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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF PARABOLIC REACTION–DIFFUSION SYSTEMS

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Abstract. This paper studies the asymptotic behaviour near blow-up points of solutions of the system

$$u_t = \triangle u + u^{p_1} v^{q_1}$$
$$v_t = \triangle v + u^{p_2} v^{q_2}$$

with nonnegative, bounded initial data. We derive estimates on the blow-up rates, then we prove a Liouville-type theorem and finally, making use of these results, we obtain the description of possible blow-up patterns.

1. Introduction

To our knowledge, the recent article of Andreucci, Herrero and Velázquez [1] is the first work exhaustively describing an asymptotic behaviour and a classification of blow-up patterns for a semilinear system, namely

$$\begin{cases} u_t = \triangle u + v^p \\ v_t = \triangle v + u^q \\ u(0) = u_0 \geq 0, \ v(0) = v_0 \geq 0, \quad x \in \mathbb{R}^N. \end{cases} \tag{1.1}$$

The main purpose of the article [1] is to extend the results concerning the possible asymptotics of solutions of a scalar equation

$$u_t = \triangle u + u^p \tag{1.2}$$

near blow up, to a particular group of systems of equations. Although theorems established by these authors are analogues of the ones provided for a single equation, we notice some important differences. The main novelty are profiles which do not appear in the scalar case. In order to obtain them, some basic properties of the system and its solutions are required. The first is an upper estimate for the blow-up rate. Then, the introduction of relating

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similarity variables is available. This leads to a rescaled system and the next step consists of deriving the Liouville theorem in order to justify linearizing around its constant solutions. Finally, a corresponding operator with respect to the spectrum and eigenfunctions is considered.

Unfortunately, in [1] the proof of the fact that rescaled solutions of system (1.1) converge to its constant solutions requires indeed strong assumptions on parameters p and q. In fact, it works only for p and q sufficiently close to some subcritical p_0, so it is possible to take advantage of the convergence of solutions of system (1.1) to the solution of a scalar equation (1.2) with p_0.

Our principal result is a complete classification of blow-up patterns for a system

$$\begin{cases} u_t = \triangle u + u^{p_1} v^{q_1} \\ v_t = \triangle v + u^{p_2} v^{q_2}, \quad x \in \mathbb{R}^N,\, t \in (0,T), \\ u(0) = u_0 \geq 0,\ v(0) = v_0 \geq 0, \quad x \in \mathbb{R}^N. \end{cases} \tag{1.3}$$

It is obvious that (1.1) is the particular case of (1.3), where $p_1 = q_2 = 0$. Moreover, our results are improved in comparison with [1] as a tight condition concerning p and q (in the statement of Liouville's theorem) is no longer necessary. We are able to achieve it as a result of adopting the method used by Giga and Kohn in [6].

We shall sketch now the main steps of our approach to the system analysis. As we have just mentioned, our first task is to derive a bound for the blow-up rate. To formulate the statement we need some notation.

Let A denote a matrix of the form

$$A = \begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix}, \tag{1.4}$$

and $d = \det\,(A - I)$.

Assuming that $d \neq 0$, we define (α, β) as a solution of the linear system

$$(A - I)(\alpha, \beta)^t = (1,1)^t; \text{ i.e.,} \tag{1.5}$$

$$\alpha = \frac{q_2 - q_1 - 1}{d}, \qquad \beta = \frac{p_1 - p_2 - 1}{d}\,. \tag{1.6}$$

Then blow up of nontrivial solutions of (1.3) with nonnegative initial values occurs if $\max(\alpha,\beta) \geq N/2$ or if $0 < \max(\alpha,\beta) < N/2$ provided that initial data are large enough (cf. [3] and [10]). We shall establish the following result:

Theorem 1. *Let* $(u(x,t), v(x,t))$ *be a solution of* (1.3) *in* $\mathbb{R}^N \times (0,T)$, $\min(\alpha,\beta) > 0$ *and one of the following conditions be satisfied:*

$$N = 1, 2 \text{ or } N \geq 3 \text{ and } \max(\alpha,\beta) \geq \frac{N-2}{4} \tag{1.7}$$

or

$$\triangle u_0 + A u_0^{1+\frac{1}{\alpha}} > 0 \tag{1.8}$$

for $\max(\alpha,\beta)=\alpha$ *and* $A=(\beta/\alpha)^{q_1/d\alpha}$. *The corresponding condition for* v_0 *is*

$$\triangle v_0 + B v_0^{1+\frac{1}{\beta}} > 0$$

for $\max(\alpha,\beta)=\beta$ *and* $B=(\alpha/\beta)^{p_2/d\beta}$. *Then there hold*

$$u(x,t) \le C(T-t)^{-\alpha}, \quad v(x,t) \le C(T-t)^{-\beta} \tag{1.9}$$

for some positive constant C.

Our proof of the fact relies on using a corresponding estimate for some supersolution to system (1.3). We use the idea of an invariant region. In fact, it works for the systems of m equations ($m \ge 2$) and the proof of that more general result is presented in [13]. The method employed in [1] seems to suit only the system in form (1.1), i.e., the completely coupled system of two equations.

To proceed further, we rescale variables using the upper bound (1.9):

$$u(x,t) = (T-t)^{-\alpha} U(y,s), \qquad v(x,t) = (T-t)^{-\beta} V(y,s), \tag{1.10}$$

where $y = x(T-t)^{-1/2}$, $s = -\log(T-t)$. Then by a standard computation

$$\begin{cases} U_s = \triangle U - \frac{1}{2} y \nabla U - \alpha U + U^{p_1} V^{q_1} \\ V_s = \triangle V - \frac{1}{2} y \nabla V - \beta V + U^{p_2} V^{q_2}. \end{cases} \tag{1.11}$$

We denote by C_α and C_β constant positive solutions of (1.11), so

$$\alpha C_\alpha^{1-p_1} = C_\beta^{q_1}, \quad \beta C_\beta^{1-q_2} = C_\alpha^{p_2}. \tag{1.12}$$

Then, we shall prove a Liouville theorem of parabolic type in the following form:

Theorem 2. *Let* $(u(x,t), v(x,t))$ *be functions satisfying for some* $M > 1$

$$\frac{\alpha}{\beta} u_0(x) \le M v_0(x)^{\alpha/\beta} \quad \text{or} \quad \frac{\beta}{\alpha} v_0(x) \le M u_0(x)^{\beta/\alpha} \tag{1.13}$$

and (U,V) *given by the formula* (1.10) *be any nonnegative, bounded solution of* (1.11), *defined for all* $(y,s) \in \mathbb{R}^N \times \mathbb{R}$. *If* $(U,V) \ne (0,0)$, *and*

$$\min(\alpha,\beta) > 0, \tag{1.14}$$

and

$$\frac{1}{d}\min(d\alpha, d\beta) \ge \frac{N-2}{4}, \; N \ge 3 \quad \text{or} \quad N = 1,2, \tag{1.15}$$

then

$$\lim_{s\to\infty}(U(y,s),V(y,s)) = (C_\alpha, C_\beta) \tag{1.16}$$

uniformly on compact sets $|y|<R<\infty$*, with constants* C_α, C_β *solving* (1.12).

