Differential and Integral Equations Volume 15, Number 2, February 2002, Pages 167–189

SLIDING MODES IN BANACH SPACES

Laura Levaggi
Dipartimento di Matematica, Università di Genova,
Via Dodecaneso, 35 - 16146 Genova, Italy

(Submitted by: Viorel Barbu)

Abstract. Using differential inclusions and viability theory we define sliding modes for (feedback) controlled semilinear differential equations in Banach spaces. We compare this definition with the equivalent control method for infinite-dimensional systems proposed by V. Utkin and Yu. Orlov. We show that if the sliding manifold satisfies suitable regularity hypotheses and the semigroup is compact, the projected evolution found by means of the equivalent control and our sliding mode do coincide.

1. Introduction

The study of discontinuous dynamical systems is a topic of much interest for both mathematical theory and engineering applications. Ordinary differential equations with discontinuous right-hand sides represent mathematical models of many physical systems, but often stem also from optimal control problems. Besides, it can prove to be useful to choose a control function which switches abruptly when the state lies on a certain surface. In fact one can observe that, under some nonsingularity hypotheses, when the state trajectories reach the discontinuity set, they are no more able to leave it. This behaviour of the system is called sliding mode and on well-chosen surfaces it shows some attractive invariance properties. For systems governed by ODEs, several tools have been developed in order to design a surface S in such a way that a sliding mode appears on it and has prescribed properties. These are mainly related to the work of V. Utkin (for an overview see [20] and the references therein) and the so-called equivalent control method, which enables one to readily write the equation of motion on S by simply substituting the discontinuous control with a particular continuous one, the equivalent control. The motion so obtained is proved to be the limit of solutions of differential equations in which the discontinuity has been cut off by

Accepted for publication: October 2000.
AMS Subject Classifications: 93B12, 34G10, 34A60.

a regularization process. In this sense it is a solution of the given dynamical system.

From a mathematical theory point of view, the first problem one has to solve when dealing with a discontinuous differential equation, is the lack of regularity. Classical theory on existence of solutions is no use in this context, and one even has to redefine the concept of solution itself. There are of course many ways to do this, but generally speaking, the most common is to introduce a set-valued function to replace the right-hand side and consider the solutions of the resulting differential inclusion. Two of the most popular definitions are the so-called Filippov and Krasovskij solutions. For any point in the state domain the multifunction is basically defined so as to take into account the closed and convex hull of the values that the right-hand side takes in the neighbourhood of the point. Filippov solutions are moreover designed so that one can reject values which are only taken up on sets of zero measure (see, e.g., [6, 7, 8]). In this framework, sliding motions can be viewed as viable solutions on the sliding surface corresponding to the differential inclusion. Necessary and sufficient conditions in order to get sliding-mode existence are therefore handed out by viability theory (a comprehensive text is of course [2]). The regularization process that substantiates the equivalent control method does not always lead to the same result as viability applied to the Filippov concept of solution (which is the most suited for a comparison). However, a complete equivalence of the two definitions of sliding mode is gotten for systems which are affine in the control, in nonsingular cases. Although there have been some promising attempts to apply sliding modes to distributed parameter systems, their theoretical generalization to infinite-dimensional differential equations is not well established. The first approaches to this problem have tried to use finite-dimensional theory in connection with Fourier transform (modal control) or the Lyapunov method [11, 13, 15, 19], leaving aside the problem of the mathematical description of sliding modes. More recently, Y. Orlov and V. Utkin [12, 14] have proposed a definition of equivalent control for systems governed by semilinear differential equations in Banach spaces, i.e., dynamical systems which can be described through an unbounded operator generating a continuous semigroup. They show that, under some hypotheses, a regularization process similar to the finite-dimensional one allows the description of the motion on the sliding surfaces through the application of the equivalent control method.

In this paper we define a concept of solution for semilinear differential equations in Banach spaces having state-discontinuous right-hand side. This is subsequently used to make sense of a control system with discontinuous feedback law and to make precise the meaning of motion on a sliding surface. Applying viability theorems in this context, necessary and sufficient conditions are shown for the existence of sliding modes. Moreover, we prove that, when the unbounded operator generates a compact semigroup, the equivalent control method leads exactly to our viable generalized solution, provided that the sliding surface satisfies some regularity property.

In Section 2 we establish the new concept of solution: we define a set-valued function F and transform a given semilinear, infinite-dimensional differential equation

$$\dot{x} + Ax = f(x), \quad x(0) = x_0 \qquad A : \mathcal{D}(A) \subset X \to X$$

into the semilinear differential inclusion on a reflexive Banach space X

$$\dot{x} + Ax \in F(x), \quad x(0) = x_0 \qquad F : X \to 2^X.$$

Regularity properties of the multifunction are proven which allow us to apply known results on the existence of viable mild solutions for the previous semilinear inclusion.

In Section 3 we specialize the results of Section 2 to feedback control systems governed by compact semigroups which present discontinuities on a linear subspace S of X and show that if $S \subset \mathcal{D}(A)$ it is possible to give an explicit tangency condition that forces viability. We show that there is a natural way to extend the definition of equivalent control and sliding mode to infinite-dimensional, semilinear differential equations ([12]), and we prove a regularization property showing new features as compared with its finite-dimensional counterpart. Finally we show that in our setting the sliding motion arising from the equivalent control method is the only viable solution of the introduced differential inclusion.

In the last section we apply the procedure to the stabilization problem for an infinite-dimensional plant.

2. Generalized solutions of semilinear differential equations

The first mathematical problem one encounters when dealing with a discontinuous differential equation, is obviously the need to define what we mean by a solution. For ordinary differential equations several types of generalized solutions have been conceived. Among them we can cite at least two: Filippov and Krasovskij solutions. Referring to the differential equation

$$\dot{x}(t) = f(t, x(t)), \qquad f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \tag{1}$$

they are obtained by replacing the discontinuous single-valued function f with the set-valued function F_F or F_K respectively:

$$F_F(t,x) = \bigcap_{\varepsilon>0} \bigcap_{m(N)=0} \overline{\mathrm{co}}\, f(t, \mathcal{B}(x,\varepsilon)\setminus N), \quad F_K(t,x) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, f(t, \mathcal{B}(x,\varepsilon)),$$

where m is the Lebesgue measure on $\mathbb{R}^n$, $\mathcal{B}(x,r)$ is the open ball of center x and radius r and $\overline{\mathrm{co}}$ denotes the closed convex hull. The second intersection in $F_F(t,x)$ is what makes the difference between the two definitions, and its aim is to reject possible misbehaviour of f on sets of zero measure in the state space. A Filippov or Krasovskij solution of (1) is an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ such that

$$\dot{x}(t) \in F_F(t, x(t)) \text{ a.e. } t \in [0,T], \qquad \dot{x}(t) \in F_K(t, x(t)) \text{ a.e. } t \in [0,T]$$

respectively (for a thorough study of this subject see [1, 6, 7, 8, 10]).

In this section we will try to extend in some way the concept of generalized solution to autonomous semilinear differential equations in Banach spaces. First of all we define the set-valued function used to set up the differential inclusion, then we prove regularity properties and summarize important viability results which will be used later on. With our definition we attempt also to apply the "rejection feature" of the Filippov solutions to the infinite-dimensional case. Instead of using zero-measure sets, we deal here with densely defined functions. This choice is motivated by our final aim, namely the mathematical description of sliding modes for infinite-dimensional control systems. In that case, we can think of the domain of the right-hand side as something which we can act on, not as a given datum. In other words we do have information on the distribution of the singularities, and of course we want to exploit them. Doing this, however, we end up with a set-valued mapping that much resembles Krasovskij's, so that we can think of it as being half way between the two. Note also that in the finite-dimensional case we could leave out a set of positive measure (consider for example the decomposition $\mathbb{R} = (\mathbb{R}\setminus\mathbb{Q}) \cup \mathbb{Q}$), so that our definition can be substantially different from Filippov's. However, the setting we have in mind is far from being so general: our discontinuities will lie on a linear subspace, and in this case the rejected set has null measure.

