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PARAMETER-ELLIPTIC BOUNDARY VALUE PROBLEMS CONNECTED WITH THE NEWTON POLYGON

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Abstract. In this paper pencils of partial differential operators depending polynomially on a complex parameter and corresponding boundary value problems with general boundary conditions are studied. We define a concept of ellipticity for such problems (for which the parameter-dependent symbol in general is not quasi-homogeneous) in terms of the Newton polygon and introduce related parameter-dependent norms. It is shown that this type of ellipticity leads to unique solvability of the boundary value problem and to two-sided a priori estimates for the solution.

1. Introduction

In the present paper we consider a pencil of partial differential operators of the form

$$P(D,\lambda)=\sum_{\alpha,k} a_{\alpha k}\lambda^k D^\alpha \tag{1.1}$$

of order $2M$ depending polynomially on the complex parameter λ and acting in the half-space $\mathbb{R}^n_+ := \{x=(x',x_n)\in\mathbb{R}^n : x_n>0\}$. We supplement this partial differential operator with boundary conditions $B_1(D),\dots,B_M(D)$ and consider the boundary value problem

$$P(D,\lambda)u=f,\quad B_j(D)u=g_j\quad (j=1,\dots,M). \tag{1.2}$$

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Here and below we use the standard multi-index notation $D^\alpha = D_1^{\alpha_1}\cdots D_n^{\alpha_n}$ with $D_j = -i\partial/\partial x_j$ and $\xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$. The aim of our investigations is to endow the classical Sobolev spaces with parameter-dependent norms, realize (1.2) as a bounded operator with respect to these norms and to prove for large λ the existence of a bounded inverse operator. Simultaneously we will obtain uniform (with respect to λ) a priori estimates for the solution of the boundary value problem.

The case where the symbol $P(\xi,\lambda) := \sum_{\alpha,k} a_{\alpha k}\lambda^k\xi^\alpha$ of the operator (1.1) is quasi-homogeneous with respect to ξ and λ (up to perturbations of lower order) has been studied intensively since the papers of Agmon [1] and Agranovich–Vishik [4] appeared. In these papers it was shown that it is possible to define parameter-dependent norms using a quasi-homogeneous weight function (depending on ξ and λ) as a Fourier multiplier for which the boundary value problem can be realized as a bounded operator which has a bounded inverse for large values of λ. These results hold under conditions on P and B_j which are called the conditions of ellipticity with parameter (or parameter-ellipticity). These results imply results on boundary value problems which are parabolic in the sense of Petrovskii where, roughly speaking, the parameter λ has to be replaced by the time derivative $\partial/\partial t$.

However, the composition of two operators which are parabolic in the sense of Petrovskii with different weights for the time derivative no longer belongs to this class of operators. The same holds for parameter-elliptic boundary value problems. Consider, for instance, the operator $(\Delta^2+\lambda)(-\Delta+\lambda)$ with appropriate boundary conditions, where Δ stands for the Laplace operator. This parameter-dependent operator has no quasi-homogeneous principal symbol in the sense of Agmon–Agranovich–Vishik, and thus this theory cannot be applied. In the particular case where we consider the composition of two operators one might try to apply the theory of parameter-ellipticity to each of the operators separately; however, this is no longer possible if we consider operators like

$$-\Delta^3 + \lambda\Delta^2 + \lambda^2\,. \tag{1.3}$$

General operators of such type appear, for instance, if we consider the resolvent of Douglis–Nirenberg systems (mixed-order systems) and the determinant of their symbol. If in (1.3) the last term λ^2 was omitted, we would obtain a typical operator of singular perturbation theory (here $\lambda = \varepsilon^{-1}$ for a small parameter ε) as has been treated in [6] and [7] with methods similar to those used in the present paper.

The main questions concerning general boundary value problems of the form (1.2) consist in finding the appropriate Sobolev spaces (i.e., parameter-dependent norms) for which these operators are bounded in the sense that they are continuous with norm bounded by a constant independent of the parameter λ and in finding conditions on P and B_j which ensure the existence of a bounded inverse operator for large λ. In particular, it is of interest to find conditions of Shapiro–Lopatinskii type (i.e., conditions on the boundary operators which may be formulated in an algebraic way) which lead to invertibility. These questions are answered in the present paper; here we restrict ourselves to the model problem, so we assume that the operators P and B_j have constant (scalar) coefficients and act in the half-space and that $B_j(\xi)$ is homogeneous in ξ. One reason for this is given in Remark 5.12. We plan to investigate variable coefficients, operators on manifolds with boundary and nonstationary problems in a subsequent paper.

Operator pencils of the form (1.1) acting in the whole space have been studied thoroughly in the monograph [9] where such operators appear by reduction of homogeneous Cauchy problems using the Laplace transform; for the resolvent of Douglis–Nirenberg systems on closed manifolds (i.e., compact manifolds without boundary) see [5]. One main tool for establishing these results was the so-called Newton polygon and the concept of N-ellipticity connected with this polygon. It turns out that in the case of boundary value problems this concept is useful, too. In particular, we will describe the parameter-dependent norms in terms of the Newton polygon and prove the invertibility for large λ using the Newton polygon approach.

Let us remark that the parameter-dependent norms and the a priori estimates appearing in the present paper are more complicated than the corresponding terms in the Agmon–Agranovich–Vishik theory of ellipticity with parameter. The (relatively) simple structure appearing in the theory of ellipticity with parameter is caused by the homogeneity of the problem which contains only one large parameter. The problems considered in the present case contain, in some sense, more than one large parameter, which makes the estimates more complicated.

The plan of this paper is as follows. In Section 2 we will define N-ellipticity for pencils of the form (1.1) and study equivalent conditions for this type of ellipticity. In Section 3 we will define the conditions of Shapiro–Lopatinskii type and N-elliptic boundary value problems, introduce parameter-dependent norms and state the main results on continuity and invertibility for the operator related to (1.2). The proof of the last result is based on estimates on the solutions of an ordinary differential equation (as is the case in the

Agmon–Agranovich–Vishik theory). For this estimate it is essential to know the behaviour of the zeros of the polynomial $P(\xi_1, \dots, \xi_{n-1}, \cdot, \lambda)$ under the condition of N-ellipticity. This behaviour is described in Section 4, and the proof of the main results can be found in Section 5.

2. N-ellipticity with parameter

Let $P(\xi, \lambda)$ be a polynomial in $\xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$ with complex coefficients,

$$P(\xi, \lambda) = \sum_{\alpha,k} a_{\alpha k} \lambda^k \xi^\alpha.$$

The Newton polygon $N(P)$ is defined as the convex hull in $\mathbb{R}^2$ of all points $(|\alpha|, k)$ with $a_{\alpha k} \neq 0$, their projections $(|\alpha|, 0)$ and $(0, k)$ and the origin. In the following, we will recall some definitions and results connected with the Newton polygon. For a more detailed discussion we refer the reader to [9] and [5].

From the definition of $N(P)$ it follows that the Newton polygon has the origin as one vertex and that the adjoining edges belong to coordinate axes.

Denote by $\Gamma_0, \dots, \Gamma_{J+1}$ the vertices of the polygon $N(P)$, starting with $\Gamma_0 = (0, 0)$ and indexed in the clockwise direction. For $j = 1, \dots, J$ we choose $r_j \geq 0$ such that the vector $(1, r_j)$ is an exterior normal to the edge $\Gamma_j \Gamma_{j+1}$ joining Γ_j and Γ_{j+1}. By convexity, we have $r_1 > \dots > r_J$. In the case where $\Gamma_1 \Gamma_2$ is horizontal we pose $r_1 = \infty$, and in the case where $\Gamma_J \Gamma_{J+1}$ is vertical we have $r_J = 0$ (see also Figure 1).

Definition 2.1. The Newton polygon $N(P)$ is called regular if it has no edge which is parallel to one of the coordinate axes but does not belong to this axis.

It follows from the definition above that $N(P)$ is regular if and only if $r_1 < \infty$ and $r_J > 0$. In other words, in this case

$$\infty > r_1 > \dots > r_J > 0\,. \tag{2.1}$$

With each polynomial P we connect the weight function of its Newton polygon defined by

$$W_P(\xi, \lambda) := \sum_{(i,k) \in N(P) \cap \mathbb{Z}^2} |\xi|^i \lambda^k\,. \tag{2.2}$$

Obviously

$$|P(\xi, \lambda)| \leq C\, W_P(\xi, \lambda) \tag{2.3}$$

holds with a constant C independent of (ξ, λ).

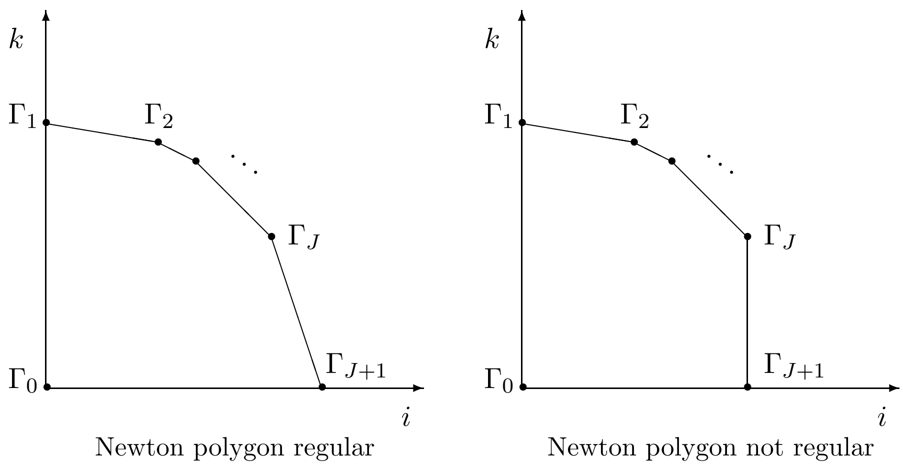

FIGURE 1. Examples of Newton polygons.

Definition 2.2. The polynomial $P(\xi,\lambda)$ is called N-elliptic with parameter in $[0,\infty)$ if

(i) the Newton polygon $N(P)$ is regular,

(ii) there exists a $\lambda_0 > 0$ such that

$$|P(\xi,\lambda)| \geq C\ W_P(\xi,\lambda) \quad \text{for } \xi \in \mathbb{R}^n \text{ and } \lambda \geq \lambda_0. \tag{2.4}$$

Here and in the following, the letter C stands for a positive constant which may vary from one time of appearance to another.

As an example, let us consider the symbol of the operator (1.3). Here we have $P(\xi,\lambda) = |\xi|^6 + \lambda|\xi|^4 + \lambda^2$, and the Newton polygon has the form indicated in Figure 2. The weight function in this example is equivalent to $1+\lambda^2+\lambda|\xi|^4+|\xi|^6$, and obviously P is N-elliptic in the sense of Definition 2.2.

There are several numbers connected with the geometry of the Newton polygon which will play an essential role for the analysis below. First of all, let us denote the coordinates of the vertices Γ_j by

$$\Gamma_j = (p_j, q_j) \quad (j = 0,\dots,J+1)\,.$$

Note that $p_0 = q_0 = p_1 = q_{J+1} = 0$. With these coordinates we have for the outer normal vectors $(1, r_j)$ introduced above the equality

$$p_j + r_j q_j = p_{j+1} + r_j q_{j+1} \quad (j = 1,\dots,J),$$

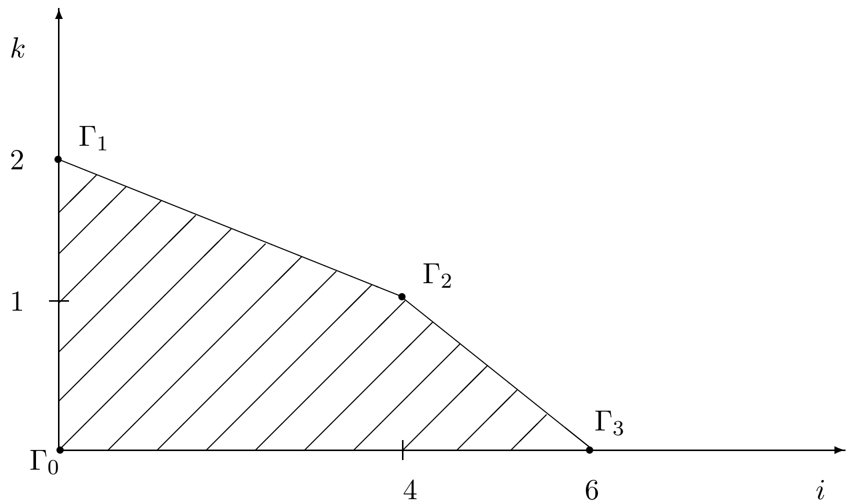

FIGURE 2. The Newton polygon for the example (1.3).

and hence $r_j = \frac{p_{j+1}-p_j}{q_j-q_{j+1}}$. It will turn out later that the numbers p_j are even. Since these numbers divided by 2 are so important for the following, we set

$$M_j := \frac{p_{j+1}}{2} \quad (j = 0, \dots, J), \qquad N_j := \frac{p_{j+1} - p_j}{2} \quad (j = 1, \dots, J).$$

We have $M_0 = 0$ and additionally define $M := M_J$.

The principal parts P_{Γ_j} and $P_{\Gamma_j\Gamma_{j+1}}$ of the polynomial $P(\xi, \lambda)$ corresponding to the vertex Γ_j and the edge $\Gamma_j\Gamma_{j+1}$, respectively, are defined by

$$P_{\Gamma_j}(\xi, \lambda) := \sum_{\substack{\alpha,k \\ (|\alpha|,k)=\Gamma_j}} a_{\alpha k}\lambda^k\xi^\alpha \quad (j = 1, \dots, J+1),$$

$$P_{\Gamma_j\Gamma_{j+1}}(\xi, \lambda) := \sum_{\substack{\alpha,k \\ (|\alpha|,k)\in\Gamma_j\Gamma_{j+1}}} a_{\alpha k}\lambda^k\xi^\alpha \quad (j = 1, \dots, J).$$

In the example of the operator (1.3) (see also Figure 2) we have $J = 2$, $r_1 = 4$, $r_2 = 2$, and the principal parts of P are given by $P_{\Gamma_1} = \lambda^2$, $P_{\Gamma_2} = \lambda|\xi|^4$, $P_{\Gamma_3} = |\xi|^6$, $P_{\Gamma_1\Gamma_2} = \lambda|\xi|^4 + \lambda^2$, $P_{\Gamma_2\Gamma_3} = |\xi|^6 + \lambda|\xi|^4$.

The following result is taken from [5].