Now we can linearize around (C_α, C_β). We introduce the functions (w_1,w_2) given by

$$U(y,s) = C_\alpha + w_1(y,s), \quad V(y,s) = C_\beta + w_2(y,s) \tag{1.17}$$

satisfying a system

$$\begin{cases} w_{1_s} & = & \triangle w_1 - \frac{1}{2}y\nabla w_1 - \alpha w_1 + q_1\alpha\ \frac{C_\alpha}{C_\beta}\ w_2 + f_1(w) \\ & \equiv & A_1 w_1 + q_1\ \frac{C_\alpha}{C_\beta}\ w_2 + f_1(w), \\ w_{2_s} & = & \triangle w_2 - \frac{1}{2}y\nabla w_2 - \beta w_2 + p_2\beta\ \frac{C_\beta}{C_\alpha}\ w_1 + f_2(w) \\ & \equiv & A_2 w_2 + p_2\ \frac{C_\beta}{C_\alpha}\ w_1 + f_2(w), \end{cases} \tag{1.18}$$

where $f_i(w) = f_i(w_1,w_2)$ and $f_i(w) = O(w^2)$ as $w\to 0$ for $i=1,2$. Setting

$$L^2_\rho(\mathbb{R}^N) = \{g\in L^2_{loc}(\mathbb{R}^N) : \int_{\mathbb{R}^N}|g(x)|^2\rho(x)\,dx<\infty,\ \rho(x)=e^{-|x|^2/4}\} \tag{1.19}$$

and $V = L^2_\rho(\mathbb{R}^N)\times L^2_\rho(\mathbb{R}^N)$,

$$H^k_\rho(\mathbb{R}^N) = \{g\in H^k_{loc}(\mathbb{R}^N) : \int_{\mathbb{R}^N}|g^{(j)}(x)|^2\rho(x)\,dx<\infty,\ \text{for all } j\in[0,k]\}, \tag{1.20}$$

we define $A : V\to V$ with domain $D(A) = H^2_\rho(\mathbb{R}^N)\times H^2_\rho(\mathbb{R}^N)$, by

$$A = \begin{pmatrix} A_1 & q_1\frac{C_\alpha}{C_\beta}I \\ p_2\frac{C_\beta}{C_\alpha}I & A_2 \end{pmatrix}. \tag{1.21}$$

We consider (1.18) as the following dynamical system in V:

$$\begin{pmatrix} w_{1_s} \\ w_{2_s} \end{pmatrix} = A\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} f_1(w) \\ f_2(w) \end{pmatrix}. \tag{1.22}$$

We separate the operator A as follows:

$$A\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = (\triangle - \frac{1}{2}y\nabla)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} + \begin{pmatrix} -\alpha & q_1\alpha\frac{C_\alpha}{C_\beta} \\ p_2\beta\frac{C_\beta}{C_\alpha} & -\beta \end{pmatrix}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \equiv A_0\begin{pmatrix}\varphi\\ \psi\end{pmatrix} + B\begin{pmatrix}\varphi\\ \psi\end{pmatrix}. \tag{1.23}$$

Then, for any element $\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in V$ we write

$$\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \sum_{\gamma} a_{\gamma}^{+} H_{\gamma} z_{+} + \sum_{\gamma} a_{\gamma}^{-} H_{\gamma} z_{-} \tag{1.24}$$

with z_+ and z_- eigenfunctions of the operator B, and c_γ and H_γ eigenfunctions of A_0 given by

$$c_\gamma H_\gamma(y) = c_{\gamma_1} H_{\gamma_1}(y_1) \dots c_{\gamma_N} H_{\gamma_N}(y_N), \tag{1.25}$$

where $\gamma = (\gamma_1, \dots, \gamma_N)$, $|\gamma| = \gamma_1 + \cdots + \gamma_N = 0, 1, 2, \dots$, $H_k(s) = \breve{H}_k(s/2)$, $\breve{H}_k$ is the standard k^{th} Hermite polynomial and c_γ is some normalization constant.

Our last theorem is the following:

Theorem 3. *Let* (u, v) *be a solution of* (1.3) *in* $\mathbb{R}^N \times (0, T)$*. Assume that the estimates* (1.9) *hold and* (U, V) *defined by* (1.10) *converges to constants* (C_α, C_β) *given by* (1.12)*, as* $s \to \infty$*. Let* $w = (w_1, w_2)$ *be given by* (1.17)*. Then the following possibilities arise.*

If $p_2 q_1 > 1$*, then either there exists an orthogonal transformation of coordinate axes* $(y_1', \dots, y_N')$ *such that*

$$w(\cdot, s) = -\frac{C}{s} z_+ \sum_{k=1}^{l} H_{[\sqrt{\Delta} - \alpha - \beta]}(y_k') + o\Big(\frac{1}{s}\Big) \quad \text{as } s \to \infty, \tag{1.26}$$

and $1 \le l \le N$, $C = C(p_i, q_i) > 0$*, or else there exists an even number* m, $m \ge [\sqrt{\Delta} - \alpha - \beta] + 1$ *where* $\Delta = (\alpha - \beta)^2 + 4\alpha\beta p_2 q_1$ *is a nonnegative number, and some* c_γ *such that*

$$w(\cdot, s) = -\Big(\sum_{|\gamma| = m} c_\gamma H_\gamma(y)\Big) e^{(\lambda_+ - \frac{m}{2})s} z_+ + o(e^{(\lambda_+ - \frac{m}{2})s}), \quad \text{as } s \to \infty, \tag{1.27}$$

where $\lambda_+ = \frac{1}{2}(\sqrt{\Delta} - \alpha - \beta)$*, or*

$$w(\cdot, s) = \Big(\sum_{|\gamma| = m} c_\gamma H_\gamma(y)\Big) e^{(\lambda_- - \frac{m}{2})s} z_- + o(e^{(\lambda_- - \frac{m}{2})s}), \quad \text{as } s \to \infty, \tag{1.28}$$

where $\lambda_- = -\frac{1}{2}(\sqrt{\Delta} + \alpha + \beta)$ *and* $m \ge 0$.

The multilinear form $\sum_{|\gamma| = m} c_\gamma x^\gamma$ *is nonnegative and nontrivial and convergence takes place in* $H_\rho^1(\mathbb{R}^N)$ *as well as in* $C_{loc}^{k,\theta}$ *for any* $k \ge 0$ *and* $\theta \in (0, 1)$.

If $p_2 q_1 \le 1$*, then only one of the cases* (1.26) *or* (1.27) *occurs, when* $\lambda_- < 0$, $\lambda_+ \le 0$ *and in* (1.26) m *is any nonnegative integer.*

Remark 1.1. To guarantee the hypothesis of Theorem 3 it suffices to assume that (1.13), (1.14) and (1.15) hold.

The plan of the paper is a natural extension of the Introduction. More precisely, we prove Theorem 2 in Section 2, and Theorem 3 (a Liouville theorem) in Section 3. The implementation of a functional frame leading to the proof of Theorem 4, as well as this proof, can be found in Section 4.

2. Estimate for the blow-up rate

As we have just mentioned in the Introduction, the proof of (1.9) is a special case of the one presented in [13] for an even number $m \geq 2$. For this reason, we shall briefly sketch the arguments leading to Theorem 1, for the convenience of the reader.

Our basic tool is the concept of an invariant region. To establish our consideration, we assume throughout the section that $\max(\alpha, \beta) = \alpha$. First, we recall the result proved in [10] and [13].

Lemma 2.1. *Let* $\max(\alpha, \beta) = \alpha$ *and* $\beta > 0$. *Then a set*

$$M = \{(u, v) : v - au^{\beta/\alpha} \leq 0,\ u, v \geq 0\}, \tag{2.1}$$

where $a = (\beta/\alpha)^{1/d\alpha}$ *and* α, β, d *are given by* (1.4)–(1.6), *is a regular invariant region for* (1.3).