In this section we focus our attention on this particular kind of differential equation in Banach spaces,

$$\dot{x}(t) + Ax(t) = f(x(t)), \quad x(0) = x_0,$$

under this general frame:

Hypothesis 2.1. X is a reflexive Banach space.

Hypothesis 2.2. The operator $-A : \mathcal{D}(A) \subset X \to X$ is closed, densely defined and generates a C_0-semigroup $K(t)$, $t \geq 0$ on X.

Hypothesis 2.3 (Growth condition)**.** The function $f : \mathcal{D}(f) \subset X \to X$ is densely defined and satisfies the following linear growth condition:

$$\|f(x)\| \leq M\|x\| + N, \qquad \forall x \in \mathcal{D}(f) \tag{2}$$

for some nonnegative constants M, N.

We are going to define a solution concept for this kind of differential problem and study it in the framework of viability.

2.1. **Multifunction definition.** Let us define the set-valued map $F : X \to 2^X$ by

$$F(x) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, f(\overline{\mathcal{B}}(x,\varepsilon) \cap \mathcal{D}(f)), \quad x \in X, \tag{3}$$

$\overline{\mathcal{B}}(x, r)$ being the closed ball of center x and radius r. We are going to prove that this is a well-defined closed and convex-valued mapping which is strongly-weakly upper semicontinuous. For this purpose recall the definition of upper semicontinuity with respect to the product topology $\tau_1 \times \tau_2$ on $X \times X$: for any $x \in X$ and for any τ_2-neighbourhood V of $F(x)$ there exists a τ_1-neighbourhood W of x such that $F(y) \subset V$, $\forall y \in W$.

Proposition 2.1. *If f satisfies* (2), *$F(x)$ is a nonvoid, closed, convex and bounded subset of X for all $x \in X$. Moreover, F is strongly-weakly upper semicontinuous and locally bounded.*

Proof. From the density of $\mathcal{D}(f)$ in X it follows immediately that every element of the intersection in (3) is not empty. To prove that $F(x)$ is itself nonvoid it is sufficient to note that by (2), $\overline{\mathrm{co}}\, f(\overline{\mathcal{B}}(x,\varepsilon) \cap \mathcal{D}(f))$ is weakly compact, because in any topological space the intersection of a decreasing family of nonempty compact sets cannot be empty. Obviously $F(x)$ is closed and convex and so weakly closed. The boundedness property is an easy consequence of the growth condition assumed on f, and the proof of the first statement is complete.

We show now that the graph of F is sequentially closed in the $s \times w$ topology and then prove that this implies the required semicontinuity property. Let $((x_n, y_n))$ be a sequence in the product space $X \times X$ such that (x_n) is strongly convergent to x_0, (y_n) has weak limit y_0 and $y_n \in F(x_n)$ for all n. We have therefore that for any $m \in \mathbb{N}$, y_0 belongs to the weak closure of the set $\{y_n : n \geq m\}$. Because of the convergence of (x_n), for any $\delta > 0$ there exists a positive integer N_δ such that $x_n \in \overline{\mathcal{B}}(x_0, \delta)$ for all $n \geq N_\delta$, and so

$$F(x_n) \subset \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, f(\overline{\mathcal{B}}(x_0, \varepsilon+\delta) \cap \mathcal{D}(f)), \quad \forall n \geq N_\delta.$$

From this we see that for any $\delta > 0$ one has

$$\begin{aligned} y_0 \in \overline{\{y_n : n \geq N_\delta\}}^w \quad &\subset \quad \overline{\bigcup_{n\geq N_\delta} \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, f(\overline{\mathcal{B}}(x_0, \varepsilon+\delta) \cap \mathcal{D}(f))}^w \\ &= \quad \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, f(\overline{\mathcal{B}}(x_0, \varepsilon+\delta) \cap \mathcal{D}(f)) \end{aligned}$$

(here $\overline{D}^w$ stands for the weak closure of set D) so that

$$y_0 \in \bigcap_{\varepsilon,\delta>0} \overline{\mathrm{co}}\, f(\overline{\mathcal{B}}(x_0, \varepsilon+\delta) \cap \mathcal{D}(f)) = F(x_0).$$

Assume now that F is not strongly-weakly u.s.c. at x_0; then we can find a weakly open neighbourhood V of $F(x_0)$ and a sequence x_n strongly converging to x_0 such that $F(x_n) \not\subset V$ $\forall n \in \mathbb{N}$. Therefore, we can choose $y_n \in F(x_n)$, $y_n \notin V$ for all n. Using property (2) it is easy to see that, possibly passing to a subsequence, there exists y_0 such that y_n weakly converges to y_0. This weak limit cannot belong to $F(x_0)$, because otherwise, as V would be a weak neighbourhood of y_0, we could find N such that $y_n \in V$ for all $n > N$, and this is absurd. So we have found a sequence $((x_n, y_n))$ in the graph of F strongly-weakly convergent to (x_0, y_0) for which $y_0 \notin F(x_0)$. We showed that this is not possible; therefore F must be u.s.c. □

A useful way to express the set-valued map F is in the form of a closed, convex hull of weak limit values of sequences in the range of the function f. This is clarified by the following:

Proposition 2.2. *If f satisfies Hypothesis* 2.3, *for all $x \in X$ we have*

$$F(x) = \overline{\mathrm{co}}\,\{z \in X : \exists \{x_n\} \subset \mathcal{D}(f),\, x_n \to x \,\text{such that}\, f(x_n) \rightharpoonup z\}. \tag{4}$$

Proof. Let us call $G(x)$ the set on the right-hand side of (4); if $z \in G(x)$, reasoning as in the first part of the proof in Proposition 2.1, it is easy to show that $z \in F(x)$.

If now $z \in F(x)$, for any $\varepsilon > 0$ we have $z \in \overline{\mathrm{co}}\, f(\overline{\mathcal{B}}(x, \varepsilon) \cap \mathcal{D}(f))$. Recall now the following characterization of the closed, convex hull of a subset A

of a Banach space X (see, e.g., [3], p. 64):

$$\overline{\mathrm{co}}\,(A) = \{y \in X \,:\, \langle \lambda, y \rangle \le \sup_{x \in A} \langle \lambda, x \rangle,\ \forall \lambda \in X^*\}. \tag{5}$$

Therefore, for any $\varepsilon > 0$, $\lambda \in X^*$ and $\delta > 0$ there exists $x_\delta = x_\delta(\varepsilon, \lambda)$ in $\overline{\mathcal{B}}(x, \varepsilon) \cap \mathcal{D}(f)$ such that $\langle \lambda, z - f(x_\delta) \rangle < \delta$. Fixing $\delta > 0$ and choosing $\varepsilon_n \to 0$, we can build the sequence $x_n = x_\delta(\varepsilon_n, \lambda)$ in the domain of f which is strongly convergent to x. From the growth property (2) satisfied by f, possibly choosing a subsequence, we can find y_δ such that $f(x_n) \rightharpoonup y_\delta$ and $\langle \lambda, z - f(x_n) \rangle < \delta$ for all n. By definition of weak convergence we then have $\langle \lambda, z - y_\delta \rangle \le \delta$. Eventually we have shown that, for all $\lambda \in X^*$ and $\delta > 0$, there exist $x_n \to x$, $x_n \in \mathcal{D}(f)$ for all n, $y_\delta \in X$ with $f(x_n) \rightharpoonup y_\delta$ such that $\langle \lambda, z \rangle \le \langle \lambda, y_\delta \rangle + \delta$; i.e., for any $\lambda \in X^*$, $\langle \lambda, z \rangle \le \sup\{\langle \lambda, y \rangle :$ y is the weak limit of $f(x_n)$ for some $x_n \to x,\ x_n \in \mathcal{D}(f)\, \forall n \in \mathbb{N}\}$. Going back to (5) we get $z \in G(x)$. □

