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APPROXIMATION AND ASYMPTOTIC BEHAVIOUR OF EVOLUTION FAMILIES*

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Abstract. Let $(A(t))_{t\geq 0}$ and $(B(t))_{t\geq 0}$ be two families of closed operators satisfying the Acquistapace–Terreni conditions or the Kato–Tanabe conditions, or assumptions of maximal regularity, and let $(U(t,s))_{t>s\geq 0}$ and $(V(t,s))_{t>s\geq 0}$ be the associated evolution families. We obtain some estimates for $\|U(t,s)-V(t,s)\|$ in terms of $\|A(\tau)^{-1}-B(\tau)^{-1}\|$ for $s\leq\tau\leq t$. We deduce some results showing that if $\|A(\tau)^{-1}-B(\tau)^{-1}\|\to 0$ sufficiently quickly as $\tau\to\infty$ then U and V have similar asymptotic behaviour.

1. Introduction

We consider long-time asymptotic properties of solutions of nonautonomous Cauchy problems of the form

$$\frac{d}{dt}u(t)=A(t)u(t)\quad (t\geq s\geq 0),\qquad u(s)=x, \tag{1.1}$$

on a Banach space X. Once one has obtained appropriate results for the homogeneous problem (1.1), it is straightforward to deduce results for the corresponding inhomogeneous problems. The properties which have been most thoroughly studied are (exponential) stability and exponential dichotomy; see the monographs [11], [13], [15], [29], [34] and [40].

If two families of operators $(A(t))_{t\geq 0}$ and $(B(t))_{t\geq 0}$ approach each other in a suitable sense as $t\to\infty$, then one may expect that the solutions of (1.1) have long-time asymptotic properties which are similar to those of the solutions of the corresponding problem for $(B(t))$. The first results of this

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type were obtained by Tanabe [39] (see [34, Section 5.8] or [40, Section 5.6]) in some cases when $B(t) = B$ is autonomous and $D(A(t)) = D(B)$ for all $t \geq 0$. Recently, Schnaubelt [37] has extended Tanabe's results. He assumed that the family of operators $A(t)$ satisfy the Acquistapace–Terreni conditions, and that they converge in a certain sense, as $t \to \infty$, to an operator B which generates a holomorphic C_0-semigroup satisfying exponential dichotomy. Then the evolution family associated with (1.1) also has exponential dichotomy.

In this paper we extend Schnaubelt's ideas in several directions. We improve his result by assuming a more natural and weaker notion of convergence of $A(t)$ to B (see Theorem 6.2 and Remark 3.5). We establish the same results with the Acquistapace–Terreni conditions replaced by the Kato–Tanabe conditions or by assumptions of L^p-maximal regularity (Theorem 6.3). In Section 7, we consider other types of asymptotic behaviour instead of exponential dichotomy.

The basic technique is to estimate the difference between the evolution family $U(t,s)$ associated with (1.1) and the semigroup $e^{(t-s)B}$ generated by B, or, more generally, the evolution family associated with a nonautonomous family of operators $B(t)$. The estimation need only be carried out for $t-s$ fixed or bounded, but for types of exponential behaviour other than dichotomy the estimates have to be iterated and then it is important that they should be as precise as possible.

We give three methods for obtaining the basic estimates, in Sections 3, 4 and 5, respectively. The first is an extension of the approach used in [37]; it applies to the Kato–Tanabe conditions and also to many cases of the Acquistapace–Terreni conditions including those where B is autonomous. The second method covers all cases of the Acquistapace–Terreni conditions, but it gives less precise estimates; it involves passing to the Yosida approximations $A_n(t)$ of $A(t)$ and extracting from the techniques of the original paper of Acquistapace and Terreni [2] specific estimates for the difference between the corresponding evolution families (see Proposition 4.4 and compare the corresponding estimate for the Kato–Tanabe conditions in [3, Section 3]). The third method uses maximal regularity to obtain the estimates.

The final section of the paper contains some comparative discussion of the various results and some examples from parabolic partial differential equations.

2. Acquistapace–Terreni and Kato–Tanabe conditions

Let $J = [0, T]$ for some $T > 0$ or $J = \mathbb{R}_+$, and let $(A(t))_{t\in J}$ be a family of closed linear operators on a complex Banach space X. We write $R(\lambda, A(t))$ for $(\lambda - A(t))^{-1}$ when λ belongs to the resolvent set $\varrho(A(t))$ of $A(t)$.

We say that the family satisfies the *Acquistapace–Terreni conditions* if there exist $\theta \in (\pi/2, \pi)$, $\omega \in \mathbb{R}$, $K, L \geq 0$ and $\mu, \nu \in (0, 1]$ with $\mu + \nu > 1$, such that

(SC) $\varrho(A(t) - \omega) \supseteq \Sigma_\theta \cup \{0\}$ and $\|R(\lambda + \omega, A(t))\| \leq \frac{K}{1+|\lambda|}$ for all $t \in J$ and $\lambda \in \Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\}$; and

(AT) $\|(\omega - A(t))R(\lambda+\omega, A(t))(R(\omega, A(t)) - R(\omega, A(s)))\| \leq \frac{L|t-s|^\mu}{|\lambda|^\nu}$ for all $s, t \in J$ and $\lambda \in \Sigma_\theta$.

Since $(\omega - A(t))R(1 + \omega, A(t))$ is invertible with inverse $R(\omega, A(t)) + I$, the norm of which is bounded by $K + 1$, it follows from (AT) that

$$\|R(\omega, A(t)) - R(\omega, A(s))\| \leq C|t - s|^\mu \tag{2.1}$$

for all $s, t \in J$, where $C = L(K + 1)$. The identity

$$\begin{aligned} &R(\lambda + \omega, A(t)) - R(\lambda + \omega, A(s)) \\ &= (\omega - A(t))R(\lambda+\omega, A(t))(R(\omega, A(t)) - R(\omega, A(s)))(\omega - A(s))R(\lambda+\omega, A(s)) \end{aligned}$$

and (AT) imply that

$$\|R(\lambda + \omega, A(t)) - R(\lambda + \omega, A(s))\| \leq \frac{C|t - s|^\mu}{|\lambda|^\nu} \tag{2.2}$$

for all $s, t \in J$ and $\lambda \in \Sigma_\theta$.

It follows from the sectorial condition (SC) that $A(t)$ generates a holomorphic semigroup $\left(e^{\tau A(t)}\right)_{\tau>0}$ given by

$$e^{\tau A(t)} = \frac{1}{2\pi i}\int_\Gamma e^{(\lambda+\omega)\tau} R(\lambda + \omega, A(t))\, d\lambda = \frac{e^{\omega\tau}}{2\pi i\tau}\int_\Gamma e^{\lambda} R(\lambda/\tau + \omega, A(t))\, d\lambda, \tag{2.3}$$

where Γ is the upward-directed path consisting of the rays $\arg\lambda = e^{\pm i\theta'}$ for any $\theta' \in (\pi/2, \theta)$, or any deformation of such a path. The following properties are standard (see [30, Section 2.1]):

(S1) $e^{\tau_1 A(t)} e^{\tau_2 A(t)} = e^{(\tau_1+\tau_2)A(t)} \quad (\tau_1, \tau_2 > 0)$;

(S2) $\|e^{\tau A(t)}\| \leq Ce^{\omega\tau} \quad (\tau > 0)$;

(S3) the map $\tau \mapsto e^{\tau A(t)}$ belongs to $C^\infty((0,\infty),\mathcal{L}(X))$ and for each $k \in \mathbb{N}$ and $\tau > 0$, $e^{\tau A(t)}$ maps X into $D(A(t)^k)$ and

$$\Big(\frac{d}{d\tau}\Big)^k \big(e^{\tau A(t)}\big) = A(t)^k e^{\tau A(t)} = \frac{e^{\omega\tau}}{2\pi i \tau^{k+1}} \int_\Gamma (\lambda + \omega\tau)^k e^\lambda R(\lambda/\tau + \omega, A(t))\, d\lambda. \tag{2.4}$$

In particular, for $\tau \in (0,T]$ (where $T > 0$ is arbitrary in the case when $J = \mathbb{R}_+$),

$$\big\|A(t)^k e^{\tau A(t)}\big\| \le \frac{C_k}{\tau^k},$$

$$\big\|e^{-\omega\tau} R(\omega, A(t)) e^{\tau A(t)} - R(\omega, A(t))\big\| = \Big\| \int_0^\tau e^{-\omega\tau} e^{rA(t)}\, dr \Big\| \le C\tau. \tag{2.5}$$

Moreover, it follows easily from (AT), (2.4), (2.3) and (2.2) that

$$\big\|(\omega - A(t)) e^{\tau A(t)} \left(R(\omega, A(t)) - R(\omega, A(s))\right)\big\| \le C|t-s|^\mu \tau^{\nu-1}, \tag{2.6}$$

$$\big\|e^{\tau A(t)} - e^{\tau A(s)}\big\| \le C|t-s|^\mu \tau^{\nu-1} \tag{2.7}$$

for all $\tau \in (0,T]$ and $s,t \in J$. Here and throughout the paper, C and C_k are constants which may vary from line to line and which may depend on T and on the parameters θ, ω, K, L, μ and ν of the Acquistapace–Terreni conditions (and on other parameters such as k) but which are otherwise independent of the particular family $(A(t))$ and are independent of time variables such as s, t and τ.

