(f,g)\|_{(s)} = \|f\|_{H^s} + \|g\|_{H^{s-1}}$$

with H^s the usual Sobolev space.

We use coordinates (t,x) on $\mathbb{R}^{1+n}$. The Fourier transform of $f(x)$ (respectively $u(t,x)$) is denoted $\widehat{f}(\xi) = \mathcal{F}f(\xi)$ (respectively $\widehat{u}(\tau,\xi) = \mathcal{F}u(\tau,\xi)$). For any $\alpha \in \mathbb{R}$ we define pseudodifferential operators Λ^α, Λ_+^α and Λ_-^α by

$$\widehat{\Lambda^\alpha f}(\xi) = \left(1+|\xi|^2\right)^{\alpha/2} \widehat{f}(\xi), \quad \widehat{\Lambda_+^\alpha u}(\tau,\xi) = \left(1+\tau^2+|\xi|^2\right)^{\alpha/2} \widehat{u}(\tau,\xi),$$

$$\widehat{\Lambda_-^\alpha u}(\tau,\xi) = \left(1 + \frac{(\tau^2-|\xi|^2)^2}{1+\tau^2+|\xi|^2}\right)^{\alpha/2} \widehat{u}(\tau,\xi).$$

Observe that the Fourier symbol of Λ_-^α is comparable to $\left(1+\big||\tau|-|\xi|\big|\right)^\alpha$. The operator $\widetilde{\square}$ in (2.2) is just $\Lambda_+\Lambda_-$, and $F^\varepsilon = \Lambda_-^\varepsilon F$.

The Sobolev and Wave Sobolev spaces H^s and $H^{s,\theta}$ are given by the norms

$$\|f\|_{H^s} = \|\Lambda^s f\|_{L^2(\mathbb{R}^n)},$$

$$\|u\|_{H^{s,\theta}} = \left\|\Lambda^s \Lambda_-^\theta u\right\|_{L^2(\mathbb{R}^{1+n})}.$$

For the basic properties of the latter, see, e.g., [8]. In particular, we shall use the fact that $H^{s,\theta}$ embeds in $C_b(\mathbb{R}, H^s)$ when $\theta > \frac{1}{2}$. Associated to $H^{s,\theta}$ is the space $\mathcal{H}^{s,\theta}$ with norm

$$\|u\|_{\mathcal{H}^{s,\theta}} = \|u\|_{H^{s,\theta}} + \|\partial_t u\|_{H^{s-1,\theta}} \sim \left\|\Lambda^{s-1}\Lambda_+\Lambda_-^\theta u\right\|_{L^2}.$$

By the above, $\mathcal{H}^{s,\theta}$ embeds in (2.1) for $\theta > \frac{1}{2}$.

In general, if a Banach space $\mathcal{X}^s$ embeds in (2.1) then it makes sense to restrict its elements to any interval $I \subset \mathbb{R}$. The resulting restriction space is denoted $\mathcal{X}_I^s$. It is always possible to define a norm on this space which makes it complete. Indeed, $\mathcal{X}_I^s$ is the quotient space $\mathcal{X}^s/{\sim_I}$, where $\sim_I$ is the equivalence relation

$$u \sim_I v \iff u(t) = v(t) \text{ for all } t \in I.$$

Since $\mathcal{X}^s$ embeds in (2.1), the equivalence classes are closed sets in $\mathcal{X}^s$, so the quotient space is complete when equipped with the norm

$$\|u\|_{\mathcal{X}_I^s} = \inf_{v \sim_I u} \|v\|_{\mathcal{X}^s}.$$

If $I = [0,T]$, we always write $\mathcal{X}_T^s$ instead of $\mathcal{X}_I^s$.

We now state the precise result.

Theorem 1. *Let $\mathcal{X}^s$ be a Banach space such that*

(i) *$\mathcal{X}^s$ embeds in $\mathcal{H}^{s,\theta}$ for some $\theta > \frac{1}{2}$,*

(ii) $|\widehat{u}| \le |\widehat{v}| \implies \|u\|_{\mathcal{X}^s} \le \|v\|_{\mathcal{X}^s}$,

(iii) *there exists $\gamma < 2$ such that*[1]

$$\|u\|_{\mathcal{X}^s} \lesssim \left\|\mathcal{F}\Lambda^{s-1}\Lambda_+\Lambda_-^\gamma u(\tau,\xi)\right\|_{L_\xi^2(L_\tau^\infty)}$$

for all u.

Let $\varepsilon \ge 0$. Then for all $(f,g) \in H^s \times H^{s-1}$ and $F \in \widetilde{\Box}\Lambda_-^{-\varepsilon}(\mathcal{X}^s)$, there is a unique $u \in C(\mathbb{R}, H^s) \cap C^1(\mathbb{R}, H^{s-1})$ which solves (1.1), *and the estimate* (2.2) *holds for all $0 < T < 1$. Moreover, if $\varepsilon > 0$, then* (2.3) *holds.*

Remarks. (1) Estimate (2.2) in the special case $\mathcal{X}^s = \mathcal{H}^{s,\theta}$ and $\varepsilon = 0$ was proved in [4].

(2) The proof gives something stronger: for all $T > 0$, there is a linear operator W_T such that

$$\Box W_T F = F \quad \text{on} \quad [0,T] \times \mathbb{R}^n,$$

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[1] We use the notation $\|v(\tau,\xi)\|_{L_\xi^2(L_\tau^\infty)} = \left(\int \|v(\cdot,\xi)\|_{L^\infty}^2 \, d\xi\right)^{1/2}$.

with vanishing initial data at $t = 0$, and W_T is bounded from

$$\mathcal{Y}^{s,\varepsilon} = \widetilde{\Box}\Lambda_-^{-\varepsilon}(\mathcal{X}^s) \tag{2.4}$$

into $\mathcal{X}^s$ for all $\varepsilon \geq 0$. Moreover, when $\varepsilon > 0$, the operator norm

$$\|W_T\|_{\mathcal{Y}^{s,\varepsilon} \to \mathcal{X}^s} \to 0 \quad \text{as} \quad T \to 0.$$

(3) The reason for the condition $\gamma < 2$ is as follows. The proof of the theorem shows that

$$C_{T,\varepsilon} \leq CT^{\min\{\alpha\varepsilon,\delta\}},$$

where $\delta = 2-\gamma-\alpha/2+\min\{0,\alpha(\theta+\varepsilon-1)\}$ and $0 \leq \alpha \leq 1$ can be chosen at will. The constant C is independent of ε and α. In the typical applications γ is close to 1; see [8].

(4) s plays no role. Indeed, if $\mathcal{X}^s$ satisfies the hypotheses of the theorem, then so does $\mathcal{X}^{s'} = \Lambda^{s-s'}\mathcal{X}^s$ for all s'.

3. APPLICATIONS

For $t \in \mathbb{R}$ we denote by τ_t the time-translation operator $\tau_t u = u(\cdot + t, \cdot)$, and for any interval $I \subseteq \mathbb{R}$ we denote restriction to the time-slab $I \times \mathbb{R}^n$ by $|_I$.

Suppose $\mathcal{X}^s$ satisfies the hypotheses of Theorem 1, is invariant under time-translation[2], and for all $\phi \in C_c^\infty(\mathbb{R})$, the multiplication map $u \mapsto \phi(t)u(t,x)$ is bounded from $\mathcal{X}^s$ into itself.

Consider a system of wave equations on $\mathbb{R}^{1+n}$ of the form

$$\Box u = \mathcal{N}(u), \tag{3.1}$$

where u takes values in $\mathbb{R}^N$ and $\mathcal{N} : \mathcal{X}^s \to \mathcal{D}'$ is (i) time-translation invariant ($\mathcal{N}$ commutes with τ_t); (ii) local in time[3]; and (iii) satisfies $\mathcal{N}(0) = 0$.

Furthermore, we assume that for some $\varepsilon > 0$, $\mathcal{N}$ satisfies, with notation as in (2.4), a Lipschitz condition

$$\|\mathcal{N}(u) - \mathcal{N}(v)\|_{\mathcal{Y}^{s,\varepsilon}} \leq A\big(\max\{\|u\|_{\mathcal{X}^s}, \|v\|_{\mathcal{X}^s}\}\big)\,\|u-v\|_{\mathcal{X}^s} \tag{3.2}$$

for all $u, v \in \mathcal{X}^s$, where A is a continuous function.

Theorem 2. *Under the above assumptions, the Cauchy problem for* (3.1) *is locally well-posed for initial data in $H^s \times H^{s-1}$, in the following sense:*

[2] That is to say, τ_t is an isomorphism of $\mathcal{X}^s$ for all t.

[3] By this we mean that if $u|_I = v|_I$, where I is an open interval, then $\mathcal{N}(u)|_I = \mathcal{N}(v)|_I$.