This implies that for a solution of (1.3) with initial data satisfying $(u_0, v_0) \in \partial M$, i.e., $v_0 = au_0^{\beta/\alpha}$, it holds that $(u(t), v(t)) \in M$ for all $t \in (0, T)$. We take advantage of this to prescribe a system (1.3) as follows. For initial values (u_0, v_0) we can take $(\bar{u}_0, \bar{v}_0)$ satisfying

$$0 \leq (u_0, v_0) \leq (\bar{u}_0, a\bar{u}^{\beta/\alpha}) = (\bar{u}_0, \bar{v}_0) \tag{2.2}$$

Next, we consider (u^*, v^*), a solution of (1.3) with $(u^*(0), v^*(0)) = (\bar{u}_0, \bar{v}_0)$. Since (u^*, v^*) belong to M, this leads to the system

$$\begin{cases} u^*_t - \triangle u^* \leq a^{q_1} (u^*)^{1+\frac{1}{\alpha}} \\ v^*_t - \triangle v^* \leq a^{q_2} (u^*)^{\frac{1+\beta}{\alpha}} \\ u^*(0) = \bar{u}_0,\ v^*(0) = \bar{v}_0. \end{cases} \tag{2.3}$$

In this way, we have obtained a supersystem corresponding to (1.3), namely

$$\begin{cases} \bar{u}_t - \triangle \bar{u} = a^{q_1} \bar{u}^{1+\frac{1}{\alpha}} \\ \bar{v}_t - \triangle \bar{v} = a^{q_2} \bar{u}^{\frac{1+\beta}{\alpha}} \\ \bar{u}(0) = \bar{u}_0,\ \bar{v}(0) = \bar{v}_0 \end{cases} \tag{2.4}$$

and

$$(u(x,t), v(x,t)) \le (\bar{u}(x,t), \bar{v}(x,t)). \tag{2.5}$$

We are now in position to prove Theorem 1. Suppose that (1.7) holds; i.e.,

$$N = 1,2 \quad \text{or} \quad N \ge 3 \quad \text{and} \quad 1 \le \frac{1+\alpha}{\alpha} \le \frac{N+2}{N-2}, \tag{2.6}$$

where $\alpha = \max(\alpha, \beta)$. Then we can apply the known result concerning a scalar equation, proved in [7] (Theorem 3.7 therein), to the first equation of (2.4). It follows that for some $C > 0$

$$\bar{u}(x,t) \le C(T-t)^{-\alpha}. \tag{2.7}$$

Thus, by (2.5)

$$u(x,t) \le C(T-t)^{-\alpha}. \tag{2.8}$$

Let us recall that $(u^*, v^*) \in M$ is by definition a solution of (1.3). Moreover, by (2.3) we assure that $u^*(x,t) \le \bar{u}(x,t) \le C(T-t)^{-\alpha}$, but this implies

$$v^*(x,t) \le a(u^*(x,t))^{\beta/\alpha} \le C'(T-t)^{-\beta}. \tag{2.9}$$

On the other hand, by (2.2), (u^*, v^*) is a supersolution to (1.3); then the upper bound (2.9) remains true for $v(x,t)$. In this way (2.8) and (2.9) establish a desired blow-up rate (1.9) under assumption (1.7).

Our next step consists in proving (1.9) using the condition (1.8). We consider the first equation of (2.4), i.e.,

$$\bar{u}_t - \triangle \bar{u} = A\bar{u}^{1+\frac{1}{\alpha}}, \quad \bar{u}(0) = \bar{u}_0, \tag{2.10}$$

where $A = a^{q_1}$ and $\bar{u}_0$ is chosen in such a way that

$$\triangle \bar{u}_0 + A\bar{u}_0^{1+\frac{1}{\alpha}} > 0 \tag{2.11}$$

for u_0 satisfying (1.8). Following ideas of [4] we introduce a function

$$F = \bar{u}_t - \delta A \bar{u}^r, \quad r = 1 + \frac{1}{\alpha}. \tag{2.12}$$

A simple computation reveals that

$$F_t - \triangle F = Ar\bar{u}^{r-1}F + \delta Ar(r-1)\bar{u}^{r-2}|\nabla \bar{u}|^2.$$

Then F satisfies

$$F_t - \triangle F - Ar\bar{u}^{r-1}F \ge 0. \tag{2.13}$$

Now we set $\delta > 0$ such that $F(0) > 0$, which is possible by (2.11) (because $\bar{u}_t(0) > 0$ and $\bar{u}(0)$ is bounded). Using (2.13) and the maximum principle we see that F is nonnegative; i.e., for some $\delta > 0$

$$\bar{u}_t \ge \delta A \bar{u}^r. \tag{2.14}$$

This, by integration, leads to $\frac{\bar{u}(x,t)^{-r+1}}{r-1} \geq \delta A(T-t)$. Then

$$\bar{u}(x,t) \leq C(T-t)^{-\alpha} \quad \text{where} \quad \alpha = \frac{1}{r-1},\ C = ((r-1)\delta A)^{-\alpha}. \tag{2.15}$$

The last inequality has the same form as (2.7), so we can complete the proof using exactly the same route as in the previous case to obtain the assertion. Finally, we obtain Theorem 1.

3. Liouville theorem

This section eventually leads to the proof of Theorem 2. To proceed, one first proves some estimates.

Lemma 3.1. *Let* $\min(\alpha,\beta) > 0$*. If* $d\alpha > d\beta$*, then a solution* (u,v) *of* (1.3) *satisfies for some constant* M

$$u(x,t) \leq M\frac{\beta}{\alpha}(v(x,t))^{\alpha/\beta}. \tag{3.1}$$

Remark 3.2. If $d\alpha \leq d\beta$, then respectively holds

$$v(x,t) \leq M\frac{\alpha}{\beta}(u(x,t))^{\beta/\alpha}. \tag{3.2}$$

Proof. We proceed by adapting an argument in [4]. We denote

$$\alpha_1 = -d\alpha, \quad \beta_1 = -d\beta \tag{3.3}$$

and introduce the functions

$$h(v) = Mv^{\alpha_1}, \quad k(u) = \frac{\alpha}{\beta}u^{\beta_1}. \tag{3.4}$$

We shall consider

$$J = h(v) - k(u). \tag{3.5}$$

We have

$$J_t - \triangle J = h'(v)u^{p_2}v^{q_2} - k'(u)u^{p_1}v^{q_1} - h''(v)|\nabla v|^2 + k''(u)|\nabla u|^2. \tag{3.6}$$

By definition of J it follows that $h'(v)\nabla v = \nabla J + k'(u)\nabla u$ and

$$|\nabla v|^2 = \left(\frac{k'(u)}{h'(v)}\right)^2 |\nabla u|^2 + b\nabla J$$

for some coefficient b. Substituting this in (3.6), we obtain

$$J_t - \triangle J - b_1\nabla J = M\alpha_1 v^{\alpha_1 - 1 + q_2}u^{p_2} - \alpha_1 u^{\beta_1 - 1 + p_1}v^{q_1}$$
$$+ \alpha_1\big[(\beta_1 - 1)u^{\beta_1 - 2} - (\alpha_1 - 1)v^{\alpha_1 - 2}\frac{u^{2\beta_1 - 2}}{Mv^{2\alpha_1 - 2}}\big]|\nabla u|^2$$

$$= \alpha_1 u^{p_2} v^{q_1}(M-1) + \alpha_1 u^{\beta_1 - 2}\big[(\beta_1 - 1) - (\alpha_1 - 1)\frac{u^{\beta_1}}{Mv^{\alpha_1}}\big]|\nabla u|^2.$$

We use again (3.5); in this way $u^{\beta_1} = \frac{\beta_1}{\alpha_1}(Mv^{\alpha_1} - J)$. Then

$$J_t - \triangle J - b_1\nabla J - b_2 J = \alpha_1 u^{p_2} v^{q_1}(M-1) + u^{\beta_1-2}(\beta_1 - \alpha_1)|\nabla u|^2 \tag{3.7}$$

where b_1, b_2 are some coefficients.

By assumption and (3.3), $\beta_1 - \alpha_1 > 0$. If we choose $M > 1$ such that

$$Mv(x,0)^{\alpha/\beta} \geq \frac{\alpha}{\beta} u(x,0), \tag{3.8}$$

then by the maximum principle and (3.7) $J > 0$. This implies (3.1). □

The above assertion plays a crucial role in deriving the Liouville theorem. It consists in the possibility of rewriting the system (1.3) to a system of inequalities. Nevertheless, both are the scalar ones. Moreover, it turns out that adapting a technique used to discuss the scalar problem is then available.