2.2. **Generalized solutions and viability results.** We remark that up to this point the only hypothesis made on f is just some kind of linear growth, so no regularity is needed to define F and to get its properties. The multifunction we have defined will now be used to extend the concept of solution for a semilinear differential equation on a Banach space having discontinuous right-hand side. More precisely:

Definition 2.1. We will define the *generalized solution* of the differential equation

$$\dot{x}(t) + Ax(t) = f(x(t)), \quad x(0) = x_0 \tag{6}$$

to be a mild solution of the differential inclusion

$$\dot{x}(t) + Ax(t) \in F(x(t)), \quad x(0) = x_0. \tag{7}$$

We recall that a mild solution of the preceding inclusion is a continuous function $x : [0, T] \to X$ defined for some $T > 0$ and such that

$$x(t) = K(t)x_0 + \int_0^t K(t - s)g(s)\, ds, \quad \forall t \in [0, T],$$

where $g : [0, T] \to X$ is in $L^1(0, T; X)$ and satisfies $g(s) \in F(x(s))$ for almost every $s \in [0, T]$.

Remark 2.1. Note that in the finite-dimensional case the difference between our definition of generalized solutions and that of Krasowskij or Filippov lies solely in the construction of the multifunction. In fact, if $\dim(X) = n < \infty$ A is a continuous linear operator on X, then we have for example that

Krasowskij solutions of (6) are absolutely continuous functions $x : [0,T] \to X$ such that

$$\begin{cases} \dot{x}(t) + Ax(t) = g(t) & \text{a.e. } t \in [0,T] \\ x(0) = x_0, \end{cases}$$

with

$$g(t) \in \bigcap_{\varepsilon>0} \overline{\text{co}}\, f(\mathcal{B}(x(t),\varepsilon)) \quad \text{a.e. } t \in [0,T], \qquad g \in L^1(0,T;X).$$

Now $-A$ is the generator of the uniformly continuous group $K(t) = e^{-At}$, $t \geq 0$; therefore the map $t \mapsto K(t)$ is differentiable in norm ([16], p. 3), and this is enough to show that there is no gap between mild solutions and strong solutions of

$$\dot{x} + Ax \in \bigcap_{\varepsilon>0} \overline{\text{co}}\, f(\mathcal{B}(x,\varepsilon)), \quad x(0) = x_0.$$

In what follows, we will be particularly interested in the following class of solutions.

Definition 2.2. If $S \subset X$ and $x_0 \in S$, a mild solution of (7) that satisfies $x(t) \in S$ for all t is called *viable* on S. S is a *viable domain* for (7) if for any $x_0 \in S$ there exists a viable solution of the differential inclusion starting from x_0.

A *generalized viable solution* of (6) is a viable solution of (7).

We now state some results of viability theory for infinite-dimensional semilinear differential inclusions proved by Cârjă and Vrabie [4]. We will see that, as in the finite-dimensional case, one can find necessary or sufficient tangency conditions to get viability. In a Banach space setting these explicitly involve the unbounded operator $-A$.

Definition 2.3. Let S be a nonempty subset of X, and τ either the strong or the weak topology on X. S satisfies the *τ-tangency condition* with respect to (7) if and only if for each $x \in S$ there exists $y \in F(x)$ which is *τ-A-tangent to S* in x; i.e., for each $\delta > 0$ and each τ-neighbourhood V of zero there exist $t \in (0,\delta)$, $p \in V$ such that $K(t)x + t(y+p) \in S$. S is said to satisfy the *bounded τ-tangency condition* if moreover there is a locally bounded function $\mathcal{M} : S \to \mathbb{R}^+$ such that p can be chosen in $V \cap \overline{\mathcal{B}}(0,\mathcal{M}(x))$.

Remark 2.2. If $\tau = s$, the tangency concept introduced above coincides with that of Shi Shouzhong [17]. The set of vectors s-A-tangent to S in x is in fact

$$T_S^A(x) = \{v \in X \,:\, \liminf_{h\to 0^+} \frac{d(K(h)x + hv, S)}{h} = 0\},$$

and the tangency condition with respect to (7) is

$$F(x) \cap T_S^A(x) \neq \emptyset, \qquad \forall x \in S.$$

If $A = 0$ the definition is the classical one with the Bouligand contingent cone $T_S(x)$ (see, e.g., [1], p. 176); an element of this cone will be briefly called *tangent* to S in x.

The following theorem contains some of the results in [4] (namely Theorem 2.2, Remark 3.1 and Theorem 5.1).

Theorem 2.1. *Let X be a reflexive Banach space, S a nonempty and locally closed subset of X, $F : S \to 2^X$ a nonempty, closed, convex and bounded valued mapping which is strongly-weakly upper semicontinuous and locally bounded. Let $-A : \mathcal{D}(A) \subset X \to X$ be the infinitesimal generator of a C_0-semigroup $K(t)$ on X. Then a necessary condition in order that S be a viable domain for* (7) *is the sequential bounded w-tangency condition, i.e.,*

$$\begin{aligned} &\textit{there exists a locally bounded function } \mathcal{M} : S \to \mathbb{R}^+ \textit{ such that } \forall x \in S \\ &\textit{there exist } y \in F(x),\ t_n \searrow 0,\ p_n \rightharpoonup 0 \textit{ with } \|p_n\| \leq \mathcal{M}(x) \textit{ for which} \\ &K(t_n)x + t_n(y + p_n) \in S \qquad \forall n \in \mathbb{N}. \end{aligned} \tag{8}$$

If $-A$ generates a compact semigroup, then the s-tangency condition is sufficient for the existence of viable solutions of (7) *on S.*

Remark 2.3. In the proof of the first part of the above result the authors of [4] show that given any viable solution of (7),

$$z(t) = K(t)x + \int_0^t K(t-s)g(s)\,ds,\ t \in [0,T],\ T > 0, \quad g(s) \in F(z(s)) \text{ a.e. } s,$$

one can construct t_n so that $\frac{1}{t_n}\int_0^{t_n} g(s)\,ds \rightharpoonup y$ and y fulfills property (8) (see the proof of Theorem 2.2, p. 414 of [4]). Arbitrarily fixing $t^* \in (0,T)$, using as starting point $z(t^*) \in S$ and the viable solution

$$z^*(t) = K(t)z(t^*) + \int_0^t K(t-s)g(s+t^*)\,ds = z(t+t^*), \quad t \in [0, T-t^*]$$

it is easy to see that for all $t \in [0,T)$ there exists $y(t) \in F(z(t))$ satisfying the sequential bounded w-tangency condition which is the weak limit of $\frac{1}{t_n}\int_t^{t+t_n} g(s)\,ds$ for some $t_n \searrow 0$. As $g \in L^1(0,T;X)$ the set of Lebesgue points of g covers almost all the interval $[0,T]$, so for almost all $s \in [0,T]$ $g(s) = y(s)$.