Let $\Delta = \{(t,s) \in J^2 : t > s\}$ and $\Delta_T = \{(t,s) \in \Delta : t \le s + T\}$ (where $T > 0$ is arbitrary when $J = \mathbb{R}_+$; note that $\Delta_T = \Delta$ when $J = [0,T]$). If $J = [0,T]$, and *a fortiori* if $J = \mathbb{R}_+$, it is shown in [2], [1] (see also [6], [42], [43]) that the family $(A(t))_{t\in J}$ "generates" an *evolution family* $(U(t,s))_{(t,s)\in\Delta}$ of bounded linear operators on X satisfying the following:

(EF1) $U(t,s) = U(t,r)U(r,s)$ whenever $0 \le s < r < t \in J$;
(EF2) U is strongly continuous on Δ;
(EF3) $\lim_{t\downarrow s} U(t,s)x = x$ if and only if $x \in \overline{D(A(s))}$;
(EF4) $U(\cdot,s) \in C^1((s,\infty)\cap J, \mathcal{L}(X))$, $U(t,s)$ maps X into $D(A(t))$ and $\frac{\partial}{\partial t} U(t,s) = A(t)U(t,s)$ whenever $(t,s) \in \Delta$;
(EF5) $\big\|U(t,s)\big\| \le C$ and $\big\|A(t)U(t,s)\big\| \le C/(t-s)$ whenever $(t,s) \in \Delta_T$;
(EF6) $\big\|U(t,s) - e^{(t-s)A(s)}\big\| \le C(t-s)^\delta$ whenever $(t,s) \in \Delta_T$, where $\delta = \mu + \nu - 1 > 0$.

Of these conditions, (EF1) and (EF2) form the abstract definition of an evolution family on an arbitrary interval in $\mathbb{R}$. When $\lim_{t\downarrow s} U(t,s)x = x$ for all $s \in J$ and $x \in X$, we put $U(s,s) = I$ and we say that U is a (strongly continuous) evolution family for $t \ge s$. For the general theory of evolution families, we refer to [11]. The properties (EF4)–(EF6) are not part of the definition of an evolution family, but they are consequences of the Acquistapace–Terreni conditions showing that U is associated with the Cauchy problem (1.1).

Instead of the Acquistapace–Terreni conditions, we may consider the conditions introduced by Kato and Tanabe [27] (see also [40, Section 5.3]), which also imply the generation of an evolution family.

We say that $(A(t))_{t\in J}$ satisfies the *Kato–Tanabe conditions* if there exist $\theta \in (\pi/2, \pi)$, $\omega \in \mathbb{R}$, K, $L \geq 0$ and α, $\nu \in (0, 1]$ such that (SC) holds and

(KT) The map $t \mapsto R(\omega, A(t))$ belongs to $C^1(J, \mathcal{L}(X))$ and its derivative satisfies $\left\| \frac{d}{dt} R(\omega, A(t)) - \frac{d}{ds} R(\omega, A(s)) \right\| \leq L|t-s|^\alpha$ for all $s, t \in J$ and
$\left\| \frac{d}{dt} \left(R(\lambda + \omega, A(t)) \right) \right\| \leq \frac{L}{1+|\lambda|^\nu}$ for all $t \in J$ and $\lambda \in \Sigma_\theta$.

Note that differentiability of $R(\omega, A(\cdot))$ implies differentiability of $R(\lambda, A(\cdot))$.

When the Kato–Tanabe conditions hold, we put $\mu = 1$. Then (2.1), (2.2) and (2.7) all hold. Moreover, it is shown in [27], [40, Section 5.3] that $(A(t))_{t\in J}$ generates an evolution family $(U(t,s))_{(t,s)\in\Delta}$ satisfying (EF1)–(EF6). (When we assume the Kato–Tanabe conditions, our constants C may depend on θ, ω, K, L, α and ν, and also on T.)

Now suppose that $B = (B(t))_{t\in J}$ is a family of closed linear operators on the Banach space X, satisfying the Acquistapace–Terreni conditions or the Kato–Tanabe conditions. Let $V := (V(t,s))_{(t,s)\in\Delta}$ be the evolution family generated by B. We shall need to consider the following property:

(D) For each $t \in J$, $V(t, \cdot) \in C^1([0,t), \mathcal{L}(X))$ and, for $0 \leq s < t$, $\frac{\partial}{\partial s} V(t,s)$ is an extension of $-V(t,s)B(s)$.

It is shown in [3, Theorem 1.6] that if B satisfies the Kato–Tanabe conditions, then (D) holds and

$$\left\| \frac{\partial}{\partial s} V(t,s) \right\| \leq C/(t-s) \qquad ((t,s) \in \Delta_T). \tag{2.8}$$

When B satisfies the Acquistapace–Terreni conditions, there is a duality technique which has been employed in [3, Section 6] to establish (D) in many cases. Consider the following condition which is dual to (AT):

(AT)′ There exist $L' \geq 0$ and $\mu', \nu' \in (0, 1]$ with $\mu' + \nu' > 1$ such that

$$\left\| \left(R(\omega, B(t)) - R(\omega, B(s)) \right) B(t) R(\lambda + \omega, B(t)) \right\| \leq \frac{L'|t-s|^{\mu'}}{|\lambda|^{\nu'}}$$

for all $s, t \in J$ and $\lambda \in \Sigma_\theta$.

Assuming that each $B(t)$ is densely defined and that (AT)′ holds (in addition to (SC) and (AT)), it is shown in [3, Theorem 6.4] that (D) and (2.8) both hold. (Here and in the subsequent results which depend on (AT)′, C may depend on the constants L', μ' and ν' as well as the constants appearing in (SC) and (AT).)

Remark 2.1. In the literature the Acquistapace–Terreni and Kato–Tanabe conditions are usually given in the case when $\omega = 0$. However, all the assertions above follow in the more general case, by rescaling. In fact, if $(A(t))$ satisfies either set of conditions, then the family $(A(t) - \omega)$ satisfies the same conditions with ω replaced by 0. If $(A(t)-\omega)$ generates an evolution family $(U_\omega(t,s))$, then $(A(t))$ generates an evolution family given by $U(t,s) = e^{\omega(t-s)}U_\omega(t,s)$. The properties described above and in Sections 3, 4 and 5 are invariant under this rescaling, if one multiplies the constants C by $e^{|\omega|T}$.

3. Estimates involving (D)

In this and the following two sections, the time interval J will be compact; i.e., $J = [0,T]$ for some fixed positive T. We consider two families of operators $(A(t))_{t\in[0,T]}$ and $(B(t))_{t\in[0,T]}$ on the same space X. We assume throughout this and the next section that each family satisfies the Acquistapace–Terreni conditions or the Kato–Tanabe conditions with constants θ_A, ω_A, K_A, L_A, μ_A, ν_A and θ_B, ω_B, K_B, L_B, μ_B, ν_B, respectively. We shall write $\theta = \min(\theta_A, \theta_B)$, $\omega = \max(\omega_A, \omega_B)$, $K = \max(K_A, K_B)$, $L = \max(L_A, L_B)$, $\mu = \min(\mu_A, \mu_B)$, $\delta = \min(\mu_A + \nu_A - 1, \mu_B + \nu_B - 1)$ and $\gamma = \delta/(1+\delta)$. Note that $0 < \delta \le 1$. The constants C may depend on these parameters and parameters appearing in (AT)′ and on T, but they do not otherwise depend on the families $(A(t))$ and $(B(t))$. Without loss of generality we shall assume in this and the following section that $\omega = 0$ (see Remark 2.1).

Let $(U(t,s))_{T\ge t>s\ge 0}$ and $(V(t,s))_{T\ge t>s\ge 0}$ be the respective evolution families. Our aim here is to show that $\|U(t,0) - V(t,0)\|$ is small for $0 < t \le T$ if $\|A(\tau)^{-1} - B(\tau)^{-1}\|$ is small for $0 \le \tau \le T$.

We write

$$\begin{aligned} g(\tau) &= \|A(\tau)^{-1} - B(\tau)^{-1}\|, \\ G_p &= \Big(\int_0^T g(\tau)^p \, d\tau\Big)^{1/p} \qquad (0 < p < \infty), \\ G_\infty &= \sup_{0\le\tau\le T} g(\tau) = \sup_{0\le\tau\le T} \|A(\tau)^{-1} - B(\tau)^{-1}\|. \end{aligned} \tag{3.1}$$

Note that it follows from (2.1) that

$$|g(\tau) - g(\tau')| \le C|\tau - \tau'|^\mu \tag{3.2}$$

for all $\tau, \tau' \geq 0$. We first observe that this regularity condition implies the following variant of the Gagliardo–Nirenberg inequality [10, p. 147], showing that the quantities G_p are comparable for different values of p.

Proposition 3.1. *Let $0 < p < \infty$. There are constants $C_p, c_p > 0$ such that*

$$c_p G_p \leq G_\infty \leq C_p \max\left(G_p^{\mu p/(1+\mu p)}, G_p\right).$$

Proof. The first inequality is trivial, with $c_p = T^{1/p}$. For the second inequality, suppose that $g(\tau) > C(T/2)^\mu$ for some $\tau \in [0, T]$ (throughout this proof, C is as in (3.2)). Then $g(\tau') \geq g(\tau) - C(T/4)^\mu \geq (1 - 2^{-\mu})\, g(\tau)$ whenever $|\tau - \tau'| \leq T/4$. Hence,

$$G_p \geq \left(\frac{T}{4}\right)^{1/p} \left(1 - 2^{-\mu}\right) g(\tau).$$

It follows that $G_\infty \leq C'_p G_p$ in this case.

On the other hand, suppose that $g(\tau) \leq C(T/2)^\mu$ for all $\tau \in [0, T]$. For each τ, either $\left[\tau - (g(\tau)/C)^{1/\mu}, \tau\right]$ or $\left[\tau, \tau + (g(\tau)/C)^{1/\mu}\right]$ is contained in $[0, T]$. Hence,

$$G_p^p \geq \int_0^{(g(\tau)/C)^{1/\mu}} (g(\tau) - Ch^\mu)^p \, dh = \frac{g(\tau)^{(1+\mu p)/\mu}}{C^{1/\mu}} \int_0^1 (1 - r^\mu)^p \, dr.$$

It follows that $g(\tau) \leq C''_p G_p^{\mu p/(1+\mu p)}$ in this case. □

In the next two propositions, we shall give some estimates for the difference of the two evolution families U and V over a short time interval from 0 to h and from $t-h$ to t. In the proofs of Theorems 3.4 and 4.7, we shall see that there are several situations in which can obtain estimates from h to $t-h$ or t. Then we choose h to give the optimal estimate overall.