Applying (3.1) in (1.3) we have

$$u_t - \triangle u \geq M_1 u^r, \qquad v_t - \triangle v \leq M_2 v^q \tag{3.9}$$

where $r = 1 + \frac{1}{\alpha}$, $q = 1 + \frac{1}{\beta}$, $M_1 = \big(\frac{\alpha}{\beta M}\big)^{q_1}$, $M_2 = \big(\frac{M\beta}{\alpha}\big)^{p_2}$. We recall that (u, v) satisfying (3.9) is simultaneously a solution of (1.3). Therefore, it satisfies bounds (1.9) whenever one of the conditions (1.7), (1.8) holds. On the other hand v is a subsolution to a scalar equation, corresponding to the second inequality of the system (3.9). Then, under assumption (1.7) or (1.8) rewritten in terms of $\beta = \frac{1}{q-1}$, we obtain the same upper bound for v by applying the related result concerning the scalar problem. Next, by (3.1)

$$u(x,t) \leq M\frac{\beta}{\alpha}(C(T-t)^{-\beta})^{\alpha/\beta} = C'(T-t)^{-\alpha}.$$

In this way we have obtained a desired estimate for the blow-up rate without the use of an invariant region. However, this method is available by Lemma 3.1, i.e., only for system of two equations.

We introduce similarity variables setting

$$\begin{aligned} &u(x,t) = (T-t)^{-\frac{1}{r-1}} U(y,s), \quad v(x,t) = (T-t)^{-\frac{1}{q-1}} V(y,s) \\ &y = x(T-t)^{-\frac{1}{2}}, \quad s = -\log(T-t), \end{aligned} \tag{3.10}$$

i.e., the change defined as in (1.10) by definitions of r and q.

We consider a corresponding system

$$\begin{cases} U_s - \triangle U + \frac{1}{2} y\nabla U + \frac{1}{r-1}U - M_1 U^r \geq 0 \\ V_s - \triangle V + \frac{1}{2} y\nabla V + \frac{1}{q-1}V - M_2 V^q \leq 0. \end{cases} \tag{3.11}$$

We use the above considerations to conclude that we can quote a result of Giga and Kohn ([6], Proposition 1) in the following way:

Lemma 3.3. *If v is a function satisfying* (3.9*b*) *and* (1.9*b*) *holds, then*

$$|\nabla^k v(x,t)| \le C(T-t)^{-\beta-\frac{k}{2}}. \tag{3.12}$$

This implies

Lemma 3.4. *If V is a bounded function satisfying* (3.11*b*) *defined on* $\mathbb{R}^{N+1}$, *then*

$$|\nabla V| + |\nabla^2 V| \le C', \quad |V_s| \le C'(1+|y|). \tag{3.13}$$

Remark 3.5. Using (3.1) and the system (1.3) we can infer that the corresponding results remain true for u and U.

Following Giga and Kohn we shall prove Theorem 2 in two main steps. First, we want to convince ourselves that as $s \to \infty$ then V and in consequence U are necessarily self-similar, i.e., independent of s. Therefore, (U,V) should approach a stationary solution of (1.11). From this point of view we shall next analyze the global nonnegative and bounded solutions of the system

$$\begin{cases} \triangle U - \frac{1}{2} y \nabla U - \alpha U + U^{p_1} V^{q_1} = 0 \\ \triangle V - \frac{1}{2} y \nabla V - \beta V + U^{p_2} V^{q_2} = 0. \end{cases} \tag{3.14}$$

It turns out that they are constants given by (1.12) or both equal to zero.

To proceed it seems convenient to rewrite (3.11b) as follows:

$$\rho V_s - \nabla \cdot (\rho \nabla V) + \frac{1}{q-1} \rho V - M_2 \rho V^q \le 0, \tag{3.15}$$

where $\rho(y) = \exp(-\frac{1}{4}|y|^2)$. We establish

Lemma 3.6. *If V solves* (3.11) *and is bounded, then*

$$\int_a^b \int_{\mathbb{R}^N} |V_s|^2 \rho \, dy \, ds \le E[V](a) - E[V](b), \tag{3.16}$$

where $a < b$ are real and

$$E[V](s) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla V|^2 \rho \, dy + \frac{1}{2(q-1)} \int_{\mathbb{R}^N} |V|^2 \rho \, dy - \frac{M_2}{q+1} \int_{\mathbb{R}^N} |V|^{q+1} \rho \, dy. \tag{3.17}$$

Proof. We derive the above assertion analogously to the proof of Proposition 3 in [6]. By multiplying (3.15) by V_s and integrating, one obtains

$$\int_{B_R} |V_s|^2 \rho \, dy \le \int_{B_R} V_s \nabla \cdot (\rho \nabla V) \, dy - \frac{1}{2(q-1)} \frac{d}{ds} \int_{B_R} |V|^2 \rho \, dy$$

$$+\frac{M_2}{q+1}\frac{d}{ds}\int_{B_R}|V|^{q+1}\rho\,dy$$
$$=-\frac{1}{2}\frac{d}{ds}\int_{B_R}|\nabla V|^2\rho\,dy+\int_{\partial B_R}V_s\frac{\partial V}{\partial r}\rho\,d\sigma$$
$$-\frac{1}{2(q-1)}\frac{d}{ds}\int_{B_R}|V|^2\rho\,dy+\frac{M_2}{q+1}\frac{d}{ds}\int_{B_R}|V|^{q+1}\rho\,dy,$$

where $B_R=\{y:|y|<R\}$. By Lemma 3.3 we observe that the surface term tends to zero as $R\to\infty$. Therefore, integration from a to b and the passage with R to the limit gives

$$\int_a^b\int_{\mathbb{R}^N}|V_s|^2\rho\,dy\,ds\le E[V](a)-E[V](b).$$

This concludes the proof. □

We set

$$V_{+\infty}(y,s)=\lim_{s_j\to\infty}V(y,s+s_j),\quad V_{-\infty}(y,s)=\lim_{s'_j\to-\infty}V(y,s+s'_j). \tag{3.18}$$

This corresponds respectively to limits $\lim_{t\to T}v(x,t)$ and $\lim_{t\to-\infty}v(x,t)$. Using the previous estimate we can assume that $V_\infty, V_{-\infty}$ are independent of s.

Lemma 3.7. *Assume that V is a bounded solution of (3.11) on $\mathbb{R}^{N+1}$ and that a sequence $s_j\to+\infty$ (respectively, $s_j\to-\infty$) is monotonically increasing (decreasing) with $s_{j+1}-s_j\to\pm\infty$ as $j\to\infty$. Let $V_j(y,s)=V(y,s+s_j)$ converges to a limit $V_{\pm\infty}$ uniformly on compact sets of $\mathbb{R}^{N+1}$ with $\nabla V_j(y,m)\to\nabla V_{\pm\infty}(y,m)$ almost everywhere in $\mathbb{R}^N$, for all integer m. Then $V_{\pm\infty}$ does not depend on s and $E[V_\infty], E[V_{-\infty}]$ are independent of the choice of $\{s_j\}$.*

Proof. This lemma is an analogue of Proposition 4 in [6]. We consider $s_j\to\infty$. By Lemma 3.6, putting $V=V_j$, $a=m$, and $b=m+s_{j+1}-s_j$, we get

$$\int_m^{m+s_{j+1}-s_j}\int_{\mathbb{R}^N}|V_{js}|^2\rho\,dy\,ds\le E[V_j](m)-E[V_j](m+s_{j+1}-s_j)$$
$$=E[V_j](m)-E[V_{j+1}](m). \tag{3.19}$$

Lemma 3.4 yields that $|\nabla V_j|<C, |V_j|<c$ for all i. Thus, we can pass with $j\to\infty$ in (3.19), and since $E[V_j](m)\to E[V_\infty](m)$, $s_{j+1}-s_j\to\infty$, we see that

$$\lim_{j\to\infty}\int_m^K\int_{\mathbb{R}^N}|V_{js}|^2\rho\,dy\,ds\le 0,\quad m<K.$$

Therefore, we must have

$$\lim_{j\to\infty}\int_m^K\int_{\mathbb{R}^N}|V_{js}|^2\rho\,dy\,ds=0. \tag{3.20}$$

It follows, by the fact that $|V_{js}|\le C(1+|y|)$ (i.e., (3.13b)) and the lower semicontuity of the integral in (3.20), that also

$$\int_m^K\int_{\mathbb{R}^N}|V_{\infty s}|^2\rho\,dy\,ds=0 \tag{3.21}$$

for even m and K. This establishes an assertion concerning V_∞.