From the preceding remark we get therefore the following:

Corollary 2.1. *In the same hypotheses of Theorem* 2.1 *any solution of* (7) *starting from* $x_0 \in S$ *and viable on* S *can be written as*

$$z(t) = K(t)x_0 + \int_0^t K(t-s)g(s)\,ds$$

with $g \in L^1(0,T;X)$ *and such that there exists a set of null measure* E *in* $[0,T]$ *for which, when* $s \notin E$, $g(s) \in F(z(s))$ *and moreover* $g(s)$ *is sequentially weakly tangent to* S *in* $z(s)$.

3. Control systems

In this section we will apply the theory of viability for semilinear differential inclusions given above to a class of infinite dimensional systems which are linear in the control. We will be concerned with controlled differential equations of the form

$$\dot{x}(t) + Ax(t) = Bu(x(t)), \quad x(0) = x_0, \tag{9}$$

where x is the state variable, u is the control variable and the following hold:

Hypothesis 3.1. $-A : \mathcal{D}(A) \subset X \to X$ is a densely defined, closed linear operator generating a compact C_0-semigroup $K(t)$, $t \geq 0$, on X.

Hypothesis 3.2. U is a Banach space.

Hypothesis 3.3. $B : U \to X$ is a continuous linear operator.

Hypothesis 3.4. $u : \mathcal{D}(u) \subset X \to U$ is a densely defined function that satisfies the growth condition (2).

Under these conditions, we obviously can define the set-valued function F starting from $f = Bu$ as in the previous subsection. From Proposition 2.1 F has all the properties required to apply Theorem 2.1.

3.1. **Sliding modes.**

We now restrict our attention to a particular class of feedback laws u.

Hypothesis 3.5. Let Y be a Banach space, $C : X \to Y$ a continuous linear operator, $C \neq 0$ and

$$\mathcal{D}(u) = X \setminus S, \quad S = \ker C.$$

This way S is a proper linear subspace of X and therefore has void interior. This obviously means $\mathcal{D}(u)$ is dense. A wide class of discontinuous controls u satisfying all these requirements is given by functions of the form

$$u(x) = N(x)\frac{Cx}{\|Cx\|},$$

where $Y \equiv U$ and N is a continuous, real-valued function satisfying (2). Such controls, the so-called *unit controls*, are used in the finite-dimensional case to induce a sliding motion on a prescribed surface S and in [15] for the application of the Lyapunov method in Hilbert spaces. The sliding mode is attained when, upon reaching the surface S, the state is henceforth constrained to remain (slide) on it. In practice S can be chosen to get either stability of the trajectory or robustness with respect to matched external disturbances or to impose other properties on the dynamics (see [20] and its references).

We attempt now to extend the concept of sliding motion to infinite-dimensional systems, using the concept of generalized solutions as introduced in Definition 2.1 and viability theory.

Lemma 3.1. *Let S be as in Hypothesis* 3.5, *$-A$ generate a C_0-semigroup $K(t)$, $t \geq 0$ on X and $x \in S \cap \mathcal{D}(A)$; then $y \in X$ is s-A-tangent to S in x if and only if $y \in S + Ax$.*

Moreover, if y is such that there exist sequences $t_n \searrow 0$, $p_n \rightharpoonup 0$ for which

$$K(t_n)x + t_n(y + p_n) \in S, \tag{10}$$

then y is s-A tangent to S in x.

Proof. It is not difficult to prove that if $x \in \mathcal{D}(A)$, we can characterize the s-A-tangent elements to S in x through the contingent cone to S in x. In fact for any $v \in X$, $h_n \in \mathbb{R}^+$ and $q_n \in X$ one obviously has

$$\begin{aligned} x + h_n(q_n + v) &= K(h_n)x + h_n(\frac{x - K(h_n)x}{h_n} + q_n + v) \\ &= K(h_n)x + h_n(\frac{x - K(h_n)x}{h_n} - Ax + q_n + v + Ax) \\ &= K(h_n)x + h_n(p_n + v + Ax). \end{aligned}$$

By definition of the generator A, the sequence $\frac{1}{h_n}(K(h_n)x - x) + Ax$ is convergent to zero for any $h_n \searrow 0$. Therefore, if v is tangent to S in x we obviously get that $Ax + v$ is s-A-tangent to S. On the other hand, if $Ax + v$ is s-A-tangent to S, from the relation $q_n = p_n + \frac{1}{h_n}(K(h_n)x - x) + Ax$, we easily have that v is tangent to S.

To complete the first part of the proof recall that the Bouligand contingent cone in this case is just S itself ([1], p. 222).

Now applying the operator C to (10) we get

$$C\,\frac{K(t_n)x - x}{t_n} + Cy + Cp_n = 0.$$

As $x \in \mathcal{D}(A)$ and C is continuous, the first term strongly converges to $-CAx$. The last summand is weakly convergent to zero because C is weakly continuous. We therefore have $C[-Ax + y] = 0$ so that $y \in Ax + S$, and the proof is complete. □

Previous results lead therefore to the following corollary.

Corollary 3.1. *Under the stated Hypotheses* 3.1 *to* 3.5, *if* $S \subset \mathcal{D}(A)$, *then for any* $x_0 \in S$ *there exist generalized viable solutions on* S *of* (9) *if and only if the following condition holds: for all* $x \in S$

$$Ax \in F(x) - S, \quad \textit{where} \quad F(x) = \bigcap_{\varepsilon > 0} \overline{\mathrm{co}}\, Bu(\overline{\mathcal{B}}(x,\varepsilon) \cap \mathcal{D}(u)).$$

Moreover, under these conditions any viable solution $x(\cdot)$ *satisfies*

$$x(t) = K(t)x_0 + \int_0^t K(t-s) f(s)\, ds, \quad t \geq 0,$$

with $f(s) \in Ax(s) + S$ *for almost every* s.

Proof. The first statement follows from Proposition 2.1, Theorem 2.1 and Lemma 3.1, the second from Corollary 2.1. □

3.2. **Equivalent control method.** Let us briefly summarize the fundamentals of the equivalent control method for finite-dimensional linear systems (a reference book for theory and applications is of course [20]). Suppose we are given the ordinary differential equation

$$\dot{x}(t) = Ax(t) + Bu(x(t)), \quad x \in \mathbb{R}^n,\ A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ u(x) \in \mathbb{R}^m\ \forall x,$$

with

$$u_j(x) = \begin{cases} u_j^+(x) & \text{if} \quad s_j(x) > 0 \\ u_j^-(x) & \text{if} \quad s_j(x) < 0 \end{cases} \qquad j = 1, \dots m \tag{11}$$

and $s_j : \mathbb{R}^n \to \mathbb{R}$ a linear function for all j. The sliding manifold S is given by $S = \{x \in \mathbb{R}^n \,:\, s(x) = 0\}$, where $s : x \in \mathbb{R}^n \mapsto (s_1(x), \dots, s_m(x)) \in \mathbb{R}^m$. Calling C the matrix associated with s, the equivalent control is obtained as a solution of the equation $C\dot{x} = 0$. Therefore in the nonsingular case when $\det(CB) \neq 0$, one has

$$CAx + CBu_{\mathrm{eq}}(x) = 0, \qquad u_{\mathrm{eq}}(x) = -(CB)^{-1}CAx,$$

and the sliding mode on the sliding surface S is the solution of the differential equation

$$\dot{x}(t) = Ax(t) - B(CB)^{-1}CAx(t).$$

By analogy, let us formally apply the equivalent control method to the infinite-dimensional system

$$\dot{x} + Ax = Bu, \quad x(0) = x_0 \in S, \quad S = \ker C, \quad C \in \mathcal{L}(X, Y) \tag{12}$$

satisfying the further assumption

Hypothesis 3.6. $S \subset \mathcal{D}(A)$.