Proposition 3.2. *There is a constant C such that*

(i) $\left\|U(h,0) - V(h,0)\right\| \leq C\left(h^\delta + \frac{g(0)^\eta}{h^\eta}\right)$ *whenever* $0 < h \leq T$ *and* $0 \leq \eta \leq 1$;

(ii) $\left\|(U(t,t-h) - V(t,t-h))U(t-h,0)\right\| \leq C\left(\frac{h^\delta + g(t)}{t-h}\right)$ *whenever* $0 < h < t \leq T$.

Proof. (i) By (EF6),

$$\left\|U(h,0) - e^{hA(0)}\right\| \leq Ch^\delta, \qquad \left\|e^{hB(0)} - V(h,0)\right\| \leq Ch^\delta.$$

By (2.3),

$$e^{hA(0)} - e^{hB(0)} = \frac{1}{2\pi i h} \int_\Gamma e^\lambda \left(R(\lambda/h, A(0)) - R(\lambda/h, B(0))\right) d\lambda$$

$$= \frac{1}{2\pi i h} \int_\Gamma e^\lambda A(0) R(\lambda/h, A(0)) \left(B(0)^{-1} - A(0)^{-1}\right) B(0) R(\lambda/h, B(0))\, d\lambda.$$

It follows that

$$\left\| e^{hA(0)} - e^{hB(0)} \right\| \le \frac{Cg(0)}{h}.$$

Moreover, $\left\| e^{hA(0)} - e^{hB(0)} \right\| \le C$, by (S2). Hence, $\left\| e^{hA(0)} - e^{hB(0)} \right\| \le Cg(0)^\eta / h^\eta$. This gives

$$\left\| U(h,0) - V(h,0) \right\| \le C\big(h^\delta + \frac{g(0)^\eta}{h^\eta}\big).$$

(ii) Note first that

$$\begin{aligned} &\left\| U(t,t-h)A(t-h)^{-1} - A(t-h)^{-1} \right\| \\ &\le \left\| \big(U(t,t-h) - e^{hA(t-h)}\big) A(t-h)^{-1} \right\| + \left\| e^{hA(t-h)} A(t-h)^{-1} - A(t-h)^{-1} \right\| \\ &\le C\big(h^\delta + h\big), \end{aligned} \tag{3.3}$$

by (EF6) and (2.5). There is a similar estimate for V. Now,

$$\begin{aligned} &\left\| (U(t,t-h) - V(t,t-h)) U(t-h,0) \right\| \\ &\le \left\| (U(t,t-h) - V(t,t-h)) A(t-h)^{-1} \right\| \, \left\| A(t-h) U(t-h,0) \right\| \\ &\le \Bigg(\left\| U(t,t-h)A(t-h)^{-1} - A(t-h)^{-1} \right\| + \left\| A(t-h)^{-1} - B(t-h)^{-1} \right\| \\ &+ \left\| B(t-h)^{-1} - V(t,t-h) B(t-h)^{-1} \right\| \\ &+ \left\| V(t,t-h) \left(B(t-h)^{-1} - A(t-h)^{-1} \right) \right\| \Bigg) \frac{C}{t-h} \\ &\le C\big(\frac{h^\delta + h + g(t-h)}{t-h}\big) \le C\big(\frac{h^\delta + g(t)}{t-h}\big) \end{aligned}$$

by (EF5), (3.3), (3.2) and the fact that $h \le T$. □

Proposition 3.3. *Suppose that* (D) *and* (2.8) *hold. Then there is a constant* C *such that*

$$\left\| V(t,h)(U(h,0) - V(h,0)) \right\| \le C\big(\frac{h^\delta + g(0)}{t-h}\big)$$

whenever $0 < h < t \le T$.

Proof. It follows from (D) that

$$V(t,h)(U(h,0) - V(h,0)) = -\left(\tfrac{\partial}{\partial h} V(t,h)\right) B(h)^{-1} (U(h,0) - V(h,0)).$$

By (2.8),

$$\big\|V(t,h)(U(h,0)-V(h,0))\big\| \le \frac{C}{t-h}\big\|B(h)^{-1}(U(h,0)-V(h,0))\big\|.$$

Now,

$$\begin{aligned}
&\big\|B(h)^{-1}(U(h,0)-V(h,0))\big\| \le \big\|\big(B(h)^{-1}-B(0)^{-1}\big)\,U(h,0)\big\| \\
&+\big\|\big(B(0)^{-1}-A(0)^{-1}\big)\,U(h,0)\big\| + \big\|A(0)^{-1}\left(U(h,0)-e^{hA(0)}\right)\big\| \\
&+\big\|A(0)^{-1}e^{hA(0)}-A(0)^{-1}\big\| + \big\|A(0)^{-1}-B(0)^{-1}\big\| \\
&+\big\|B(0)^{-1}-B(0)^{-1}e^{hB(0)}\big\| + \big\|B(0)^{-1}\left(e^{hB(0)}-V(h,0)\right)\big\| \\
&+\big\|\big(B(0)^{-1}-B(h)^{-1}\big)\,V(h,0)\big\| \\
&\le C\big(h^{\mu}+g(0)+h^{\delta}+h+g(0)+h+h^{\delta}+h^{\mu}\big) \le C\big(h^{\delta}+g(0)\big),
\end{aligned}$$

using (2.1), (EF5), (EF6) and (2.5), and the facts that $h \le T$ and $\delta \le \mu \le 1$. The result follows immediately. □

The following result gives an estimate for $\big\|U(t,0)-V(t,0)\big\|$ in many cases. Note that we make no assumptions that the operators are densely defined.

Theorem 3.4. *Suppose that $(A(t))_{t\in[0,T]}$ satisfies either the Acquistapace–Terreni or the Kato–Tanabe conditions, and $(B(t))_{t\in[0,T]}$ satisfies either the Acquistapace–Terreni conditions and* (AT)$'$ *or the Kato–Tanabe conditions. There is a constant C such that*

$$\big\|U(t,0)-V(t,0)\big\| \le C\min\left(G_\infty\left(1+|\log G_\infty|\right),1\right)$$

whenever $T/2 \le t \le T$.

Proof. First, we suppose additionally that $(B(t))$ satisfies (D) and (2.8). We shall write $W(t,s)=\frac{\partial}{\partial s}V(t,s)$. Then the function $\tau \mapsto V(t,\tau)U(\tau,s)$ is differentiable on (s,t) with derivative

$$\begin{aligned}
&W(t,\tau)U(\tau,0)+V(t,\tau)A(\tau)U(\tau,0) = W(t,\tau)\left(I-B(\tau)^{-1}A(\tau)\right)U(\tau,0)\\
&=W(t,\tau)(A(\tau)^{-1}-B(\tau)^{-1})A(\tau)U(\tau,0).
\end{aligned}$$

Hence,

$$\begin{aligned}
&V(t,t-h)\left(U(t-h,h)-V(t-h,h)\right)U(h,0)\\
&\qquad = \int_h^{t-h} W(t,\tau)(A(\tau)^{-1}-B(\tau)^{-1})A(\tau)U(\tau,0)\,d\tau. \quad (3.4)
\end{aligned}$$

Then, from this equation, (2.8) and (EF5),

$$\big\|V(t,t-h)(U(t-h,h)-V(t-h,h))U(h,0)\big\| \le C\int_h^{t-h}\frac{g(\tau)}{(t-\tau)\tau}\,d\tau \tag{3.5}$$

$$\le CG_\infty\int_h^{t-h}\frac{d\tau}{(t-\tau)\tau} = \frac{2CG_\infty}{t}\log(\frac{t}{h}-1) \tag{3.6}$$

whenever $0 < h < T/4$ and $T/2 \le t \le T$. Observe that

$$U(t,0)-V(t,0) = V(t,t-h)(U(t-h,h)-V(t-h,h))U(h,0)$$
$$+ V(t,h)\big(U(h,0)-V(h,0)\big) + (U(t,t-h)-V(t,t-h))\,U(t-h,0).$$

Now, Proposition 3.2 (ii), (3.6) and Proposition 3.3 give

$$\big\|U(t,0)-V(t,0)\big\| \le C\big(h^\delta + G_\infty + G_\infty\log(T/h)\big) \tag{3.7}$$

whenever $0 < h < T/4$ and $T/2 \le t \le T$.

If $G_\infty < 4^{-\delta}$, we let $h = TG_\infty^{1/\delta}$. This gives

$$\big\|U(t,0)-V(t,0)\big\| \le CG_\infty\,(1+|\log G_\infty|)$$

in this case. If $G_\infty \ge 4^{-\delta}$, one simply uses the estimate

$$\big\|U(t,0)-V(t,0)\big\| \le C.$$

Since we may interchange the roles of $A(t)$ and $B(t)$, the only case which has not been covered is when $(A(t))$ and $(B(t))$ satisfy the Acquistapace–Terreni conditions, and $(B(t))$ satisfies (AT)$'$ but the domains are not dense.

It is tedious but elementary to see that the Yosida approximations $A_n(t) := nA(t),\ R(n,A(t))$ and $B_n(t)$ satisfy the Acquistapace–Terreni conditions and $B_n(t)$ satisfy (AT)$'$ with constants which are independent of $n \ge 1$ (see Lemma 4.1), and therefore they generate evolution families U_n and V_n. Noting also that $g(t)$ is unchanged when $A(t)$ and $B(t)$ are replaced by $A_n(t)$ and $B_n(t)$, it follows from the first case above that

$$\big\|(U_n(t,0)-V_n(t,0))\big\| \le C\,(\min(G_\infty\,(1+|\log G_\infty|)\,,1)),$$

where C is independent of n. Letting $n\to\infty$ gives the result (see Proposition 4.4). □

Remarks 3.5. 1. When $(B(t))$ is autonomous, i.e., $B(t) = B$ for all t, one has $V(t,s) = e^{(t-s)B}$, and (AT)$'$, (D) and (2.8) all hold for elementary reasons. This case has been considered by Schnaubelt [37] (assuming also that $A(t)$ and B are densely defined). Instead of (3.7), he obtained an estimate of the form

$$\left\|U(t,0) - V(t,0)\right\| \le C\big(h^\delta + \frac{G_{\infty,\alpha}}{h}\big),$$

where $G_{\infty,\alpha} = \sup_{0\le\tau\le T}\left\|(-B)^\alpha\left(A(\tau)^{-1} - B^{-1}\right)\right\|$. Here, $\alpha \in (0,1]$ and $(-B)^\alpha$ is the fractional power of $-B$, and it is assumed that $D(A(t)) \subseteq D((-B)^\alpha)$.