Lemma 2.3. *For a polynomial $P(\xi,\lambda)$ the following conditions are equivalent:*

(i) *P is N-elliptic with parameter in $[0,\infty)$.*

(ii) *There exists a $\lambda_0>0$, numbers $r_1,\dots,r_J$ satisfying (2.1) and numbers $N_1,\dots N_J$ such that*

$$C\prod_{j=1}^{J}\big(\Lambda_j(\xi,\lambda)\big)^{2N_j}\le|P(\xi,\lambda)|\le C'\prod_{j=1}^{J}\big(\Lambda_j(\xi,\lambda)\big)^{2N_j}\quad(\xi\in\mathbb{R}^n,\lambda\in[\lambda_0,\infty))$$

holds for positive constants C and C', where

$$\Lambda_j(\xi,\lambda):=|\xi|+\lambda^{1/r_j}\quad(j=1,\dots,J).\tag{2.5}$$

(iii) *The polygon $N(P)$ is regular, and we have*

$$P_{\Gamma_j}(\xi,\lambda)\neq0\quad(j=1,\dots,J+1),$$
$$P_{\Gamma_j\Gamma_{j+1}}(\xi,\lambda)\neq0\quad(j=1,\dots,J)$$

for all $(\xi,\lambda)\in\mathbb{R}^n\times[0,\infty)$ with $|\xi|>0$ and $\lambda>0$.

The principal part P_{Γ_j} is of the form

$$P_{\Gamma_j}(\xi,\lambda)=\pi_j(\xi)\lambda^{q_j},\tag{2.6}$$

where π_j is a homogeneous polynomial of order $p_j=2M_{j-1}$. As we have $p_1=0$, we may assume that $\pi_1(\xi)=1$. The principal part $P_{\Gamma_j\Gamma_{j+1}}$ is $(1,r_j)$-homogeneous in (ξ,λ) of degree $p_j+r_jq_j=p_{j+1}+r_jq_{j+1}$ in the sense that

$$P_{\Gamma_j\Gamma_{j+1}}(\alpha\xi,\alpha^{r_j}\lambda)=\alpha^{p_j+r_jq_j}P_{\Gamma_j\Gamma_{j+1}}(\xi,\lambda)\quad(\alpha>0).$$

Each term in this polynomial contains the factor $\lambda^{q_{j+1}}$, and it is natural to pose

$$P_j(\xi,\lambda)=\lambda^{-q_{j+1}}P_{\Gamma_j\Gamma_{j+1}}(\xi,\lambda).\tag{2.7}$$

For the reason of $(1,r_j)$-homogeneity, the polynomial P_j can be written in the form

$$P_j(\xi,\lambda)=\pi_j(\xi)\lambda^{q_j-q_{j+1}}+\dots+\pi_{j+1}(\xi)\quad(j=1,\dots,J).$$

With respect to ξ, this is a polynomial of order $p_{j+1}=2M_j$. The polynomials P_j will be called the edge principal parts of P. Note that P_j is quasi-homogeneous in (ξ,λ), but in general it does not satisfy the condition of parameter-ellipticity. Such polynomials have been studied in [6]–[8] where a definition similar to the following one can be found:

Definition 2.4. Let $0 \le m_1 \le m_2$ be integers, $r > 0$ and

$$R(\xi,\lambda) = \sum_{j=m_1}^{m_2} R_j(\xi)\lambda^{(m_2-j)/r}$$

be a $(1,r)$-homogeneous polynomial in ξ and λ, where R_j is homogeneous of degree j. Then R is called weakly parameter-elliptic in $[0,\infty)$ if the inequality

$$|R(\xi,\lambda)| \ge C\,|\xi|^{m_1}\,(|\xi| + \lambda^{1/r})^{m_2-m_1}$$

holds for all $(\xi,\lambda) \in \mathbb{R}^n \times [0,\infty)$.

Remark 2.5. It was shown in [6], Lemma 3.2, that R is weakly parameter-elliptic in $[0,\infty)$ if and only if the following conditions are satisfied:
(i) $R_{m_1}(\xi) \ne 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$.
(ii) $R_{m_2}(\xi) \ne 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$.
(iii) $R(\xi,\lambda) \ne 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and all $\lambda > 0$.

Theorem 2.6. *The polynomial P is N-elliptic with parameter in* $[0,\infty)$ *if and only if all the edge principal parts* (2.7) *are weakly parameter-elliptic in* $[0,\infty)$.

Proof. We see from the equivalence of (i) and (iii) in Lemma 2.3 and from (2.6) and (2.7) that P is N-elliptic if and only if we have

$$\begin{aligned} \pi_j(\xi) &\ne 0 \quad (\xi \ne 0,\ j = 1,\dots,J+1), \\ P_j(\xi,\lambda) &\ne 0 \quad (\xi \ne 0,\ \lambda > 0,\ j = 1,\dots,J)\,. \end{aligned}$$

But due to Remark 2.5 this is equivalent to the weak parameter-ellipticity of all edge principal parts P_j. □

Let us assume for the remainder of this section that P is N-elliptic. We know from the previous theorem that the polynomials π_j are homogeneous elliptic polynomials of degree p_j for $j = 2,\dots,J+1$. In the case $n \ge 3$ this implies that the numbers p_j, $j = 2,\dots,J+1$ are even and $M_j = p_{j+1}/2$ and $N_j = (p_{j+1} - p_j)/2$ are integers. For $n = 2$ we will assume this in the following without further stipulation. In the same way $P_j(\xi',\cdot,\lambda)$ has no real roots for $\xi' \ne 0$ and $\lambda \ge 0$, and the number of roots of P_j in the upper (lower) half-plane of the complex plane is independent of (ξ',λ). As $P_j(\xi',\tau,0) = \pi_{j+1}(\xi',\tau)$, the number of roots of $P_j(\xi',\cdot,\lambda)$ with positive imaginary part equals $M_j = p_{j+1}/2$.

Now let us consider the problem (1.2) acting in the half-space. We shall introduce coordinates (x',t) such that the half-space is defined by the condition $x' \in \mathbb{R}^{n-1}$, $t \ge 0$. The dual variables will be $\xi = (\xi',\tau)$ with $\xi' \in \mathbb{R}^{n-1}$.

In these variables we pose

$$Q_j(\tau,\lambda) := \frac{P_j(0,\tau,\lambda)}{\pi_j(0,\tau)} \quad (j = 1,\dots,J). \tag{2.8}$$

As the polynomials π_j are homogeneous and elliptic, the equality $\pi_j(0,\tau) = c_j\tau^{p_j}$ holds for some nonvanishing constant c_j. In the same way the $(1,r_j)$-homogeneous polynomial $P_j(0,\tau,\lambda)$ is of the form

$$P_j(0,\tau,\lambda) = c_{j+1}\tau^{p_{j+1}} + \cdots + c_j\tau^{p_j}\lambda^{q_j-q_{j+1}}.$$

Therefore, (2.8) is a $(1,r_j)$-homogeneous polynomial of (τ,λ) and of order $p_{j+1} - p_j = 2N_j$ with respect to τ.

Due to Theorem 2.6 and the definition of weak parameter-ellipticity, the inequality

$$|P_j(\xi,\lambda)| \geq C\, |\xi|^{p_j}(|\xi| + \lambda^{\frac{1}{r_j}})^{p_{j+1}-p_j}$$

follows. Setting $\xi = (0,\tau)$ and dividing by $c_j\tau^{p_j}$ we obtain

$$|Q_j(\tau,\lambda)| \geq C\, (|\tau| + \lambda^{\frac{1}{r_j}})^{p_{j+1}-p_j}.$$

Therefore the polynomial (2.8) has no real roots. However, here we have to impose an additional condition on the number of zeros with positive imaginary part:

Definition 2.7. We say that the polynomial $P_j(\xi,\lambda)$ satisfies the Vishik–Lyusternik condition if the polynomial $Q_j(\cdot,1)$ has exactly N_j roots in $\mathbb{C}_+$.

Remark 2.8. a) The name of this condition is connected with the theory of singular perturbations. Replacing λ by ε^{-r_j} and multiplying $P_j(\xi,\lambda)$ by $\varepsilon^{r_jq_j}$, we obtain the symbol of an operator pencil with small parameter in front of the highest derivative, as was considered by Vishik and Lyusternik in [12]. The condition of Definition 2.7 is exactly the condition of regular degeneration introduced in [12].

b) For $j = 1$ and $n > 2$ the Vishik–Lyusternik condition is satisfied automatically, because

$$P_1(\xi,\lambda) \neq 0 \quad \text{for all } \xi \in \mathbb{R}^n,\ \lambda \geq 0 \text{ with } |\xi| + \lambda > 0,$$

i.e., the polynomial $P(\xi,\lambda)$ is elliptic with parameter in the sense of Agmon and Agranovich–Vishik.

c) The Vishik–Lyusternik condition holds if the number r_j is even (in this case if τ is a zero then $-\tau$ is a zero, too).

Definition 2.9. Let the polynomial $P(\xi,\lambda)$ be N-elliptic with parameter. We say that P satisfies the Vishik–Lyusternik condition if for $j=1,\dots,J$ the edge polynomial P_j satisfies the condition of Definition 2.7.

3. MAIN RESULTS

3.1. **Shapiro–Lopatinskii conditions.** We will see in Section 4 that the polynomials P_j and Q_j determine the behaviour of the zeros of $P(\xi',\cdot,\lambda)$ for $\xi'\in\mathbb{R}^{n-1}\setminus\{0\}$ and sufficiently large $\lambda>0$. Lemma 2.3 describes N-ellipticity for P as the nonvanishing of the leading parts P_{Γ_j} and $P_{\Gamma_j\Gamma_{j+1}}$. A similar approach will be used if we consider the boundary value problem $(P,B_1,\dots,B_M)$. Here the basic quasi-homogeneous operators will be $P_j(D,\lambda)$ and $Q_j(D_n,\lambda)$, which are of order $p_{j+1}=2M_j$ and $p_{j+1}-p_j=2N_j$, respectively. We will define N-ellipticity for the boundary value problem $(P,B_1,\dots,B_M)$ by conditions of Shapiro–Lopatinskii type for the operators P_j and Q_j, supplemented by properly chosen groups of boundary operators $B_1(D),\dots,B_M(D)$.

We start with a preliminary remark on ordinary differential equations. Let $A(\tau)$ be a complex polynomial of degree $2m$ and $B_1(\tau),\dots,B_m(\tau)$ be polynomials of degree m_j. Assume that A has no real roots and exactly m roots $\tau_1,\dots,\tau_m$ in $\mathbb{C}_+$, and define $A_+(\tau):=\prod_{j=1}^m(\tau-\tau_j)$. We are interested in the ordinary differential equation on the half-line given by

$$\begin{aligned} A(D_t)w(t)&=0 \quad (t>0),\\ B_j(D_t)w(t)&=g_j \quad (j=1,\dots,m),\\ w(t)&\to 0 \quad (t\to+\infty). \end{aligned} \tag{3.1}$$

The following equivalence is well known from the theory of elliptic boundary value problems.

Lemma 3.1. *The following conditions are equivalent:*

(i) *For every $(g_1,\dots,g_m)\in\mathbb{C}^m$ the problem* (3.1) *has a unique solution.*

(ii) *The Lopatinskii matrix* $\operatorname{Lop}(A,B_1,\dots,B_m):=\big(\overline{b}_{ik}\big)_{i,k=1,\dots,m}$ *is invertible. Here $\overline{B}_k(\tau)=\sum_{i=1}^m\overline{b}_{ik}\tau^{i-1}$ is the remainder of $B_i(\tau)$ modulo $A_+(\tau)$.*

(iii) *Let γ be a closed contour in $\mathbb{C}_+$ enclosing $\tau_1,\dots,\tau_m$. Then there exist polynomials $N_1(\tau),\dots,N_m(\tau)$ of order $\le m$ such that*

$$\frac{1}{2\pi i}\int_\gamma\frac{B_i(\tau)N_k(\tau)}{A_+(\tau)}\,\mathrm{d}\,\tau=\delta_{ik}\quad (i,k=1,\dots,m).$$

Now we come back to the boundary value problem $(P, B_1, \ldots, B_M)$ given by (1.2). For the following, we will assume $m_i := \operatorname{ord} B_i$ $(i = 1, \ldots, M)$ satisfies the inequalities

$$m_1 \leq \cdots \leq m_{M_1} < m_{M_1+1} \leq \cdots \leq m_{M_2} < m_{M_2+1} \leq \cdots \leq m_{M_J} < 2M. \quad (3.2)$$

Definition 3.2. The boundary value problem $(P(D, \lambda), B_1(D), \ldots, B_M(D))$ is called N-elliptic with parameter in $[0, \infty)$ if the following conditions hold:

(i) $P(\xi, \lambda)$ is N-elliptic with parameter in $[0, \infty)$ in the sense of Definition 2.2.

(ii) $P(\xi, \lambda)$ satisfies the Vishik–Lyusternik condition (Definition 2.9).

(iii) For each $j = 1, \ldots, J$, $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$ and $\lambda \geq 0$ the boundary value problem $\big(P_j(\xi', D_t, \lambda), B_1(\xi', D_t), \ldots, B_{M_j}(\xi', D_t)\big)$ satisfies the equivalent conditions of Lemma 3.1.

(iv) For each $j = 1, \ldots, J$ the boundary value problem

$$\big(Q_j(D_t, 1), B_{M_{j-1}+1}(0, D_t), \ldots, B_{M_j}(0, D_t)\big)$$

satisfies the equivalent conditions of Lemma 3.1.

Remark 3.3. a) Taking $\lambda = 0$ in Definition 3.2(iii), we obtain that for each $j = 1, \ldots, J$ the (homogeneous) boundary value problem $(\pi_{j+1}(D), B_1(D), \ldots, B_{M_j}(D))$ satisfies the classical Shapiro–Lopatinskii condition.

b) Let us call the boundary value problem $(P_j(D), B_1(D), \ldots, B_{M_j}(D))$ weakly parameter-elliptic in $[0, \infty)$ if P_j is weakly parameter-elliptic in the sense of Definition 2.4, satisfies the Vishik–Lyusternik condition, and if conditions 3.2 (iii) and (iv) are satisfied for j, where Q_j is defined in (2.8). This definition was introduced in [7] without using the term weak parameter-ellipticity. By definition we obtain that $(P, B_1, \ldots, B_M)$ is N-elliptic with parameter if and only if each of the boundary value problems $(P_j(D), B_1(D), \ldots, B_{M_j}(D))$ is weakly parameter-elliptic. A similar concept was implicitly used in [8].

Example 3.4. Let $P(\xi, \lambda)$ be N-elliptic with parameter in $[0, \infty)$ in the sense of Definition 2.2, and assume that $P(\xi, \lambda)$ satisfies the Vishik–Lyusternik condition. Then the Dirichlet boundary value problem $(P(D, \lambda), B_1(D), \ldots, B_M(D))$ with $B_j(D) = (\partial/\partial x_n)^{j-1}$ in $\mathbb{R}^n_+$ is N-elliptic in the sense of Definition 3.2. It can easily be checked directly that the conditions of 3.2 are satisfied, or one can use the result from [6] where it was shown that the Dirichlet boundary problem corresponding to the edge operator P_j is weakly parameter-elliptic in the sense of Remark 3.3 b).

3.2. **Functional spaces.** Now we want to introduce parameter-dependent norms for the classical L_2-Sobolev spaces for which the boundary value problem (1.2) has a realization as a bounded operator which is – for sufficiently large λ – invertible with bounded inverse. Here the norms of these operators can be estimated by a constant independent of λ which implies uniform a priori estimates for the solution of (1.2).