(I) (***Local existence***) *For all* $(f,g) \in H^s \times H^{s-1}$ *there exist a* $T > 0$ *and a* $u \in \mathcal{X}^s_T$ *which solves* (3.1) *on* $S_T = (0,T) \times \mathbb{R}^n$ *with initial data* (f,g)*. Moreover,* T *can be chosen to depend continuously on* $\|f\|_{H^s} + \|g\|_{H^{s-1}}$*.*

(II) (***Uniqueness***) *If* $T > 0$ *and* $u, u' \in \mathcal{X}^s_T$ *are two solutions of* (3.1) *on* S_T *with the same initial data* (f,g)*, then* $u = u'$*.*

(III) (***Continuous dependence on initial data***) *If, for some* $T > 0$*,* $u \in \mathcal{X}^s_T$ *solves* (3.1) *on* S_T *with initial data* (f,g)*, then for all* $(f',g') \in H^s \times H^{s-1}$ *sufficiently close to* (f,g)*, there exists a* $u' \in \mathcal{X}^s_T$ *which solves* (3.1) *on* S_T *with initial data* (f',g')*, and*

$$\left\|u - u'\right\|_{\mathcal{X}^s_T} \le C \left\|(f - f', g - g')\right\|_{(s)}.$$

If, moreover, $\mathcal{N}$ *is* C^∞ *as a map from* $\mathcal{X}^s$ *into* $\mathcal{Y}^{s,\varepsilon}$*, then we have*

(IV) (***Smooth dependence on initial data***) *Suppose* $u \in \mathcal{X}^s_T$ *solves* (3.1) *on* S_T *for some* $T > 0$*, with initial data* $(f,g) \in H^s \times H^{s-1}$*, and that* (f_δ, g_δ) *is a smooth perturbation of the initial data; i.e.,*

$$\delta \mapsto (f_\delta, g_\delta), \qquad \mathbb{R} \to H^s \times H^{s-1}$$

is C^∞ *and takes the value* (f,g) *at* $\delta = 0$*. Let* u_δ *be the corresponding solution of* (3.1) *(by (III),* $u_\delta \in \mathcal{X}^s_T$ *for* $|\delta| < \delta_0$*). Then the map* $\delta \mapsto u_\delta$ *from* $[-\delta_0, \delta_0]$ *into* $\mathcal{X}^s_T$ *is* C^∞*.*

We write $\mathcal{X}^\sigma = \Lambda^{s-\sigma}\mathcal{X}^s$ *and* $\mathcal{Y}^{\sigma,\varepsilon} = \Lambda^{s-\sigma}\mathcal{Y}^{s,\varepsilon}$ *for* $\sigma \in \mathbb{R}$*. If for all* $\sigma > s$ *there is a continuous function* A_σ *such that*

$$\|\mathcal{N}(u)\|_{\mathcal{Y}^{\sigma,\varepsilon}} \le A_\sigma\big(\|u\|_{\mathcal{X}^s}\big)\,\|u\|_{\mathcal{X}^\sigma} \tag{3.3}$$

for all $u \in \mathcal{X}^s \cap \mathcal{X}^\sigma$*, then we have*

(V) (***Persistence of higher regularity***) *If* $\sigma > s$ *and* $u \in \mathcal{X}^s_T$ *solves* (3.1) *on* S_T *with initial data* $(f,g) \in H^\sigma \times H^{\sigma-1}$ *for some* $T > 0$*, then*

$$u \in C\big([0,T], H^\sigma\big) \cap C^1\big([0,T], H^{\sigma-1}\big).$$

Remark. Typically, proving that $\mathcal{N}$ is C^∞ is no harder than proving it is locally Lipschitz. As an example, relevant for wave maps, consider

$$\mathcal{N}(u) = \Gamma(u) Q_0(u,u),$$

where u is real-valued, $\Gamma : \mathbb{R} \to \mathbb{R}$ is C^∞ and Q_0 is the bilinear "null form"

$$Q_0(u,v) = -\partial_t u\, \partial_t v + \nabla_x u \cdot \nabla_x v.$$

Fix $s > \frac{n}{2}$ and set $\mathcal{X} = \mathcal{H}^{s,\theta}$, $\mathcal{X}' = H^{s,\theta}$ and $\mathcal{Y} = \widetilde{\Box}\Lambda_-^{-\varepsilon}(\mathcal{X}) = H^{s-1,\theta+\varepsilon-1}$. One can show, for appropriate $\theta > \frac{1}{2}$ and $\varepsilon > 0$, that

(i) Q_0 is bounded, hence C^∞, from $\mathcal{X} \times \mathcal{X}$ to $\mathcal{Y}$;
(ii) $\Phi_\Gamma : u \mapsto \Gamma(u) - \Gamma(0)$ is a locally bounded map of $\mathcal{X}'$ to itself;
(iii) multiplication is bounded, hence C^∞, from $\mathcal{X}' \times \mathcal{Y}$ to $\mathcal{Y}$.

It remains to prove that Φ_Γ is C^∞ as a map of $\mathcal{X}'$, but this follows from (ii) (which is valid for any C^∞ function Γ) and the fact that $\mathcal{X}'$ is an algebra. Indeed, since

$$\Gamma(v) - \Gamma(u) = \int_0^1 \Gamma'\big(u + t(v-u)\big)(v-u)\,dt,$$

Φ_Γ is locally Lipschitz. Then, since

$$\Gamma(v) - \Gamma(u) - \Gamma'(u)(v-u) = \int_0^1 \big\{\Gamma'\big(u+t(v-u)\big) - \Gamma'(u)\big\}\ dt \cdot (v-u)$$

we have

$$\big\|\Gamma(v) - \Gamma(u) - \Gamma'(u)(v-u)\big\|_{\mathcal{X}'} \le C(u,v)\,\|v-u\|_{\mathcal{X}'}\,,$$

where

$$C(u,v) \le C \int_0^1 \big\|\Gamma'\big(u+t(v-u)\big) - \Gamma'(u)\big\|_{\mathcal{X}'}\ dt.$$

But by the above, $\Phi_{\Gamma'}$ is locally Lipschitz, so $C(u,v) = O(\|v-u\|_{\mathcal{X}'})$. Thus Φ_Γ is C^1, and by induction C^∞.

4. Abstract local well-posedness

Theorem 2 is conveniently proved in an abstract setting, which we discuss here.

Observe that if $\mathcal{X}^s$ is a space satisfying the assumptions of the last section, and $\mathcal{I}$ denotes the set of compact intervals $I \subseteq \mathbb{R}$, then the family $\{\mathcal{X}^s_I\}_{I \in \mathcal{I}}$ has the following properties:

(S1) $\mathcal{X}^s_I$ embeds in $C(I, H^s) \cap C^1(I, H^{s-1})$ for all $I \in \mathcal{I}$;
(S2) the solution of $\Box u = 0$ with $(u, \partial_t u)\big|_{t=0} = (f,g) \in H^s \times H^{s-1}$ satisfies

$$\|u\|_{\mathcal{X}^s_T} \le C\,\|(f,g)\|_{(s)}$$

for all $0 < T < 1$ and all (f,g);
(S3) τ_t is an isometry from $\mathcal{X}^s_I$ onto $\mathcal{X}^s_{-t+I}$ for all $t \in \mathbb{R}$ and $I \in \mathcal{I}$;
(S4) if $I \subseteq J$, then $|_I : \mathcal{X}^s_J \to \mathcal{X}^s_I$ is norm decreasing;
(S5) whenever I and J are two overlapping intervals ($I \cap J$ has nonempty interior) and $u \in \mathcal{X}^s_I$, $v \in \mathcal{X}^s_J$ agree on the overlap ($u(t) = v(t)$ for all $t \in I \cap J$), then

$$\|w\|_{\mathcal{X}^s_{I\cup J}} \le C_{I,J}\left(\|u\|_{\mathcal{X}^s_I} + \|v\|_{X^s_J}\right),$$

where $w(t) = u(t)$ for $t \in I$ and $w(t) = v(t)$ for $t \in J$.

Note that (S2) holds by Theorem 1; we write $\mathcal{X}_T^s$ instead of $\mathcal{X}_{[0,T]}^s$. (S5) holds by the assumption that $u \mapsto \phi(t)u(t,x)$ is bounded on $\mathcal{X}^s$ for all $\phi \in C_c^\infty(\mathbb{R})$.

We now consider an abstract family of spaces with these properties.

Theorem 3. *Let $s \in \mathbb{R}$. Let $\{\mathcal{X}_I^s\}_{I \in \mathcal{I}}$ be a family of Banach spaces satisfying (S1–5). Consider the system* (3.1)*, where $\mathcal{N}$ is an operator which*

- (N1) *maps $\mathcal{X}_{[a,b]}^s$ into $\mathcal{D}'\big((a,b) \times \mathbb{R}^n\big)$ for all $-\infty < a < b < \infty$;*
- (N2) *is time-translation invariant: $\mathcal{N} \circ \tau_t = \tau_t \circ \mathcal{N}$;*
- (N3) *is local in time: $\mathcal{N}(u|_{[a,b]}) = \mathcal{N}(u)|_{(a,b)}$ whenever $[a,b] \subset I$ and $u \in \mathcal{X}_I^s$;*
- (N4) *satisfies $\mathcal{N}(0) = 0$.*

Assume further that for all $0 < T < 1$ and $u \in \mathcal{X}_T^s$, there exists $v \in \mathcal{X}_T^s$ (necessarily unique) which solves $\Box v = \mathcal{N}(u)$ on $S_T = (0,T) \times \mathbb{R}^n$ with vanishing initial data at $t = 0$; we write $v = W\mathcal{N}(u)$ and assume that

$$\|W\mathcal{N}(u)\|_{\mathcal{X}_T^s} \leq C_T A\big(\|u\|_{\mathcal{X}_T^s}\big)\,\|u\|_{\mathcal{X}_T^s} \tag{4.1}$$

and, more generally,

$$\big\|W\big(\mathcal{N}(u) - \mathcal{N}(v)\big)\big\|_{\mathcal{X}_T^s} \leq C_T A\big(\max\{\|u\|_{\mathcal{X}_T^s}, \|v\|_{\mathcal{X}_T^s}\}\big)\,\|u - v\|_{\mathcal{X}_T^s} \tag{4.2}$$

for all $0 < T < 1$ and $u, v \in \mathcal{X}_T^s$, where

$$\lim_{T \to 0^+} C_T = 0 \tag{4.3}$$

and A is a continuous function.

Then the system (3.1) *is locally well-posed for initial data in $H^s \times H^{s-1}$, in the sense that properties (I–III) of Theorem 2 hold.*

Remarks. (1) (*Lifespan.*) By local existence (I), and properties (S4) and (N3), for given (f,g) the set $E(f,g)$ consisting of all $T > 0$ for which there exists $u \in \mathcal{X}_T^s$ solving (3.1) on S_T with data (f,g), is a nonempty interval. Then, by local existence (I) and uniqueness (II), as well as properties (S3–5) and (N2,3), it follows that this interval is open. Moreover, by continuous dependence on initial data (III), the *lifespan* $T^* = \sup E(f,g)$ is a lower-semicontinuous function of (f,g).