2. In the proof of Theorem 3.4, we could estimate the integral in (3.5) in other ways, for example by $C_p G_p/h^{1/p}$ for $1 \le p < \infty$. Combined with Proposition 3.1, this leads to an estimate

$$\left\|U(t,0) - V(t,0)\right\| \le C_p \min\left(G_p^{\delta p/(1+\delta p)}, 1\right). \tag{3.8}$$

However, combining Theorem 3.4 with Proposition 3.1 gives

$$\left\|U(t,0) - V(t,0)\right\| \le C_{p,\eta} \min\left(G_p^{\eta}, 1\right),$$

whenever $0 < \eta < \mu p/(1+\mu p)$. This is sharper than (3.8) except in the case when $\nu = 1$.

3. It is immediate that Theorem 3.4 remains valid when the time interval $[T/2, T]$ is replaced by $[T_0, T]$ for any $T_0 \in (0, T]$ (with a different value of C). However, it is not valid if $[T/2, T]$ is replaced by $(0, T]$ (see Remark 4.5).

We note the following simple result in the case of bounded perturbations.

Proposition 3.6. *Suppose that $(A(t))_{t\in[0,T]}$ satisfies either the Acquistapace–Terreni or the Kato–Tanabe conditions, and $(B(t))_{t\in[0,T]}$ satisfies either the Acquistapace–Terreni conditions and* (AT)′ *or the Kato–Tanabe conditions. Suppose in addition that for each $t \ge 0$, $D(B(t)) = D(A(t))$, $D(B(t))$ is dense in X and $A(t) - B(t)$ is a bounded operator. There is a constant C such that*

$$\left\|U(t,0) - V(t,0)\right\| \le C \int_0^t \left\|A(\tau) - B(\tau)\right\| d\tau$$

whenever $0 \le t \le T$.

Proof. It follows from (D) and (3.4) that

$$U(t,0)x - V(t,0)x = \lim_{h\downarrow 0} \int_h^{t-h} V(t,\tau)(A(\tau) - B(\tau))U(\tau,0)x \, d\tau.$$

The estimate follows. □

4. Acquistapace–Terreni conditions without (AT)$'$

The aim of this section is to obtain an estimate similar to that in Theorem 3.4, assuming the Acquistapace–Terreni conditions without the additional assumption (AT)$'$. If we do not assume (AT)$'$, we have to proceed in a different way, and the penalty is that the estimate becomes somewhat weaker. The proof of Theorem 4.7 uses Yosida approximations, which also appear in the construction of the evolution family in [2].

So let $(A(t))_{t\in[0,T]}$ be a family of closed linear operators which satisfy the Acquistapace–Terreni conditions. We will again assume that $\omega = 0$, without loss of generality. For $n \geq 1$ (not necessarily an integer) and $t \in [0,T]$, define the *Yosida approximation* $A_n(t) \in \mathcal{L}(X)$ by

$$A_n(t) = nA(t)R(n, A(t)).$$

Note that $A_n(t)^{-1} = A(t)^{-1} - n^{-1}$ and $\|A_n(t)\| = \|n(nR(n, A(t)) - I)\| \leq (K+1)n$.

The proof of the following lemma is tedious but elementary (see [2, Lemmas 4.1 and 4.2]).

Lemma 4.1. *There is a constant c (depending only on the angle θ in* (SC) *and* (AT)*) such that $(A_n(t))_{t\in[0,T]}$ satisfies the Acquistapace–Terreni conditions with constants θ, cK, cL, μ and ν. In particular, the family $(A_n(t))_{t\in[0,T]}$ generates an evolution family $(U_n(t,s))_{T\geq t>s\geq 0}$ satisfying* (EF1)–(EF6) *with constants independent of $n \geq 1$.*

We shall need to estimate $\|U(t,0) - U_n(t,0)\|$ for $0 < t \leq T$.

Lemma 4.2. *There are constants C and C_k ($k \in \mathbb{N}$) such that*

- (i) $\|e^{\tau A_n(t)} - e^{\tau A(t)}\| \leq \frac{C}{n^\eta \tau^\eta}$ *whenever $t \in [0,T]$, $\tau > 0$, $n \geq 1$, $0 \leq \eta \leq 1$;*
- (ii) $\|(A_n(t)e^{\tau A_n(t)} - A(t)e^{\tau A(t)})A(t)^{-1}\| \leq \frac{C}{n^\eta \tau^\eta}$ *whenever $t \in [0,T]$, $\tau > 0$, $n \geq 1$, $0 \leq \eta \leq 1$;*
- (iii) $\|(A_n(t)^k e^{\tau A_n(t)} - A(t)^k e^{\tau A(t)})(A(t)^{-1} - A(s)^{-1})\| \leq \frac{C_k (t-s)^\mu}{n^\eta \tau^{k-\nu+\eta}}$ *whenever $k \in \mathbb{N}$, $0 \leq s \leq t \leq T$, $\tau > 0$, $n \geq 1$ and $0 \leq \eta \leq 1$.*

Proof. The following formula is easily deduced from (2.4):

$$\begin{aligned} &A_n(t)^k e^{\tau A_n(t)} - A(t)^k e^{\tau A(t)} \\ &\qquad = \frac{1}{2\pi i \tau^k} \int_\Gamma \frac{\lambda^k e^\lambda}{\lambda + n\tau} A(t) R\Big(\frac{\lambda n}{\lambda + n\tau}, A(t)\Big) A(t) R(\lambda/\tau, A(t))\, d\lambda. \end{aligned}$$

To obtain the estimates, one uses that $\|A(t)R(\lambda', A(t))\| \leq K+1$ for $\lambda' = \lambda n/(\lambda + n\tau)$ and $\lambda' = \lambda/\tau$ (from (SC), since $\lambda' \in \Sigma_\theta$ whenever $\lambda \in \Sigma_\theta$), and there exists $c > 0$ such that $|\lambda + n\tau| \geq c|\lambda|^{1-\eta} n^\eta \tau^\eta$ for all $\lambda \in \Sigma_\theta$ and $\eta \in [0,1]$. For (iii), the estimate (AT) is also used. □

Proposition 4.3. *Let $\eta \in (0, \delta)$. There is a constant C_η (depending only on η, T and the Acquistapace–Terreni constants) such that*

$$\left\|(U(t,0) - U_n(t,0))\, A(0)^{-1}\right\| \leq \frac{C_\eta}{n^\eta t^\eta} \tag{4.1}$$

whenever $n \geq 1$ and $0 < t \leq T$.

Proof. We follow the method of [2], but we do not vary the initial vector x when we replace $A(t)$ by its Yosida approximations. We fix $n \geq 1$. For $t \in [0,T]$, let $L(t) = A(t)e^{tA(t)}A(0)^{-1}$. Then $L : (0,T] \to \mathcal{L}(X)$ is continuous [2, Proposition 2.1] and

$$\left\|L(t)\right\| \leq \left\|e^{tA(t)}\right\| + \left\|A(t)e^{tA(t)}\left(A(0)^{-1} - A(t)^{-1}\right)\right\| \leq C,$$

by (S2) and (2.6). Here, C may depend on T and the Acquistapace–Terreni constants, but is independent of t. For $0 \leq \tau < t \leq T$, let

$$Q(t,\tau) = A(t)^2 e^{(t-\tau)A(t)}\left(A(t)^{-1} - A(\tau)^{-1}\right).$$

Let $B_\eta := B_\eta([0,T], X)$ be the space of all measurable functions $f : [0,T] \to X$ such that $\left\|f\right\|_{B_\eta} := \sup_{t\in[0,T]}\left\|t^\eta f(t)\right\| < \infty$. For $f \in B_\eta$, let

$$(Qf)(t) = \int_0^t Q(t,\tau)f(\tau)\,d\tau.$$

It is shown in [2, Proposition 2.6] that Q is a bounded linear operator on B_η, $I - Q$ is invertible, and $\left\|(I-Q)^{-1}\right\|_{\mathcal{L}(B_\eta)} \leq C$, where C may depend on η, T and the Acquistapace–Terreni constants, but not otherwise on $(A(t))$. Moreover, for each $x \in X$,

$$U(t,0)A(0)^{-1}x = A(t)^{-1}\left[(I-Q)^{-1}L(\cdot)x\right](t). \tag{4.2}$$

Similarly,

$$U_n(t,0)A(0)^{-1}x = A_n(t)^{-1}\left[(I-Q_n)^{-1}L_n(\cdot)x\right](t), \tag{4.3}$$

where

$$\begin{aligned} L_n(t) &= A_n(t)e^{tA_n(t)}A(0)^{-1}, \\ Q_n(t,\tau) &= A_n(t)^2 e^{(t-\tau)A_n(t)}\left(A_n(t)^{-1} - A_n(\tau)^{-1}\right), \\ (Q_n f)(t) &= \int_0^t Q_n(t,\tau)f(\tau)\,d\tau. \end{aligned}$$

The estimates given above for L and $(I-Q)^{-1}$ apply to L_n and $(I-Q_n)^{-1}$. Thus,

$$\sup_{n\geq 1}\sup_{0<t\leq T}\|L_n(t)\| < \infty, \qquad \sup_{n\geq 1}\|(I-Q_n)^{-1}\|_{\mathcal{L}(B_\eta)} < \infty. \tag{4.4}$$

We have to estimate $\|L_n(\cdot)x - L(\cdot)x\|_{B_\eta}$ for $x \in X$ and also $\|Q_n - Q\|_{\mathcal{L}(B_\eta)}$. Now,

$$\begin{aligned}\|L_n(t) - L(t)\| &\leq \|\big(A_n(t)e^{tA_n(t)} - A(t)e^{tA(t)}\big)A(t)^{-1}\| \\ &\quad + \|\big(A_n(t)e^{tA_n(t)} - A(t)e^{tA(t)}\big)\big(A(0)^{-1} - A(t)^{-1}\big)\| \\ &\leq \frac{C}{n^\eta t^\eta} + \frac{Ct^\mu}{n^\eta t^{1-\nu+\eta}} \leq \frac{C}{n^\eta t^\eta}.\end{aligned}$$

In the penultimate line, we have used Lemma 4.2 (ii) and (iii), and in the final line we have used the fact that $t^{\mu+\nu-1} = t^\delta \leq T^\delta$. Hence,

$$\|L_n(\cdot)x - L(\cdot)x\|_{B_\eta} \leq \frac{C\|x\|}{n^\eta}. \tag{4.5}$$

By Lemma 4.2 (iii),

$$\|Q_n(t,\tau) - Q(t,\tau)\| \leq \frac{C(t-\tau)^{\delta-1-\eta}}{n^\eta}.$$

For $f \in B_\eta$, it follows that

$$\|t^\eta\left((Q_nf)(t) - (Qf)(t)\right)\| \leq \frac{Ct^\eta}{n^\eta}\int_0^t \frac{(t-\tau)^{\delta-1-\eta}}{\tau^\eta}\|\tau^\eta f(\tau)\|\,d\tau \leq \frac{C}{n^\eta}\|f\|_{B_\eta},$$

since $0 < \eta < \delta$ and $0 < t \leq T$. Thus,

$$\|Q_n - Q\|_{\mathcal{L}(B_\eta)} \leq \frac{C}{n^\eta} \tag{4.6}$$

(see [2, Proposition 5.4]).