As usual, we will define the L_2-Sobolev spaces, first in the whole space $\mathbb{R}^n$, using the Fourier transform F. Recall that in the Agmon–Agranovich–Vishik theory of ellipticity with parameter the parameter-dependent norm is defined via weight functions of the form $(|\xi|^2 + \lambda^2)^m$, i.e., in this theory we have homogeneous (or quasi-homogeneous) weight functions. In the case of the present paper, however, we don't have homogeneity, and we will introduce more complicated norms adapted to the boundary value problem.

For this we fix a tuple $\mathbf{s} = (s_1, \dots, s_J)$ of real numbers and define

$$\Psi_{\mathbf{s}}(\xi, \lambda) := \prod_{j=1}^{J} \big(\Lambda_j(\xi, \lambda)\big)^{s_j} \tag{3.3}$$

(recall that Λ_j is defined in (2.5)). For $s_j = 2N_j$ the function $\Psi_{\mathbf{s}}$ appears in Lemma 2.3. We will endow the Sobolev space $H^{s_1+\dots+s_J}(\mathbb{R}^n)$ with the parameter-dependent norm

$$\|u\|_{\mathbf{s},\mathbb{R}^n} := \big\|F^{-1}\Psi_{\mathbf{s}}(\xi, \lambda)Fu(\xi)\big\|_{L_2(\mathbb{R}^n)}. \tag{3.4}$$

We will write $H_{\mathbf{s}}(\mathbb{R}^n)$ if we consider $H^{s_1+\dots+s_J}(\mathbb{R}^n)$ endowed with the norm (3.4). The space $H_{\mathbf{s}}(\mathbb{R}^{n-1})$ is defined in the same way, replacing $\Psi_{\mathbf{s}}(\xi, \lambda)$ by $\Psi_{\mathbf{s}}(\xi', \lambda) := \Psi_{\mathbf{s}}(\xi', 0, \lambda)$.

For the description of the trace spaces below we will need "shifted" weight functions. More precisely, we define for $\mathbf{s} \in \mathbb{R}^J$ with $s_j \geq 0$ and for $0 \leq \kappa \leq s_1 + \dots + s_J$ the function

$$\Psi_{\mathbf{s}}^{(-\kappa)}(\xi, \lambda) := \Lambda_K(\xi, \lambda)^{s_1+\dots+s_K-\kappa} \prod_{\ell=K+1}^{J} \Lambda_\ell^{s_\ell}(\xi, \lambda), \tag{3.5}$$

where the index K is chosen such that $s_1 + \dots + s_{K-1} < \kappa \leq s_1 + \dots + s_K$ (with obvious modification if $\kappa \leq s_1$). For the corresponding spaces and norms we will write $H_{\mathbf{s}}^{(-\kappa)}$ and $\|\cdot\|_{\mathbf{s}}^{(-\kappa)}$, respectively.

Now let us consider the Sobolev space $H_{\mathbf{s}}(\mathbb{R}^n_+)$ which, following the general theory (see, e.g., [13]), may be defined as the quotient space $H_{\mathbf{s}}(\mathbb{R}^n)/H_{\mathbf{s}}(\mathbb{R}^n)_-$ where $H_{\mathbf{s}}(\mathbb{R}^n)_-$ stands for the subspace of all distributions in $H_{\mathbf{s}}(\mathbb{R}^n)$ with support contained in the set $\{x \in \mathbb{R}^n : x_n \le 0\}$. Note that we have the equivalence $\Psi_{\mathbf{s}}(\xi,\lambda) \approx |\tilde{\Psi}_{\mathbf{s}}(\xi,\lambda)|$ with

$$\tilde{\Psi}_{\mathbf{s}}(\xi,\lambda) := \prod_{j=1}^{J} \big[i\xi_n + (|\xi'|^2 + \lambda^{2/r_j})^{1/2}\big]^{2s_j} .$$

Here the symbol $\approx$ means that the quotient of the left-hand side and the right-hand side can be estimated from above and from below by positive constants not depending on ξ or λ. As $\tilde{\Psi}_{\mathbf{s}}$ can be extended as a holomorphic function with polynomial growth to the lower half-plane $\operatorname{Im}\xi_n < 0$, the norm in $H_{\mathbf{s}}(\mathbb{R}^n_+)$ as a quotient space is equivalent to the norm

$$\|u\|_{\mathbf{s},\mathbb{R}^n_+} := \|\tilde{\Psi}_{\mathbf{s}}(D,\lambda)u_0\|_{L_2(\mathbb{R}^n_+)} \quad (u \in H_{\mathbf{s}}(\mathbb{R}^n_+)) ,$$

where u_0 is an arbitrary representative of u. Here the pseudodifferential operator $\tilde{\Psi}_{\mathbf{s}}(D,\lambda)$ is defined, as usual, by $\tilde{\Psi}_{\mathbf{s}}(D,\lambda)u_0 := F^{-1}\tilde{\Psi}_{\mathbf{s}}(\xi,\lambda)Fu_0(\xi)$.

We will mainly consider the case where $s_1 + \cdots + s_J$ is a nonnegative integer. Here the binomial formula tells us that

$$\Psi_{\mathbf{s}}(\xi,\lambda) \approx \Big(\sum_{\ell=0}^{s_1+\cdots+s_J} \xi_n^{2\ell} \big(\Psi_{\mathbf{s}}^{(-\ell)}(\xi',\lambda)\big)^2 \Big)^{\frac{1}{2}}.$$

Therefore, in this case we may use

$$\|u\|_{\mathbf{s},\mathbb{R}^n_+} := \Big(\sum_{\ell=0}^{s_1+\cdots+s_J} \int_{\mathbb{R}^{n-1}} \big(\Psi_{\mathbf{s}}^{(-\ell)}(\xi',\lambda)\big)^2 \|D_t^\ell (F'u)(\xi',\cdot)\|^2_{L_2(\mathbb{R}_+)} \,\mathrm{d}\,\xi' \Big)^{\frac{1}{2}} \quad (3.6)$$

as an equivalent norm in $H_{\mathbf{s}}(\mathbb{R}^n_+)$, where F' stands for the partial Fourier transform with respect to the first $n-1$ variables.

The norm (3.4) was investigated in [6] where also the trace operators $\gamma_\ell : H^s(\mathbb{R}^n_+) \to H^{s-1/2}(\mathbb{R}^{n-1})$ mapping u to $(D_t^\ell u)(x',0)$ (for $s > \ell + 1/2$) were considered as operators in the parameter-dependent norms introduced above. The following result was shown in [6] (for integer s_j, but the proof for real s_j is literally the same).

Lemma 3.5. *Let $s_1 + \cdots + s_J > \frac{1}{2}$. For every $\lambda_0 > 0$ and every $\ell \in \mathbb{Z}$ with $0 \le \ell < s_1 + \cdots + s_J - \frac{1}{2}$ there exists a constant $C > 0$ independent of u and λ such that*

$$\|\gamma_\ell u\|^{(-\ell-1/2)}_{\mathbf{s},\mathbb{R}^{n-1}} \le C\|u\|_{\mathbf{s},\mathbb{R}^n_+} \quad (u \in H_{\mathbf{s}}(\mathbb{R}^n_+),\ \lambda \ge \lambda_0) .$$

3.3. **Continuity and invertibility results.** Now let us consider the operator corresponding to the boundary value problem (1.2) acting in the Sobolev spaces with parameter-dependent norms introduced in the previous subsection. We start with the continuity result, where there is no need to assume N-ellipticity.

Theorem 3.6. *Let $N(P)$ be regular, let $\mathbf{s} \in \mathbb{R}^J$ be a tuple of nonnegative real numbers with $s_1 + \cdots + s_J > m_M + \frac{1}{2}$, and set $t_j := s_j - 2N_j$. Then the operator*

$$(P, B_1, \ldots, B_M)\colon H_{\mathbf{s}}(\mathbb{R}^n_+) \to H_{\mathbf{t}}(\mathbb{R}^n_+) \times \prod_{j=1}^{M} H_{\mathbf{s}}^{(-m_j-1/2)}(\mathbb{R}^{n-1}) \tag{3.7}$$

is continuous and there exists a constant $C > 0$ such that for every $\lambda \geq 0$ the inequality

$$\|Pu\|_{\mathbf{t},\mathbb{R}^n_+} + \sum_{j=1}^{M} \|B_j u\|^{(-m_j-1/2)}_{\mathbf{s},\mathbb{R}^{n-1}} \leq C\ \|u\|_{\mathbf{s},\mathbb{R}^n_+}$$

holds.

Proof. We use the fact that there exists a bounded linear extension operator, i.e., a bounded operator $E_0\colon H_{\mathbf{s}}(\mathbb{R}^n_+) \to H_{\mathbf{s}}(\mathbb{R}^n)$ with $R_0 E_0 u = u$ for all $u \in H_{\mathbf{s}}(\mathbb{R}^n_+)$, where R_0 stands for the operator of restriction onto $\mathbb{R}^n_+$.

For $u \in H_{\mathbf{s}}(\mathbb{R}^n_+)$ we set $u_0 := E_0 u \in H_{\mathbf{s}}(\mathbb{R}^n)$. By the definition of W_P (see (2.2)) and by the equivalence $W_P(\xi,\lambda) \approx \prod_{j=1}^J \Lambda_j(\xi,\lambda)^{2N_j}$ we obtain, using Plancherel's theorem,

$$\begin{aligned}\|Pu\|_{\mathbf{t},\mathbb{R}^n_+} \leq \|Pu_0\|_{\mathbf{t},\mathbb{R}^n} &= \Big\| \prod_{j=1}^{J} \Lambda_j(\xi,\lambda)^{s_j-2N_j} P(\xi,\lambda)(Fu_0)(\xi) \Big\|_{L_2(\mathbb{R}^n)} \\ &\leq C \Big\| \prod_{j=1}^{J} \Lambda_j^{s_j}(\xi,\lambda)(Fu_0)(\xi) \Big\|_{L_2(\mathbb{R}^n)} = C\|u_0\|_{\mathbf{s},\mathbb{R}^n} \leq C\|u\|_{\mathbf{s},\mathbb{R}^n_+}\,.\end{aligned}$$

Now let us consider the boundary operators. By homogeneity, we have

$$|B_j(\xi)| \leq C|\xi|^{m_j} \leq C\ (\lambda^{1/r_\ell} + |\xi|)^{m_j}$$

for every $\ell = 1, \ldots, J$. Therefore, using the definition of the shifted weight function, we obtain

$$|B_j(\xi)|\ \Psi_{\mathbf{s}}^{(-m_j)}(\xi,\lambda) \leq C\ \Psi_{\mathbf{s}}(\xi,\lambda)\,.$$

Now we apply Lemma 3.5 and get $\|B_j u\|^{(-m_j-1/2)}_{\mathbf{s},\mathbb{R}^{n-1}} \le C\|u\|_{\mathbf{s},\mathbb{R}^n_+}$, which finishes the proof of the theorem. □

Now we come to the main result of the present paper.

Theorem 3.7. *Let* $(P, B_1, \dots, B_M)$ *be N-elliptic with parameter in* $[0,\infty)$. *Let* $\mathbf{s} \in \mathbb{R}^J$ *be a tuple of real numbers satisfying*

$$\begin{aligned} &s_1 + \cdots + s_k \in [m_{M_k} + 1/2, m_{M_k+1} + 1/2] \quad (k = 1, \dots, J-1)\,, \\ &s_1 + \cdots + s_J \in (m_M + 1/2, \infty)\,. \end{aligned} \tag{3.8}$$

Assume for simplicity that $s_1 + \cdots + s_J$ *is an integer, and set* $t_j := s_j - 2N_j$. *Then there exists a* $\lambda_0 > 0$ *such that for every* $\lambda \ge \lambda_0$ *the operator* (3.7) *is invertible with bounded inverse in the sense that for every*

$$(f, g_1, \dots, g_M) \in H_{\mathbf{t}}(\mathbb{R}^n_+) \times \prod_{j=1}^{M} H_{\mathbf{s}}^{(-m_j-1/2)}(\mathbb{R}^{n-1})$$

there exists a unique solution $u \in H_{\mathbf{s}}(\mathbb{R}^n_+)$ *of the boundary value problem* $Pu = f$, $B_j u = g_j$ $(j = 1, \dots, M)$, *and the a priori estimate*

$$\|u\|_{\mathbf{s},\mathbb{R}^n_+} \le C \left(\|f\|_{\mathbf{t},\mathbb{R}^n_+} + \sum_{j=1}^{M} \|g_j\|^{(-m_j-1/2)}_{\mathbf{s},\mathbb{R}^{n-1}} \right) \tag{3.9}$$

holds with a constant $C = C(\lambda_0)$ *independent of* u *or* λ.

Note that the a priori estimate is two-sided due to Theorem 3.6. The main step in the proof of Theorem 3.7 is to show the following estimate.

Theorem 3.8. *Let* $(P, B_1, \dots, B_M)$ *be N-elliptic with parameter in* $[0,\infty)$. *Let* $\mathbf{s} \in \mathbb{R}^J$ *be a tuple of real numbers satisfying* (3.8). *Then there exists a* $\lambda_0 > 0$ *such that for all* $\lambda \ge \lambda_0$ *and all* $\xi' \in \mathbb{R}^{n-1}$ *the ordinary differential equation*

$$P(\xi', D_t, \lambda) w(t) = 0 \quad (t > 0), \tag{3.10}$$

$$B_j(\xi', D_t) w(0) = h_j \quad (j = 1, \dots, M) \tag{3.11}$$

$$w(t) \to 0 \quad (t \to \infty)$$

is uniquely solvable for every $(h_1, \dots, h_M) \in \mathbb{C}^M$, *and for its solution* $w = w(t, \xi', \lambda)$ *the estimate*

$$\Psi_{\mathbf{s}}^{(-\ell)}(\xi', \lambda)\, \|D_t^\ell w(\cdot, \xi', \lambda)\|_{L_2(\mathbb{R}_+)} \le C \sum_{j=1}^{M} \Psi_{\mathbf{s}}^{(-m_j-1/2)}(\xi', \lambda)\, |h_j|$$

holds for $\ell = 0, 1, \dots$ *with a constant* $C = C(\lambda_0)$ *independent of* ξ' *and* λ.

Sections 4 and 5 are devoted to the proof of this theorem (see Subsection 5.3). Here we derive Theorem 3.7 from Theorem 3.8.