(2) (*Higher regularity.*) Set $\mathcal{X}_T^\sigma = \Lambda^{s-\sigma}\mathcal{X}_T^s$. If, for any $\sigma > s$, there is a continuous A_σ such that

$$\|W\mathcal{N}(u)\|_{\mathcal{X}_T^\sigma} \leq C_T A_\sigma\big(\|u\|_{\mathcal{X}_T^s}\big)\,\|u\|_{\mathcal{X}_T^\sigma} \tag{4.4}$$

and

$$\begin{aligned}\big\|W\big(\mathcal{N}(u)-\mathcal{N}(v)\big)\big\|_{\mathcal{X}_T^\sigma} &\le C_T A_\sigma\big(\max\{\|u\|_{\mathcal{X}_T^s},\|v\|_{\mathcal{X}_T^s}\}\big)\,\|u-v\|_{\mathcal{X}_T^\sigma}\\ &\quad + C_T A_\sigma\big(\max\{\|u\|_{\mathcal{X}_T^\sigma},\|v\|_{\mathcal{X}_T^\sigma}\}\big)\,\|u-v\|_{\mathcal{X}_T^s} \qquad (4.5)\end{aligned}$$

for all $u, v \in \mathcal{X}_T^s \cap \mathcal{X}_T^\sigma$, where C_T is the constant appearing in (4.1) and (4.2), then (V) of Theorem 2 holds.

(3) (*Smooth dependence on data.*) Suppose $S = W\mathcal{N}$ is not just Lipschitz but C^∞ as a map of $\mathcal{X}_T^s$. Recall that the k-th derivative $S^{(k)}(u)$, $u \in \mathcal{X}_T^s$, is a k-linear map from $\mathcal{X}_T^s \times \cdots \times \mathcal{X}_T^s$ into $\mathcal{X}_T^s$; let $\big\|S^{(k)}(u)\big\|_{(T)}$ denote its operator norm. In view of (4.5),

$$\big\|S'(u)\big\|_{(T)} \le C_T A\big(\|u\|_{\mathcal{X}_T^s}\big). \qquad (4.6)$$

Suppose there exist, for $k = 2, 3, \dots$, B_k continuous and increasing so that

$$\sup_{0<T<1}\ \sup_{\|u\|_{\mathcal{X}_T^s}\le R} \big\|S^{(k)}(u)\big\|_{(T)} \le B_k(R). \qquad (4.7)$$

Then (IV) of Theorem 2 holds.

(4) If $\mathcal{N}$ is multilinear, then so is $W\mathcal{N}$, so if the latter is bounded on some Banach space, it is trivially C^∞ on that space. Thus, by the previous remark, the dependence on initial data is C^∞.

In this connection, we mention an interesting observation due to Keel–Tao [2, Section 8], concerning the feasibility of proving well-posedness for wave maps in the critical data space by an iteration argument. For simplicity we take $n = 2$, but this is not essential.

A wave map $u : \mathbb{R}^{1+2} \to S^1 \subseteq \mathbb{C}$ satisfies the equation

$$\Box u + u(\partial_\mu u \cdot \partial^\mu u) = 0, \qquad (4.8a)$$

where $\cdot$ is the Euclidean inner product on $\mathbb{R}^2 = \mathbb{C}$. Consider initial data

$$u\big|_{t=0} = 1, \qquad \partial_t u\big|_{t=0} = ig, \qquad (4.8b)$$

where i is the imaginary unit and $g \in L^2$ is real-valued.

Observe that if (4.8) is well-posed for $g \in L^2$, then the solutions stay on S^1 and hence are wave maps. This is certainly true for any smooth solution,[4] and therefore true in general by an approximation argument, using the continuous dependence on initial data.

[4] If $u : \mathbb{R}^{1+2} \to \mathbb{R}^2$ is a smooth solution of (4.8) then $\phi = u \cdot u - 1$ solves the linear equation $\Box\phi + \phi(\partial_\mu u \cdot \partial^\mu u) = 0$ with vanishing initial data, so by uniqueness, ϕ must vanish.

If (4.8) could be proved well-posed for $g \in L^2$ by an iteration argument in some Banach space, the dependence on the initial data would necessarily be C^∞, since in this case the operator $\mathcal{N}$ is trilinear.

As it turns out, (4.8) is well-posed for $g \in L^2$, but the dependence on the data is not even C^2. In fact, since $\theta \mapsto e^{i\theta}$, $\mathbb{R} \to S^1$ is a geodesic, the solution of (4.8) is given by $u = e^{iv}$, where $\Box v = 0$ with initial data $(0, g)$; clearly $(u, \partial_t u)$ belongs to $C(\mathbb{R}, \dot{H}^1 \times L^2)$ and depends continuously on g.

Thus $u_\varepsilon = e^{i\varepsilon v}$ solves (4.8) with g replaced by εg. But since $\dot{H}^1$ is not an algebra, one would not expect the map $\varepsilon \to u_\varepsilon|_{t=1}$, $\mathbb{R} \to \dot{H}^1$ to be twice differentiable at $\varepsilon = 0$ for all choices of g, and indeed it is not, as proved in [2, Proposition 8.3].

5. Proof of Theorem 1

In this section, (i–iii) refer to the hypotheses of the theorem. With notation as in (2.4), observe that by (i),

$$\mathcal{Y}^{s,\varepsilon} \subseteq H^{s-1,\theta+\varepsilon-1} \tag{5.1}$$

for all $\varepsilon \geq 0$, and that $\mathcal{Y}^{s,\varepsilon} \subseteq \mathcal{Y}^{s,0}$.

We shall use the fact that the symbol of Λ_- is comparable to $1 + \big||\tau| - |\xi|\big|$. More precisely, there is a constant $c > 0$ such that

$$c^{-1}\big(1 + \big||\tau| - |\xi|\big|\big) \leq \left(1 + \frac{(\tau^2 - |\xi|^2)^2}{1 + |\tau|^2 + |\xi|^2}\right)^{\frac{1}{2}} \leq c\big(1 + \big||\tau| - |\xi|\big|\big). \tag{5.2}$$

Fix two bump functions $\chi, \phi \in C_c^\infty(\mathbb{R})$ such that $0 \leq \chi, \phi \leq 1$, $\chi = 1$ on $[-2, 2]$, and $\phi = 1$ on $[-2c, 2c]$ with support in $[-4c, 4c]$, where c is the constant in (5.2).

Fix $0 \leq \alpha \leq 1$. Given $0 < T < 1$, write, for any $F \in H^{s-1,\theta-1}$,

$$F = \phi\big(T^\alpha \Lambda_-\big)F + \big\{I - \phi\big(T^\alpha \Lambda_-\big)\big\}\, F = F_1 + F_2,$$

where I denotes the identity operator. In view of (5.2),

$$\big||\tau| - |\xi|\big| \leq 4c^2 T^{-\alpha} \quad \text{for} \quad (\tau, \xi) \in \operatorname{supp} \widehat{F_1}, \tag{5.3}$$

$$\big||\tau| - |\xi|\big| \geq T^{-\alpha} \quad \text{for} \quad (\tau, \xi) \in \operatorname{supp} \widehat{F_2}. \tag{5.4}$$

Now define $u = \chi(t)u_0 + \chi(t/T)u_1 + u_2$, where

$$u_0 = \partial_t W(t) f + W(t) g \quad \text{with} \quad W(t) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}},$$

$$u_1 = -\int_0^t W(t - t')F_1(t', \cdot)\, dt', \quad \widehat{u}_2 = (\tau^2 - |\xi|^2)^{-1} \widehat{F}_2.$$

Observe that $F_1 \in H^{s-1,0} \subseteq L^1_{\text{loc}}(\mathbb{R}, H^{s-1})$, so u_1 is well-defined.

Lemma 1. *u solves (1.1) on $[0,T] \times \mathbb{R}^n$.*

Proof. The only point which is not evident is that $u_2\big|_{t=0} = \partial_t u_2\big|_{t=0} = 0$. This is clearly true when $F \in \mathcal{S}$, since then $F_2 \in \mathcal{S}$, so u_2 is necessarily given by Duhamel's formula. The general case then follows by density, since clearly $F \mapsto u_2$ is linear and bounded from $H^{s-1,\theta-1}$ into $\mathcal{H}^{s,\theta}$, and the latter space embeds in (2.1). □

Using hypothesis (iii) only, we shall prove the following lemmas.

Lemma 2. *For all $(f,g) \in H^s \times H^{s-1}$,*

$$\|\chi(t)u_0\|_{\mathcal{X}^s} \le C\,\|(f,g)\|_{(s)}\,.$$

Lemma 3. *For all $\varepsilon \ge 0$ and $F \in H^{s-1,\theta+\varepsilon-1}$,*

$$\|\chi(t/T)u_1\|_{\mathcal{X}^s} \le CT^\delta\,\|F\|_{H^{s-1,\theta+\varepsilon-1}}\,,$$

where

$$\delta = 2 - \gamma - \alpha/2 + \min\big\{0, \alpha(\theta+\varepsilon-1)\big\}, \tag{5.5}$$

γ is as in hypothesis (iii), and C is independent of ε and α.

Since by (i) we have (5.1), and since by (5.4) and (ii) we clearly have

$$\|u_2\|_{\mathcal{X}^s} \le CT^{\alpha\varepsilon}\,\|F\|_{\mathcal{Y}^{s,\varepsilon}}\,, \tag{5.6}$$

the theorem follows.

We turn to the proofs of Lemmas 2 and 3.

Some notation: for $\gamma \in \mathbb{R}$, let $\mathcal{D}^\gamma$ be defined by $\widehat{\mathcal{D}^\gamma\chi}(\tau) = (1+|\tau|^2)^{\gamma/2}\widehat{\chi}(\tau)$. In what follows, $p \lesssim q$ means that $p \le Cq$ for some positive constant C independent of α and ε.

5.1. **Proof of Lemma 3.** We write $F_1 = F_{1,1} + F_{1,2}$, where $\widehat{F_{1,1}}(\tau,\xi)$ and $\widehat{F_{1,2}}(\tau,\xi)$ are supported in the regions $|\xi| \le T^{-\alpha}$ and $|\xi| \ge T^{-\alpha}$ respectively. Let $u_{1,j}$ be defined as u_1, but with F_1 replaced by $F_{1,j}$ for $j = 1,2$.

The following lemma, proved in Section 5.3, characterizes $u_{1,1}$ and $u_{1,2}$.