For $t \in (0,T]$, we obtain from (4.2) and (4.3) the following:

$$\begin{aligned}&\|t^\eta\left(U(t,0) - U_n(t,0)\right)A(0)^{-1}x\| \\ &\leq \|A(\cdot)^{-1}\left[(I-Q)^{-1}L(\cdot)x\right] - \left(A(\cdot)^{-1} - n^{-1}\right)\left[(I-Q_n)^{-1}L_n(\cdot)x\right]\|_{B_\eta} \\ &\leq \|A(\cdot)^{-1}[(I-Q)^{-1}L(\cdot)x - (I-Q_n)^{-1}L(\cdot)x]\|_{B_\eta} \\ &+ \|A(\cdot)^{-1}[(I-Q_n)^{-1}(L(\cdot)x - L_n(\cdot)x)]\|_{B_\eta} + \frac{1}{n}\|(I-Q_n)^{-1}L_n(\cdot)x\|_{B_\eta} \\ &\leq K\|(I-Q)^{-1}(Q_n-Q)(I-Q_n)^{-1}L(\cdot)x\|_{B_\eta} \\ &+ K\|(I-Q_n)^{-1}(L(\cdot)x - L_n(\cdot)x)\|_{B_\eta} + \frac{1}{n}\|(I-Q_n)^{-1}L_n(\cdot)x\|_{B_\eta} \leq \frac{C\|x\|}{n^\eta}.\end{aligned}$$

Here, we have used (SC), (4.4), (4.5), (4.6), and the inequality $n \geq n^\eta$. □

Proposition 4.4. *Suppose that $(A(t))_{t\in[0,T]}$ satisfies the Acquistapace–Terreni conditions, and let $\rho \in (0, \gamma\delta)$. There is a constant C_ρ such that*

$$\|U(t,0) - U_n(t,0)\| \leq \frac{C_\rho}{n^\rho}$$

whenever $n \geq 1$ and $T/2 \leq t \leq T$.

Proof. Let $\eta = \rho/\gamma \in (0, \delta)$, and let $h \in (0, T/4]$. Then

$$\begin{aligned} &U(t,0) - U_n(t,0) \\ &\quad = (U(t,h) - U_n(t,h))A(h)^{-1}A(h)U(h,0) + U_n(t,h)\,(U(h,0) - U_n(h,0)). \end{aligned}$$

By Proposition 4.3 (with $A(\tau)$ replaced by $A(\tau + h)$, and t by $t - h$),

$$\|(U(t,h) - U_n(t,h))\,A(h)^{-1}\| \leq \frac{C}{n^\eta (t-h)^\eta} \leq \frac{C4^\eta}{n^\eta T^\eta}.$$

By (EF5),

$$\|U_n(t,h)\| \leq C, \qquad \|A(h)U(h,0)\| \leq \frac{C}{h}.$$

By Proposition 3.2 (with V replaced by U_n and $g(0)$ by $1/n$),

$$\|U(h,0) - U_n(h,0)\| \leq C\big(h^\delta + \frac{1}{n^\eta h^\eta}\big).$$

Since $h \leq T/4$, this gives

$$\|U(t,0) - U_n(t,0)\| \leq C\big(h^\delta + \frac{1}{n^\eta h}\big).$$

Choosing $h = n^{-\eta/(1+\delta)}(T/4)$, we obtain the result. □

Remark 4.5. While Proposition 4.4 shows that $\|U(t,0) - U_n(t,0)\| \to 0$ as $n \to \infty$ uniformly for t in compact subsets of $(0, T]$, the convergence may not be uniform for small t. This may occur even in the autonomous case when $A(t) = A$ for all t. If A_n is the Yosida approximation of A, then $\lim_{t\downarrow 0}\|e^{tA_n}\| = 1$. However, there exists an operator A generating a bounded holomorphic C_0-semigroup such that $\limsup_{t\downarrow 0}\|e^{tA}\| > 1$ [7, Remark 3.7.10].

Now suppose that $(A(t))_{t\in[0,T]}$ and $(B(t))_{t\in[0,T]}$ both satisfy the Acquistapace–Terreni conditions. The first step in the proof of Theorem 4.7 is to observe that condition (D) holds in the case when $(B(t))_{t\geq 0}$ is a uniformly bounded family in $\mathcal{L}(X)$.

Lemma 4.6. *Suppose that $(B(t))_{t\geq 0}$ is a uniformly bounded family in $\mathcal{L}(X)$ satisfying the Acquistapace–Terreni conditions. Then* (D) *holds.*

Proof. First, note that it follows from (EF3) and (EF4) that

$$\|V(t,0)-I\| = \Big\|\int_0^t B(\tau)V(\tau,0)\,d\tau\Big\| \to 0$$

as $t \downarrow 0$. Moreover,

$$\|B(t)-B(0)\| \le \|B(t)\|\,\|B(0)^{-1}-B(t)^{-1}\|\,\|B(0)\| \to 0$$

as $t \downarrow 0$. Let $t>0$. For $h>0$,

$$\|V(t,h)-V(t,0)\| = \|V(t,h)(I-V(h,0))\| \to 0$$

as $h \downarrow 0$. It follows from (EF4) that

$$\frac{1}{h}\,(V(t,h)-V(t,0)) = -\frac{1}{h}\int_0^h V(t,h)B(\tau)V(\tau,0)\,d\tau \to -V(t,0)B(0)$$

in norm as $h \downarrow 0$.

A similar argument shows left-differentiability. □

Theorem 4.7. *Suppose that $(A(t))_{t\in[0,T]}$ and $(B(t))_{t\in[0,T]}$ satisfy the Acquistapace–Terreni conditions.*

(1) *Let $\eta \in (0,\gamma^2)$. There is a constant C_η such that*

$$\|U(t,0)-V(t,0)\| \le C_\eta \min\,(G_1^\eta, 1)$$

whenever $T/2 \le t \le T$.

(2) *Let $\eta \in (0,\gamma\delta/(1+\gamma\delta))$. There is a constant C'_η such that*

$$\|U(t,0)-V(t,0)\| \le C'_\eta \min\,(G_\infty^\eta, 1)$$

whenever $T/2 \le t \le T$.

Proof. 1. Consider the Yosida approximations $(A_n(t))$ and $(B_n(t))$, and let $h \in (0,t/2]$. By Lemma 4.6, (3.4), (EF5) and the fact that $\|B_n(\tau)\| \le Cn$,

$$\begin{aligned} &\big\|(U_n(t,h)-V_n(t,h))\,U_n(h,0)\big\| &(4.7)\\ &= \Big\|\int_h^t V_n(t,\tau)B_n(\tau)\left(A_n(\tau)^{-1}-B_n(\tau)^{-1}\right)A_n(\tau)U_n(\tau,0)\,d\tau\Big\| \\ &\le Cn\int_h^t \frac{\|A(\tau)^{-1}-B(\tau)^{-1}\|}{\tau}\,d\tau \le \frac{CnG_1}{h}. &(4.8)\end{aligned}$$

Moreover, by Proposition 3.2 (i),

$$\big\|V_n(t,h)\,(U_n(h,0)-V_n(h,0))\big\| \le C\big(h^\delta + \frac{G_\infty}{h}\big).$$

Hence,

$$\|U_n(t,0)-V_n(t,0)\| \le C\big(\frac{nG_1+G_\infty}{h} + h^\delta\big)$$

whenever $0 < h \le T/2 \le t \le T$. Now we choose $h = (nG_1 + g(0))^{1/(1+\delta)}$, obtaining

$$\left\|U_n(t,0) - V_n(t,0)\right\| \le C\left(nG_1 + G_\infty\right)^\gamma \le C\left(n^\gamma G_1^\gamma + G_\infty^\gamma\right),$$

when $nG_1 + G_\infty \le (T/2)^{1+\delta}$. On the other hand, this estimate also holds when $nG_1 + G_\infty > (T/2)^{1+\delta}$ since the left-hand side is bounded uniformly for $n \ge 1$ and $t \in [0,T]$. Now,

$$\begin{aligned}
&\left\|U(t,0) - V(t,0)\right\| \\
&\quad\le \left\|U(t,0) - U_n(t,0)\right\| + \left\|U_n(t,0) - V_n(t,0)\right\| + \left\|V_n(t,0) - V(t,0)\right\| \\
&\quad\le C_\rho\Big(\frac{1}{n^\rho} + n^\gamma G_1^\gamma + G_\infty^\gamma\Big),
\end{aligned}$$

whenever $n \ge 1$ and $0 < \rho < \gamma\delta$, by Proposition 4.4. If $G_1 \le 1$, choosing $n = G_1^{-\gamma/(\rho+\gamma)}$ gives

$$\left\|U(t,0) - V(t,0)\right\| \le C_\rho\big(G_1^{\rho\gamma/(\rho+\gamma)} + G_\infty^\gamma\big).$$

By Proposition 3.1, $G_\infty^\gamma \le CG_1^{\mu\gamma/(1+\mu)} \le CG_1^{\gamma^2} \le G_1^\eta$ if $G_1 \le 1$ and $\eta < \gamma^2$, since $\mu \ge \delta$. As ρ varies between 0 and $\gamma\delta$, $\rho\gamma/(\rho+\gamma)$ varies between 0 and $\delta\gamma/(1+\delta) = \gamma^2$. Hence,

$$\left\|U(t,0) - V(t,0)\right\| \le C_\eta G_1^\eta$$

if $G_1 \le 1$. Moreover, $\left\|U(t,0) - V(t,0)\right\|$ is bounded uniformly for $t \in (0,T]$, so this completes the proof.