Proof of Theorem 3.7. As in the proof of Theorem 3.6, we fix a continuous extension operator $E_0\colon H_{\mathbf{t}}(\mathbb{R}^n_+) \to H_{\mathbf{t}}(\mathbb{R}^n)$. We are looking for a solution u of the form $u = u_1 + u_2$ with

$$u_1 := R_0 F^{-1}\, \frac{(FE_0 f)(\xi)}{P(\xi,\lambda)}\,,$$

where R_0 again stands for the operator of restriction to $\mathbb{R}^n_+$. With the same steps as in the proof of Theorem 3.6, replacing $P(\xi,\lambda)$ by $1/P(\xi,\lambda)$ and using N-ellipticity for P, we easily see that

$$\|u_1\|_{\mathbf{s},\mathbb{R}^n_+} \le C\, \|f\|_{\mathbf{t},\mathbb{R}^n_+}\,. \tag{3.12}$$

Taking partial Fourier transform F' with respect to $(x_1,\dots,x_{n-1})$, we obtain the ordinary differential equation (3.10)-(3.11) for $w(t,\xi',\lambda) := (F'u_2)(t,\xi',\lambda)$ with $h_j = h_j(\xi',\lambda) := (F'g_j)(\xi') - (F'B_ju_1)(\xi',\lambda)$. Due to Lemma 3.5, Theorem 3.6 and (3.12), we have

$$\|h_j\|^{(-m_j-1/2)}_{\mathbf{s},\mathbb{R}^{n-1}} \le C\big(\|u_1\|_{\mathbf{s},\mathbb{R}^n_+} + \|g_j\|^{(-m_j-1/2)}_{\mathbf{s},\mathbb{R}^{n-1}}\big) \le C\big(\|f\|_{\mathbf{t},\mathbb{R}^n_+} + \|g_j\|^{(-m_j-1/2)}_{\mathbf{s},\mathbb{R}^{n-1}}\big).$$

Now we apply Theorem 3.8 to obtain that for sufficiently large λ the problem (3.10)–(3.11) has a unique solution $w(t,\xi',\lambda)$, and we can define $u_2 := (F')^{-1}w$. Using the equivalent norm (3.6) and the estimate of Theorem 3.8, we get

$$\begin{aligned}
\|u_2\|_{\mathbf{s},\mathbb{R}^n_+} &\le C\Big(\sum_{\ell=0}^{s_1+\dots+s_J} \int_{\mathbb{R}^{n-1}} \big[\Psi_{\mathbf{s}}^{(-\ell)}(\xi',\lambda)\|D_t^\ell w(\cdot,\xi,\lambda)\|_{L_2(\mathbb{R}_+)}\big]^2\, \mathrm{d}\,\xi'\Big)^{1/2}\\
&\le C\Big(\sum_{j=1}^{M} \int_{\mathbb{R}^{n-1}} \big[\Psi_{\mathbf{s}}^{(-m_j-1/2)}(\xi',\lambda)|h_j(\xi',\lambda)|\big]^2\, \mathrm{d}\,\xi'\Big)^{1/2}\\
&\le C\sum_{j=1}^{M} \|h_j\|^{(-m_j-1/2)}_{\mathbf{s},\mathbb{R}^{n-1}} \le C\,\Big(\|f\|_{\mathbf{t},\mathbb{R}^n_+} + \sum_{j=1}^{M}\|g_j\|^{(-m_j-1/2)}_{\mathbf{s},\mathbb{R}^{n-1}}\Big)\,.
\end{aligned}$$

From this and (3.12) we obtain the solvability of the boundary value problem for $\lambda \ge \lambda_0$ in the spaces indicated in the theorem and the a priori estimate (3.9). For an arbitrary solution u of the boundary value problem $Pu = f$, $B_ju = g_j$ the function $F'(u-u_1)$ satisfies the ordinary differential equation (3.10)–(3.11), and thus the a priori estimate holds for u, too, which also shows the uniqueness of the solution. □

Example 3.9. Let $P(\xi,\lambda)$ be N-elliptic with parameter in the sense of Definition 2.2, and consider the Dirichlet boundary value problem connected with $P(D,\lambda)$. In this case we have $m_j = j-1$ for $j=1,\dots,M$, and condition (3.8) is satisfied for

$$s_1 := M_1, \quad s_j := M_j - M_{j-1} = N_j \quad (j=2,\dots,J)\,. \tag{3.13}$$

As the Dirichlet boundary value problem corresponding to P is N-elliptic (see Example 3.4), we may apply Theorem 3.7 to the canonical choice (3.13) of s_j. For $J=2$, we obtain

$$\begin{aligned}
&\|u\|_{(M_1,M_2-M_1),\mathbb{R}^n_+}\\
&\le C\Big(\|f\|_{(-M_1,-M_2+M_1),\mathbb{R}^n_+} + \sum_{j=1}^{M_1}\|g_j\|_{(M_1-j+1/2,M_2-M_1),\mathbb{R}^{n-1}}\\
&+\sum_{j=M_1+1}^{M_2}\|g_j\|_{(0,M_2-j+1/2),\mathbb{R}^{n-1}}\Big)\,,
\end{aligned}$$

where we used the definition of the shifted weight function; see (3.5). This a priori estimate has some similarity with the estimate obtained in [6] for the Dirichlet boundary value problem connected with weakly parameter-elliptic operator pencils. Contrary to the estimate in [6] the a priori estimate above does not contain an additional L_2-term on the right-hand side and guarantees uniqueness of the solution.

4. The zeros of the symbol

The first important step in proving the ODE estimate of Theorem 3.8 is to study the zeros of the algebraic equation (in τ)

$$P(\xi',\tau,\lambda) = 0, \tag{4.1}$$

assuming for the remainder of this section that P is N-elliptic with parameter in $[0,\infty)$ and that the Vishik–Lyusternik condition holds. As this polynomial is not quasi-homogeneous with respect to ξ and λ, two main questions arise:

a) What is the behaviour of the moduli of the zeros of (4.1) for $|\xi'|+\lambda\to\infty$?

b) Are the zeros of (4.1) close (in some appropriate sense) to the zeros of a quasi-homogeneous polynomial?

These questions will be answered in the present section (see Theorem 4.4 and Corollary 4.5), where the answer to question b) of course will imply the answer to a). It will turn out that the answers depend on the relation between $|\xi'|$ and λ as $|\xi'|+\lambda\to\infty$.

Throughout this section, we will constantly use the following elementary fact on the zeros of polynomials (cf., e.g., [10], pp. 105 ff.):

Lemma 4.1. *Let $P_0(\tau)=\sum_{j=0}^k c_j^0\tau^j$ be a complex polynomial of order $\ell\le k$ (note that $c_k^0=0$ is possible). Then for every $\delta>0$ there exists an $r>0$ such that for every polynomial $P(\tau)=\sum_{j=0}^k c_j\tau^j$ with*

$$\max_{j=1,\dots,k}|c_j-c_j^0|<r$$

and for every root τ_j^0 of P_0 there exists a root τ_j of P with $|\tau_j-\tau_j^0|<\delta$.

Now let us come back to the polynomial (4.1). Due to Lemma 2.3, we have $P(\xi',\tau,0)=\pi_{J+1}(\xi',\tau)\neq 0$ for all $(\xi',\tau)\neq 0$, and thus the leading coefficient of $P(\xi',\cdot,\lambda)$ (which is a polynomial of order $2M$) is a nonvanishing constant. Therefore we can choose $2M$ branches of roots depending continuously on (ξ',λ). Due to Definition 2.2 (ii), there exists a $\lambda_0>0$ such that $P(\xi',\cdot,\lambda)$ has no real roots for $\lambda\ge\lambda_0$.

Lemma 4.2. *For large-enough λ the polynomial $P(\xi',\cdot,\lambda)$ has exactly M roots in $\mathbb{C}_+$.*

Proof. For $\lambda\ge\lambda_0$ the number of roots of $P(\xi',\cdot,\lambda)$ does not depend on (ξ',λ). As we have

$$\frac{P(\xi',\tau,\lambda)}{|\xi'|^{2M}}=\pi_{J+1}\Big(\frac{\xi'}{|\xi'|},\frac{\tau}{|\xi'|}\Big)+\sum_{\substack{\alpha,k\\|\alpha|<2M}}a_{\alpha k}\frac{\lambda^k}{|\xi'|^{2M-|\alpha|}}\Big(\frac{\xi'}{|\xi'|}\Big)^{\alpha'}\Big(\frac{\tau}{|\xi'|}\Big)^{\alpha_n},$$

for $|\xi'|\gg|\lambda|$ the polynomial $|\xi'|^{-2M}P(\xi',\cdot,\lambda)$ is a small perturbation in the sense of Lemma 4.1 of $\pi_{J+1}(\xi'/|\xi'|,\tau/|\xi'|)$. From the ellipticity of π_{J+1} our statement follows. □

Note that due to the proof of the previous lemma, for $|\xi'|\gg\lambda$ the roots of $P(\xi',\cdot,\lambda)$ are close to the roots of

$$\pi_{J+1}\Big(\frac{\xi'}{|\xi'|},\frac{\tau}{|\xi'|}\Big)=|\xi'|^{-2M}\pi_{J+1}(\xi',\tau),$$

which is an elliptic homogeneous polynomial, and therefore they are of order $O(|\xi'|)$, and their (positive) imaginary part can be estimated from below by a constant times $|\xi'|$. This already gives us an answer to the questions of the beginning of this section for the case that $|\xi'|\gg\lambda$.

Now we want to describe the behaviour of the zeros of (4.1) for all (ξ', λ) belonging to $G := G_\rho := \mathbb{R}^{n-1} \times [\rho, \infty)$, where ρ is sufficiently large. As was already mentioned, this behaviour depends on the relation between $|\xi'|$ and λ. Therefore, we use a finite partition of G which describes this relation and which is directly connected with the Newton polygon. We fix $\varepsilon > 0$ and write

$$G = \bigcup_{j=1}^{J+1} G(\Gamma_j) \cup \bigcup_{j=1}^{J} G(\Gamma_j \Gamma_{j+1}) \tag{4.2}$$

where $G(\Gamma_j) = G_{\varepsilon,\rho}(\Gamma_j)$ and $G(\Gamma_j\Gamma_{j+1}) = G_{\varepsilon,\rho}(\Gamma_j\Gamma_{j+1})$ are defined by

$$\begin{aligned} G(\Gamma_1) &:= \{(\xi', \lambda) \in G : \varepsilon^{-1}|\xi'|^{r_1} < \lambda\}, \\ G(\Gamma_j) &:= \{(\xi', \lambda) \in G : \varepsilon^{-1}|\xi'|^{r_j} < \lambda < \varepsilon|\xi'|^{r_{j-1}}\} \quad (j = 2, \dots, J), \\ G(\Gamma_{J+1}) &:= \{(\xi', \lambda) \in G : \lambda < \varepsilon|\xi'|^{r_J}\}, \\ G(\Gamma_j\Gamma_{j+1}) &:= \{(\xi', \lambda) \in G : \varepsilon|\xi'|^{r_j} \le \lambda \le \varepsilon^{-1}|\xi'|^{r_j}\} \quad (j = 1, \dots, J). \end{aligned}$$

Remark 4.3. a) This covering was introduced in [9], Chapter 4, Section 2. Note that the domains $G(\Gamma_j)$ are nonempty and that (4.2) defines a partition of G by disjoint sets provided that

$$\rho > \varepsilon^{(r_j + r_{j+1})/(r_{j+1} - r_j)} \quad (j = 1, \dots, J)\,. \tag{4.3}$$

In the following, we will consider only ε and ρ satisfying (4.3). Without loss of generality, we may also assume that $\varepsilon < 1$ and $\rho > 1$.

b) In the domain $G(\Gamma_j\Gamma_{j+1})$ we have, by definition, $|\xi'| \approx \lambda^{1/r_j}$. The regions $G(\Gamma_j)$ related to the vertexes Γ_j are in some sense intermediate cases.

It was shown in [9] that for $(\xi', \lambda) \in G(\Gamma_j\Gamma_{j+1})$ an estimate of the form

$$|P(\xi, \lambda) - P_{\Gamma_j\Gamma_{j+1}}(\xi, \lambda)| \le C\varepsilon |P_{\Gamma_j\Gamma_{j+1}}(\xi, \lambda)|$$

holds. In other words, $P_{\Gamma_j\Gamma_{j+1}}$ is the principal part of P in the domain $G(\Gamma_j\Gamma_{j+1})$. A similar estimate holds for $(\xi', \lambda) \in G(\Gamma_j)$.

We shall answer the questions a) and b) from the beginning of this section for each domain $G(\Gamma_j\Gamma_{j+1})$ and $G(\Gamma_j)$ separately. As in the case of weakly parameter-elliptic symbols (see [6]) the roots will be split into several groups. The "main" group will be determined by the principal part $P_{\Gamma_j\Gamma_{j+1}}$ or P_{Γ_j}, i.e., by the zeros of the polynomials P_j and π_j. The "additional" groups will be determined by the polynomials Q_ℓ for $\ell \ge j + 1$.

It also should be mentioned that in the case $n = 1$ our results about the roots can be deduced from the representation of the roots in the form of

Puiseux series. The covering we use, in some sense, is the replacement of these series for $n > 1$.

Denote by $\tau_1^0(\xi',\lambda),\dots,\tau_{M_j}^0(\xi',\lambda)$ the zeros of $P_j(\xi',\cdot,\lambda)$ in $\mathbb{C}_+$. Note that $\tau_k^0(\xi',0)$ are the zeros of $\pi_{j+1}(\xi',\cdot)$ in $\mathbb{C}_+$.

We denote by $\tau_{M_{\ell-1}+1}^1(\lambda),\dots,\tau_{M_\ell}^1(\lambda)$ the zeros of $Q_\ell(\cdot,\lambda)$ in $\mathbb{C}_+$.

Theorem 4.4. *Suppose that $P(\xi,\lambda)$ is N-elliptic with parameter in $[0,\infty)$ and that the Vishik–Lyusternik condition (Definition 2.9) holds. Then for every $\delta > 0$ there exists an $\varepsilon_0 = \varepsilon_0(\delta) > 0$ and a $\rho_0 = \rho_0(\delta,\varepsilon_0) > 0$ such that for all $(\xi',\lambda) \in G_{\rho_0}$ the following statements hold:*

a) *Let $j \in \{1,\dots,J\}$ and $(\xi',\lambda) \in G_{\varepsilon_0,\rho_0}(\Gamma_j\Gamma_{j+1})$. Then for a suitable numbering of the roots $\tau_k(\xi',\lambda)$ of the polynomial $P(\xi',\cdot,\lambda)$ we have*

$$|\tau_k(\xi',\lambda) - \tau_k^0(\xi',\lambda)| \le \delta\Lambda_j(\xi',\lambda) \ (k = 1,\dots,M_j), \tag{4.4}$$

$$|\tau_k(\xi',\lambda) - \tau_k^1(\lambda)| \le \delta\lambda^{1/r_\ell} \ (k = M_{\ell-1}+1,\dots,M_\ell;\ \ell = j+1,\dots,J). \tag{4.5}$$

b) *Let $(\xi',\lambda) \in G_{\varepsilon_0,\rho_0}(\Gamma_{j+1})$ for some $j \in \{0,\dots,J\}$. Then the statement in* a) *holds if in* (4.4) *$\tau_k^0(\xi',\lambda)$ is replaced by $\tau_k^0(\xi',0)$ and $\Lambda_j = \Lambda_j(\xi',\lambda)$ is replaced by $|\xi'| = \Lambda_j(\xi',0)$.*

Note in part b) that for $j = 0$ the first group of zeros does not appear due to the definition $M_0 = 0$.