Now, we discuss properties of $E[V_\infty]$. We argue by contradiction. Let $\breve{s}_j$ be a sequence such that $\breve{s}_j$ the satisfies assumptions of Lemma 3.7, $\breve{V}_j(y,s)=V(y,s+\breve{s}_j)$, $\breve{V}_\infty=\lim_{\breve{s}_j\to\infty}\breve{V}_j$ and $E[V_\infty]\ne E[\breve{V}_\infty]$. Let $E[V_\infty]\le E[\breve{V}_\infty]$, $s_j<\breve{s}_j$ (we can obtain it passing to a subsequence). We use Lemma 3.6 setting in (3.16) $a=s_j$ and $b=\breve{s}_j$. Then

$$\begin{aligned}\int_{s_j}^{\breve{s}_j}\int_{\mathbb{R}^N}|V_s|^2\rho\,dy\,ds&\le E[V](s_j)-E[V](\breve{s}_j)\\&=E[V_j](0)-E[\breve{V}_j](0)\to E[V_\infty]-E[\breve{V}_\infty]<0.\end{aligned} \tag{3.22}$$

Thus, for j large enough, this is a contradiction with the nonnegativity of the integral in (3.22). We conclude that $E[V_\infty]=E[\breve{V}_\infty]$; i.e., $E[V_\infty]$ is independent of $\breve{s}_j$. We complete the proof considering the case $s_j\to-\infty$ similarly. □

Finally, we infer

Proposition 3.8. *If V is a bounded global solution of* (3.11) *on $\mathbb{R}^{N+1}$, then both the limits V_∞ and $V_{-\infty}$ exist and are independent of s. Moreover, $V(y,s+s_j)\to V_{\pm\infty}$ as $s_j\to\pm\infty$ uniformly on compact subsets of $\mathbb{R}^N$ with*

$$\lim_{s\to\pm\infty}\nabla V(y,s)=0\quad\textit{for almost every } y.$$

Proof. It suffices to guarantee that the assumptions of Lemma 3.7 are satisfied. However, this follows from bounds (3.13) by a standard diagonal argument. Indeed, we have $|\nabla V|\le C'$ and $|V_s|\le C'(1+|y|)$; then for some subsequence $\{s_j\}, s_j\to\infty$, $V(y,s+s_j)\to V_\infty(y,s)$ uniformly on compact sets. The bound $|\nabla^2V|\le C$ assures that $\nabla V(y,m+s_j)\to\nabla V_\infty(y,m)$ almost everywhere for every integer m and some subsequence s_j. Clearly, we can take such a subsequence to $s_{j+1}-s_j\to\infty$, also. Then, applying Lemma 3.7, we obtain the desired conclusion. □

Our next task consists in characterizing the bounded global nonnegative solutions of the stationary system (3.14), since by the second equation of

(1.11) if V is independent of s then U must be also stationary. We classify such solutions analyzing first V.

Proposition 3.9. *If* (U,V) *is a bounded, nonnegative global solution of* (3.14) *in* $\mathbb{R}^N$*, and* V *satisfies the inequality*

$$\triangle V - \frac{y}{2}\nabla V - \frac{V}{q-1} + M_2 V^q \geq 0, \tag{3.23}$$

then $U = V \equiv 0$ *or* $(U,V) \equiv (C_\alpha, C_\beta)$*, whenever* $N = 1,2$ *or* $N \geq 3$ *and* $1 < q \leq \frac{N+2}{N-2}$ *(i.e.,* $\beta \geq \frac{N-2}{4}$*).*

Proof. As we have mentioned V is a subsolution of a corresponding scalar equation, namely $V(y) \leq \bar{V}(y)$ where

$$\triangle \bar{V} - \frac{y}{2}\nabla \bar{V} - \frac{\bar{V}}{q-1} + M_2 \bar{V}^q = 0. \tag{3.24}$$

Applying the known result to the above equation (cf. Theorem 1 of [6], also [5]) we conclude that the only bounded, global solutions of (3.24) are $\bar{V} \equiv 0$ or $\bar{V} \equiv K$, where $K = (\frac{\beta}{M_2})^\beta = [M_2(q-1)]^{-\frac{1}{q-1}}$ provided $N = 1,2$ or $N \geq 3$ and $1 < q < \frac{N+2}{N-2}$.

Similarly, as before we rewrite (3.23) into the form

$$\nabla \cdot (\rho \nabla V) - \frac{\rho}{q-1} V + M_2\, \rho\, |V|^q \geq 0 \tag{3.25}$$

with $\rho(y) = \exp(-\frac{1}{4}|y|^2)$. Multiplying (3.25) by $-V$, integrating over $\mathbb{R}^N$ and integrating by parts in the first term, we obtain

$$\int_{\mathbb{R}^N} |\nabla V|^2 \rho\, dy + \frac{1}{q-1}\int_{\mathbb{R}^N} |V|^2 \rho\, dy - M_2 \int_{\mathbb{R}^N} |V|^{q+1} \rho\, dy \leq 0. \tag{3.26}$$

This yields

$$\int_{\mathbb{R}^N} |\nabla V|^2 \rho\, dy \leq \int_{\mathbb{R}^N} |V|^2 \rho \big(M_2 |V|^{q-1} - \frac{1}{q-1}\big)\, dy. \tag{3.27}$$

Then, by the fact that $|V| \leq [\frac{1}{M_2(q-1)}]^{\frac{1}{q-1}}$ or $|V| = 0$, we get

$$\int_{\mathbb{R}^N} |\nabla V|^2 \rho\, dy \leq 0 \quad \text{or} \quad V \equiv 0. \tag{3.28}$$

Therefore, V is necessarily some constant. Clearly, by (3.14b) U is also a constant given by $U = \beta V^{1-q_2}$. Since the only constant solutions of (3.14) are (C_α, C_β) or $U = V \equiv 0$, the proof is complete. □

By Lemma 3.1, Proposition 3.8 and 3.9 apply and establish Theorem 2 whenever the upper bounds (1.9) are satisfied; i.e., (1.7) or (1.8) holds. It remains to consider the case when $d\alpha \leq d\beta$. By Remark 3.2 we can execute an entirely parallel analysis to prove Propositions 3.8 and 3.9 with V replaced by U. Eventually, we prove Theorem 2 assuming $\alpha \geq \frac{N-2}{4}$ for $N \geq 3$ or $N = 1, 2$. Summarizing, we obtain our Liouville theorem under hypothesis (1.15), i.e.,

$$\frac{1}{d}\min(d\alpha, d\beta) \geq \frac{N-2}{4}, \quad N \geq 3 \quad \text{or} \quad N = 1, 2$$

and (1.14), i.e., $\min(\alpha, \beta) > 0$.

Remark 3.10. If $d > 0$ then (1.15) yields (1.14) and (1.7); otherwise, we can only conclude (1.7).

Therefore, using the above considerations, the proof of Theorem 2 is complete.