Lemma 4. *Given $0 < T < 1$, let $u_{1,j}$, $j = 1,2$ be defined as above. There exist sequences $f_j^+, f_j^- \in H^s$ and $g_j \in C([0,1], H^{s-1})$ such that*

$$\operatorname{supp}\widehat{f_j^\pm} \subseteq \{\xi : |\xi| \ge T^{-\alpha}\}, \quad \operatorname{supp}\widehat{g_j(\rho)} \subseteq \{\xi : |\xi| \le T^{-\alpha}\},$$

$$\left\|f_j^\pm\right\|_{H^s}, \sup_{0\le\rho\le 1}\|g_j(\rho)\|_{H^{s-1}} \lesssim T^{\alpha(1/2-j)}\,\|F_1\|_{H^{s-1,0}} \tag{5.7}$$

for $j = 1, 2, \dots,$ *and*

$$u_{1,1} = \sum\nolimits_1, \qquad u_{1,2} = \sum\nolimits_2 + E, \tag{5.8}$$

where

$$\sum\nolimits_1 = \sum_{j=1}^{\infty} \frac{t^{j+1}}{j!} \int_0^1 e^{it(2\rho-1)\sqrt{-\Delta}} g_j(\rho)\, d\rho,$$

$$\sum\nolimits_2 = \sum_{j=1}^{\infty} \frac{t^j}{j!} \left(e^{it\sqrt{-\Delta}} f_j^+ + e^{-it\sqrt{-\Delta}} f_j^- \right)$$

and, given $\gamma \geq 1$,

$$\sup_{\tau \in \mathbb{R}} \left| \mathcal{F}\Lambda^s \Lambda_-^\gamma \{\chi E\}(\tau,\xi) \right| \lesssim \left(\left\| \widehat{\mathcal{D}^{\gamma-1}\chi} \right\|_{L^\infty} + \left\| \widehat{\mathcal{D}^{\gamma}(t\chi)} \right\|_{L^\infty} \right) \int \left| \mathcal{F}\Lambda^{s-1} F_{1,2}(\lambda,\xi) \right| \, d\lambda \tag{5.9}$$

for all $\xi \in \mathbb{R}^n$ *and* $\chi \in C_c^\infty(\mathbb{R})$. *Moreover,*

$$\widehat{\chi u_{1,2}}(\tau,\xi) = \frac{1}{2\pi} \int \int_0^1 \int_0^1 (1-\rho) \widehat{\chi}''(\tau - \mu) \widehat{F_{1,2}}(\lambda,\xi)\, d\sigma\, d\rho\, d\lambda, \tag{5.10}$$

where

$$\mu = b + \sigma(a-b), \quad a = |\xi| + \rho(\lambda - |\xi|), \quad b = -|\xi| + \rho(\lambda + |\xi|) \tag{5.11}$$

and $\chi \in C_c^\infty(\mathbb{R})$.

We need two more lemmas.

Lemma 5. *If* $\sigma \in \mathbb{R}$, $\gamma \geq 0$, $f \in H^\sigma$ *and* $\chi \in C_c^\infty(\mathbb{R})$, *then*

$$\left\| \mathcal{F}\Lambda^\sigma \Lambda_-^\gamma \{\chi(t) e^{\pm it\sqrt{-\Delta}} f\}(\tau,\xi) \right\|_{L_\xi^2(L_\tau^\infty)} \lesssim \left\| \widehat{\mathcal{D}^\gamma \chi} \right\|_{L^\infty} \|f\|_{H^\sigma}. \tag{5.12}$$

If $-1 \leq \rho \leq 1$, $g \in H^\sigma$ *and* $\operatorname{supp} \widehat{g} \subseteq \{\xi : |\xi| \leq r\}$, *then*

$$\left\| \mathcal{F}\Lambda^\sigma \Lambda_-^\gamma \{\chi(t) e^{i\rho t\sqrt{-\Delta}} g\}(\tau,\xi) \right\|_{L_\xi^2(L_\tau^\infty)} \lesssim \left(\left\| \widehat{\mathcal{D}^\gamma \chi} \right\|_{L^\infty} + r^\gamma \|\widehat{\chi}\|_{L^\infty} \right) \|g\|_{H^\sigma}. \tag{5.13}$$

Proof. This is a triviality. To prove (5.12), simply note that the Fourier transform of $\chi(t)e^{\pm it\sqrt{-\Delta}} f$ is $\widehat{\chi}(\tau \mp |\xi|)\widehat{f}(\xi)$. To prove (5.13), observe that the Fourier transform of $\chi(t)e^{i\rho t\sqrt{-\Delta}} g$ is $\widehat{\chi}(\tau - \rho|\xi|)\widehat{g}(\xi)$ and that

$$\left| |\tau| - |\xi| \right| \leq \left| \tau - \rho|\xi| \right| + (1 - |\rho|)|\xi| \leq \left| \tau - \rho|\xi| \right| + r$$

for $\xi \in \operatorname{supp} \widehat{g}$ and $-1 \leq \rho \leq 1$. $\square$

Lemma 6. *Assume that the inequality*

$$\|u\|_{\mathcal{X}^s} \lesssim \left\|\mathcal{F}\Lambda^{s-1}\Lambda_+\Lambda_-^\gamma u(\tau,\xi)\right\|_{L^2_\xi(L^\infty_\tau)}$$

of hypothesis (iii) of Theorem 1 holds for some $\gamma \in \mathbb{R}$. *Then for all* $\chi \in C_c^\infty(\mathbb{R})$ *and* $f \in H^s$,

$$\left\|\chi(t)e^{\pm it\sqrt{-\Delta}}f\right\|_{\mathcal{X}^s} \lesssim \left\|\widehat{\mathcal{D}^\gamma\chi}\right\|_{L^\infty}\|f\|_{H^s} + \left\|\widehat{\mathcal{D}^{\gamma+1}\chi}\right\|_{L^\infty}\|f\|_{H^{s-1}}. \tag{5.14}$$

Moreover, if $-1 \le \rho \le 1$, $g \in H^{s-1}$ *and* $\operatorname{supp}\widehat{g} \subseteq \{\xi : |\xi| \le r\}$, *where* $r \ge 1$, *then*

$$\left\|\chi(t)e^{i\rho t\sqrt{-\Delta}}g\right\|_{\mathcal{X}^s} \lesssim \left(r^{\gamma+1}\|\widehat{\chi}\|_{L^\infty} + r\left\|\widehat{\mathcal{D}^\gamma\chi}\right\|_{L^\infty} + \left\|\widehat{\mathcal{D}^{\gamma+1}\chi}\right\|_{L^\infty}\right)\|g\|_{H^{s-1}}. \tag{5.15}$$

Proof. Observe that

$$\begin{aligned}&\left\|\mathcal{F}\Lambda^{s-1}\Lambda_+\Lambda_-^\gamma u(\tau,\xi)\right\|_{L^2_\xi(L^\infty_\tau)}\\ &\qquad\lesssim \left\|\mathcal{F}\Lambda^{s}\Lambda_-^\gamma u(\tau,\xi)\right\|_{L^2_\xi(L^\infty_\tau)} + \left\|\mathcal{F}\Lambda^{s-1}\Lambda_-^{\gamma+1} u(\tau,\xi)\right\|_{L^2_\xi(L^\infty_\tau)}\end{aligned}$$

for any $u(t,x)$. Thus, (5.14) and (5.15) follow immediately from (5.12) and (5.13) of Lemma 5, respectively. □

We are now in a position to finish the proof of Lemma 3.

Estimate for $u_{1,1}$. We first estimate $u_{1,1}$ using (5.8). By (5.3),

$$\|F_1\|_{H^{s-1,0}} \lesssim T^{\min\{0,\alpha(\theta+\varepsilon-1)\}}\|F\|_{H^{s-1,\theta+\varepsilon-1}}. \tag{5.16}$$

It is easily checked that

$$\left\|\mathcal{F}\mathcal{D}^r\left\{t^j\chi(t/T)\right\}\right\|_{L^\infty} \le T^{1-r+j}\left\|\mathcal{F}\mathcal{D}^r(t^j\chi)\right\|_{L^\infty} \tag{5.17}$$

for $r \ge 0$, $j \ge 0$ and $0 < T < 1$. Combining (5.15) of Lemma 6 with (5.7), (5.16) and (5.17) yields

$$\left\|\chi(t/T)\sum\nolimits_1\right\|_{\mathcal{X}^s} \le C_{\chi,\gamma}T^\delta\|F_1\|_{H^{s-1,\theta+\varepsilon-1}},$$

where δ is given by (5.5) and

$$C_{\chi,\gamma} \lesssim \sum_{j=1}^\infty \frac{1}{j!}\left\|\mathcal{F}\mathcal{D}^{\gamma+1}(t^{j+1}\chi)\right\|_{L^\infty} \lesssim \sum_{j=1}^\infty \frac{1}{j!}\left(\left\|t^{j+1}\chi\right\|_{L^1} + \left\|(t^{j+1}\chi)'''\right\|_{L^1}\right).$$

Since χ is compactly supported, $C_{\chi,\gamma} < \infty$.

Estimate for $u_{1,2}$. Next we estimate $u_{1,2}$ using (5.8). In view of hypothesis (iii) of the theorem,

$$\|\chi(t/T)u_{1,2}\|_{\mathcal{X}^s} \lesssim A + B,$$

where

$$A = \left\| \mathcal{F}\Lambda^s \Lambda_-^\gamma \{\chi(t/T) u_{1,2}\}(\tau,\xi) \right\|_{L_\xi^2(L_\tau^\infty)(|\tau| \le C|\xi|)},$$
$$B = \left\| \mathcal{F}\Lambda^{s-1} \Lambda_-^{\gamma+1} \{\chi(t/T) u_{1,2}\}(\tau,\xi) \right\|_{L_\xi^2(L_\tau^\infty)(|\tau| \ge C|\xi|)}$$

and $C > 1$ is a constant which will be specified later.