Imposing the sliding condition $C\dot{x} = 0$, the definition of the equivalent control requires that for all $x \in S$ there exists a unique element $u_{\mathrm{eq}}(x) \in U$ such that

$$CBu_{\mathrm{eq}}(x) = CAx.$$

If we suppose that the operator $CB : U \to Y$ has a continuous inverse $(CB)^{-1}$ and $CA(S) \subset CB(U)$, we get an explicit definition of u_{eq}, and the sliding motion has to satisfy the following differential equation:

$$\dot{x} = [B(CB)^{-1}C - I]Ax, \quad x(0) = x_0. \tag{13}$$

Of course, we have to specify in which sense we have to find the solutions of the preceding infinite-dimensional equation. In what follows we will prove that the operator

$$\tilde{A} = [B(CB)^{-1}C - I]A$$

generates another C_0-semigroup, so that the solutions of (13) can be understood in the classical sense.

From now on, we impose some further hypotheses:

Hypothesis 3.7. The continuous linear operator $CB : U \to Y$ has linear continuous inverse $(CB)^{-1} : CB(U) \subset Y \to U$.

Hypothesis 3.8. $CA(S) \subset CB(U)$.

Under these conditions we can define for all $x \in S$ the *equivalent control* u_{eq},

$$u_{\mathrm{eq}}(x) = (CB)^{-1}CAx.$$

Before we start the proof, we make some remarks and comments on the invertibility hypothesis we have imposed on CB. Assuming that we can define $(CB)^{-1} : CB(U) \to U$ as a continuous operator implies that B has closed range. In fact obviously $CB(U)$ is isomorphic to U, hence closed. But using the continuity of C we can write

$$C(\overline{B(U)}) \subset \overline{CB(U)} \subset CB(U) \subset C(\overline{B(U)});$$

that is, $CB(U) = C(\overline{B(U)})$. This is impossible unless $B(U) = \overline{B(U)}$: let $x = \lim Bu_n \in \overline{B(U)}$; then $Cx = CBu$ for some $u \in U$, so that there exists $s \in S$ such that $\lim B(u_n - u) = s$. Applying $(CB)^{-1}C$ to the preceding identity we have $\lim u_n = u$, and therefore $x = Bu$. Note also that from the same argument we get $B(U) \cap S = \{0\}$.

Let us now consider the operator $Q = B(CB)^{-1}C$ that appears in (13). Then $Q : \mathcal{D}(Q) \subset X \to X$ with

$$\mathcal{D}(Q) = \{x \in X \,:\, Cx \in CB(U)\} = \{x \in X \,:\, x \in B(U) + S\} = S \oplus B(U).$$

It is easy to prove that this direct sum defines a closed subspace of X. Let $y_n = x_n + Bu_n$ be a sequence in $\mathcal{D}(Q)$ having strong limit y; applying the operator C we have $CBu_n \to Cy$, so that there must exist $u \in U$ such that $Cy = CBu$ because of the closedness of $CB(U)$. Therefore, using $(CB)^{-1}$ we get the convergence of u_n to u and that of x_n to $y - Bu$, and the proof is complete. Of course Q is a continuous operator, and moreover it is a projection on $B(U)$; in fact, if $x = Bu + s \in \mathcal{D}(Q)$, $u \in U$, $s \in S$ we have

$$Qx = B(CB)^{-1}CBu = Bu \in \mathcal{D}(Q), \qquad Q^2x = Qx.$$

Then $P = (I - Q) : \mathcal{D}(P) = \mathcal{D}(Q) \subset X \to X$ is a projection on S. Coming back to equation (13), we can rewrite it as

$$\dot{x} = \tilde{A}x, \quad x(0) = x_0 \qquad \tilde{A} = -PA.$$

Now $\tilde{A}x$ is well defined for all $x \in S$ because we assumed $CA(S) \subset CB(U)$, so that $A(S) \subset \mathcal{D}(P)$ and besides $\tilde{A}x \in S$, P being a projection on S. We have chosen S as a closed subspace of X, so S itself is a Banach space with the norm induced by X. Denoting by A_S the restriction of A on S (this is meaningful since $S \subset \mathcal{D}(A)$) we can write $\tilde{A} = -PA_S$ and think of it as a linear mapping from S in itself. Does it generate a semigroup on S?

Proposition 3.1. *If $-A$ generates a C_0-semigroup and Hypotheses* 3.5–3.8 *are satisfied, $\tilde{A} \in \mathcal{L}(S)$, so it is the generator of the C_0-semigroup $\tilde{K}(t) = e^{\tilde{A}t}$.*

Proof. Since $-A$ is a generator, it is a closed operator and so is $-A_S$ because S is a closed subspace of X. Therefore $-A_S$ is a closed operator with closed domain, and this means it is bounded (see, e.g., [9], p. 166); the claim follows from the facts that $\tilde{A} = -PA_S$ and P is continuous. □

This result proves that in case the equivalent control can be defined through the continuous operator $(CB)^{-1}$, starting from any point $x_0 \in S$ the evolution $x(t) = \tilde{K}(t)x_0$ is meaningful and satisfies $x(t) \in S$ for all $t \geq 0$. Moreover, x is continuously differentiable, and from the construction of $\tilde{K}(t)$ we have $\dot{x}(t) \in S$. Which is the link that relates this solution to the control system we started from? From a physical point of view, the lack of determinism that appears when we are dealing with a discontinuous differential equation can be viewed as a result of model imperfections. The mathematical description of the real-life control system we are modelling does not take into account all small imperfections such as delays, hysteresis, saturation and so on that are affecting the control device. Using a more appropriate feedback law we would therefore get round the problem of existence of the solution of the differential equation and get what is called a *real* sliding motion, according to the terminology of [20]. This of course can be done paying the price of a real motion that does not effectively slide on a surface but remains in a boundary layer of it. What is most important is that the real motions obtained using real controls converge to a well-defined ideal one when the imperfections tend to zero. We will prove that in our setting this is true and that the limit value is exactly the solution given by the above equivalent control method.

Definition 3.1. Let $\delta > 0$ be a parameter used to model any kind of imperfections, x_δ^0 a point in X and $u_\delta : X \to U$ a regular function such that the equation

$$\dot{x} + Ax = Bu_\delta, \quad x(0) = x_\delta^0 \tag{14}$$

has a unique solution $x_\delta(t)$ in the strong sense (for sufficient conditions on u_δ see, e.g., [16]). This function will be called a *δ-trajectory from x_δ^0*.

The real sliding motions we have in mind will be special instances of δ-trajectories which will lie in some δ-vicinity of S. Before going into the details, we prove a useful lemma, which will make sense of many manipulations involving not-everywhere-defined operators in the proof of the main result.

Lemma 3.2. *If $x_\delta(t)$ is a δ-trajectory from a point $x_\delta^0 \in \mathcal{D}(Q)$ the following conditions are equivalent: for $t \geq 0$*

(a) *$x_\delta(t) \in \mathcal{D}(Q)$;*

(b) $\dot{x}_\delta(t) \in \mathcal{D}(Q)$;
(c) $Ax_\delta(t) \in \mathcal{D}(Q)$.