The final result of this section is

Theorem 4. *Let* (U, V) *be a nonnegative, bounded global solution of* (1.11) *and* (1.14), (1.15) *hold. Then under one of the following assumptions,*

$$d\beta < d\alpha, \quad M < \Big(\frac{q+1}{2}\Big)^{1/p_2}\Big(\frac{\alpha}{\beta}\Big)^{1+\frac{1}{d\beta}} \quad \text{or} \tag{3.29}$$

$$d\beta > d\alpha, \quad M' < \Big(\frac{r+1}{2}\Big)^{1/q_1}\Big(\frac{\beta}{\alpha}\Big)^{1+\frac{1}{d\alpha}},$$

where $M, M' > 1$ *are such that*

$$M v_0(x)^{\alpha/\beta} \geq \frac{\alpha}{\beta}\, u_0(x), \quad M' u_0(x)^{\beta/\alpha} \geq \frac{\beta}{\alpha}\, v_0(x), \tag{3.30}$$

the following possibilities arise:

$$(U, V) \equiv (C_\alpha, C_\beta) \text{ or} \tag{3.31}$$

$$(U, V) \equiv (0, 0) \text{ or} \tag{3.32}$$

$$(U, V) \to (C_\alpha, C_\beta) \quad \text{as} \quad s \to -\infty$$

$$\text{and} \quad (U, V) \to (0, 0) \quad \text{as} \quad s \to +\infty. \tag{3.33}$$

Proof. We shall analyze the case when (3.29a) holds. Then, by Lemma 3.6 and a relation (3.16) with $a = -\tau$, $b = \tau$

$$\int_{-\tau}^{\tau}\int_{\mathbb{R}^N} |V_s|^2 \rho\, dy\, ds \leq E[V](-\tau) - E[V](\tau).$$

Using Proposition 3.8, we pass with $\tau \to \infty$. Then denoting $V_{\pm\infty} = \lim_{s\to\pm\infty} V(y, s)$, we obtain

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^N} |V_s|^2 \rho\, dy\, ds \leq E[V_{-\infty}] - E[V_{+\infty}]. \tag{3.34}$$

Since $V_{\pm\infty}$ are independent of s, by Proposition 3.9 it must occur that $V_{\pm\infty} \equiv C_\beta$ or 0. Clearly, $E[0] = 0$ whereas (3.29a) yields

$$E[C_\beta] = (\frac{1}{2}\beta C_\beta^2 - \frac{M_2}{q+1}C_\beta^{q+1}) \int_{\mathbb{R}^N} \rho\, dy > 0, \text{ by } C_\beta = (\beta^{p_1-1}\alpha^{-p_2})^{1/d}. \tag{3.35}$$

Assume $V_{-\infty} = 0$; then by (3.34) and (3.35) necessarily $V_{+\infty} = 0$. This implies $V_s = 0$, so $V \equiv 0$, and using (1.11) also $U \equiv 0$; i.e., (3.32) is satisfied.

If $V_{-\infty} = C_\beta$, then $V_{+\infty} = C_\beta$ gives that V is independent of s. Arguing as in the previous case, it follows that $(U, V) \equiv (C_\alpha, C_\beta)$, i.e., (3.31).

In the only remaining case we have $(V_{-\infty}, V_{+\infty}) = (C_\beta, 0)$, and then using (1.11) we see that (3.33) emerges. The analysis when the hypothesis (3.29b) holds is paralleled by Remark 3.2, the analogues of Propositions 3.8–3.9 and Lemma 3.6 with the respective energy E'. Thus, the assertion is proved. □

4. Classification of blow-up patterns

This section is devoted to discussion of the possible behaviors of solutions of (1.3) near blow-up points.

To proceed we develop the ideas mentioned in Introduction. Under assumptions that yield the conclusion of Theorem 2, we make the use of the convergence $(U, V) \to (C_\alpha, C_\beta)$ as $\tau \to \infty$ to constants given by (1.12), to define

$$w_1 = U - C_\alpha, \quad w_2 = V - C_\beta. \tag{4.1}$$

This leads to the following system:

$$\begin{cases} w_{1s} &= \triangle w_1 - \frac{1}{2}y\nabla w_1 - \alpha(w_1 + C_\alpha) + (w_1 + C_\alpha)^{p_1}(w_2 + C_\beta)^{q_1} \\ &= \triangle w_1 - \frac{1}{2}y\nabla w_1 - \alpha w_1 + q_1\alpha\frac{C_\alpha}{C_\beta}w_2 + f_1(w) \\ &\equiv A_0 w_1 - \alpha w_1 + q_1\alpha\frac{C_\alpha}{C_\beta} + f_1(w) \\ w_{2s} &= \triangle w_2 - \frac{1}{2}y\nabla w_2 - \beta(w_2 + C_\beta) + (w_1 + C_\alpha)^{p_2}(w_2 + C_\beta)^{q_2} \\ &= \triangle w_2 - \frac{1}{2}y\nabla w_2 - \beta w_2 + p_2\beta\frac{C_\beta}{C_\alpha}w_1 + f_2(w) \\ &\equiv A_0 w_2 - \beta w_2 + p_2\beta\frac{C_\beta}{C_\alpha}w_1 + f_2(w) \end{cases} \tag{4.2}$$

where $f_i(w) = O(w^2)$ as $w \to 0$ and $w = (w_1, w_2)$.

We may rewrite (4.2) in a more convenient form:

$$\begin{pmatrix} w_{1s} \\ w_{2s} \end{pmatrix} = (\triangle - \frac{1}{2}y\nabla)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} -\alpha & q_1\alpha\frac{C_\alpha}{C_\beta} \\ p_2\beta\frac{C_\beta}{C_\alpha} & -\beta \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} f_1(w) \\ f_2(w) \end{pmatrix} = Aw + f(w). \tag{4.3}$$

A linear operator $A : V \to V$ with domain $D(A) = H^2_\rho(\mathbb{R}^N) \times H^2_\rho(\mathbb{R}^N)$ (spaces are defined in (1.19)–(1.20)) is not self-adjoint in V, where

$$||g||^2_\rho = \int_{\mathbb{R}^N} (g_1^2 + g_2^2) e^{-|x|^2/4}\, dx, \quad g = (g_1, g_2). \tag{4.4}$$

Now, we consider $A_0 = \triangle - \frac{1}{2} y \nabla$. Then A_0 is self-adjoint in $L^2_\rho(\mathbb{R}^N)$, has eigenvalues of the form $-|\gamma|/2$, where $\gamma = (\gamma_1, \ldots, \gamma_N), \gamma_i$ is a nonnegative integer for any $i = 1, 2, \ldots$ with corresponding eigenfunctions $c_\gamma H_\gamma(y)$ defined by (1.25) and a relation $|c_\gamma|^2 ||H_\gamma||^2_\rho = 1$. Since $\{H_\gamma\}$ is an orthonormal basis in $L^2_\rho(\mathbb{R}^N)$, it is natural to introduce an isomorphism between V and $l^2(\mathbb{R}^2) = \{(a_\gamma, b_\gamma) \in \mathbb{R}^2 : \sum_\gamma (|a_\gamma|^2 + |b_\gamma|^2) < \infty\}$. To this end we set

$$\begin{pmatrix} f \\ g \end{pmatrix} \longmapsto \begin{pmatrix} \{f_\gamma\} \\ \{g_\gamma\} \end{pmatrix} \in l^2(\mathbb{R}^2), \tag{4.5}$$

where $f_\gamma = \langle f, H_\gamma \rangle_\rho$, $g_\gamma = \langle g, H_\gamma \rangle_\rho$, $|\gamma| = 0, 1, 2, \ldots$ for any $(f, g) \in V$. Therefore, we have

$$\left(A \begin{pmatrix} f \\ g \end{pmatrix} \right)_\gamma = \begin{pmatrix} -\frac{|\gamma|}{2} - \alpha & q_1 \alpha \frac{C_\alpha}{C_\beta} \\ \\ p_2 \beta \frac{C_\beta}{C_\alpha} & -\frac{|\gamma|}{2} - \beta \end{pmatrix} \begin{pmatrix} f_\gamma \\ g_\gamma \end{pmatrix}. \tag{4.6}$$