Corollary 4.5. a) *Let $(\xi',\lambda) \in G(\Gamma_j\Gamma_{j+1})$ for some $j = 1,\dots,J$. Then, with the numbering of Theorem* 4.4*, we have*

$$|\tau_k(\xi',\lambda)| \approx \Lambda_j(\xi',\lambda) \quad (k = 1,\dots,M_j),$$
$$|\tau_k(\xi',\lambda)| \approx \Lambda_\ell(\xi',\lambda) \quad (k = M_{\ell-1}+1,\dots,M_\ell;\ \ell = j+1,\dots,J)$$

and

$$|\operatorname{Im}\tau_k(\xi',\lambda)| \ge C\Lambda_j(\xi',\lambda) \quad (k = 1,\dots,M_j),$$
$$|\operatorname{Im}\tau_k(\xi',\lambda)| \ge C\Lambda_l(\xi',\lambda) \quad (k = M_{\ell-1}+1,\dots,M_\ell;\ \ell = j+1,\dots,J).$$

b) *Let $(\xi',\lambda) \in G(\Gamma_{j+1})$ for some $j \in \{0,\dots,J\}$. Then the statement in* a) *holds if $\Lambda_j(\xi',\lambda)$ is replaced by $\Lambda_j(\xi',0) = |\xi'|$.*

Proof of Theorem 4.4. The idea of the proof is to show in each case that the polynomial $P(\xi',\cdot,\lambda)$ is, after division by a suitable factor, a small perturbation (in the sense of Lemma 4.1) of one of the corresponding polynomials P_j and Q_j.

We start with showing that there exists an $\varepsilon_0 > 0$ such that for all $\rho \ge 1$ satisfying (4.3) and all $(\xi',\lambda) \in \bigcup_{j=0}^J G_{\varepsilon_0,\rho}(\Gamma_{j+1})$ the stated inequalities hold.

So we assume that $(\xi', \lambda) \in G_{\varepsilon,\rho}(\Gamma_{j+1})$ for some $j \in \{0, \dots, J\}$ and some fixed ε and ρ.

We start with the construction of the first group $\tau_1, \dots, \tau_{M_j}$ of zeros (which appears only for $j \geq 1$). For this, we write $P = P_{\Gamma_{j+1}} + (P - P_{\Gamma_{j+1}})$ and obtain

$$\frac{P(\xi', \tau, \lambda)}{|\xi'|^{p_{j+1}} \lambda^{q_{j+1}}} = P_j\Big(\frac{\xi'}{|\xi'|}, \frac{\tau}{|\xi'|}, 0\Big) + \frac{(P - P_{\Gamma_{j+1}})(\xi', \tau, \lambda)}{|\xi'|^{p_{j+1}} \lambda^{q_{j+1}}}\,, \tag{4.6}$$

noting that $P_j(\xi', \tau, 0) = \lambda^{-q_{j+1}} P_{\Gamma_{j+1}}(\xi', \tau, \lambda)$ and using the homogeneity of P_j. We want to estimate the coefficients of the last term in (4.6), considered as a polynomial in $\tau/|\xi'|$. For this we write

$$\begin{aligned} \frac{(P - P_{\Gamma_{j+1}})(\xi', \tau, \lambda)}{|\xi'|^{p_{j+1}} \lambda^{q_{j+1}}} &= \sum_{(|\alpha|, k) \in N(P) \setminus \Gamma_{j+1}} a_{\alpha k} \frac{(\xi')^{\alpha'} \tau^{\alpha_n} \lambda^k}{|\xi'|^{p_{j+1}} \lambda^{q_{j+1}}} \\ &= \sum_{(|\alpha|, k) \in N(P) \setminus \Gamma_{j+1}} a_{\alpha k} |\xi'|^{|\alpha| - p_{j+1}} \lambda^{k - q_{j+1}} \Big(\frac{\xi'}{|\xi'|}\Big)^{\alpha'} \Big(\frac{\tau}{|\xi'|}\Big)^{\alpha_n}. \end{aligned} \tag{4.7}$$

Now we want to show that (for $j \geq 1$)

$$|\xi'|^{i - p_{j+1}} \lambda^{k - q_{j+1}} \leq \varepsilon + \varepsilon^{1/r_j} \quad \text{for } (i, k) \in N(P) \setminus \Gamma_{j+1}\,. \tag{4.8}$$

To prove (4.8), we use that for all $(i, k) \in N(P)$ the inequalities

$$i + r_j k \leq p_{j+1} + r_j q_{j+1}\,, \tag{4.9}$$

$$i + r_{j+1} k \leq p_{j+1} + r_{j+1} q_{j+1} \tag{4.10}$$

hold. We also note that, by definition of $G(\Gamma_{j+1})$ and due to the inequalities $\lambda \geq \rho > 1$ and $\varepsilon < 1$, we have

$$|\xi'| \geq \Big(\frac{\lambda}{\varepsilon}\Big)^{1/r_j} \geq \varepsilon^{-1/r_j} > 1\,. \tag{4.11}$$

First let $(i, k) \in N(P)$ with $k > q_{j+1}$. Then we use (4.9) and get

$$|\xi'|^i \lambda^k \leq |\xi'|^{p_{j+1}} \lambda^{q_{j+1}} \Big(\frac{\lambda}{|\xi'|^{r_j}}\Big)^{k - q_{j+1}} \leq \varepsilon^{k - q_{j+1}} |\xi'|^{p_{j+1}} \lambda^{q_{j+1}} \leq \varepsilon |\xi'|^{p_{j+1}} \lambda^{q_{j+1}}\,.$$

Similarly, for $(i, k) \in N(P)$ with $k < q_{j+1}$ we use (4.10) to obtain

$$|\xi'|^i \lambda^k \leq |\xi'|^{p_{j+1}} \lambda^{q_{j+1}} \Big(\frac{|\xi'|^{r_{j+1}}}{\lambda}\Big)^{q_{j+1} - k} \leq \varepsilon |\xi'|^{p_{j+1}} \lambda^{q_{j+1}}\,.$$

Finally, let $(i, k) \in N(P)$ with $k = q_{j+1}$ and $i < p_{j+1}$. Using (4.11), we see

$$|\xi'|^i \lambda^k = |\xi'|^{p_{j+1}} \lambda^{q_{j+1}} |\xi'|^{i - p_{j+1}} \leq |\xi'|^{p_{j+1}} \lambda^{q_{j+1}} |\xi'|^{-1} \leq \varepsilon^{1/r_j} |\xi'|^{p_{j+1}} \lambda^{q_{j+1}}\,.$$

So inequality (4.8) is shown for all $(i,k)\in N(P)\setminus\Gamma_{j+1}$.

From (4.8) we see that the coefficients of the right-hand side of (4.7), considered as a polynomial in $\tau/|\xi'|$, tend to zero for $\varepsilon\to 0$. Therefore, the left-hand side of (4.6) is a small perturbation of $P_j(\xi'/|\xi'|,\tau/|\xi'|,0)$ in the sense of Lemma 4.1. From this lemma we see that for every $\delta>0$ there exists an $\varepsilon_0>0$ such that for all $(\xi',\lambda)\in G_{\varepsilon_0,\rho}(\Gamma_{j+1})$ there exist zeros $\tau_1(\xi',\lambda),\dots,\tau_{M_j}(\xi',\lambda)$ of $P(\xi',\cdot,\lambda)$ satisfying

$$|\tau_k(\xi',\lambda)-\tau_k^0(\xi',0)|\le\delta|\xi'|\quad(k=1,\dots,M_j)\,.$$

So we have constructed the first M_j roots of $P(\xi',\cdot,\lambda)$.

Now we fix $\ell\in\{j+1,\dots,J\}$ and set $\Lambda:=\lambda^{1/r_l}$ and $z:=\tau/\Lambda$. In the equality

$$\begin{aligned}P(\xi',\tau,\lambda)=P_{\Gamma_\ell\Gamma_{\ell+1}}(0,\tau,\lambda)&+P_{\Gamma_\ell\Gamma_{\ell+1}}(\xi',\tau,\lambda)-P_{\Gamma_\ell\Gamma_{\ell+1}}(0,\tau,\lambda)\\&+P(\xi',\tau,\lambda)-P_{\Gamma_\ell\Gamma_{\ell+1}}(\xi',\tau,\lambda)\end{aligned}$$

we divide both sides by Λ^{d_ℓ} with $d_\ell:=p_\ell+q_\ell r_\ell(=p_{\ell+1}+q_{\ell+1}r_\ell)$. Using $P_\ell=\lambda^{-q_{\ell+1}}P_{\Gamma_\ell\Gamma_{\ell+1}}$ and $P_\ell(\xi',\tau,\lambda)=\Lambda^{p_{\ell+1}}P_\ell(\xi'/\lambda,z,1)$, we obtain

$$\begin{aligned}\Lambda^{-d_\ell}P(\xi',\tau,\lambda)=P_\ell(0,z,1)&+\Big[P_\ell\Big(\frac{\xi'}{\Lambda},z,1\Big)-P_\ell(0,z,1)\Big]\\&+\Lambda^{-d_\ell}\Big[P(\xi',\tau,\lambda)-P_{\Gamma_\ell\Gamma_{\ell+1}}(\xi',\tau,\lambda)\Big].\end{aligned}\tag{4.12}$$

To estimate the second term of the sum on the right-hand side, we expand P_ℓ in a Taylor series with respect to ξ'. We obtain

$$P_\ell\Big(\frac{\xi'}{\Lambda},z,1\Big)-P_\ell\Big(0,z,1\Big)=\sum_{|\beta'|\ge1}\frac{1}{(\beta')!}\Big(\frac{\xi'}{\Lambda}\Big)^{\beta'}(\partial_{\xi'}^{\beta'}P_\ell)(0,z,1).\tag{4.13}$$

As $r_j>r_\ell$ for $\ell>j$, we may estimate ξ'/Λ by

$$\frac{|\xi'|}{\lambda^{1/r_\ell}}\le\frac{|\xi'|}{\lambda^{1/r_{j+1}}}\le\varepsilon^{1/r_{j+1}}\,,\tag{4.14}$$

where we used the definition of $G(\Gamma_{j+1})$. Therefore the right-hand side of (4.13) is a polynomial in $z=\tau/\Lambda$ whose coefficients tend to zero for $\varepsilon\to0$.

To estimate the last term on the right-hand side of (4.12), we write

$$\Lambda^{-d_\ell}a_{\alpha k}(\xi')^{\alpha'}\tau^{\alpha_n}\lambda^k=\Lambda^{-d_\ell+|\alpha|+kr_\ell}a_{\alpha k}\Big(\frac{\xi'}{\Lambda}\Big)^{\alpha'}\Big(\frac{\tau}{\Lambda}\Big)^{\alpha_n}.$$

For $(|\alpha|,k)\in N(P)\setminus\Gamma_\ell\Gamma_{\ell+1}$ we have $|\alpha|+kr_\ell<d_\ell$. According to (4.14) and using $1/\Lambda\le|\xi'|/\Lambda$, we see that the last term in (4.12) is a polynomial in $z=\tau/\Lambda$ whose coefficients tend to zero for $\varepsilon\to0$. Now we

can apply Lemma 4.1 to the right-hand side of (4.12) and the polynomial $P_\ell(0,\tau/\Lambda,1)=\pi_\ell(0,\tau/\Lambda)Q_\ell(\tau/\Lambda,1)$. We obtain that there exists an $\varepsilon_0>0$ such that for all $(\xi',\lambda)\in G_{\varepsilon_0,\rho}(\Gamma_{j+1})$ and for every root $\tau_k^1(1)$ of $Q_\ell(\cdot,1)$ there exists a root $\tau_k(\xi',\lambda)/\Lambda$ of the right-hand side of (4.12), considered as a polynomial in τ/Λ, with

$$|\Lambda^{-1}\tau_k(\xi',\lambda)-\tau_k^1(1)|\le\delta,$$

and therefore, using the homogeneity of Q_ℓ,

$$|\tau_k(\xi',\lambda)-\tau_k^1(\lambda)|\le\delta\lambda^{1/r_\ell}.$$

This finishes the proof of the stated inequalities if $(\xi',\lambda)\in G_{\varepsilon_0,\rho}(\Gamma_{j+1})$ for some j .

In the second part of the proof we show that there exists a $\rho_0>0$ such that for all $(\xi',\lambda)\in\bigcup_{j=1}^J G_{\varepsilon_0,\rho_0}(\Gamma_j\Gamma_{j+1})$ the inequalities of the theorem hold, with ε_0 being given in part a) of the proof. Again we start with the construction of the first group $\tau_1,\dots,\tau_{M_j}$ of zeros. We write (4.1) in the form

$$P_{\Gamma_j\Gamma_{j+1}}+(P-P_{\Gamma_j\Gamma_{j+1}})=0 \tag{4.15}$$

and make the transformations $\xi'=\Lambda_j\omega'$, $\lambda=\Lambda_j^{r_j}\nu$, $\tau=\Lambda_j z$. After division by $\Lambda_j^{p_j+r_jq_j}$, using the homogeneity of $P_{\Gamma_j\Gamma_{j+1}}$, equation (4.15) can be rewritten in the form

$$P_{\Gamma_j\Gamma_{j+1}}(\omega',z,\nu)+\sum_{(|\alpha|,k)\in N(P)\setminus\Gamma_j\Gamma_{j+1}}\Lambda_j^{|\alpha|+r_jk-p_j-r_jq_j}a_{\alpha k}(\omega')^{\alpha'}\nu^k z^{\alpha_n}=0\,. \tag{4.16}$$

As the integer tuples on the edge $\Gamma_j\Gamma_{j+1}$ are exactly the pairs (i,k) for which $i+r_jk$ is maximal, there exists a constant $\kappa>0$ such that for all integer tuples $(i,k)\in N(P)\setminus\Gamma_j\Gamma_{j+1}$, we have, $i+r_jk\le p_j+r_jq_j-\kappa$. Therefore, the sum in (4.16) is a polynomial in z whose coefficients can be estimated by a constant times

$$\Lambda_j^{-\kappa}=(|\xi'|+\lambda^{1/r_j})^{-\kappa}\le\lambda^{-\kappa/r_j}$$

which tends to zero for $\lambda\to\infty$. So we see that the left-hand side of (4.16), considered as a polynomial in z, is a small perturbation (in the sense of Lemma 4.1) of the polynomial $P_{\Gamma_j\Gamma_{j+1}}(\omega',z,\nu)=\nu^{q_{j+1}}P_j(\omega',z,\nu)$. From Lemma 4.1 we obtain that there exists a $\rho_0>0$ such that for all $(\xi',\lambda)\in G_{\varepsilon_0,\rho_0}(\Gamma_j\Gamma_{j+1})$ and for every root $\tau_k^0(\omega',\nu)$ of $P_j(\omega',\cdot,\nu)$ there exists a root $z_k=\tau_k(\xi',\lambda)/\Lambda_j$ of the left-hand side of (4.16) with

$$|\Lambda_j^{-1}\tau_k(\xi',\lambda)-\tau_k^0(\Lambda_j^{-1}\xi',\Lambda_j^{-r_j}\lambda)|\le\delta.$$

Now it remains to notice that, by homogeneity,

$$\Lambda_j \tau_k^0(\Lambda_j^{-1}\xi', \Lambda_j^{-r_j}\lambda) = \tau_k^0(\xi', \lambda),$$

and therefore

$$|\tau_k(\xi', \lambda) - \tau_k^0(\xi', \lambda)| \le \delta\Lambda_j \quad (k = 1, \dots, M_j).$$

This finishes the construction of the first group of roots.