To estimate A we use the expression (5.8) for $u_{1,2}$. By (5.12) of Lemma 5, as well as (5.7), (5.16) and (5.17),

$$\left\| \mathcal{F}\Lambda^s \Lambda_-^\gamma \left\{ \chi(t/T) \sum\nolimits_2 \right\} (\tau,\xi) \right\|_{L_\xi^2(L_\tau^\infty)} \lesssim C_{\chi,\gamma} T^\delta \left\| F_1 \right\|_{H^{s-1,\theta+\varepsilon-1}},$$

where

$$C_{\chi,\gamma} \lesssim \sum_{j=1}^\infty \frac{1}{j!} \left\| \mathcal{F}\mathcal{D}^\gamma (t^j \chi) \right\|_{L^\infty} < \infty.$$

Next, by (5.9) and (5.3),

$$\left\| \mathcal{F}\Lambda^s \Lambda_-^\gamma \{\chi(t/T) E\}(\tau,\xi) \right\|_{L_\xi^2(L_\tau^\infty)} \lesssim \left(\left\| \widehat{\mathcal{D}^{\gamma-1}\chi} \right\|_{L^\infty} + \left\| \widehat{\mathcal{D}^\gamma (t\chi)} \right\|_{L^\infty} \right) T^{-\alpha/2} \left\| F_1 \right\|_{H^{s-1,0}},$$

which combined with (5.16) and (5.17) yields

$$\left\| \mathcal{F}\Lambda^s \Lambda_-^\gamma \{\chi(t/T) E\}(\tau,\xi) \right\|_{L_\xi^2(L_\tau^\infty)} \lesssim C_{\chi,\gamma} T^\delta \left\| F_1 \right\|_{H^{s-1,\theta+\varepsilon-1}},$$

where

$$C_{\chi,\gamma} \lesssim \left\| \widehat{\mathcal{D}^{\gamma-1}\chi} \right\|_{L^\infty} + \left\| \widehat{\mathcal{D}^\gamma (t\chi)} \right\|_{L^\infty}.$$

For B we use the expression (5.10) for $\widehat{\chi u_{1,2}}(\tau,\xi)$. Recall that $|\tau| \ge C|\xi|$ in the definition of B, where C is yet to be determined. We claim that if we take $C = 8(1+c^2)$, where c is the constant appearing in (5.2) and (5.3), then with μ given by (5.11),

$$|\tau| \ge C|\xi| \implies \big| |\tau| - |\xi| \big| \sim |\tau| \sim |\tau - \mu|$$

for all λ such that $(\lambda,\xi) \in \operatorname{supp} \widehat{F_{1,2}}$, and all $0 \le \rho, \sigma \le 1$. Indeed,

$$|\mu| \le |b| + |a-b| \le 4|\xi| + \big| |\lambda| - |\xi| \big|,$$

and we have

$$\big| |\lambda| - |\xi| \big| \le 4c^2 T^{-\alpha} \le 4c^2 |\xi|$$

for $(\lambda,\xi) \in \operatorname{supp} \widehat{F_{1,2}}$. Thus,

$$B \lesssim \left\| \mathcal{F}\mathcal{D}^{\gamma+1} \{t^2 \chi(t/T)\} \right\|_{L^\infty} \int \left| \mathcal{F}\Lambda^{s-1} F_{1,2}(\lambda,\xi) \right| d\lambda,$$

and using (5.16) and (5.17), we get

$$B \lesssim C_{\chi,\gamma} T^{\delta} \left\| F_1 \right\|_{H^{s-1,\theta+\varepsilon-1}},$$

where $C_{\chi,\gamma} \lesssim \left\| \mathcal{F}\mathcal{D}^{\gamma+1}(t^2\chi) \right\|_{L^\infty}$.

5.2. **Proof of Lemma 2.** We will show, using hypothesis (iii) of Theorem 1, that for all $\chi \in C_c^\infty(\mathbb{R})$ and $(f,g) \in H^s \times H^{s-1}$,

$$\left\| \chi(t)\partial_t W(t) f \right\|_{\mathcal{X}^s} \lesssim \left\| \widehat{\mathcal{D}^\gamma \chi} \right\|_{L^\infty} \left\| f \right\|_{H^s} + \left\| \widehat{\mathcal{D}^{\gamma+1}\chi} \right\|_{L^\infty} \left\| f \right\|_{H^{s-1}} \tag{5.18}$$

and

$$\left\| \chi(t) W(t) g \right\|_{\mathcal{X}^s} \lesssim \left(\left\| \widehat{\mathcal{D}^\gamma \chi} \right\|_{L^\infty} + \left\| \widehat{\mathcal{D}^{\gamma+1}(t\chi)} \right\|_{L^\infty} \right) \left\| g \right\|_{H^{s-1}} + \left\| \widehat{\mathcal{D}^{\gamma+1}\chi} \right\|_{L^\infty} \left\| g \right\|_{H^{s-2}}. \tag{5.19}$$

Clearly this implies Lemma 2.

We apply Lemma 6. Evidently, (5.14) implies (5.18), so it remains to prove (5.19). For this, we split $g = g_1 + g_2$, where $\widehat{g_1}$ is supported in the region $|\xi| < 1$ and $\widehat{g_2}$ is supported in $|\xi| \ge 1$. Since

$$(-\Delta)^{-\frac{1}{2}} \sin(t\sqrt{-\Delta}) = t \int_0^1 e^{it(2\rho-1)\sqrt{-\Delta}} \, d\rho,$$

we have

$$\chi(t)(-\Delta)^{-\frac{1}{2}} \sin(t\sqrt{-\Delta}) g_1 = \int_0^1 t\chi(t) e^{it(2\rho-1)\sqrt{-\Delta}} g_1 \, d\rho.$$

By (5.15),

$$\left\| t\chi(t) e^{it(2\rho-1)\sqrt{-\Delta}} g_1 \right\|_{\mathcal{X}^s} \lesssim \left\| \mathcal{F}\mathcal{D}^{\gamma+1}(t\chi) \right\|_{L^\infty} \left\| g_1 \right\|_{H^{s-1}}$$

for $0 \le \rho \le 1$, and it follows that

$$\left\| \chi(t) W(t) g_1 \right\|_{\mathcal{X}^s} \le \left\| \widehat{\mathcal{D}^{\gamma+1}(t\chi)} \right\|_{L^\infty} \left\| g_1 \right\|_{H^{s-1}}.$$

This proves (5.19) with g replaced by its low-frequency part g_1.

Since $\left\| (-\Delta)^{-\frac{1}{2}} g_2 \right\|_{H^s} \le 2 \left\| g \right\|_{H^{s-1}}$, the estimate (5.19) with g replaced by g_2 follows immediately from (5.14).

5.3. **Proof of Lemma 4.** We write $u_1 = -(2i)^{-1}(-\Delta)^{-\frac{1}{2}}(v_+ - v_-)$, where

$$v_\pm(t) = \int_0^t e^{\pm i(t-t')\sqrt{-\Delta}} F_1(t',\cdot)\,dt'.$$

As in [4, Lemma 4.4],

$$\widehat{v_\pm(t)}(\xi) = \frac{e^{\pm it|\xi|}}{2\pi} \int \frac{e^{it(\tau\mp|\xi|)}-1}{i(\tau\mp|\xi|)} \widehat{F_1}(\tau,\xi)\,d\tau \tag{5.20}$$

$$= \frac{e^{\pm it|\xi|}}{2\pi} \sum_{j=1}^{\infty} \frac{t^j}{j!} \int i^{j-1}(\tau\mp|\xi|)^{j-1}\widehat{F_1}(\tau,\xi)\,d\tau. \tag{5.21}$$

Formula for $u_{1,1}$. We now prove the formula for $u_{1,1}$ in (5.8). By (5.21), we have

$$\begin{aligned}\widehat{u_{1,1}(t)}(\xi) &= \frac{1}{4\pi}\sum_{j=1}^{\infty}\frac{t^j}{j!}\int i^j\,|\xi|^{-1}\big(\beta_{j,\tau}(|\xi|)-\beta_{j,\tau}(-|\xi|)\big)\widehat{F_{1,1}}(\tau,\xi)\,d\tau \\ &= \frac{1}{2\pi}\sum_{j=1}^{\infty}\frac{t^j}{j!}\int_0^1\int i^j\beta'_{j,\tau}\big((2\rho-1)\,|\xi|\big)\widehat{F_{1,1}}(\tau,\xi)\,d\tau\,d\rho,\end{aligned}$$

where $\beta_{j,\tau}(r) = e^{itr}(\tau-r)^{j-1}$. Since

$$\beta'_{j,\tau}(r) = ite^{itr}(\tau-r)^{j-1} - e^{itr}(j-1)(\tau-r)^{j-2},$$

where the second term occurs only for $j \geq 2$, we get

$$u_{1,1}(t) = \frac{1}{2\pi}\sum_{j=1}^{\infty}\frac{t^{j+1}}{j!}\int_0^1 i^{j+1}\Big(1-\frac{j}{j+1}\Big)e^{it(2\rho-1)\sqrt{-\Delta}}k_j(\rho)\,d\rho,$$

where $k_j(\rho)$ is given by

$$\widehat{k_j(\rho)}(\xi) = \int \big(\tau-(2\rho-1)\,|\xi|\big)^{j-1}\widehat{F_{1,1}}(\tau,\xi)\,d\tau.$$

If we set

$$g_j = \frac{i^{j+1}}{2\pi}\left(1-\frac{j}{j+1}\right)k_j,$$

then

$$u_{1,1}(t) = \sum_{j=1}^{\infty}\frac{t^{j+1}}{j!}\int_0^1 e^{it(2\rho-1)\sqrt{-\Delta}}g_j(\rho)\,d\rho.$$

Notice that

$$\big|\tau-(2\rho-1)\,|\xi|\big| \leq |\tau| + |\xi| \leq \big||\tau|-|\xi|\big| + 2\,|\xi|$$

for $0 \le \rho \le 1$. But if $(\tau,\xi) \in \operatorname{supp} \widehat{F_{1,1}}$, then $|\xi| \le T^{-\alpha}$, and by (5.3) we also have $\bigl||\tau| - |\xi|\bigr| \lesssim T^{-\alpha}$. It follows that

$$\left|\widehat{g_j(\rho)}(\xi)\right| \lesssim T^{\alpha(1-j)} \int \left|\widehat{F_{1,1}}(\tau,\xi)\right| d\tau \lesssim T^{\alpha(1/2-j)} \left(\int \left|\widehat{F_{1,1}}(\tau,\xi)\right|^2 d\tau \right)^{\frac{1}{2}},$$

whence $\|g_j(\rho)\|_{H^{s-1}} \lesssim T^{\alpha(1/2-j)} \|F_1\|_{H^{s-1,0}}$ for $0 \le \rho \le 1$.