It remains to deal with an operator given by (4.6). The spectrum consists of $-\frac{|\gamma|}{2} + \lambda_-$ and $-\frac{|\gamma|}{2} + \lambda_+$, where $\lambda_- = \frac{1}{2}(-\alpha - \beta - \sqrt{\Delta})$, $\lambda_+ = \frac{1}{2}(-\alpha - \beta + \sqrt{\Delta})$ and $\Delta = (\alpha - \beta)^2 + 4\alpha\beta p_2 q_1$. Since $\lambda_- \lambda_+ = \alpha\beta(1 - p_2 q_1)$, we take into account two cases. If $p_2 q_1 < 1$, then by the fact that $\min(\alpha, \beta) > 0$ both values λ_- and λ_+ are negative. Otherwise, $\lambda_- < 0$ and $\lambda_+ > 0$.

Let us remark that λ_- and λ_+ are the eigenfunctions of the operator B, namely

$$B = \begin{pmatrix} -\alpha & q_1 \alpha \frac{C_\alpha}{C_\beta} \\ p_2 \beta \frac{C_\beta}{C_\alpha} & -\beta \end{pmatrix}. \tag{4.7}$$

The corresponding eigenfunctions are given by

$$\begin{pmatrix} 1 \\ \frac{C_\beta}{2} \frac{\alpha - \beta - \sqrt{\Delta}}{q_1 \alpha C_\alpha} \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{C_\beta}{2} \frac{\alpha - \beta + \sqrt{\Delta}}{q_1 \alpha C_\alpha} \end{pmatrix}.$$

We choose functions z_- and z_+ parallel respectively to the ones above to assure that

$$B z_- = \lambda_- z_-, \quad B z_+ = \lambda_+ z_+, \quad |z_-| = |z_+| = 1. \tag{4.8}$$

Now, we represent V in the form

$$V = V_- \oplus V_+, \quad V_{-/+} = \{(a_\gamma, b_\gamma) = c_\gamma z_{-/+}, \sum_\gamma |c_\gamma|^2 < \infty\} \tag{4.9}$$

so V_- and V_+ correspond to subspaces spanned on $\{H_\gamma z_-\}$ and $\{H_\gamma z_+\}$ respectively. It is to be noticed that we cannot expect that V_- and V_+ are orthogonal unless we have some additional assumptions, namely

$$C_\alpha = p_2 \beta C_\beta. \tag{4.10}$$

Nevertheless, since $\{H_\gamma z_-\} \cup \{H_\gamma z_+\}$ constitutes a basis of V we are able to expand $(f, g) \in V$ as follows:

$$\begin{pmatrix} f \\ g \end{pmatrix} = \sum_\gamma a_\gamma^- H_\gamma z_- + \sum_\gamma a_\gamma^+ H_\gamma z_+. \tag{4.11}$$

Finally, we obtain

$$\begin{aligned} A(H_\gamma z_-) &= -\frac{1}{2}(|\gamma| + \alpha + \beta + \sqrt{\Delta}) H_\gamma z_- \\ A(H_\gamma z_+) &= \frac{1}{2}(\sqrt{\Delta} - \alpha - \beta - |\gamma|) H_\gamma z_+, \end{aligned} \tag{4.12}$$

where as before $\Delta = (\alpha - \beta)^2 + 4\alpha\beta p_2 q_1$ and $\gamma = (\gamma_1, \dots, \gamma_N)$, $\gamma_i \geq 0$.

It is worth noticing that if $p_2 q_1 < 1$, then all eigenvalues of A are negative. In this case we may repeat an analysis presented in [15] for examining the scalar case. In fact, we need only the part concerning the situation when $m \geq 3$ therein.

If $p_2 q_1 > 1$, there are positive eigenvalues corresponding to γ such that $|\gamma| < \sqrt{\Delta} - \alpha - \beta$ and $|\gamma|$ is an integer. The neutral mode, i.e., an eigenvalue equal to zero, occurs for $|\gamma| = [\sqrt{\Delta} - \alpha - \beta]$, where $[\cdot]$ denotes an entire part of the number. The corresponding eigenfunctions are $H_\gamma z_+$ with γ as above.

The following results can be proved using entirely parallel arguments as in [1] or [15]. Therefore, we shall sketch the proofs and refer to those works for details.

Assume that $p_2 q_1 > 1$, the assumptions of Theorem 2 are satisfied and $w = (w_1, w_2)$ defined by (4.1) is represented as follows:

$$w = \sum_\gamma a_\gamma^- H_\gamma z_- + \sum_\gamma a_\gamma^+ H_\gamma z_+. \tag{4.13}$$

Lemma 4.1. *If w as above is given by* (4.13), *then*

$$\lim_{s \to \infty} \frac{\sum_{|\gamma| < [\sqrt{\Delta} - \alpha - \beta]} |a_\gamma^+(s)|}{||w(\cdot, s)||_\rho} = 0. \tag{4.14}$$

Proof. As in [1], we desire to obtain a contradiction. If (4.14) does not hold, then assuming the limit equal to 1 we have for $s \geq \bar{\tau}$

$$\sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(s)| > ||w(\cdot,s)||_\rho. \tag{4.15}$$

By (4.3)

$$w_s = Aw + f(w), \tag{4.16}$$

where $f(w) = 0(w^2)$ as $w \to 0$. We associate to A a corresponding semigroup $S(s)$ to write

$$w(\cdot,s) \quad = \quad S(s-\bar{\tau})w(\cdot,\bar{\tau}) + \int_{\bar{\tau}}^s S(s-\tau)f(w(\cdot,\tau))\,d\tau \tag{4.17}$$

for $s > \bar{\tau}$. By regularizing effect, for some $L > 0$

$$||w(\cdot,s)||_{\rho,r} \leq C||w(\cdot,\bar{\tau})||, \text{ for } s \geq \bar{\tau} + L. \tag{4.18}$$

Then, for $2L \leq s - \bar{\tau} \leq 3L$

$$\int_{\bar{\tau}}^s S(s-\tau)f(w(\cdot,\tau))\,d\tau = \int_{\bar{\tau}}^{s-L} \cdots + \int_{s-L}^s \cdots \equiv I_1 + I_2 \tag{4.19}$$

and

$$||I_2|| \leq c\Big(\sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(\bar{\tau})|\Big)^2. \tag{4.20}$$

Then there exists $\delta > 0$ such that

$$||I_1|| \leq C\Big(\sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(\bar{\tau})|\Big)^{1+\delta}. \tag{4.21}$$

Then (4.17) yields

$$\begin{aligned}\sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(s)| \geq & \sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(\bar{\tau})| e^{(s-\tau)/2} \\ & - C\Big[\Big(\sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(\bar{\tau})|\Big)^2 + \Big(\sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(\bar{\tau})|\Big)^{1+\delta}\Big]\end{aligned}$$

so for $2L < s - \bar{\tau} < 3L$

$$\sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(s)| \geq \sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(\bar{\tau})| \tag{4.22}$$

and

$$||w(\cdot,s)|| \leq C||w(\cdot,\bar{\tau})|| \;. \tag{4.23}$$

Next, we introduce a function T,

$$T(s) = \sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(s)|^2(||\varphi_+(\cdot,s)||^2 + ||w_-(\cdot,s)||^2)^{-1}, \tag{4.24}$$

where $w = w_+ + w_-$ and $w_{+/-} \in V_{+/-}$,

$$\varphi_+ = w_+ - \sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} H_\gamma z_+ \langle H_\gamma z_+, w_+\rangle. \tag{4.25}$$

Then $T(s) \geq \theta_1 > 0$ for some $\theta_1 > 0$, $2L < s - \bar{\tau} < 3L$.