The construction of the roots $\tau_k(\xi', \lambda)$ with $k \ge M_j + 1$ can be made in exactly the same way as in the proof of part a), only replacing (4.14) by

$$\frac{|\xi'|}{\Lambda} = \frac{|\xi'|}{\lambda^{1/r_j}}\, \lambda^{1/r_j - 1/r_\ell} < \varepsilon_0^{-1/r_j} \lambda^{1/r_j - 1/r_\ell}. \tag{4.17}$$

As $r_j > r_\ell$ for $j < \ell$, the right-hand side of (4.17) can be made arbitrarily small for fixed ε_0 and $\lambda \ge \rho_0$ with sufficiently large ρ_0. This finishes the proof of (4.4)–(4.5) if $(\xi', \lambda) \in G_{\varepsilon_0,\rho_0}(\Gamma_j\Gamma_{j+1})$ for some j and thus the proof of the theorem. □

5. Estimates for the ODE problem

5.1. **The basic solutions.** Throughout this section, we will assume that $(P, B_1, \dots, B_M)$ is N-elliptic with parameter in $[0, \infty)$. Section 4 describes the behaviour of the zeros of $P(\xi', \cdot, \lambda)$ for large λ. As this behaviour depends on the subdomain $G(\Gamma_j)$ or $G(\Gamma_j\Gamma_{j+1})$ to which (ξ', λ) belongs, let us consider each subdomain separately. So we will consider fixed ε_0 and ρ_0 given in Theorem 4.4 and a fixed index $j \in \{1, \dots, J\}$ and assume throughout this subsection that (ξ', λ) belongs to $G_{\varepsilon_0,\rho_0}(\Gamma_j\Gamma_{j+1})$. We will indicate the necessary changes for (ξ', λ) belonging to one of the subdomains $G(\Gamma_1), \dots, G(\Gamma_{J+1})$ at the end of this subsection. (To be precise, the notions introduced below additionally depend on the index j; as we consider this index as fixed, we will omit this dependence in our notation.)

In the following we will define for a polynomial $P(\tau)$ the polynomial $P_+(\tau) := \prod_{j=1}^{\ell}(\tau - \tau_j)$ where $\tau_1, \dots, \tau_\ell$ are the zeros of P with positive imaginary part. In Section 4 we have seen that the polynomial $P_+(\xi', \cdot, \lambda)$ can be factored as

$$P_+(\xi', \tau, \lambda) = \Big[\prod_{k=1}^{M_j}(\tau - \tau_k(\xi', \lambda))\Big] \cdot \prod_{\ell=j+1}^{J} \Big[\prod_{k=M_{\ell-1}+1}^{M_\ell}(\tau - \tau_k(\xi', \lambda))\Big]. \tag{5.1}$$

Here the first product is close to $P_{j,+}(\xi', \tau, \lambda)$ in the sense of Theorem 4.4, and the product corresponding to the index ℓ is close to $Q_{\ell,+}(\tau, \lambda)$. Here and below, we will always assume that the zeros of P are numbered in the sense of Theorem 4.4.

As a first step on the way to finding solutions of the ODE problem (3.10)–(3.11), we will consider the problems given by one of the operators appearing in the factorization (5.1) and some of the boundary operators. For instance, for $p \in \{1, \dots, M_j\}$ we will look for the solution W_p of

$$\prod_{k=1}^{M_j} (D_t - \tau_k(\xi', \lambda)) W_p(t) = 0 \quad (t > 0),$$
$$B_i(\xi', D_t) W_p(0) = \delta_{ip} \quad (i = 1, \dots, M_j)\,.$$

The unique solvability of this problem for sufficiently large λ will be a consequence of the fact that $\prod_{k=1}^{M_j} (\tau - \tau_k(\xi', \lambda))$ is close to P_j and of the condition on $(P_j, B_1, \dots, B_{M_j})$ appearing in the definition of N-ellipticity (Definition 3.2 (iii)). Similarly, we will find solutions W_p for $p \in \{M_{\ell-1}+1, \dots, M_\ell\}$ corresponding to the other factors in (5.1), now using the condition on Q_ℓ. We will call $W_1, \dots, W_M$ the basic solutions.

Of course, every basic solution W_p satisfies $P(\xi', D_t, \lambda) W_p(t) = 0$, i.e., equation (3.10), but it will not satisfy the boundary conditions (3.11). We will see in the subsequent subsection how to construct a solution of (3.10)–(3.11) in terms of the basic solutions.

The perturbation arguments below will be based on the following lemma.

Lemma 5.1. *Assume that* $(A^0(D_t), B_1^0(D_t), \dots, B_m^0(D_t))$ *is a boundary value problem in* $\mathbb{R}_+$ *satisfying the conditions of Lemma* 3.1. *Let* $\tau_1^0, \dots, \tau_m^0$ *be the zeros of* A^0 *in* $\mathbb{C}_+$, *and write* $B_j^0(\tau) = \sum_{\ell=0}^{m_j} b_{j\ell}^0 \tau^\ell$. *Fix a contour* $\gamma^0 \subset \mathbb{C}_+$ *enclosing* $\tau_1^0, \dots, \tau_m^0$ *and polynomials* $N_1^0, \dots, N_m^0$ *such that*

$$\frac{1}{2\pi i} \int_{\gamma^0} \frac{B_k^0(\tau) N_\ell^0(\tau)}{A_+^0(\tau)} \,\mathrm{d}\,\tau = \delta_{k\ell} \quad (k, \ell = 1, \dots, m)\,.$$

Then there exists a $\delta > 0$ *with the following property:*

Let $A_+(\tau) = \prod_{j=1}^m (\tau - \tau_j)$ *and* $B_j(\tau) = \sum_{\ell=0}^{m_j} b_{j\ell} \tau^\ell$ *be polynomials with*

$$|\tau_j - \tau_j^0| < \delta \quad (j = 1, \dots, m)$$
$$|b_{j\ell} - b_{j\ell}^0| < \delta \quad (j = 1, \dots, m\,;\ \ell = 1, \dots, m_j)\,.$$

Then γ_0 *encloses* $\tau_1, \dots, \tau_m$, *and there exist polynomials* $N_1, \dots, N_m$ *such that*

$$\frac{1}{2\pi i} \int_{\gamma^0} \frac{B_k(\tau) N_\ell(\tau)}{A_+(\tau)} \,\mathrm{d}\,\tau = \delta_{k\ell} \quad (k, \ell = 1, \dots, m)\,. \tag{5.2}$$

Moreover, if $|N_\ell^0(\tau)| < C_\ell$ on γ^0, then we may assume the same estimate for N_ℓ.

Proof. For sufficiently small $\delta > 0$, the contour γ^0 encloses $\tau_1, \dots, \tau_m$. The fact that there exist $N_1, \dots, N_m$ with (5.2) is equivalent to the invertibility of the Lopatinskii matrix $\operatorname{Lop}(A, B_1, \dots, B_m) := \big(\overline{b}_{ik}\big)_{i,k=1,\dots,m}$ (cf. Lemma 3.1). To show that for sufficiently small δ this matrix is invertible, it suffices to note that the entries of the Lopatinskii matrix depend continuously on the coefficients of B_j and A_+. The coefficients of the polynomials N_ℓ can be expressed explicitly in terms of the coefficients of the inverse of the Lopatinskii matrix (cf. [2]) and therefore depend continuously on the coefficients of this matrix. From this we see that for sufficiently small δ the polynomials $N_1, \dots, N_m$ with the stated property exist and that we have $N_\ell(\tau) \to N_\ell^0(\tau)$ for $\delta \to 0$. As γ^0 is compact, this convergence is uniform for $\tau \in \gamma^0$, which proves the last statement of the lemma. □

We will apply this result to the factors appearing in (5.1) as perturbations of P_j and Q_ℓ for $\ell = j+1, \dots, J$. For this we fix a contour γ_j^0 in $\mathbb{C}_+$ enclosing the zeros

$$\tau_1^0\Big(\frac{\xi'}{\Lambda_j}, \frac{\lambda}{\Lambda_j^{r_j}}\Big), \dots, \tau_{M_j}^0\Big(\frac{\xi'}{\Lambda_j}, \frac{\lambda}{\Lambda_j^{r_j}}\Big)$$

of $P_j(\Lambda_j^{-1}\xi', \cdot, \Lambda_j^{-r_j}\lambda)$. By homogeneity and compactness, we may assume that γ_j^0 is independent of ξ' and λ. Similarly, for $\ell \in \{j+1, \dots, J\}$ we fix a contour $\gamma_\ell^1 \subset \mathbb{C}_+$ enclosing the zeros $\tau^1_{M_{\ell-1}+1}(1), \dots, \tau^1_{M_\ell}(1)$ of $Q_j(\cdot, 1)$.

Proposition 5.2. *There exist ε_0 and ρ_0 such that for $(\xi', \lambda) \in G_{\varepsilon_0,\rho_0}(\Gamma_j\Gamma_{j+1})$ the following properties hold:*

a) *The contour γ_j^0 encloses $\Lambda_j^{-1}\tau_1(\xi', \lambda), \dots, \Lambda_j^{-1}\tau_{M_j}(\xi', \lambda)$, and there exist functions $N_1(\xi', \tau, \lambda), \dots, N_{M_j}(\xi', \tau, \lambda)$, depending polynomially on τ and being bounded by a constant independent of ξ' and λ for all $\tau \in \gamma_j^0$ such that*

$$\frac{1}{2\pi i} \int_{\gamma_j^0} \frac{B_k(\Lambda_j^{-1}\xi', \tau) N_i(\xi', \tau, \lambda)}{\prod_{p=1}^{M_j}(\tau - \Lambda_j^{-1}\tau_p(\xi', \lambda))} \, \mathrm{d}\,\tau = \delta_{ki} \quad (k, i = 1, \dots, M_j)\,.$$

b) *For $\ell = j+1, \dots, J$ the contour γ_ℓ^1 encloses $\lambda^{-1/r_\ell}\tau_k(\xi', \lambda)$ for $k = M_{\ell-1}+1, \dots, M_\ell$, and there exist $N_{M_{\ell-1}+1}(\xi', \tau, \lambda), \dots, N_{M_\ell}(\xi', \tau, \lambda)$, depending polynomially on τ and being bounded on γ_ℓ^1 by a constant independent of*

ξ' and λ such that

$$\frac{1}{2\pi i}\int\limits_{\gamma_\ell^1}\frac{B_k(\lambda^{-1/r_\ell}\xi',\tau)N_i(\xi',\tau,\lambda)}{\prod_{p=M_{\ell-1}+1}^{M_\ell}(\tau-\lambda^{-1/r_\ell}\tau_p(\xi',\lambda))}\,\mathrm{d}\,\tau=\delta_{ki}\quad(k,i=M_{\ell-1}+1,\ldots,M_\ell)\,.$$

Proof. a) First we remark that for $p=1,\ldots,M_j$

$$\tau_p^0\Big(\frac{\xi'}{\Lambda_j},\frac{\lambda}{\Lambda_j^{r_j}}\Big)=\frac{\tau_p^0(\xi',\lambda)}{\Lambda_j}$$

by homogeneity. Theorem 4.4 now tells us that we may apply Lemma 5.1 to the boundary value problem

$$P_j\Big(\frac{\xi'}{\Lambda_j},D_t,\frac{\lambda}{\Lambda_j^{r_j}}\Big),B_1\Big(\frac{\xi'}{\Lambda_j},D_t\Big),\ldots,B_{M_j}\Big(\frac{\xi'}{\Lambda_j},D_t\Big)$$

(which satisfies the conditions of Lemma 3.1 due to the definition of N-ellipticity) and the boundary value problem

$$\prod_{p=1}^{M_j}\Big(D_t-\frac{\tau_p(\xi',\lambda)}{\Lambda_j}\Big),B_1\Big(\frac{\xi'}{\Lambda_j},D_t\Big),\ldots,B_{M_j}\Big(\frac{\xi'}{\Lambda_j},D_t\Big)\,.$$

From Lemma 5.1 we obtain the desired result.

b) In the same way we can apply Lemma 5.1 to the boundary value problem $Q(D_t,1),B_{M_{\ell-1}+1}(0,D_t),\ldots,B_{M_\ell}(0,D_t)$ and its small perturbation

$$\prod_{m=M_{\ell-1}+1}^{M_\ell}\Big(D_t-\frac{\tau_m(\xi',\lambda)}{\lambda^{1/r_\ell}}\Big),B_{M_{\ell-1}+1}\Big(\frac{\xi'}{\lambda^{1/r_\ell}},D_t\Big),\ldots,B_{M_\ell}\Big(\frac{\xi'}{\lambda^{1/r_\ell}},D_t\Big)\,.$$

Here the fact that the second boundary value problem is a small perturbation of the first one follows from the estimate $\lambda^{-1/r_\ell}|\xi'|<\delta$, which holds for sufficiently large λ as $r_\ell<r_j$ and $|\xi'|\approx\lambda^{1/r_j}$. □

Definition 5.3. We define the basic solution $W_k=W_k(t,\xi',\lambda)$ by

$$W_k(t,\xi',\lambda):=\frac{1}{2\pi i}\int\limits_{\gamma_j^0}\frac{N_k(\xi',\tau,\lambda)e^{i\Lambda_j t\tau}}{\prod_{p=1}^{M_j}(\tau-\Lambda_j^{-1}\tau_p(\xi',\lambda))}\,\mathrm{d}\,\tau\quad(k=1,\ldots,M_j),\tag{5.3}$$

$$W_k(t,\xi',\lambda):=\frac{1}{2\pi i}\int\limits_{\gamma_\ell^1}\frac{N_k(\xi',\tau,\lambda)e^{i\lambda_\ell^{1/r_\ell}t\tau}}{\prod_{p=M_{\ell-1}+1}^{M_\ell}(\tau-\lambda_\ell^{-1/r_\ell}\tau_p(\xi',\lambda))}\,\mathrm{d}\,\tau$$

$$(k=M_{\ell-1}+1,\ldots,M_\ell;\ \ell=j+1,\ldots,J).\tag{5.4}$$

Lemma 5.4. a) *For $k = 1, \dots, M$ the basic solution W_k satisfies*

$$P(\xi', D_t, \lambda) W_k(t, \xi', \lambda) = 0 .$$

b) *Let $k \in \{1, \dots, M_j\}$. Then we have*

$$|B_i(\xi', D_t) W_k(0, \xi', \lambda)| \le C\, \Lambda_j^{m_i} \quad (i = 1, \dots, M) \tag{5.5}$$

and $B_i(\xi', D_t) W_k(0, \xi', \lambda) = \delta_{ik} \Lambda_j^{m_i}$ for $i = 1, \dots, M_j$.

c) *Let $k \in \{M_{\ell-1} + 1, \dots, M_\ell\}$ for some $\ell \ge j+1$. Then we have*

$$|B_i(\xi', D_t) W_k(0, \xi', \lambda)| \le C\, \lambda_\ell^{m_i/r_\ell} \quad (i = 1, \dots, M)$$

and $B_i(\xi', D_t) W_k(0, \xi', \lambda) = \delta_{ik} \lambda_\ell^{m_i/r_\ell}$ for $i = M_{\ell-1} + 1, \dots, M_\ell$.

d) *For $k = 1, \dots, M_j$ and $r = 0, 1, 2, \dots$ we have $\|D_t^r W_k(\cdot, \xi', \lambda)\|_{L_2(\mathbb{R}_+)} \le C\, \Lambda_j^{r-1/2}$. For $k = M_{\ell-1}+1, \dots, M_\ell$ the same estimate holds with Λ_j replaced by λ_ℓ^{1/r_ℓ}.*

Proof. Applying $P(\xi', D_t, \lambda)$ to (5.3), we obtain the integrand

$$\Lambda_j^{-M_j} \frac{N_k(\xi', \tau, \lambda) P(\xi', \Lambda_j \tau, \lambda)}{\prod_{m=1}^{M_j} (\Lambda_j \tau - \tau_m(\xi', \lambda))}\, e^{i \Lambda_j t \tau}$$

which is a holomorphic function of τ. This shows $PW_k = 0$ for $k \le M_j$. For $k > M_j$ the proof of part a) is the same.