First formula for $u_{1,2}$. Next we prove the formula for $u_{1,2}$ in (5.8). By (5.20) and (5.21),

$$u_{1,2}(t) = \sum_{j=1}^{\infty} \frac{t^j}{j!} \left(e^{it\sqrt{-\Delta}} f_j^+ + e^{-it\sqrt{-\Delta}} f_j^- \right) + E_+(t) + E_-(t),$$

where

$$\begin{aligned}
\widehat{f_j^+}(\xi) &= (4\pi |\xi|)^{-1} \int_0^\infty i^j (|\tau| - |\xi|)^{j-1} \widehat{F_{1,2}}(\tau,\xi)\, d\tau, \\
\widehat{f_j^-}(\xi) &= -(4\pi |\xi|)^{-1} \int_{-\infty}^0 i^j (|\xi| - |\tau|)^{j-1} \widehat{F_{1,2}}(\tau,\xi)\, d\tau, \\
\widehat{E_+(t)}(\xi) &= -\frac{1}{4\pi |\xi|} \int_{-\infty}^0 \frac{e^{it\tau} - e^{it|\xi|}}{|\tau| + |\xi|} \widehat{F_{1,2}}(\tau,\xi)\, d\tau, \\
\widehat{E_-(t)}(\xi) &= -\frac{1}{4\pi |\xi|} \int_0^\infty \frac{e^{it\tau} - e^{-it|\xi|}}{|\tau| + |\xi|} \widehat{F_{1,2}}(\tau,\xi)\, d\tau.
\end{aligned}$$

It follows easily from (5.3) that $\left\| f_j^\pm \right\|_{H^s} \lesssim T^{\alpha(1/2-j)} \|F_1\|_{H^{s-1,0}}$.

Next, observe that

$$\widehat{\chi E_+}(\tau,\xi) = -\frac{1}{4\pi |\xi|} \int_{-\infty}^0 \frac{\widehat{\chi}(\tau - \lambda) - \widehat{\chi}(\tau - |\xi|)}{|\lambda| + |\xi|} \widehat{F_{1,2}}(\lambda,\xi)\, d\lambda.$$

Since

$$\frac{\widehat{\chi}(\tau - \lambda) - \widehat{\chi}(\tau - |\xi|)}{|\lambda| + |\xi|} = \int_0^1 \widehat{\chi}'\bigl(\tau - |\xi| + \rho(|\lambda| + |\xi|)\bigr)\, d\rho$$

for $\lambda < 0$, it is easy to see, by considering separately the two cases $\bigl||\tau| - |\xi|\bigr| \le 2(|\lambda| + |\xi|)$ and $\bigl||\tau| - |\xi|\bigr| > 2(|\lambda| + |\xi|)$, that for a given $\gamma \ge 1$,

$$\bigl(1 + \bigl||\tau| - |\xi|\bigr|\bigr)^\gamma \frac{\left|\widehat{\chi}(\tau - \lambda) - \widehat{\chi}(\tau - |\xi|)\right|}{|\lambda| + |\xi|} \lesssim \left\|\widehat{\mathcal{D}^{\gamma-1}\chi}\right\|_{L^\infty} + \left\|\widehat{\mathcal{D}^{\gamma}(t\chi)}\right\|_{L^\infty}$$

for all $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and $\lambda < 0$. We conclude that

$$\left|\mathcal{F}\Lambda^s \Lambda_-^\gamma \{\chi E_+\}(\tau,\xi)\right|$$

$$\lesssim \Big(\big\|\widehat{\mathcal{D}^{\gamma-1}\chi}\big\|_{L^\infty} + \big\|\widehat{\mathcal{D}^{\gamma}(t\chi)}\big\|_{L^\infty} \Big) \int \big|\mathcal{F}\Lambda^{s-1}F_{1,2}(\lambda,\xi)\big| \, d\lambda.$$

The same estimate holds for E_-, so $E = E_+ + E_-$ satisfies (5.9).

Second formula for $u_{1,2}$. Finally, we prove (5.10). Using (5.20), we write

$$\widehat{u_{1,2}(t)}(\xi) = \frac{1}{4\pi\,|\xi|} \int \left\{ \frac{e^{it\tau} - e^{it|\xi|}}{\tau - |\xi|} - \frac{e^{it\tau} - e^{-it|\xi|}}{\tau + |\xi|} \right\} \widehat{F_{1,2}}(\tau,\xi)\, d\tau.$$

Thus,

$$\begin{aligned}
&\widehat{\chi u_{1,2}}(\tau,\xi) \\
&= \frac{1}{4\pi\,|\xi|} \int \left\{ \frac{\widehat{\chi}(\tau-\lambda) - \widehat{\chi}(\tau - |\xi|)}{\lambda - |\xi|} - \frac{\widehat{\chi}(\tau-\lambda) - \widehat{\chi}(\tau + |\xi|)}{\lambda + |\xi|} \right\} \widehat{F_{1,2}}(\lambda,\xi)\, d\lambda \\
&= -\frac{1}{4\pi\,|\xi|} \int\int_0^1 \big\{ \widehat{\chi}'(\tau - a) - \widehat{\chi}'(\tau - b) \big\} \widehat{F_{1,2}}(\lambda,\xi)\, d\rho\, d\lambda \\
&= \frac{1}{2\pi} \int \int_0^1 \int_0^1 (1-\rho)\widehat{\chi}''\big(\tau - b + \sigma(b-a)\big) \widehat{F_{1,2}}(\lambda,\xi)\, d\sigma\, d\rho\, d\lambda,
\end{aligned}$$

where $a = |\xi| + \rho(\lambda - |\xi|)$ and $b = -|\xi| + \rho(\lambda + |\xi|)$.

6. Proof of Theorem 3

We may assume that the function A in (4.1) and (4.2) is increasing, and that C_T is an increasing and continuous function of T.

6.1. **Local existence.** Given (f,g) and $0 < T < 1$, let u_0 be the solution of the homogeneous wave equation with initial data (f,g), and set

$$u_j = u_0 + W\mathcal{N}(u_{j-1}), \quad j = 1,2,\dots.$$

By (S2), there is a constant C such that

$$\|u_0\|_{\mathcal{X}^s_T} \le C\big(\|f\|_{H^s} + \|g\|_{H^{s-1}}\big). \tag{6.1}$$

Combined with (4.1) this gives

$$\|u_j\|_{\mathcal{X}^s_T} \le R/2 + C_T A\big(\|u_{j-1}\|_{\mathcal{X}^s_T}\big)\, \|u_{j-1}\|_{\mathcal{X}^s_T}$$

for $j \ge 1$, where R is twice the right-hand side of (6.1). By (4.3), we may choose T so small that $2C_T A(R) \le 1$. Since A is increasing, it now follows by induction that $\|u_j\|_{\mathcal{X}^s_T} \le R$ for all j. It then follows by (4.2) that

$$\|u_{j+1} - u_j\|_{\mathcal{X}^s_T} \le \frac{1}{2}\, \|u_j - u_{j-1}\|_{\mathcal{X}^s_T} \tag{6.2}$$

for $j \geq 1$. Thus, (u_j) is a Cauchy sequence in $\mathcal{X}_T^s$, and we let u be its limit. In view of (4.2), $\mathcal{N}(u_j) \to \mathcal{N}(u)$ in the sense of distributions on $S_T = (0,T) \times \mathbb{R}^n$. Since $\Box u_j = \mathcal{N}(u_{j-1})$ on S_T with initial data (f,g), by passing to the limit we conclude that $\Box u = \mathcal{N}(u)$ on S_T with the same data.

6.2. **Uniqueness.** Assume that $T > 0$ and $u, u' \in \mathcal{X}_T^s$ are two solutions of (3.1) on S_T with the same initial data (f,g). It suffices to prove that the set

$$E = \{t \in [0,T] : u(\rho) = u'(\rho) \text{ for all } \rho \in [0,t]\}$$

is open in $[0,T]$, since E is obviously closed and nonempty.

Assume that $t \in E$, $t < T$. By (S4) we may consider u and u' to be elements of $\mathcal{X}_{[t,T]}^s$, and by (N3) they are both solutions of (3.1) on $(t,T) \times \mathbb{R}^n$ with the same initial data at time t (since $t \in E$). Next, by (S3) and (N2), $\tau_t u, \tau_t u' \in \mathcal{X}_{T-t}^s$ solve (3.1) on $(0, T-t) \times \mathbb{R}^n$ with identical initial data at time 0.

By the above, it suffices to prove that $\varepsilon \in E$ for some arbitrarily small $\varepsilon > 0$. But by (4.2),

$$\left\| u - u' \right\|_{\mathcal{X}_\varepsilon^s} \leq B(\varepsilon) \left\| u - u' \right\|_{\mathcal{X}_\varepsilon^s},$$

where $B(\varepsilon) = C_\varepsilon A\big(\max\{\|u\|_{X_\varepsilon^s}, \|u'\|_{X_\varepsilon^s}\}\big)$, and in view of (4.3) and property (S4), $\lim_{\varepsilon \to 0^+} B(\varepsilon) = 0$.

6.3. **Continuous dependence on initial data.**
Step 1. We prove that (III) follows from a weaker condition. We denote by $u(f,g) \in \mathcal{X}_T^s$ the solution obtained in Section 6.1. Recall that $T = T(f,g) > 0$ is continuous, and

$$C_{T(f,g)} A\big(\|u(f,g)\|_{\mathcal{X}_{T(f,g)}^s}\big) \leq \frac{1}{2}. \tag{6.3}$$

We claim that (III) follows from

(III$'$) *The map $(f,g) \mapsto u(f,g)$ is Lipschitz, in the sense that*

$$\left\| u(f,g) - u(f',g') \right\|_{\mathcal{X}_T^s} \lesssim \left\| f - f' \right\|_{H^s} + \left\| g - g' \right\|_{H^{s-1}} \tag{6.4}$$

for all initial data pairs (f,g) and (f',g') in $H^s \times H^{s-1}$, where $T = \min\{T(f,g), T(f',g')\}$.