2. The integral in (4.8) may alternatively be estimated by $G_\infty \log(t/h)$. This gives

$$\left\|U_n(t,0) - V_n(t,0)\right\| \le C\Big(nG_\infty \log\big(\frac{t}{h}\big) + \frac{G_\infty}{h} + h^\delta\Big)$$

whenever $0 < h \le T/2 \le t \le T$. When $G_\infty < 1/4$, we may choose $h = TG_\infty^{1/(1+\delta)}$ and obtain

$$\left\|U_n(t,0) - V_n(t,0)\right\| \le C\left(nG_\infty\left|\log G_\infty\right| + G_\infty^\gamma\right).$$

Hence,

$$\left\|U(t,0) - V(t,0)\right\| \le C\Big(nG_\infty\left|\log G_\infty\right| + G_\infty^\gamma + \frac{1}{n^\rho}\Big)$$

whenever $n \ge 1$ and $0 < \rho < \gamma\delta$. Now choosing $n = (G_\infty\left|\log G_\infty\right|)^{-1/(1+\rho)}$ gives

$$\left\|U(t,0) - V(t,0)\right\| \le C\big(\left(G_\infty\left|\log G_\infty\right|\right)^{\rho/(1+\rho)} + G_\infty^\gamma\big).$$

For $\eta \in (0, \gamma\delta/(1+\gamma/\delta))$, we may choose $\rho \in (0, \gamma\delta)$ such that $\rho/(1+\rho) > \eta$. Since $(x|\log x|)^{\rho/(1+\rho)} \le Cx^\eta$ and $x^\gamma < x^\eta$ for $0 < x < 1/4$, this gives

$$\big\|U(t,0) - V(t,0)\big\| \le C'_\eta G_\infty^\eta$$

if $G_\infty < 1/4$. Since the left-hand side is bounded, this completes the proof.

5. Maximal regularity

In this section, we estimate the difference of two evolution families under the condition that the generating families of operators have L^p-maximal regularity. The estimates of Sections 3 and 4 are based on estimates of integrals such as those in (3.4) and (4.7), and maximal regularity provides an alternative way to estimate that integral.

As an example, consider the case when $B = B(t)$ is independent of t and bounded. Then, as in (4.7), we have

$$\begin{aligned}&\Big(U(t-h,h) - e^{(t-h)B}\Big)\, U(h,0)x \\ &= \int_h^t Be^{(t-\tau)B}(A(\tau)^{-1} - B^{-1})A(\tau)U(\tau,0)x\,d\tau = \big((Be^{\cdot B}) * f\big)(t),\end{aligned}$$

where $f(t) := 1_{[h,T]}(t)(A(t)^{-1} - B^{-1})A(t)U(t,0)x$ $(x \in X)$. In the proof of Theorem 3.7, we estimated this convolution in terms of the norm of B. If the operator B is unbounded, then the kernel $Be^{\cdot B}$ is singular. In [37], an assumption was added which ensured that the convolution is absolutely convergent. In Sections 3 and 4, we estimated the convolution by changing the upper limit of the integral to $t-h$, so that the convolution kernel $Be^{\cdot B}$ is bounded. However, the singular kernel may still define a bounded convolution operator on $L^p(0,T;X)$, and this is exactly the definition of L^p-maximal regularity in the autonomous case. The definition has been extended to the nonautonomous case in [23] and [24], and we recall it here.

Let $T > 0$ and $p \in (1,\infty)$ be fixed, and let $(A(t))_{t\in[0,T]}$ be a family of closed linear operators on a Banach space X. We define a multiplication operator $\mathcal{A}_p$ on $L^p := L^p(0,T;X)$ by $D(\mathcal{A}_p) := \{f \in L^p(0,T;X) : f(t) \in D(A(t))$ almost everywhere and $t \mapsto A(t)f(t) \in L^p(0,T;X)\}$ and

$$(\mathcal{A}_p f)(t) := A(t)f(t) \qquad (f \in D(\mathcal{A}_p),\ t \in [0,T]).$$

Next, we define the right-shift generator $\mathcal{D}_p$ on $L^p(0,T;X)$ by

$$D(\mathcal{D}_p) := W_0^{1,p}(0,T;X) := \big\{f \in W^{1,p}(0,T;X) : f(0) = 0\big\}$$

and

$$(\mathcal{D}_p f)(t) := -\frac{d}{dt}f(t) \qquad (f \in D(\mathcal{D}_p),\ t \in [0,T]).$$

By abuse of notation, we will in the following simply write $\mathcal{A}$ and $\mathcal{D}$ instead of $\mathcal{A}_p$ and $\mathcal{D}_p$.

Definition 5.1. We say that the family $(A(t))_{t\in[0,T]}$ has L^p*-maximal regularity* if the sum $\mathcal{A}+\mathcal{D} : D(\mathcal{A})\cap D(\mathcal{D}) \to L^p(0,T;X)$ is bijective and is a closed operator on $L^p(0,T;X)$.

Remark 5.2. A family $(A(t))_{t\in[0,T]}$ of closed linear operators on a Banach space X has L^p-maximal regularity for some $1 < p < \infty$ if and only if for every $f \in L^p(0,T;X)$ there exists a unique $u \in W^{1,p}(0,T;X)\cap D(\mathcal{A})$ such that

$$\begin{cases} \frac{d}{dt}u(t) - A(t)u(t) = -f(t) \quad \text{for almost all } t \in [0,T], \\ u(0) = 0 \end{cases} \tag{5.1}$$

(i.e., the inhomogeneous nonautonomous Cauchy problem (5.1) has a strong solution u for every inhomogeneity $f \in L^p(0,T;X)$), and u depends continuously on f). Then $(\mathcal{A}+\mathcal{D})^{-1} \in \mathcal{L}(L^p)$. This follows from the definition and the closed graph theorem.

In the following, we let p' be the conjugate of $p : \frac{1}{p}+\frac{1}{p'} = 1$. We consider $L^{p'} := L^{p'}(0,T;X')$, where X' is the dual space of X, and the left-shift generator $\tilde{\mathcal{D}}$ on $L^{p'}(0,T;X')$, defined by $D(\tilde{\mathcal{D}}) = \{g \in W^{1,p'}(0,T;X') : g(T) = 0\}$ and

$$(\tilde{\mathcal{D}}g)(t) := \frac{d}{dt}g(t) \qquad (g \in D(\tilde{\mathcal{D}}),\ t \in [0,T]).$$

We identify $L^{p'}(0,T;X')$ with a norming subspace of the dual space $(L^p(0,T;X))'$ in the standard way, and then $\tilde{\mathcal{D}}$ is the part of the adjoint $\mathcal{D}'$ of $\mathcal{D}$ in $L^{p'}(0,T;X')$.

Given a family $(A(t))_{t\in[0,T]}$ of closed linear operators on a Banach space X and $p \in (1,\infty)$, consider the following hypotheses:

(MR1) The family $(A(t))_{t\in[0,T]}$ has L^p-maximal regularity.

(MR2) The family $(A(t))_{t\in[0,T]}$ satisfies (MR1), and there exists a bounded evolution family $(U(t,s))_{T\geq t>s\geq 0}$ such that for every $f \in L^p(0,T;X)$

$$(\mathcal{A}+\mathcal{D})^{-1}f = -U * f,$$

where $(U * f)(t) := \int_0^t U(t,s)f(s)\,ds$ $(t \in [0,T])$.

(MR3) The family $(A(t))_{t\in[0,T]}$ satisfies (MR1), and the adjoint of $(\mathcal{A}+\mathcal{D})^{-1}$ maps $L^{p'}(0,T;X')$ into $D(\tilde{\mathcal{D}})$, and $D(\mathcal{A})\cap D(\mathcal{D})$ is a core for $\mathcal{A}$.

Remarks 5.3. 1. If a family $(A(t))_{t\in[0,T]}$ of closed linear operators on a Banach space satisfies (MR1) and (MR2), we say that $(A(t))_{t\in[0,T]}$ *generates* the evolution family $(U(t,s))_{T\geq t\geq s\geq 0}$, which is uniquely determined by (MR2). We do not know whether (MR2) is automatically implied by (MR1).

2. Note that (MR2) implies that $\mathcal{A}+\mathcal{D}$ is the generator of a semigroup which is strongly continuous for $t>0$ and which is the evolution semigroup associated with the evolution family U (see [11]). Moreover, the adjoint of $(\mathcal{A}+\mathcal{D})^{-1}$ leaves $L^{p'}(0,T;X')$ invariant. So if it maps $L^p(0,T;X)'$ into $D(\mathcal{D}')$, then it maps $L^{p'}(0,T;X')$ into $D(\tilde{\mathcal{D}})$.

3. If $A(t)=A$ is independent of $t\in[0,T]$, and A is densely defined and has L^p-maximal regularity for some $1<p<\infty$ (i.e., A satisfies (MR1)), then A is the generator of a holomorphic semigroup $\left(e^{\tau A}\right)_{\tau>0}$, and (MR2) and (MR3) hold. Moreover, the operator A has L^q-maximal regularity for every $1<q<\infty$, by [19, Theorem 4.2] or [25, Proposition 2.4].