Now we apply $B_i(\xi', D_t)$ to (5.3). We get

$$\begin{aligned} B_i(\xi', D_t) W_k(0, \xi', \lambda) &= \frac{1}{2\pi i} \int_{\gamma_j^0} \frac{B_i(\xi', \Lambda_j \tau) N_k(\xi', \tau, \lambda)}{\prod_{p=1}^{M_j} (\tau - \Lambda_j^{-1} \tau_p(\xi', \lambda))}\, \mathrm{d}\,\tau \\ &= \Lambda_j^{m_i} \frac{1}{2\pi i} \int_{\gamma_j^0} \frac{B_i(\Lambda_j^{-1} \xi', \tau) N_k(\xi', \tau, \lambda)}{\prod_{p=1}^{M_j} (\tau - \Lambda_j^{-1} \tau_p(\xi', \lambda))}\, \mathrm{d}\,\tau . \end{aligned}$$

Estimating the integral by a constant (see Corollary 4.5), we obtain (5.5). For $i \le M_j$ we apply Proposition 5.2 to see that the integral equals δ_{ik} which finishes the proof of part b). The proofs of c) and d) can be made in an analogous way. □

Now let us indicate the necessary changes for the case that (ξ', λ) belongs to $G(\Gamma_{j+1})$ for some $j = 0, \dots, J$. We have to replace the contour γ_0 by the contour $\tilde{\gamma}_0$ enclosing the zeros $\tau_1^0(|\xi'|^{-1}\xi'), \dots, \tau_{M_j}^0(|\xi'|^{-1}\xi')$ of $P_j(|\xi'|^{-1}\xi', \cdot, 0)$. Similarly, in all statements $\Lambda_j = \Lambda_j(\xi', \lambda)$ has to be replaced by $|\xi'| = \Lambda_j(\xi', 0)$. With these changes, the results follow with the same proofs.

5.2. **Unique solvability of the ODE problem.** Now we come back to the ODE problem (3.10)–(3.11). The aim of this subsection is to show unique solvability of this problem for large λ and to construct the solution in terms of the basic solutions introduced above. We assume throughout this subsection that the boundary value problem $(P, B_1, \dots, B_M)$ is N-elliptic in the sense of Definition 3.2.

We are looking for a solution (for sufficiently large λ) of (3.10)–(3.11) in the form

$$w(t,\xi',\lambda) = \sum_{k=1}^{M} c_k(\xi',\lambda) W_k(t,\xi',\lambda) \tag{5.6}$$

where the functions $W_1, \dots, W_M$ are the basic solutions defined in Definition 5.3. Due to Lemma 5.4 a), every function of the form (5.6) satisfies (3.10). The boundary conditions (3.11) are satisfied if and only if the linear equation system

$$H(\xi',\lambda) \begin{pmatrix} c_1(\xi',\lambda) \\ \vdots \\ c_M(\xi',\lambda) \end{pmatrix} = \begin{pmatrix} h_1 \\ \vdots \\ h_M \end{pmatrix} \tag{5.7}$$

is satisfied, where the $M \times M$ matrix $H(\xi',\lambda) = \big(h_{ik}(\xi',\lambda)\big)_{i,k=1,\dots,M}$ is given by

$$h_{ik}(\xi',\lambda) := B_i(\xi', D_t) W_k(0,\xi',\lambda)\,. \tag{5.8}$$

For the estimates below the following notation will turn out to be useful: let $(\xi',\lambda) \in G(\Gamma_j\Gamma_{j+1}) \cup G(\Gamma_{j+1})$ for some j. Then we for $k = 1, \dots, M$ we define

$$\mu_k(\xi',\lambda) := \begin{cases} \Lambda_j(\xi',\lambda) & \text{if } k \le M_j \text{ and } (\xi',\lambda) \in G(\Gamma_j\Gamma_{j+1}), \\ |\xi'| & \text{if } k \le M_j \text{ and } (\xi',\lambda) \in G(\Gamma_{j+1}), \\ \lambda^{1/r_\ell} & \text{if } M_{\ell-1}+1 \le k \le M_\ell. \end{cases}$$

Remark 5.5. a) An elementary calculation shows that we have for all $(\xi',\lambda) \in G$ and for $k \in \{M_{\ell-1}+1, \dots, M_\ell\}$ the equivalence $\mu_k(\xi',\lambda) \approx \Lambda_\ell(\xi',\lambda)$. By (2.1), we have

$$\mu_1 = \mu_2 = \dots = \mu_{M_j} < \mu_{M_j+1} = \dots = \mu_{M_{j+1}} < \mu_{M_{j+1}+1} = \dots .$$

b) Due to Corollary 4.5 we have for the zeros of $P(\xi',\cdot,\lambda)$ the equivalence

$$|\tau_k(\xi',\lambda)| \approx \mu_k(\xi',\lambda)\,,$$

so we can see that the factors μ_k describe the growth rate of the zeros for $|\xi'| + \lambda \to \infty$.

To prove the invertibility of the matrix $H(\xi', \lambda)$, we first show an estimate on the elements of this matrix. Here S_M stands for the group of all permutations of the set $\{1, \ldots, M\}$.

Lemma 5.6. a) *For every permutation* $\sigma \in S_M$ *we have the inequality*

$$\prod_{\ell=1}^{M} \mu_\ell^{m_{\sigma(\ell)}} \le \prod_{\ell=1}^{M} \mu_\ell^{m_\ell} . \tag{5.9}$$

More precisely, for $\sigma \neq \mathrm{id}$ *let* i *be the first index with* $\sigma(i) > i$ *and let* $k := \sigma^{-1}(i)$. *Then we have*

$$\prod_{\ell=1}^{M} \mu_\ell^{m_{\sigma(\ell)}} \le \Big(\frac{\mu_i}{\mu_k}\Big)^{m_{\sigma(i)}-m_i} \prod_{\ell=1}^{M} \mu_\ell^{m_\ell} .$$

b) *For every* $\sigma \in S_M \setminus \{\mathrm{id}\}$ *and for every* $\tau > 0$ *there exists a* $\lambda_0 > 0$ *such that for all* $\lambda \ge \lambda_0$ *the inequality*

$$\Big| \prod_{i=1}^{M} h_{\sigma(i),i} \Big| \le \tau \prod_{i=1}^{M} \mu_i^{m_i} \tag{5.10}$$

holds.

Proof. a) We show by induction on M that the statement in a) holds. As in the case $M = 1$ the statement is trivial; suppose it is already proved for sets of $M' < M$ elements and permutations σ' of these sets. Now we consider the set $\{1, 2, \ldots, M\} \setminus \{i\}$ and define the permutation σ' of this set by $\sigma'(\ell) := \sigma(\ell)$ for $\ell \neq i, k$ and $\sigma'(k) := \sigma(i)$. We rewrite the left-hand side of (5.9) in the form

$$\Big[\prod_{\ell \neq i,k} \mu_\ell^{m_{\sigma'(\ell)}} \mu_k^{m_{\sigma'(k)}} \Big] \mu_k^{-m_{\sigma(i)}} \mu_k^{m_i} \mu_i^{m_{\sigma(i)}} .$$

Due to the induction assumption the term in square brackets can be estimated by

$$\prod_{\ell \neq i} \mu_\ell^{m_\ell} = \mu_i^{-m_i} \prod_{\ell=1}^{M} \mu_\ell^{m_\ell} .$$

Thus the quotient of the left-hand side and the right-hand side of (5.9) is equal to $(\mu_i/\mu_k)^{m_{\sigma(i)}-m_i}$, which proves a).

b) We distinguish two cases:

Case (i). Here we assume that σ is reduced to permutations of the sets $\{1, \ldots, M_j\}$ and $\{M_{\ell-1}+1, \ldots, M_\ell\}$; i.e., the restriction of σ to each of these sets is a permutation of this set. Since one of the reduced permutations differs from the identity, the corresponding term in (5.10) equals zero due to Lemma 5.4.

Case (ii). Now we assume that σ is not reduced to the sets above. Then there exists a (minimal) index i and an $\ell \in \{j, \dots, J\}$ such that $i \le M_\ell$ and $\sigma(i) > M_\ell$. In this case we obtain, using Lemma 5.4 and part a),

$$\Big| \prod_{p=1}^{M} h_{\sigma(p),p} \Big| \le C \prod_{p=1}^{M} \mu_p^{m_{\sigma(p)}} \le C \prod_{p=1}^{M} \mu_p^{m_p} \left(\frac{\mu_i}{\mu_k} \right)^{m_{\sigma(i)} - m_i} \tag{5.11}$$

where $k := \sigma^{-1}(i)$ $(> i)$. If $k \le M_\ell$ we have $h_{ik} = h_{\sigma(k),k} = 0$ by Lemma 5.4. If $k > M_\ell$ we have $\mu_k = \lambda^{1/r_{\ell+1}}$ and $\mu_i \approx \lambda^{1/r_\ell}$. As the exponent $m_{\sigma(i)} - m_i$ is not less than 1 due to condition (3.2), the last factor in (5.11) can be made arbitrarily small if λ is chosen large enough. Thus we see that for every $\tau > 0$ there exists a $\lambda_0 > 0$ such that for $\lambda \ge \lambda_0$ the inequality (5.10) holds. □

Theorem 5.7. *Let the boundary value problem $(P, B_1, \dots, B_M)$ be N-elliptic. Then there exists a $\lambda_0 > 0$ such that for all $\lambda \ge \lambda_0$ and all $\xi' \in \mathbb{R}^{n-1}$ the following statements hold.*

a) *The matrix $H(\xi', \lambda)$ defined in* (5.8) *is nonsingular and its determinant can be estimated by*

$$|\det H(\xi', \lambda)| \ge C \prod_{i=1}^{M} \mu_i^{m_i} \,. \tag{5.12}$$

b) *For the coefficients of the inverse matrix*

$$H^{-1}(\xi', \lambda) =: \big(g_{rs}(\xi', \lambda) \big)_{r,s=1,\dots,M}$$

the estimate

$$|g_{rs}(\xi', \lambda)| \le C \begin{cases} \mu_r^{-m_r} \prod_{\ell=s}^{r-1} \mu_\ell^{m_{\ell+1} - m_\ell} & \text{if } s \le r \,, \\ \mu_r^{-m_r} \prod_{\ell=r+1}^{s} \mu_\ell^{m_{\ell-1} - m_\ell} & \text{if } s > r \end{cases} \tag{5.13}$$

holds with a constant $C = C(\lambda_0)$ independent of ξ' and λ.

Proof. a) We use the Leibniz product for the determinant of $H = H(\xi', \lambda)$, which we write in the form

$$\det H = \prod_{i=1}^{M} h_{ii} + \sum_{\sigma \in S_M \setminus \{\mathrm{id}\}} \operatorname{sign}(\sigma) \prod_{i=1}^{M} h_{\sigma(i),i} \,. \tag{5.14}$$

By Lemma 5.4 the first term on the right-hand side equals the right-hand side of (5.12) with $C = 1$. To estimate the other terms, we use Lemma 5.6 b). If we choose τ small enough, we obtain from this lemma that the matrix H is invertible and that the inequality (5.12) holds.

b) We consider only the case $s \leq r$; for $s > r$ the proof can be made in a completely analogous way. We write (assuming that λ is sufficiently large)

$$g_{rs} = \frac{\det H^{sr}}{\det H}$$

where the $(M-1) \times (M-1)$ matrix H^{sr} is obtained by omitting the s-th row and the r-th column of the matrix H. Again we will use the Leibniz formula for $\det H^{sr}$. Due to Lemma 5.4 and the definition of μ_k, we have

$$|h_{ik}| \leq C\, \mu_k^{m_i} .$$

In the same way as in the proof of Lemma 5.6 a), we obtain that each term in the Leibniz sum for $\det H^{sr}$ can be estimated by

$$C \prod_{\ell=1}^{s-1} \mu_\ell^{m_\ell} \cdot \prod_{\ell=s}^{r-1} \mu_\ell^{m_{\ell+1}} \cdot \prod_{\ell=r+1}^{M} \mu_\ell^{m_\ell} .$$

(Note for the second product that starting with column s the index of the row is shifted by one as the s-th row in the matrix H is omitted. So we obtain $\mu_\ell^{m_{\ell+1}}$ instead of $\mu_\ell^{m_\ell}$ for $\ell = s, \dots, r-1$.) From this and (5.12) the desired estimate for $|g_{rs}|$ follows. For $s > r$ the index of the column is shifted by one for $\ell = r, \dots, s-1$, which leads to the second line in (5.13). □

Corollary 5.8. a) *For sufficiently large λ the basic solutions*

$$W_1(\cdot, \xi', \lambda), \dots, W_M(\cdot, \xi', \lambda)$$

are linearly independent.

b) *For sufficiently large λ the ordinary differential equation* (3.10)–(3.11) *is uniquely solvable for every $\xi' \in \mathbb{R}^{n-1}$ and every $(h_1, \dots, h_M) \in \mathbb{C}^M$.*

Proof. Part a) follows immediately from the invertibility of $H(\xi', \lambda)$ for large λ and the definition of $H(\xi', \lambda)$. For b) we only have to note that (under the condition of N-ellipticity) the space of all stable solutions of (3.10) has dimension M. Therefore $W_1, \dots, W_M$ is a basis of this space and every stable solution of (3.10) has the form (5.6). Due to this, unique solvability of (3.10)–(3.11) is equivalent to the invertibility of $H(\xi', \lambda)$. □

5.3. **Proof of Theorem 3.8.** Now we want to prove Theorem 3.8. We already know from Corollary 5.8 that the boundary value problem (3.10)–(3.11) is uniquely solvable and we still have to prove the estimate on the solution w. Again we assume throughout this subsection that $(P, B_1, \dots, B_M)$ is N-elliptic, and we fix a tuple $\mathbf{s} \in \mathbb{R}^J$ of real numbers satisfying (3.8). First we rewrite the inequality of Theorem 5.7 b) in terms of the weight functions $\Psi_{\mathbf{s}}$ defined in Subsection 3.2.