We define the projection operators, respectively P_- and P_+, on V_- and V_+ and Q from V_+ on the space of functions orthogonal to $H_\gamma z_+$ with $|\gamma| < [\sqrt{\Delta} - \alpha - \beta] - 1$. Then by definition $\varphi_+ = Qw_+$, and by (4.16)

$$(\varphi_+)_\tau = A(\varphi_+) + QP_+(f(w)). \tag{4.26}$$

We want to estimate T from below. To this end we compute

$$\begin{aligned}\frac{dT}{ds} &= 2 \sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} a_\gamma^+(s)\dot{a}_\gamma^+(||\varphi_+(\cdot,s)||^2 + ||w_-(\cdot,s)||^2)^{-1} \\ &- 2\sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} (a_\gamma^+(s))^2(||\varphi_+(\cdot,s)||^2 + ||w_-(\cdot,s)||^2)^{-2}(\langle\varphi_+, (\varphi_+)_s\rangle + \langle w_-, (w_-)_s\rangle) \\ &\sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} (\sqrt{\Delta} - \alpha - \beta - |\gamma|)|a_\gamma^+(s)|^2(||\varphi_+(\cdot,s)||^2 + ||w_-(\cdot,s)||^2)^{-1} \\ &- 2 \sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(s)|^2(||\varphi_+(\cdot,s)||^2 + ||w_-(\cdot,s)||^2)^{-2} \\ &(\langle\varphi_+, A(\varphi_+)_s\rangle\langle\varphi_+, QP_+(f(w))\rangle + \langle w_-, P_-(f(w))\rangle\langle w_-, Aw_-\rangle).\end{aligned} \tag{4.27}$$

Using the estimates

$$\langle w_-, Aw_-\rangle \leq -c||w_-||^2, \quad \langle\varphi_+, A(\varphi_+)_s\rangle \leq 0$$

and noting that $\sqrt{\Delta} - \alpha - \beta - |\gamma| \geq 1$ as $|\gamma| < [\sqrt{\Delta} - \alpha - \beta]$, we have

$$\begin{aligned}\tfrac{1}{2}\frac{dT}{ds} &\geq \sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(s)|^2(||\varphi_+(\cdot,s)||^2 + ||w_-(\cdot,s)||^2)^{-1} \\ &\left(\tfrac{1}{2} - (\langle\varphi_+, QP_+(f(w))\rangle + \langle w_-, P_-(f(w))\rangle(||\varphi_+(\cdot,s)||^2 + ||w_-(\cdot,s)||^2)^{-1}\right).\end{aligned} \tag{4.28}$$

It is possible to establish

$$||w(\cdot,s)^3|| \leq C||w(\cdot,s)||^3. \tag{4.29}$$

Therefore, by $f(w) = O(w^2)$

$$\begin{aligned} \tfrac{1}{2}\frac{dT}{ds} &\geq \sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(s)|^2(||\varphi_+(\cdot,s)||^2 + ||w_-(\cdot,s)||^2)^{-1} \\ &\quad \big[\tfrac{1}{2} - C||w(\cdot,s)||^3(||\varphi_+(\cdot,s)||^2 + ||w_-(\cdot,s)||^2)^{-1}\big] \\ &\geq T(s)\big(\tfrac{1}{2} - CT(s)||w(\cdot,s)||\big). \end{aligned} \tag{4.30}$$

We make use of the last inequality, and by a continuation argument

$$T(s) \geq \theta_2 > 0, \text{ for } s > \bar{\tau} + L. \tag{4.31}$$

Proceeding further, for $||w|| < \varepsilon$

$$\frac{\partial}{\partial s} \sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(s)| \geq \frac{1}{2} \sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(s)| - ||QP_+f(w)|| \tag{4.32}$$

$$\geq \frac{1}{2} \sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(s)| - C||w||^2 \geq \big(\frac{1}{2} - C_1\varepsilon\big) \sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(s)|.$$

If we integrate in time, then we see that

$$\sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(s)| \geq \sum_{|\gamma|<[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(\bar{\tau}+L)| e^{C_2(s-\bar{\tau}-L)} \tag{4.33}$$

for $s \geq \bar{\tau} + L$ and $C_2 = \frac{1}{2} - C_1\varepsilon$. Thus, for $\varepsilon \to 0, w(\cdot,s)$ cannot be bounded. This is a desired contradiction. □

Lemma 4.2. *Let w be as in the previous lemma. If*

$$\lim_{s\to\infty} \frac{\sum_{|\gamma|\neq[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(s)|}{\sum_{|\gamma|=[\sqrt{\Delta}-\alpha-\beta]} |a_\gamma^+(s)|} = 0, \tag{4.34}$$

then (1.26) *in Theorem* 3 *occurs.*

Proof. We focus our attention on the case when neutral modes dominate in (4.13). Following [15] we put

$$\chi = \sum_{|\gamma|=[\sqrt{\Delta}-\alpha-\beta]} a_\gamma^+(s) H_\gamma z_+. \tag{4.35}$$

Next, separately for every $\sqrt{\Delta} - \alpha - \beta > 0$ we define matrices $G = (G_{ij})$ of dimension N^m, where $m = [\sqrt{\Delta} - \alpha - \beta]$, whose entries correspond to such

γ that $|\gamma| = m$. Namely

$$G_{i_1\dots i_m} = \begin{cases} \sqrt{m}\langle\chi, H_m(y_i)z_+\rangle H_0^{N-1} & \text{if} i_1 = \dots = i_m = i \\ \vdots & \\ \langle\chi, \prod_{j=1}^m H_1(y_{i_j})z_0\rangle H_0^{N-m} & \text{if} i_j \neq i_k \text{ for all } j \neq k. \end{cases} \tag{4.36}$$

Repeating steps of the analysis in [15] we obtain by a tedious computation

$$\dot{G} = \nu G^2 + \delta H(G) \tag{4.37}$$

with $\nu > 0$ and $H(G) = O(||G||^2)$.

A key point consists in obtaining an evolution equation for $\lambda_k(s)$ eigenvalues of $G(s)$. It turns out that

$$\dot{\lambda}_k = \theta\lambda_k^2 + \delta O(||\lambda||^2), \tag{4.38}$$

where $||\lambda||^2 = \sum_k \lambda_k^2$, $\theta > 0$ and depend on p_i, q_i. Since

$$\lambda_k(s) = \begin{cases} \frac{-1}{\theta s} + o\left(\frac{1}{s}\right) & \text{as} \quad s \to \infty,\ k = 1, \dots, l \\ o\left(\frac{1}{s}\right) & \text{as} \quad s \to \infty,\ k = l+1, \dots, N, \end{cases}$$

for some $l \geq 1$, in the relating system of coordinates, for some $K > 0$

$$G_{i_1\dots i_m} = \begin{cases} \frac{-1}{Ks} + o\left(\frac{1}{s}\right) & \text{for} \quad i_1 = \dots = i_m, i_1 = 1, \dots, l \\ o\left(\frac{1}{s}\right) & \text{otherwise.} \end{cases}$$

This yields

$$\sum_{|\gamma| = [\sqrt{\Delta} - \alpha - \beta]} a_\gamma^+(s) H_\gamma z_+ = -\frac{c}{s} z_+ \sum_{k=1}^{i} H_{[\sqrt{\Delta}-\alpha-\beta]}(y_k) + o\left(\frac{1}{s}\right) \quad \text{as} \quad s \to \infty.$$

Thus, the proof is complete. □

As long as the Lemmas 4.1–4.2 are established, the arguments presented in [15] apply. Finally, we conclude that Theorem 3 follows.

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