In the next lemma, we put

$$c_p := \|\mathcal{D}(\mathcal{A}+\mathcal{D})^{-1}\|_{\mathcal{L}(L^p)}, \tag{5.2}$$

if (MR1) holds for some $1<p<\infty$; if (MR3) holds, we put

$$\tilde{c}_p := \|\tilde{\mathcal{D}}((\mathcal{A}+\mathcal{D})^{-1})^{\tilde{}}\|_{\mathcal{L}(L^{p'})}, \tag{5.3}$$

where $((\mathcal{A}+\mathcal{D})^{-1})^{\tilde{}}$ is the restriction of $((\mathcal{A}+\mathcal{D})^{-1})'$ to $L^{p'}(0,T;X')$. The quantities c_p and $\tilde{c}_p$ are always finite, by the closed graph theorem.

Lemma 5.4. *Assume that $(A(t))_{t\in[0,T]}$ satisfies the hypotheses* (MR1) *and* (MR2)*. Let $M := \sup_{T\geq t>s\geq 0}\|U(t,s)\|$.*

(a) *For every $s\in[0,T)$ and every $x\in X$, the derivative $\frac{\partial}{\partial t}U(t,s)x$ exists for almost every $t\in(s,T)$, and*

$$\Big(\int_{s+h}^T \|\tfrac{\partial}{\partial t}U(t,s)x\|^p\,dt\Big)^{1/p} \leq \tfrac{c_p M}{h^{1/p'}}\|x\| \tag{5.4}$$

whenever $0<h<T-s$.

(b) *Assume in addition that $(A(t))_{t\in[0,T]}$ satisfies the hypothesis* (MR3)*. Then for every $t\in(0,T]$ and every $x'\in X'$, the derivative $\frac{\partial}{\partial s}U(t,s)'x'$ exists for almost every $s\in(0,t)$, and*

$$\Big(\int_0^{t-h} \|\tfrac{\partial}{\partial s}U(t,s)'x'\|^{p'}\,ds\Big)^{1/p'} \leq \tfrac{\tilde{c}_p M}{h^{1/p}}\|x'\| \tag{5.5}$$

whenever $0<h<t$.

Proof. (a) Let $s \in [0, T)$ and $h \in (0, T - s)$. Note that, for fixed $x \in X$ and every $T \geq t \geq s + h$, we have

$$U(t, s)x = \frac{1}{h} \int_s^{s+h} U(t, \tau)U(\tau, s)x \, d\tau = (U * f)(t),$$

where $f(t) := h^{-1}U(t, s)x1_{(s,s+h]}(t)$ $(t \in [0, T])$. Since we may choose $h > 0$ arbitrarily small, the first statement follows from hypothesis (MR2). Moreover, by (MR1) and (MR2),

$$\begin{aligned}\Big(\int_{s+h}^T \| \frac{\partial}{\partial t} U(t, s)x \|^p \, dt \Big)^{1/p} &\leq \Big(\int_s^T \| \frac{\partial}{\partial t} (U * f)(t) \|^p \, dt \Big)^{1/p} \\ = \| \mathcal{D}(\mathcal{A} + \mathcal{D})^{-1} f \|_{L^p} &\leq \frac{c_p}{h} \Big(\int_s^{s+h} \| U(t, s)x \|^p \, dt \Big)^{\frac{1}{p}} \leq \frac{c_p M}{h^{1/p'}} \| x \|.\end{aligned}$$

The proof of (b) is similar, since

$$(((\mathcal{A} + \mathcal{D})^{-1})' g)(s) = \int_s^T U(t, s)' g(t) \, dt$$

for $g \in L^{p'}(0, T; X')$. □

In the rest of this section, $(A(t))_{t \in [0,T]}$ and $(B(t))_{t \in [0,T]}$ will be two families of closed linear operators on X which both satisfy (MR1)–(MR3) for some $1 < p < \infty$. Denote by $(U(t, s))_{T \geq t > s \geq 0}$ (respectively, $(V(t, s))_{T \geq t > s \geq 0}$) the evolution families generated by $(A(t))_{t \in [0,T]}$ (respectively, $(B(t))_{t \in [0,T]}$).

For technical reasons we will in addition assume that

(R) the set $\varrho(\mathcal{A}) \cap \varrho(\mathcal{B})$ is nonempty.

We do not know whether (R) follows automatically from (MR1)–(MR3). Assuming (R), we may, by rescaling, assume without loss of generality in this section that $0 \in \varrho(\mathcal{A}) \cap \varrho(\mathcal{B})$. We further assume, with no essential loss, that $0 \in \varrho(A(t)) \cap \varrho(B(t))$ for all $t \in [0, T]$ and we define G_∞ by (3.1). For the general case, i.e., for arbitrary $\mu \in \varrho(\mathcal{A}) \cap \varrho(\mathcal{B})$, the constants in this section might change by multiplying with $e^{|\mathrm{Re}\, \mu| T}$ (compare with Remark 2.1). We put

$$\begin{aligned} M &:= \sup\{ \|U(t, s)\|, \|V(t, s)\| : 0 \leq s < t \leq T \}, \\ c_p &:= \max(\|\mathcal{D}(\mathcal{A} + \mathcal{D})^{-1}\|_{\mathcal{L}(L^p)}, \|\mathcal{D}(\mathcal{B} + \mathcal{D})^{-1}\|_{\mathcal{L}(L^p)}), \\ \tilde{c}_p &:= \max(\|\tilde{\mathcal{D}}((\mathcal{A} + \mathcal{D})^{-1})\tilde{}\|_{\mathcal{L}(L^{p'})}, \|\tilde{\mathcal{D}}((\mathcal{B} + \mathcal{D})^{-1})\tilde{}\|_{\mathcal{L}(L^{p'})}). \end{aligned}$$

The constants C in the following results of this section may depend on these constants, p and T.

Lemma 5.5. *There exists a constant C such that*

$$\|(\mathcal{A}+\mathcal{D})^{-1}-(\mathcal{B}+\mathcal{D})^{-1}\|_{\mathcal{L}(L^p)}\leq CG_\infty.$$

Proof. First observe that, since $L^{p'}(0,T;X')$ is a norming subspace of $L^p(0,T;X)'$, the hypothesis (MR3) implies that the operator $((\mathcal{B}+\mathcal{D})^{-1}\mathcal{D},D(\mathcal{D}))$, and *a fortiori* $((\mathcal{B}+\mathcal{D})^{-1}\mathcal{D},D(\mathcal{B})\cap D(\mathcal{D}))$, extends in a unique way to a bounded linear operator $\mathcal{E}$ on $L^p(0,T;X)$. By (MR3) again, the operator $I-\mathcal{E}$ is a bounded extension of $((\mathcal{B}+\mathcal{D})^{-1}\mathcal{B},D(\mathcal{B}))$, and

$$\|I-\mathcal{E}\|_{\mathcal{L}(L^p)}\leq\tilde{c}_p+1.$$

Next, observe that

$$(\mathcal{A}+\mathcal{D})^{-1}-(\mathcal{B}+\mathcal{D})^{-1}=(I-\mathcal{E})\left(\mathcal{A}^{-1}-\mathcal{B}^{-1}\right)\mathcal{A}(\mathcal{A}+\mathcal{D})^{-1},$$

which implies, by (MR1) and (MR3), the inequality

$$\begin{aligned}&\|(\mathcal{A}+\mathcal{D})^{-1}-(\mathcal{B}+\mathcal{D})^{-1}\|_{\mathcal{L}(L^p)}\\ &\leq\|I-\mathcal{E}\|_{\mathcal{L}(L^p)}\|\mathcal{A}^{-1}-\mathcal{B}^{-1}\|_{\mathcal{L}(L^p)}\|\mathcal{A}(\mathcal{A}+\mathcal{D})^{-1}\|_{\mathcal{L}(L^p)}\leq(\tilde{c}_p+1)G_\infty(c_p+1).\end{aligned}$$

Lemma 5.6. *There exists a constant C such that*

$$\sup_{t\in[0,T]}\sup_{\|x'\|\leq 1}\Big(\int_0^t\|U(t,s)'x'-V(t,s)'x'\|^{p'}\,ds\Big)^{1/p'}\leq CG_\infty^{1/p'}.$$

Proof. For $T\geq t>s\geq 0$, we put $K(t,s):=U(t,s)-V(t,s)$. By (MR1) and (MR2), the convolution operator

$$(\mathcal{K}f)(t):=\int_0^t K(t,s)f(s)\,ds\qquad(f\in L^p(0,T;X))$$

is bounded from $L^p(0,T;X)$ into $W^{1,p}(0,T;X)$ (with norm less than $2c_p$), and *a fortiori* into $L^\infty(0,T;X)$. By the Gagliardo–Nirenberg inequality (see Proposition 3.1, [10, p. 147] and [31, Theorem 1, p. 69]), (MR2) and Lemma 5.5, there exist constants C (varying from line to line) such that

$$\|\mathcal{K}f\|_{L^\infty}\leq C\|\mathcal{K}f\|_{L^p}^{1/p'}\|\mathcal{K}f\|_{W^{1,p}}^{1/p}\leq CG_\infty^{1/p'}\,\|f\|_{L^p}$$

for every $f\in L^p(0,T;X)$. On the other hand,

$$\begin{aligned}\|\mathcal{K}\|_{\mathcal{L}(L^p,L^\infty)}&=\sup_{\|f\|_p\leq 1}\sup_{t\in[0,T]}\Big\|\int_0^t K(t,s)f(s)\,ds\Big\|\\ &=\sup_{t\in[0,T]}\sup_{\|x'\|\leq 1}\Big(\int_0^t\|K(t,s)'x'\|^{p'}\,ds\Big)^{1/p'}.\end{aligned}$$

The claim follows. □

Theorem 5.7. *Let* $(A(t))_{t\in[0,T]}$ *and* $(B(t))_{t\in[0,T]}$ *be two families of closed linear operators on* X *satisfying* (MR1)–(MR3) *for some* $1 < p < \infty$*, and also satisfying* (R)*. Then there exists a constant* C *such that*

$$\|U(t,0) - V(t,0)\| \le C G_\infty^{1/(pp')}$$

whenever $T/2 \le t \le T$*.*

Proof. Recall from Lemma 5.6 that there exists c_1 such that for every $x' \in X'$

$$\sup_{t\in[0,T]} \Big(\int_0^t \|U(t,s)'x' - V(t,s)'x'\|^{p'}\, ds \Big)^{1/p'} \le c_1 G_\infty^{1/p'} \|x'\|.$$