With hypotheses as in (III), set

$$T_* = \inf_{0 \leq t \leq T} T\big(u(t), \partial_t u(t)\big).$$

In view of (S1) and the continuity of T, $T_* > 0$. Pick $0 < \varepsilon < T_*/2$ such that $T = M\varepsilon$ for some integer M, and set $t_j = j\varepsilon$, $f_j = u(t_j)$ and $g_j = \partial_t u(t_j)$.

Assume that (III′) holds. In view of (S3) and (N2), for $j = 0, 1, \dots, M-2$ there is a ball B_j in $H^s \times H^{s-1}$, centered at (f_j, g_j), with the following property: for all $(\phi_j, \psi_j) \in B_j$ there exists $v_j \in \mathcal{X}^s_{[t_j, t_{j+2}]}$ which solves (3.1) on $(t_j, t_{j+2}) \times \mathbb{R}^n$ with initial data (ϕ_j, ψ_j) at time t_j, and satisfies

$$\|u - v_j\|_{\mathcal{X}^s_{[t_j,t_{j+2}]}} \lesssim \|f_j - \phi_j\|_{H^s} + \|g_j - \psi_j\|_{H^{s-1}} . \tag{6.5}$$

By (S1), this implies

$$\|f_{j+1} - \phi_{j+1}\|_{H^s} + \|g_{j+1} - \psi_{j+1}\|_{H^{s-1}} \leq C\big(\|f_j - \phi_j\|_{H^s} + \|g_j - \psi_j\|_{H^{s-1}}\big), \tag{6.6}$$

where we have set $\phi_{j+1} = v(t_{j+1})$ and $\psi_{j+1} = \partial_t v(t_{j+1})$.

Thus, if we make B_{M-3} so small that $CB_{M-3} \subseteq B_{M-2}$, and then make B_{M-4} so small that $CB_{M-4} \subseteq B_{M-3}$, etc., we find that if we start with data $(\phi_0, \psi_0) \in B_0$, then v_0 exists and $(\phi_1, \psi_1) = \big(v_0(t_1), \partial_t u_0(t_1)\big) \in B_1$, so v_1 exists, and so on.

By translation invariance and uniqueness, the different v_j agree on the intersection of their domains, so by (S5) we get a solution $v \in \mathcal{X}^s_T$ of (3.1) on $(0,T) \times \mathbb{R}^n$ with data (ϕ_0, ψ_0), and $\|u - v\|_{\mathcal{X}^s_T} \lesssim \sum_{j=0}^{M-2} \|u - v_j\|_{\mathcal{X}^s_{[t_j,t_{j+2}]}}$. But in view of (6.5) and (6.6),

$$\|u - v_j\|_{\mathcal{X}^s_{[t_j,t_{j+2}]}} \lesssim \|f - \phi_0\|_{H^s} + \|g - \psi_0\|_{H^{s-1}} .$$

Step 2. We prove (III′). In view of (6.1) and (4.2), it suffices to prove

$$C_T A\big(\max\{\|u(f,g)\|_{\mathcal{X}^s_T}, \big\|u(f',g')\big\|_{\mathcal{X}^s_T}\}\big) \leq \frac{1}{2}, \tag{6.7}$$

where $T = \min\{T(f,g), T(f',g')\}$. But by (6.3) and (S4),

$$A\big(\max\{\|u(f,g)\|_{\mathcal{X}^s_T}, \big\|u(f',g')\big\|_{\mathcal{X}^s_T}\}\big) \leq \frac{1}{2}\left(\min\{C_{T(f,g)}, C_{T(f',g')}\}\right)^{-1} .$$

Since we assume that C_T is increasing in T, (6.7) follows.

6.4. **Persistence of higher regularity.** We prove the assertion in Remark (2) following Theorem 3.

Step 1. We show that (V) follows from

- (I′) *Let $\sigma \geq s$. For all $(f,g) \in H^\sigma \times H^{\sigma-1}$ there exist a $T > 0$ and a $u \in \mathcal{X}^s_T \cap C\big([0,T], H^\sigma\big) \cap C^1\big([0,T], H^{\sigma-1}\big)$ which solves* (3.1) *on $S_T = (0,T) \times \mathbb{R}^n$ with initial data (f,g). Moreover, T can be chosen to depend continuously on $\|f\|_{H^s} + \|g\|_{H^{s-1}}$.*

Assume that (I′) holds for a fixed $\sigma \geq s$, and let us denote the existence time by $T(f,g)$. This function may depend on s and σ, but these are fixed quantities.

With hypotheses as in (V), set

$$T_* = \inf_{0 \leq t \leq T} T\big(u(t), \partial_t u(t)\big),$$

and choose $0 < \varepsilon < T_*/2$ so that $T = M\varepsilon$ for some integer M. Then set $t_j = j\varepsilon$, $f_j = u(t_j)$ and $g_j = \partial_t u(t_j)$. By (I′) and translation invariance, for $j = 0, 1, \ldots, M-2$ there exists

$$u_j \in \mathcal{X}^s_{[t_j, t_{j+2}]} \cap C\big([t_j, t_{j+2}], H^\sigma\big) \cap C^1\big([t_j, t_{j+2}], H^{\sigma-1}\big)$$

which solves (3.1) on $(t_j, t_{j+2}) \times \mathbb{R}^n$ with initial data (f_j, g_j). By uniqueness, each u_j agrees with u on $[t_j, t_{j+2}]$. We conclude that

$$u \in C\big([0,T], H^\sigma\big) \cap C^1\big([0,T], H^{\sigma-1}\big).$$

Step 2. We prove (I′). If we fix $\sigma \geq s$, we may assume that the function A_σ appearing in (4.4) and (4.5) is identical with the function A appearing in (4.1) and (4.2). Recall that A is assumed to be increasing, and that $\mathcal{X}^\sigma_T = \Lambda^{s-\sigma} \mathcal{X}^s_T$ by definition.

As in Section 6.1, $\|u_j\|_{\mathcal{X}^s_T} \leq R$, where $R = 2C\big(\|f\|_{H^s} + \|g\|_{H^{s-1}}\big)$, C is the constant appearing in (6.1) and $T > 0$ is chosen so small that $2C_T A(R) \leq 1$. Another induction, using (4.4) and (6.1), gives $\|u_j\|_{\mathcal{X}^\sigma_T} \leq R'$, where $R' = 2C\big(\|f\|_{H^\sigma} + \|g\|_{H^{\sigma-1}}\big)$. Furthermore, by (6.2) we have $\|u_j - u_{j-1}\|_{\mathcal{X}^s_T} \leq 2^{-j} R$.

Thus (4.5) yields $B_{j+1} \leq B_j/2 + K2^{-j}$, where $B_j = \|u_j - u_{j-1}\|_{\mathcal{X}^\sigma_T}$ and $K = C_T A(R') R$. By induction, this implies that $B_j \leq B_0 2^{-j} + 2Kj2^{-j}$ for $j \geq 0$. Thus, (u_j) is Cauchy in $\mathcal{X}^\sigma_T$.

6.5. **Smooth dependence on initial data.** We prove the assertion in Remark (3) following Theorem 3.

Fix $r \geq 1$. We prove $\delta \mapsto u_\delta \in \mathcal{X}^s_T$ is C^r. By a finite time-step argument as in 6.4, it suffices to prove this for a $T > 0$ which depends continuously on $E = \sup_{\delta \in I} \|(f_\delta, g_\delta)\|_{(s)}$, where $I = [-\delta_0, \delta_0]$.

Denote the iterates by $u_j(\delta) \in \mathcal{X}^s_T$. Thus, $u_0(\delta)$ solves the homogeneous wave equation with initial data (f_δ, g_δ), and for $j \geq 1$,

$$u_{j+1}(\delta) = u_0(\delta) + S\big(u_j(\delta)\big), \tag{6.8}$$

where $S = W\mathcal{N}$. If we set $u_{-1} \equiv 0$, this is valid for $j \geq -1$.

Since $C^r\big(I, \mathcal{X}_T^s\big)$ is a Banach space when equipped with the norm

$$\|u\| = \sum_{0\le k\le r} \sup_{\delta\in I} \big\|u^{(k)}(\delta)\big\|_{\mathcal{X}_T^s},$$

it suffices to show that u_j is Cauchy in this norm for some $T(E) > 0$.

By (4.7) and the mean value theorem,

$$\big\|S^{(k)}(u) - S^{(k)}(v)\big\|_{(T)} \le B\big(\max\{\|u\|_{\mathcal{X}_T^s}, \|v\|_{\mathcal{X}_T^s}\}\big)\, \|u-v\|_{\mathcal{X}_T^s} \tag{6.9}$$

for $k = 1, \dots, r$, where $B = \max_{2\le k\le r+1} B_k$.

Let C be the constant in (6.1), choose T so that $C_T A(2CE) = \frac{1}{2}$, and set

$$E_k = \sup_{\delta\in I} \Big\|(d/d\delta)^k (f_\delta, g_\delta)\Big\|_{(s)}$$

for $k = 1, \dots, r$.

By induction, as in Section 6.1, we have

$$\sup_\delta \|u_j(\delta)\| \le 2CE. \tag{6.10}$$

Taking one derivative in (6.8), we get

$$u'_{j+1} = u'_0 + S'(u_j)(u'_j),$$

so by (4.6),

$$\big\|u'_{j+1}(\delta)\big\|_{\mathcal{X}_T^s} \le CE_1 + C_T A(2CE)\, \big\|u'_j(\delta)\big\|_{\mathcal{X}_T^s},$$

and since $C_T A(2CE) = \frac{1}{2}$, we conclude that

$$\sup_\delta \big\|u'_j(\delta)\big\| \le 2CE_1.$$

Taking two derivatives in (6.8) gives

$$u''_{j+1} = u''_0 + S''(u_j)(u'_j, u'_j) + S'(u_j)(u''_j).$$

Thus, using (4.7),

$$\big\|u''_{j+1}(\delta)\big\|_{\mathcal{X}_T^s} \le CE_2 + B(2CE)(2CE_1)^2 + \frac{1}{2}\, \big\|u''_j(\delta)\big\|_{\mathcal{X}_T^s},$$

so

$$\sup_\delta \big\|u''_j(\delta)\big\| \le 2CE_2 + 2B(2CE)(2CE_1)^2.$$

Continuing like this, one finds that for $k = 0, \dots, r$ and all j,

$$\sup_\delta \big\|u_j^{(k)}(\delta)\big\| \le C_k(E, E_1, \dots, E_k), \tag{6.11}$$

where C_k is some continuous function.