Proof. From (14) we always have $\dot{x_\delta}(t) + Ax_\delta(t) \in \mathcal{D}(Q)$, so (b) and (c) are equivalent. (a) $\Rightarrow$ (b) is rather obvious since $\dot{x_\delta}(t)$ is the limit of incremental ratios of x_δ, which are in $\mathcal{D}(Q)$ (closed) by (a). To get the converse, it is enough to recall that

$$x_\delta(t) = x_\delta(0) + \int_0^t \dot{x_\delta}(s)\,ds \in x_\delta^0 + t\,\overline{\mathrm{co}}(\mathcal{D}(Q)) = \mathcal{D}(Q)$$

because $x_\delta^0 \in \mathcal{D}(Q)$ and this is a closed linear subspace of X. □

We are now ready to prove the following:

Proposition 3.2. *Let us assume that $-A$ is the generator of a C_0-semigroup and Hypotheses 3.5–3.8 are satisfied. For any $\delta > 0$ let $x_\delta^0 \in \mathcal{D}(Q)$ be such that $\|x_\delta^0 - x_0\|$ tends to zero as $\delta \to 0$. If the corresponding δ-trajectories $x_\delta(t)$ are such that $x_\delta(t) \in \mathcal{D}(Q)$ for all t and*

$$\begin{aligned} &\|Qx_\delta\| \to 0 \text{ uniformly on compact subsets of } [0,+\infty], \\ &\|PAQx_\delta\|_{L^1_{\mathrm{loc}}(0,+\infty;X)} \to 0, \end{aligned} \tag{15}$$

then for all t we have $\|x_\delta(t) - x(t)\| \to 0$ for $\delta \to 0$ with $x(t) = \tilde{K}(t)x_0$ and the convergence is uniform on any bounded interval of the positive half line.

Proof. From (14), we have

$$\dot{x_\delta}(t) + Ax_\delta(t) = Q[\dot{x_\delta}(t) + Ax_\delta(t)],$$

because Q is a projection on the range of B. As $S \subset \mathcal{D}(A)$ and $x_\delta(t) \in D(A)$ for all t, we obviously have $Qx_\delta(t) \in \mathcal{D}(A)$ for all t because $\mathcal{D}(A)$ is a linear space. Denoting for brevity $y^S = Py$, $y^B = Qy$ for any $y \in \mathcal{D}(Q) \equiv \mathcal{D}(P)$ and using Lemma 3.2, we then have

$$\dot{x_\delta}^S = \tilde{A}x_\delta^S - PAx_\delta^B,$$

which means

$$x_\delta^S(t) = \tilde{K}(t)x_\delta^S(0) - \int_0^t \tilde{K}(t-s)PAx_\delta^B(s)\,ds.$$

$\tilde{K}(t)$ is a C_0-semigroup; therefore, there exist $\omega \in \mathbb{R}$ and $M \geq 1$ such that $\|\tilde{K}(t)\| \leq Me^{\omega t}$, so that

$$\|x^S_\delta(t)-x(t)\| \leq \|\tilde{K}(t)\|(\|x^0_\delta - x_0\| + \|Qx_\delta(0)\|) + \int_0^t \|\tilde{K}(t-s)\|\,\|PAx^B_\delta(s)\|\,ds$$
$$\leq Me^{\omega t}(\|x^0_\delta - x_0\| + \|Qx^0_\delta\|) + Me^{\omega t}\|PAQx_\delta\|_{L^1}.$$

By hypothesis the right-hand side converges to zero, and this is enough to complete the proof because $\|x_\delta(t) - x(t)\| = \|x^S_\delta(t) - x(t) + Qx_\delta(t)\|$. □

Let us make some comments on this proposition: as we said before, the regularization property of the ideal sliding mode makes use of trajectories of a regularized equation which do not slide on S, but live in a neighbourhood of it. The "distance" from ideality of $x_\delta(t)$ is here measured by (15): we know that $x(t) \in S$, so in the limit we want to annihilate any component of the δ-trajectory on $B(U)$. This is not however sufficient to get the overall convergence, due to the unboundedness of A. We have to make sure that the component on $B(U)$ has no effect on the evolution of the projection on S, and this is accomplished by adding the second part in (15).

3.3. **Generalized solutions vs. equivalent control method.** As seen before, we have two different ways to describe a motion on S, either by using generalized solutions or the equivalent control. Of course it is natural to wonder whether these concepts lead to different evolutions. We are going to prove that under the stated conditions of existence and uniqueness of u_{eq}, the two schemes are completely equivalent. That is, there exists only one generalized solution which is viable on S, and it coincides with the one arising from the application of the equivalent control method.

Let us suppose we are given the control system

$$\dot{x}(t) + Ax(t) = Bu(x(t)), \quad x(0) = x_0, \tag{16}$$

satisfying Hypotheses 3.1 to 3.8. The generalized solutions of (16) are by definition mild solutions of

$$\dot{x}(t) + Ax(t) \in F(x(t)), \quad x(0) = x_0, \tag{17}$$

with (see Proposition 2.2)

$$F(x) = \overline{\text{co}}\,\{z \in X \,:\, \exists\,(x_n) \subset X,\ x_n \notin S\ \forall n,\ x_n \to x \text{ and } Bu(x_n) \rightharpoonup z\}.$$

Thanks to our set of hypotheses, the expression of $F(x)$ can be much more simplified. Indeed, it is easy to show that the linearity of B and the closedness of its range imply that if $Bu(x_n) \rightharpoonup z$ then $z = B\overline{u}$ for some $\overline{u} \in U$, and using $(CB)^{-1}$ we get $u(x_n) \rightharpoonup \overline{u}$. Therefore, we can say that

$$\begin{aligned} F(x) &= \overline{\mathrm{co}}\, B(\{\overline{u} \in U \,:\, \exists\, (x_n) \subset X,\; x_n \notin S\, \forall n\; x_n \to x \text{ and } u(x_n) \rightharpoonup \overline{u}\}) \\ &= B(\overline{\mathrm{co}}\, \{\overline{u} \in U \,:\, \exists\, (x_n) \subset X,\; x_n \notin S\, \forall n\; x_n \to x \text{ and } u(x_n) \rightharpoonup \overline{u}\}), \end{aligned} \tag{18}$$

where the last equality depends upon the same argument.

As in Corollary 3.1, there exist generalized viable solutions on S of equation (16) if and only if the following condition is satisfied:

$$\forall x \in S \text{ there exists } y \in F(x) \text{ such that } Cy = CAx.$$

Under our hypotheses we have a unique well-defined equivalent control, and the tangency condition can be written as

$$\forall x \in S \text{ there exists } y \in F(x) \text{ such that } Cy = CBu_{\mathrm{eq}}(x),$$

so that, remembering (18), the tangential condition is satisfied if and only if $Bu_{\mathrm{eq}}(x) \in F(x)$ for all $x \in S$. Note here the analogy with the finite-dimensional case, where a necessary condition in order to have motion along the surface S is that all the components of the vector u_{eq} lie between values $u_i^+(x)$ and $u_i^-(x)$ of (11) (see [7], p. 56 and [20], p. 45).

If the above condition is fulfilled, we will get only one viable solution, which by Corollary 3.1 will be written as

$$x^*(t) = K(t)x_0 + \int_0^t K(t-s)Bu_{\mathrm{eq}}(x^*(s))\, ds. \tag{19}$$

This is just a mild solution of a differential inclusion, so it is continuous but not necessarily differentiable. It is not difficult to show that it satisfies (13) and therefore is much more regular.

Proposition 3.3. *If Hypotheses* 3.1–3.8 *hold, the viable solution* (19) *coincides with* $x(t) = \tilde{K}(t)x_0$.