Lemma 5.9. *For $\lambda \geq \lambda_0$ with λ_0 given in Theorem 5.7 the estimate*

$$|g_{rs}(\xi', \lambda)| \leq C\mu_r^{-m_r}(\xi', \lambda)\, \frac{\Psi_{\mathbf{s}}^{(-m_s-1/2)}(\xi', \lambda)}{\Psi_{\mathbf{s}}^{(-m_r-1/2)}(\xi', \lambda)} \tag{5.15}$$

holds with a constant $C = C(\lambda_0)$ independent of ξ' and λ.

Proof. Again we consider only the case $r \geq s$, as the proof for the opposite case can be made in the same way. Let $S \in \{1, \dots, J\}$ be the index for which

$$s \in \{M_{S-1} + 1, \dots, M_S\}\,.$$

We then have, using (3.8),

$$s_1 + \cdots + s_{S-1} \leq m_s + 1/2 \leq s_1 + \cdots + s_S\,.$$

Analogously we choose $R \in \{1, \dots, J\}$ with

$$r \in \{M_{R-1} + 1, \dots, M_R\}\,. \tag{5.16}$$

For better readability, let us introduce the abbreviation

$$\nu(\ell) := m_{M_\ell + 1} \quad (\ell = 1, \dots, J)\,.$$

Due to Remark 5.5, we can replace μ_ℓ on the right-hand side of (5.13) by the corresponding Λ_j and obtain

$$|g_{rs}| \leq C\mu_r^{-m_r}\Lambda_S^{\nu(S)-m_s}\Big(\prod_{\ell=S+1}^{R-1}\Lambda_\ell^{\nu(\ell)-\nu(\ell-1)}\Big)\Lambda_R^{m_r-\nu(R-1)}\,. \tag{5.17}$$

By definition we have

$$\frac{\Psi_{\mathbf{s}}^{(-m_s-1/2)}}{\Psi_{\mathbf{s}}^{(-m_r-1/2)}} = \Lambda_S^{s_1+\cdots+s_S-m_s-1/2}\Big(\prod_{\ell=S+1}^{R-1}\Lambda_\ell^{s_\ell}\Big)\Lambda_R^{-s_1-\cdots-s_{R-1}+m_r+1/2}\,. \tag{5.18}$$

From (5.17) and (5.18) we see that

$$|g_{rs}|\Big(\mu_r^{-m_r}\frac{\Psi_{\mathbf{s}}^{(-m_s-1/2)}}{\Psi_{\mathbf{s}}^{(-m_r-1/2)}}\Big)^{-1} \leq C\Lambda_S^{\nu(S)+1/2-s_1-\cdots-s_S} \tag{5.19}$$

$$\times \Big(\prod_{\ell=S+1}^{R-1} \Lambda_\ell^{\nu(\ell)-\nu(\ell-1)-s_\ell} \Big) \Lambda_R^{s_1+\dots+s_{R-1}-\nu(R-1)-1/2} . \tag{5.20}$$

Because of (3.8), the first exponent on the right-hand side of (5.20) is nonnegative, and we may estimate

$$\Lambda_S^{\nu(S)+1/2-s_1-\dots-s_S} \Lambda_{S+1}^{\nu(S+1)-\nu(S)-s_{S+1}} \le \Lambda_{S+1}^{\nu(S+1)+1/2-s_1-\dots-s_{S+1}} .$$

Again the last exponent is nonnegative. Proceeding in this way, we see that the right-hand side of (5.20) is not greater than

$$C\Lambda_{R-1}^{\nu(R-1)+1/2-s_1-\dots-s_{R-1}} \Lambda_R^{s_1+\dots+s_{R-1}-\nu(R-1)-1/2} \le C ,$$

which finishes the proof of inequality (5.15) for the case $s \le r$. □

Theorem 5.10. *Let $(P, B_1, \dots, B_M)$ be N-elliptic and $\mathbf{s} \in \mathbb{R}^J$ be a tuple satisfying* (3.8)*. Then for sufficiently large λ the inequality*

$$\|D_t^\ell W_k(\cdot, \xi', \lambda)\|_{L_2(\mathbb{R}_+)} \, |g_{rs}(\xi', \lambda)| \, \frac{\Psi_{\mathbf{s}}^{(-\ell)}(\xi', \lambda)}{\Psi_{\mathbf{s}}^{(-m_s-1/2)}(\xi', \lambda)} \le C$$

holds for $\ell = 0, 1, 2, \dots$ and $k = 1, \dots, M$.

Proof. Let $\ell \in \{M_{L-1}+1, \dots, M_L\}$ and R be given by (5.16). For $R \ge L$ we have by definition

$$\frac{\Psi_{\mathbf{s}}^{(-\ell)}}{\Psi_{\mathbf{s}}^{(-m_r-1/2)}} = \Lambda_L^{s_1+\dots+s_L-\ell} \Big(\prod_{k=L+1}^{R-1} \Lambda_k^{s_k} \Big) \Lambda_R^{-s_1-\dots-s_{R-1}+m_r+1/2} .$$

On the right-hand side all exponents except the last one are nonnegative, so we can estimate

$$\frac{\Psi_{\mathbf{s}}^{(-\ell)}}{\Psi_{\mathbf{s}}^{(-m_r-1/2)}} \le \Lambda_R^{s_1+\dots+s_{R-1}-\ell} \Lambda_R^{-s_1-\dots-s_{R-1}+m_r+1/2} = \Lambda_R^{m_r+1/2-\ell} . \tag{5.21}$$

From Lemma 5.4 and Lemma 5.9 we obtain, using $\mu_r = \Lambda_R$,

$$\begin{aligned} &\|D_t^\ell W_k(\cdot, \xi', \lambda)\|_{L_2(\mathbb{R}_+)} |g_{rs}(\xi', \lambda)| \frac{\Psi_{\mathbf{s}}^{(-\ell)}(\xi', \lambda)}{\Psi_{\mathbf{s}}^{(-m_s-1/2)}(\xi', \lambda)} \\ &\le C\Lambda_R^{-m_r+\ell-1/2} \frac{\Psi_{\mathbf{s}}^{(-\ell)}}{\Psi_{\mathbf{s}}^{(-m_r-1/2)}} \le C, \end{aligned}$$

which had to be shown. For $R < L$ we get

$$\frac{\Psi_{\mathbf{s}}^{(-\ell)}}{\Psi_{\mathbf{s}}^{(-m_r-1/2)}} = \Lambda_R^{-s_1-\cdots-s_R+m_r+1/2} \prod_{k=R+1}^{L-1} \Lambda_k^{-s_k} \Lambda_L^{-s_1-\cdots-s_{L-1}-\ell}.$$

Here all exponents except the first are nonpositive, and we may replace Λ_k for $k \geq R+1$ by Λ_R, again obtaining the estimate (5.21). □

Now the proof of Theorem 3.8 follows easily:

Proof of Theorem 3.8. We know from Corollary 5.8 that there exists a $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ the problem (3.10)–(3.11) is uniquely solvable with the solution $w = w(t, \xi', \lambda)$ being given by

$$w(t, \xi', \lambda) = \sum_{k=1}^{M} c_k(\xi', \lambda) W_k(t, \xi', \lambda).$$

Here c_k satisfies the linear equation system (5.7). From the estimates of Lemma 5.4 d) and Theorem 5.10, we get

$$\begin{aligned} \|D_t^\ell w(\cdot, \xi', \lambda)\|_{L_2(\mathbb{R}_+)} &\leq \sum_{k=1}^{M} |c_k| \; \|D_t^\ell W_k(\cdot, \xi', \lambda)\|_{L_2(\mathbb{R}_+)} \\ &\leq \sum_{k,j=1}^{M} |g_{kj}| \|D_t^\ell W_k(\cdot, \xi', \lambda)\|_{L_2(\mathbb{R}_+)} \; |h_j| \leq C \sum_{j=1}^{M} \frac{\Psi_{\mathbf{s}}^{(-m_j-1/2)}}{\Psi_{\mathbf{s}}^{(-\ell)}} |h_j| \end{aligned}$$

and therefore the estimate of Theorem 3.8. □

5.4. **Generalizations and comments.** Throughout this paper, we assume N-ellipticity with parameter to hold along the ray $[0, \infty)$. As in the classical theory of ellipticity with parameter developed by Agmon–Agranovich–Vishik, one can also define N-ellipticity in a closed sector $\mathcal{L} \subset \mathbb{C}$ with vertex at the origin. For this one has to replace inequality (2.4) in Definition 2.2 by

$$|P(\xi, \lambda)| \geq C \; W_P(\xi, \lambda) \quad \text{for } \xi \in \mathbb{R}^n \text{ and } \lambda \in \mathcal{L} \text{ with } |\lambda| \geq \lambda_0 . \qquad (2.4')$$

Moreover, in Definition 2.7 the polynomial $Q_j(\cdot, 1)$ has to be replaced by $Q_j(\cdot, \lambda)$ and the condition of Definition 2.7 has to hold for all $\lambda \in \mathcal{L}$ with $|\lambda| = 1$. Similarly, in Definition 3.2 (iv) the operator $Q_j(D_t, 1)$ has to be replaced by $Q_j(D_t, \lambda)$ with $\lambda \in \mathcal{L}$, $|\lambda| = 1$. Finally, the inequality $\lambda \geq 0$ in Definition 3.2 (iii) has to be replaced by $\lambda \in \mathcal{L}$.

With exactly the same proofs as above, one can show the following result.

Theorem 3.7′. *Let* $(P, B_1, \dots, B_M)$ *be N-elliptic in the sector* $\mathcal{L}$ *as indicated above. Let* $\mathbf{s} \in \mathbb{R}^J$ *satisfy* (3.8), *assume that* $s_1 + \cdots + s_J$ *is integer and set* $t_j := s_j - 2N_j$. *Then there exists a* $\lambda_0 > 0$ *such that for all* $\lambda \in \mathcal{L}$ *with* $|\lambda| \geq \lambda_0$ *the operator* (3.7) *is invertible, and the a priori estimate* (3.9) *holds uniformly for all* $\lambda \in \mathcal{L}$ *with* $|\lambda| \geq \lambda_0$ *where the constant* C *does not depend on* u *or* λ.

Note that in the case where $\mathcal{L} = \{z \in \mathbb{C} : |\arg z| \leq \theta\} \cup \{0\}$ with some $\theta \in (0, \infty)$ this implies that the uniform estimate holds in the shifted sector $\lambda_0 + \mathcal{L}$. For $\theta \geq \pi/2$ this leads to N-parabolic problems.

Remark 5.11. a) Consider the boundary value problem (1.2) with $g_j = 0$, i.e., with homogeneous boundary conditions. Under the assumptions of Theorem 3.7 (or 3.7′), this boundary value problem defines an unbounded, closed operator $P_B(\lambda)$ in $H_{\mathbf{t}}(\mathbb{R}^n_+)$ with the domain

$$D(P_B(\lambda)) := \{u \in H_{\mathbf{s}}(\mathbb{R}^n_+) : B_j u = 0 \quad \text{for } j = 1, \dots, M\}$$

acting by $P_B(\lambda)u := P(D, \lambda)u$ for $u \in D(P_B(\lambda))$. This operator is called the $H_{\mathbf{t}}$-realization of (1.2). From Theorem 3.7 we see that for large λ this operator has a bounded inverse, and the norm of $P_B(\lambda)^{-1}$ as a bounded operator in $H_{\mathbf{t}}$ can be estimated by a constant times $|\lambda|^{-\sum_j 2N_j/r_j} = |\lambda|^{-q_1}$. If $\sum_j t_j = 0$, the space $H_{\mathbf{t}}(\mathbb{R}^n_+)$ coincides with the space $L_2(\mathbb{R}^n_+)$ with equivalent norms (the equivalence constants depending on λ). In the particular case where we may set $t_j := 0$ (i.e., $s_j := 2N_j$) for all j, we obtain the standard parameter-independent L_2-norm.

One of the first questions in spectral theory of N-elliptic boundary value problems is the question of multiple completeness of the root functions. For polynomial operator pencils which are elliptic with parameter in the sense of Agmon–Agranovich–Vishik, this was proved in [3]. We hope to prove multiple completeness for the operator $P_B(\lambda)$ in a forthcoming paper.

b) For the Dirichlet problem, the canonical choice of s_j satisfying (3.8) is given by $s_j = N_j$. In this case we obtain $\mathbf{t} = -\mathbf{s}$. In the case of homogeneous Dirichlet boundary conditions, we get from Theorem 3.7 an estimate for the inverse of the operator $P_B(\lambda)$ which now can be considered as a bounded operator from $H_{\mathbf{s}}(\mathbb{R}^n_+)$ to $H_{-\mathbf{s}}(\mathbb{R}^n_+)$. An estimate in these spaces (also called energy estimate) seems to be more natural than an estimate of the L_2-realization as discussed above. In fact, such energy estimates frequently appear in the theory of singular perturbations; cf., e.g., [11].

Remark 5.12. Looking through the proof of Theorem 3.8 in the last two sections, one can see that the unique solution $w(t, \xi', \lambda)$ of (3.10)–(3.11) is given in the form (5.6), i.e., in terms of the basic solutions W_k. The definition of W_k (and thus of w) depends on the subdomain of the partition

$$G = \bigcup_{j=1}^{J+1} G(\Gamma_j) \cup \bigcup_{j=1}^{J} G(\Gamma_j \Gamma_{j+1})$$

(see (4.2)) to which (ξ', λ) belongs.

If we want to treat boundary value problems of the form (1.2) with variable coefficients, the standard method is to use microlocalization and the theory of pseudodifferential operators. But due to the piecewise definition of w mentioned above, we first have to introduce a partition of unity in the (ξ', λ)-space which corresponds to the partition (4.2) of G. For this one first has to enlarge the subdomains $G(\Gamma_j)$ and $G(\Gamma_j \Gamma_{j+1})$ slightly to obtain an open covering. This can be done by introducing several small parameters instead of one fixed parameter ε. The construction of a partition of unity with desired properties is not trivial; for the case $n = 2$ it was done in Chapter 4 of [9].

Due to this difficulty, the application of microlocalization techniques is not completely standard, and so we prefer to treat variable coefficients (and nonstationary problems) in a separate paper.

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