By (4.5) and (6.10), and the fact that $C_T A(2CE) = \frac{1}{2}$, we have

$$\|u_{j+1}(\delta) - u_j(\delta)\|_{\mathcal{X}_T^s} \le \frac{1}{2} \|u_j(\delta) - u_{j-1}(\delta)\|_{\mathcal{X}_T^s}$$

for $j \ge 0$, and it follows by induction that

$$\sup_\delta \|u_j(\delta) - u_{j-1}(\delta)\|_{\mathcal{X}_T^s} \le CE2^{-j}.$$

Next, since

$$u'_{j+1} - u'_j = S'(u_j)(u'_j - u'_{j-1}) + \left[S'(u_j) - S'(u_{j-1})\right](u'_{j-1})$$

for $j \ge 0$,we have

$$\begin{aligned} \left\|u'_{j+1}(\delta) - u'_j(\delta)\right\|_{\mathcal{X}_T^s} \le \frac{1}{2} &\left\|u'_j(\delta) - u'_{j-1}(\delta)\right\|_{\mathcal{X}_T^s} \\ &+ B(2CE) \|u_j(\delta) - u_{j-1}(\delta)\|_{\mathcal{X}_T^s} C_1(E, E_1). \end{aligned}$$

Thus, redefining C_1,

$$\left\|u'_{j+1}(\delta) - u'_j(\delta)\right\|_{\mathcal{X}_T^s} \le \frac{1}{2} \left\|u'_j(\delta) - u'_{j-1}(\delta)\right\|_{\mathcal{X}_T^s} + C_1(E, E_1)2^{-j},$$

and it follows by induction[5] that

$$\sup_\delta \left\|u'_j(\delta) - u'_{j-1}(\delta)\right\|_{\mathcal{X}_T^s} \le C_1(E, E_1)(1 + j)2^{-j},$$

where once more we have redefined C_1.

Since

$$\begin{aligned} u''_{j+1} - u''_j = &\left[S''(u_j) - S''(u_{j-1})\right](u'_j, u'_j) \\ &+ S''(u_{j-1})(u'_j - u'_{j-1}, u'_j) + S''(u_{j-1})(u'_{j-1}, u'_j - u'_{j-1}) \\ &+ \left[S'(u_j) - S'(u_{j-1})\right](u''_j) + S'(u_j)(u''_j - u''_{j-1}), \end{aligned}$$

for $j \ge 0$, we have, redefining C_2,

$$\begin{aligned} \left\|u''_{j+1}(\delta) - u''_j(\delta)\right\|_{\mathcal{X}_T^s} &\le \frac{1}{2} \left\|u''_j(\delta) - u''_{j-1}(\delta)\right\|_{\mathcal{X}_T^s} \\ &+ C_2(E, E_1, E_2) \left(\|u_j(\delta) - u_{j-1}(\delta)\|_{\mathcal{X}_T^s} + \left\|u'_j(\delta) - u'_{j-1}(\delta)\right\|_{\mathcal{X}_T^s}\right). \end{aligned}$$

Thus, redefining C_2 again,

$$\left\|u''_{j+1}(\delta) - u''_j(\delta)\right\|_{\mathcal{X}_T^s} \le \frac{1}{2} \left\|u''_j(\delta) - u''_{j-1}(\delta)\right\|_{\mathcal{X}_T^s} + C_2(E, E_1, E_2)(1 + j)2^{-j}.$$

[5]Here, and below, we use the following induction argument. Suppose B_j is a sequence of nonnegative numbers such that $B_{j+1} \le B_j/2 + P(j)2^{-j}$ for $j \ge 0$, where P is a polynomial with nonnegative coefficients. Then there is a polynomial Q such that $B_j \le Q(j)2^{-j}$ for all j. In fact, one can take $Q(t) = \int_0^t 2P(r)\,dr + B_0$.

It follows by induction that

$$\sup_\delta \left\| u_j''(\delta) - u_{j-1}''(\delta) \right\|_{\mathcal{X}_T^s} \le C_2(E, E_1, E_2)(1 + j^2)2^{-j},$$

where C_2 has been redefined yet again.

Continuing in this manner, one finds that for $k = 0, 1, \ldots, r$,

$$\sup_\delta \left\| u_j^{(k)}(\delta) - u_{j-1}^{(k)}(\delta) \right\|_{\mathcal{X}_T^s} \le C_k(E, E_1, \ldots, E_k)(1 + j^k)2^{-j},$$

where C_k has been redefined.

7. Proof of Theorem 2

As noted already, (S1–5) of Section 4 are satisfied. Since $\mathcal{N}$ is local in time and maps $\mathcal{X}^s$ into $\mathcal{D}'(\mathbb{R}^{1+n})$, (N1) of Theorem 3 holds; (N2–4) are obviously satisfied.

Let $0 < T < 1$ and $u \in \mathcal{X}_T^s$. Then u is an equivalence class in $\mathcal{X}^s$ (Section 2), and we denote by $\widetilde{u}$ an arbitrary representative in $\mathcal{X}^s$ of this equivalence class. By assumption, $\mathcal{N}(\widetilde{u})$ belongs to $\mathcal{Y}^{s,\varepsilon}$, so by Theorem 1, there is a unique $v \in \mathcal{X}_T^s$ which solves $\Box v = \mathcal{N}(\widetilde{u})$ on $(0, T) \times \mathbb{R}^n$ with vanishing initial data at $t = 0$. But v is independent of the choice of representative $\widetilde{u}$ of u, by uniqueness of solutions of (1.1) and the fact that $\mathcal{N}$ is local in time, so we may write $v = W\mathcal{N}(u)$.

Let us prove (4.2). Let $u, v \in \mathcal{X}_T^s$, and let $\widetilde{u}, \widetilde{v} \in \mathcal{X}^s$ be any two representatives of u and v respectively. By Theorem 1 followed by (3.2), we have

$$\begin{aligned}\left\| W\big(\mathcal{N}(u) - \mathcal{N}(v)\big) \right\|_{\mathcal{X}_T^s} &\le C_{T,\varepsilon} \left\| \Lambda_+^{-1} \Lambda_-^{\varepsilon-1} \big(\mathcal{N}(\widetilde{u}) - \mathcal{N}(\widetilde{v})\big) \right\|_{\mathcal{X}^s} \\ &\le C_{T,\varepsilon} A\big(\max\{\|\widetilde{u}\|_{\mathcal{X}^s}, \|\widetilde{v}\|_{\mathcal{X}^s}\}\big) \|\widetilde{u} - \widetilde{v}\|_{\mathcal{X}^s}.\end{aligned}$$

We may assume that A is increasing, whence

$$\left\| W\big(\mathcal{N}(u) - \mathcal{N}(v)\big) \right\|_{\mathcal{X}_T^s} \le C_{T,\varepsilon} A\big(\|\widetilde{w}\|_{\mathcal{X}^s} + \|\widetilde{v}\|_{\mathcal{X}^s}\big) \|\widetilde{w}\|_{\mathcal{X}^s},$$

where $\widetilde{w} = \widetilde{u} - \widetilde{v}$. Now pass to the limit as $\|\widetilde{w}\|_{\mathcal{X}^s} \to \|u - v\|_{\mathcal{X}_T^s}$ and $\|\widetilde{v}\|_{\mathcal{X}^s} \to \|v\|_{\mathcal{X}_T^s}$. Thus (4.2) holds with $A(R)$ replaced by $A(3R)$.

Thus, the hypotheses of Theorem 3 are satisfied, so (I–III) hold.

It remains to prove (IV) and (V). Let $W_T : \mathcal{Y}^{s,\varepsilon} \to \mathcal{X}^s$ be as in Remark (2) following Theorem 1. Given (f, g), let u_0 be the solution of the homogeneous wave equation with data (f, g). As in the proof of Theorem 1,

$$\|\chi(t)u_0\|_{\mathcal{X}^s} \le C \|(f, g)\|_{(s)}. \tag{7.1}$$

Consider the sequence $u_j \in \mathcal{X}^s$ of iterates, given inductively by

$$u_j = \chi(t)u_0 + W_T\mathcal{N}(u_{j-1}).$$

If $\mathcal{N}$ is C^∞, then $S = W_T\mathcal{N} : \mathcal{X}^s \to \mathcal{X}^s$ is C^∞, and the argument in Section 6.5 proves (IV).

Finally, we prove that (3.3) implies property (V). As shown in Section 6.4, it suffices to prove property (I′) stated therein. Fix $\sigma \geq s$. We may assume that (3.3) holds with $A_\sigma = A$, where A is the function appearing in (3.2).

Given $(f,g) \in H^\sigma \times H^{\sigma-1}$, choose $0 < T < 1$ so small that $2C_T A(R) \leq 1$, where R is twice the right-hand side of (7.1). A simple induction argument, essentially like the one in Section 6.1, reveals that $\|u_j\|_{\mathcal{X}^s} \leq R$ for all j and that (u_j) is a Cauchy sequence in $\mathcal{X}^s$ whose limit u solves (3.1) on $(0,T)\times\mathbb{R}^n$ with initial data (f,g).

Another induction, using (3.3), shows that $\|u_j\|_{\mathcal{X}^\sigma} \leq R'$ for all j, where $R' = 2C\big(\|f\|_{H^\sigma} + \|g\|_{H^{\sigma-1}}\big)$. Thus, by hypothesis (i) of Theorem 1, the sequence of iterates is bounded in the Hilbert space $\mathcal{H}^{\sigma,\theta}$; hence it converges weakly in that space, so it converges in the sense of distributions to an element of $\mathcal{H}^{\sigma,\theta}$. Thus, the limit $u \in \mathcal{X}^s$ must in fact belong to $\mathcal{H}^{\sigma,\theta}$, whence $u \in C\big([0,T],H^\sigma\big) \cap C^1\big([0,T],H^{\sigma-1}\big)$.

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