Proof. We proved in Proposition 3.1 that for any $x_0 \in S$ there exists a unique C^1 solution $x(t)$ of

$$\dot{x} = \tilde{A}x = -(I-Q)Ax, \quad x(0) = x_0, \qquad x(t) \in S\ \forall t.$$

Therefore, it is a mild solution of the same differential equation; that is,

$$x(t) = K(t)x_0 + \int_0^t K(t-s)QA(x(s))\, ds \quad \text{ and } x(t) \in S\ \forall t,$$

which means $x(t)$ and $x^*(t)$ satisfy the same integral equation. By Gronwall's inequality we can prove that they are identical. In fact $\|K(t)\| \leq Me^{\omega t}$ for some $M > 0$, $\omega \in \mathbb{R}$ and $-QA$ is continuous on S, so

$$\|x(t) - x^*(t)\| \leq Me^{\omega t}\|QA\|_{\mathcal{L}(S,X)} \int_0^t \|x(s) - x^*(s)\|\, ds,$$

and the thesis follows. □

4. Stabilization as an example of application

In this section we are going to show how previous results can be applied to some problems of exponential stabilization. To use the theory developed up to this point, we have to restrict the class of systems we can study. In particular from now on we will suppose the following:

Hypothesis 4.1. The infinite-dimensional differential system is governed by an unbounded operator generating a compact semigroup, and the operator B is linear and continuous.

Before we start, let us briefly review some classical results concerning this class of operators. As usual we will call $\sigma(A)$ the spectrum of A, $\rho(A) = \mathbb{C}\setminus\sigma(A)$ and $R(\lambda; A) = (A - \lambda I)^{-1} : X \to D(A)$ for $\lambda \in \rho(A)$.

Theorem 4.1. *Let $A : \mathcal{D}(A) \subset X \to X$ be the infinitesimal generator of a continuous semigroup $K(t)$, $t \geq 0$, such that there exists $t_0 > 0$ for which $K(t)$ is compact for $t > t_0$. Then*

(i) *the resolvent $R(\lambda; A)$ is compact for every $\lambda \in \rho(A)$;*
(ii) *the spectrum of A consists only of eigenvalues which are isolated, have finite multiplicities and can have only infinity as limit point;*
(iii) *for any $(\alpha, \beta) \in \mathbb{R}^2$ the set $\sigma(A) \cap \{\lambda \in \mathbb{C} : \alpha \leq \mathcal{R}e\,\lambda \leq \beta\}$ is finite;*
(iv) *A satisfies the spectrum-determined growth assumption; that is,*

$$\begin{aligned}\sup \mathcal{R}e\,\sigma(A) &= \inf\{\omega \in \mathbb{R} : \|K(t)\| \leq Me^{\omega t} \text{ for some } M > 0,\ \forall t \geq 0\} \\ &= \omega_0 < +\infty.\end{aligned}$$

Proof. For the parts (i) and (iii) see, e.g., [16], Theorem 3.3, p. 48, Corollary 3.7, p. 51; (ii) is proven for instance in [9], Theorem 6.29, p. 187, while (iv) is a result by Zabczyk [21]. □

For any $\delta < \omega_0$ the spectrum of A can be separated into two parts:

$$\sigma_1(A) = \sigma(A) \cap \{\lambda \in \mathbb{C} : \delta \leq \mathcal{R}e\,\lambda \leq \omega_0\}, \quad \sigma_2(A) = \sigma(A)\setminus\sigma_1(A),$$

with $\sigma_1(A)$ finite. Therefore if $\delta \in \rho(A)$ there exists a Jordan curve enclosing an open set containing $\sigma_1(A)$ in its interior and $\sigma_2(A)$ in its exterior. Applying now a known result on the separation of the spectrum (see [9], Theorem 6.17, p. 178), we can find a decomposition of X in the direct sum $M_1 \oplus M_2$ such that

- $A(\mathcal{D}(A) \cap M_1) \subset M_1$, $A(\mathcal{D}(A) \cap M_2) \subset M_2$;
- if P_1 is the projection on M_1 along M_2 one has $P_1(\mathcal{D}(A)) \subset \mathcal{D}(A)$, so that P_1 and A commute;
- calling $A_i : \mathcal{D}(A) \cap M_i \to M_i$ the part of A on M_i one has $A_1 \in \mathcal{L}(M_1)$;
- $\sigma(A_1) = \sigma_1(A)$ and $\sigma(A_2) = \sigma_2(A)$;
- if $\sigma_1(A)$ consists of eigenvalues then M_1 is finite dimensional.

It is easy to show that the commutation property of A with P_1 and $P_2 = I - P_1$ is inherited from that of $K(t)$ [5]. It is therefore possible to construct two semigroups $K_1(t)$ and $K_2(t)$ on M_1 and M_2 respectively, such that A_i generates $K_i(t)$ and $K_i(t)$ is again compact. By (iv) of Theorem 4.1, choosing $\delta < 0$, we get

$$\sup \mathcal{R}e\, \sigma(A_2) \leq \delta < 0$$

so that there exist $\omega_2 > 0$, $M > 0$ such that

$$\|K_2(t)\| \leq M e^{-\omega_2 t}, \qquad \forall t \geq 0;$$

i.e., $K_2(t)$ is an exponentially stable semigroup. Then the system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \tag{20}$$

which can be decomposed into the form

$$\begin{cases} \dot{x_1} = A_1 x_1 + B_1 u, & x_1(0) = x_1^0 \\ \dot{x_2} = A_2 x_2 + B_2 u, & x_2(0) = x_2^0 \end{cases}$$

with $B_i = P_i B$, $x_i^0 = x_i(0)$, is exponentially stabilizable if and only if the system

$$\dot{x_1} = A_1 x_1 + B_1 u, \quad x_1(0) = x_0^1 \tag{21}$$

is exponentially stabilizable [18].

In our setting M_1 is finite dimensional, and therefore if $n = \dim M_1$, there exists an isomorphism $\varphi : M_1 \to \mathbb{R}^n$. The above equation can be transformed into an ordinary differential equation, as soon as we recognize the decomposition of the control space into $U = U_1 \oplus \ker B_1$ with U_1 finite dimensional. This is possible because $\varphi \circ B_1 \in \mathcal{L}(U, \mathbb{R}^n)$, so that $\ker B_1$ has finite codimension $m = \dim \operatorname{Im} B_1 \leq n$, and therefore is complementable in

U. Through isomorphisms φ and $\psi : \mathbb{R}^m \to U_1$ our differential equation (21) is equivalent to

$$\dot{y} = Dy + Ev, \quad y(0) = y_0, \quad D \in \mathbb{R}^{n\times n},\ E \in \mathbb{R}^{n\times m},\ y \in \mathbb{R}^n,\ v \in \mathbb{R}^m, \quad (22)$$

and the controllability of the couple (D, E) is sufficient for the stabilizability of the overall system (this method has been used in [12]).

Using well-known techniques for stabilization of systems through discontinuity surfaces, fixing a desired decay ω, we can find a matrix G such that GE is nonsingular, and the equivalent control method gives a solution y_{eq} of (22) which is exponentially stable with fixed asymptotic behaviour (see e.g. [20], Chapter 7).

This obviously means that we can find a linear and continuous operator $C_1 = \psi \circ G \circ \varphi : M_1 \to U_1$ such that $C_1 B_1 = \psi \circ G \circ E \circ \psi^{-1}$ is onto and has continuous inverse. We are therefore able to show the following:

Proposition 4.1. *If Hypothesis* 4.1 *holds and the finite-dimensional system* (22) *is exponentially stabilizable, then one can construct a linear subspace S of X and a discontinuous feedback u such that the system*

$$\dot{x} + Ax = Bu, \quad x(0) = x_0$$

has a unique exponentially stable generalized solution viable on S.

Proof. Let us choose a feedback control $u_1 : M_1 \to U_1$ discontinuous on $S_1 = \ker C_1$ and having linear growth. For the reduced system (21) the sliding motion on the surface S_1 is a solution of

$$\dot{x_1} + A_1 x_1 \in F_1(x_1), \quad x_1(0) = x_0^1, \quad F_1(x_1) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, B_1 u_1(\overline{\mathcal{B}}(x_1, \varepsilon) \setminus S_1) \quad (23)$$

viable on S_1. Using Proposition 3.3 it coincides with $\varphi^{-1}(y_{\mathrm{eq}}(t))$ for all t, so it decays exponentially with rate ω.

It is now easy to go back to the given differential equation (20). Suppose we can define a feedback control $u(x) = u_1(P_1 x)$ with u_1 as before and such that furthermore $Bu(x) \in M_1$ for all x. Then the set-valued function which defines the associated differential inclusion, i.e.,

$$F(x) = \overline{\mathrm{co}}\, \{z \in X \,:\, \exists\, x_n \to x \text{ with } P_1 x_n \notin S_1\, \forall n,\ Bu(x_n) \rightharpoonup z\},$$

has values in M_1. In fact $Bu(x_n) \in M_1$ for any n and M_1 is weakly closed. Using again the decomposition of X, it is easy to prove that the set of mild solutions of

$$\dot{x} + Ax \in F(x), \quad x(0) = x_0 \quad (24)$$

coincides with the mild solution set of system

$$\begin{cases} \dot{x_1} + A_1 x_1 \in F_1(x_1), & x_1(0) = x_1^0 \\ \dot{x_2} + A_2 x_2 = 0, & x_2(0) = x_2^0. \end{cases} \tag{25}$$

In fact, if

$$x(t) = K(t)x_0 + \int_0^t K(t-s)f(s)\,ds, \qquad f(s) \in F(x(s)) \text{ a.e. } s,$$

is a mild solution of (24), projecting this equation and calling $x_i(t) = P_i x(t)$, thanks to the commutation property of P_i and $K(t)$ on M_i, we get

$$x_1(t) = P_1 K(t)x_0 + \int_0^t P_1 K(t-s)f(s)\,ds = K_1(t)x_0^1 + \int_0^t K_1(t-s)f(s)\,ds$$

and

$$x_2(t) = P_2 K(t)x_0 + \int_0^t P_2 K(t-s)f(s)\,ds = K_2(t)x_0^2,$$

because $F(x) \subset M_1$ for all x. From Proposition 2.2 we obviously get $F(x) = F_1(x_1)$, so we have $x(t) = x_1(t) + x_2(t)$ with the pair $(x_1(t), x_2(t))$ solving (25). The converse can be shown analogously.

The last open issue is the possibility of finding $u_1 : M_1 \to U_1$ such that $Bu_1(z) \in M_1$ for all z, u_1 with linear growth and discontinuous on S_1. This can be easily done if we show that there exists at least one $v \in U_1$ such that $Bv \in M_1$. In this case it is sufficient for example to take $u_1(z) = L(z)v$ with $L : M_1 \to \mathbb{R}$ bounded and discontinuous on S_1 (for instance $L(x) = \|C_1 z\|^{-1}\langle C_1 z, \lambda\rangle$ for some $\lambda \in U^*$ would do the job). To see that one such v must exist, suppose for the sake of contradiction that we can find a basis for U_1, say $\{v_1, \dots, v_m\}$ (remember that $\dim U_1 = m$), such that $P_2 B v_i = z_i$ with $\{z_1, \dots, z_m\}$ linearly independent. As $U = U_1 \oplus \ker P_1 B$ we have $P_1 B v_i \neq 0$ for all i, so there exist $k \geq 1$, $w_l \in M_1$ for all $l = 1, \dots, k$ such that $B : U_1 \to \operatorname{span}\{z_1, \dots, z_m\} \oplus \operatorname{span}\{w_1, \dots, w_k\}$ is onto. This is clearly impossible, so there must exist a linear combination $v = \sum_{i=1}^m \alpha_i v_i \neq 0$ such that $P_2 B v = 0$; i.e., $Bv \in M_1$ as we wanted. □

Acknowledgments. The author would like to thank Professor Tullio Zolezzi for the many fruitful discussions that have been essential for the realization of this paper.

References

[1] J.P. Aubin and A. Cellina, "Differential Inclusions," Grundlehren der mathematischen Wissenschaften **264**, Springer-Verlag, 1984.

[2] J.P. Aubin, "Viability Theory," Systems and Control: Foundations and Applications, Birkhäuser, 1991.

[3] J.P. Aubin and H. Frankowska, "Set-Valued Analysis," Birkhäuser, 1990.

[4] O. Cârjă and I.I. Vrabie, *Some new viability results for semilinear differential inclusions*, No. Dea Nonlinear Differential Equations Appl. **4** (1997), 401–424.

[5] R.F. Curtain and A.J. Pritchard, "Infinite Dimensional Linear System Theory," Lecture Notes in Control and Information Sciences **8**, Springer-Verlag, 1978.

[6] A.F. Filippov, *Differential equations with discontinuous right-hand side*, Amer. Math. Soc. Transl., **42** (1964), 199–231.

[7] A. Filippov, "Differential Equations with Discontinuous Righthand Sides," Mathematics and its Applications (Soviet series), Kluwer Academic Publishers, 1988.

[8] O. Hájek, *Discontinuous differential equations, I and II*, J. Differential Equations, **32** (1979), 149–170, 171–185.

[9] T. Kato, "Perturbation Theory for Linear Operators," Grundlehren der mathematischen Wissenschaften **132**, Springer-Verlag, 1966.

[10] N.N. Krasovskij and A.I. Subbotin, *Game-theoretical control problems*, Springer Series in Soviet Mathematics, Springer-Verlag, 1988.

[11] Yu.V. Orlov, *Application of Lyapunov method in distributed systems*, Automat. Remote Control, **4** (1983), 22–28.

[12] Yu.V. Orlov, *Discontinuous unit feedback control of uncertain infinite-dimensional systems*, IEEE Trans. Aut. Control, **45** (2000), 834–843.

[13] Yu.V. Orlov and V.I. Utkin, *Use of sliding modes in distributed system control problems*, Automat. Remote Control, **9** (1982), 36–46.

[14] Yu.V. Orlov and V.I. Utkin, *Sliding mode control in indefinite-dimensional systems*, Automatica, **23** (1987), 753–757.

[15] Yu.V. Orlov and V.I. Utkin, *Unit sliding mode control in infinite-dimensional systems*, Appl. Math. Comput. Sci., **8** (1998), 7–20.

[16] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Applied Mathematical Sciences, **44**, Springer-Verlag, 1983.

[17] Shi Shuzhong, *Viability theorems for a class of differential-operator inclusions,* J. Differential Equations, **79** (1989), 232–257.

[18] R. Triggiani, *On the stabilizability problem in Banach space*, J. Math. Anal. Appl., **52** (1975), 383–403.

[19] V.I. Utkin, *Sliding mode control of infinite-dimensional plants*, in Proceedings of the International Workshop on Variable Structure Systems and their Applications, Sarajevo, Yugoslavija, 1990.

[20] V.I. Utkin, "Sliding Modes in Control Optimization," Communications and Control Engineering Series, Springer-Verlag, 1992.

[21] J. Zabczyk, *Remarks on the algebraic Riccati equation in Hilbert space*, Appl. Math. Optim., **2** (1975/1976), 251–258.