By (MR3) and Lemma 5.4 (b), there exists a constant c_2 such that for every $x' \in X'$

$$\sup_{t\in[\frac12 T,T]} \Big(\int_0^{t-\frac14 T} \|\tfrac{\partial}{\partial s}(U(t,s)'x' - V(t,s)'x')\|^{p'}\, ds \Big)^{1/p'} \le c_2 \|x'\|. \tag{5.6}$$

The Gagliardo–Nirenberg inequality (see Proposition 3.1, [10, p. 147] and [31, Theorem 1, p. 69]; note that the inequality holds uniformly over intervals of length at least $T/4$) implies that

$$\|U(t,s)'x' - V(t,s)'x'\| \le C G_\infty^{1/(pp')} \|x'\| \tag{5.7}$$

whenever $0 \le s \le T/4$ and $T/2 \le t \le T$. The claim follows from this. □

Suppose that the families $(A(t))_{t\in[0,T]}$ and $(B(t))_{t\in[0,T]}$ satisfy (MR3) for some $1 < q < \infty$ as well as for p. Then we may replace the parameter p' by q' in equation (5.6) in the above proof, so the functions $U(t,\cdot)'x' - V(t,\cdot)x'$ belong to $L^{p'}(0, t-\frac14 T; X') \cap W^{1,q'}(0, t-\frac14 T; X')$. If we apply the Gagliardo–Nirenberg inequality to this case then we obtain the following estimate, in which $r > 1/(pp')$ if $q < p$.

Corollary 5.8. *In addition to the assumptions of Theorem 5.7, assume that the families* $(A(t))_{t\in[0,T]}$ *and* $(B(t))_{t\in[0,T]}$ *satisfy* (MR3) *for some* $1 < q < \infty$ *as well as for* p*. Then there is a constant* C *(depending on the constant in Theorem 5.7 and on* $\tilde c_q$*) such that*

$$\|U(t,0) - V(t,0)\| \le C G_\infty^r,$$

whenever $T/2 \le t \le T$*, where* $r = \max\{\frac{1}{pp'}, \frac{q'-1}{p'q'-p'+q'}\}$*.*

6. Exponential dichotomy

In this and the following section we will apply the results of Sections 3, 4 and 5 (i.e., the estimates obtained therein) to study the asymptotic behaviour of evolution families defined on half-lines. In particular, the time interval J will now be unbounded; i.e., $J = \mathbb{R}_+$.

The type of asymptotic behaviour which we will study in this section is exponential dichotomy. We consider only evolution families $U = (U(t,s))_{t \geq s \geq 0}$ which are strongly continuous for $t \geq s$, with $U(s,s) = I$. Recall that U is said to be *exponentially bounded* if there exist constants K and ω such that $\|U(t,s)\| \leq K e^{\omega(t-s)}$ whenever $t \geq s \geq 0$, and that an evolution family $(U(t,s))_{t \geq s \geq a}$ is said to have *exponential dichotomy* on $[a,\infty)$ if there exist a strongly continuous projection-valued function $P : [a,\infty) \to \mathcal{L}(X)$ and constants M and $\alpha > 0$ such that the following hold whenever $t \geq s \geq a$:

(ED1) $P(t)U(t,s) = U(t,s)P(s)$;
(ED2) $\|U(t,s)P(s)\| \leq M e^{-\alpha(t-s)}$;
(ED3) $U(t,s)$ maps $\ker P(s)$ onto $\ker P(t)$, and $\|U(t,s)x\| \geq M e^{\alpha(t-s)} \|x\|$ for all $x \in \ker P(s)$.

The following proposition is a variant of [11, Theorem 5.23], [21, Theorem 7.6.10] or [37, Proposition 2.3] for evolution families defined on $\mathbb{R}_+$. This theory relies on the assumption that the evolution families are strongly continuous for $t \geq s$.

Proposition 6.1. *Let $(U(t,s))_{t \geq s \geq 0}$ and $(V(t,s))_{t \geq s \geq 0}$ be two exponentially bounded, strongly continuous evolution families. Suppose that $(V(t,s))_{t \geq s \geq 0}$ has exponential dichotomy on $[0,\infty)$ and assume further that for some $T > 0$*

$$\lim_{s \to \infty} \sup_{t \in [s+\frac{1}{2}T, s+T]} \|U(s+t,s) - V(s+t,s)\| = 0. \tag{6.1}$$

Then U has exponential dichotomy on $[a,\infty)$ for some $a \geq 0$.

Proof. Assume that $(V(t,s))_{t \geq s \geq 0}$ has exponential dichotomy on $\mathbb{R}_+$, and let P, M and α be as in (ED1)–(ED3).

On $X = P(0)X \oplus (I - P(0))X$ we define a multiplication semigroup $(S(t))_{t \geq 0}$ by $S(t)x := S(t)(x_0 + x_1) := e^{-\alpha t}x_0 + e^{\alpha t}x_1$ ($x_0 \in P(0)X$, $x_1 \in (I - P(0))X$), and we define the evolution family $(\bar{V}(t,s))_{t \geq s}$ on $\mathbb{R}$ by

$$\bar{V}(t,s) = \begin{cases} V(t,s) & (t \geq s \geq 0), \\ V(t,0)S(-s) & (t \geq 0 \geq s), \\ S(t-s) & (0 \geq t \geq s). \end{cases}$$

It is straightforward to check that $(\bar{V}(t,s))_{t\geq s}$ has exponential dichotomy on $\mathbb{R}$, with the same constants M and α, and with projection $\bar{P}(\cdot)$ where

$$\bar{P}(t) = \begin{cases} P(t) & (t \geq 0), \\ P(0) & (t < 0). \end{cases}$$

For every $a \geq T$ we define an evolution family $(\bar{U}_a(t,s))_{t\geq s}$ on $\mathbb{R}$ by

$$\bar{U}_a(t,s) = \begin{cases} U(t,s) & (t \geq s \geq a), \\ U(t,a)\bar{V}(a,s) & (t \geq a \geq s), \\ \bar{V}(t,s) & (a \geq t \geq s). \end{cases}$$

Then we obtain the following estimates for $\big\|\bar{U}_a(s+T,s) - \bar{V}(s+T,s)\big\|$:
First case: $s \leq a - T$. By definition of $\bar{U}_a$ and $\bar{V}$ we have $\big\|\bar{U}_a(s+T,s) - \bar{V}(s+T,s)\big\| = 0$.
Second case: $a - T \leq s \leq a - \frac{1}{2}T$. We obtain

$$\begin{aligned} &\big\|\bar{U}_a(s+T,s) - \bar{V}(s+T,s)\big\| \\ &\leq \big\|U(s+T,a)V(a,s) - U(s+T,a)U(a,s)\big\| + \big\|U(s+T,s) - V(s+T,s)\big\| \\ &\leq (K_1+1)g(s), \end{aligned}$$

where $g(s) := \sup_{t\in[s+\frac{1}{2}T,s+T]}\big\|U(s+t,s)-V(s+t,s)\big\|$, and $K_1 := \sup\{\|U(s+t,s)\|, \|V(s+t,s)\| : s \geq 0, t \in [0,T]\}$.
Third case: $a - \frac{1}{2}T \leq s \leq a$. We obtain

$$\begin{aligned} \big\|\bar{U}_a(s+T,s) - \bar{V}(s+T,s)\big\| &= \big\|U(s+T,a)V(a,s) - V(s+T,a)V(a,s)\big\| \\ &\leq K_1 g(s), \end{aligned}$$

where g and K_1 are as above.
Fourth case: $s \geq a$. We obtain simply

$$\big\|\bar{U}_a(s+T,s) - \bar{V}(s+T,s)\big\| \leq g(s).$$

Putting all the estimates together, we obtain

$$\sup_{s\in\mathbb{R}}\big\|\bar{U}_a(s+T,s) - \bar{V}(s+T,s)\big\| \leq (K_1+1)\sup_{s\geq a-T} g(s).$$

The claim now follows from the assumption and [11, Theorem 5.23], [21, Theorem 7.6.10] or [37, Proposition 2.3]. □

Theorem 6.2. *Let* $(A(t))_{t\geq 0}$ *and* $(B(t))_{t\geq 0}$ *be two families of closed,* densely defined *linear operators on* X *which satisfy the Acquistapace–Terreni conditions or the Kato–Tanabe conditions. Assume that the evolution family*

$(V(t,s))_{t\geq s\geq 0}$ *generated by* $(B(t))_{t\geq 0}$ *has exponential dichotomy on* $[a,\infty)$ *for some* $a\geq 0$. *If*

$$\lim_{t\to\infty} \|R(\omega,A(t)) - R(\omega,B(t))\| = 0,$$

then the evolution family $(U(t,s))_{t\geq s\geq 0}$ *generated by* $(A(t))_{t\geq 0}$ *has exponential dichotomy on* $[b,\infty)$ *for some* $b\geq 0$.

Proof. Replacing the evolution family $(V(t,s))_{t\geq s\geq 0}$ by $(V(t+a,s+a))_{t\geq s\geq 0}$ (and similarly for U), we may without loss of generality assume that $a=0$. Note that the assumption that $A(t)$ and $B(t)$ are densely defined implies that U and V are strongly continuous for $t\geq s$, by (EF3). The claim now follows directly from Proposition 6.1 and either Theorem 3.4 (in the case of Kato–Tanabe conditions) or Theorem 4.7 (in the case of Acquistapace–Terreni conditions), applied to $e^{-\omega(t-s)}U(t,s)$ and $e^{-\omega(t-s)}V(t,s)$. □

Now we turn towards the corresponding result for maximal regularity. We need an extension of the definition of Section 5 to families on $\mathbb{R}_+$.

A family $(A(t))_{t\geq 0}$ of closed linear operators is said to satisfy (MR1) (respectively, (MR3)) *uniformly on* $\mathbb{R}_+$ if for some $T>0$ the families $(A(s+t))_{t\in[0,T]}$ satisfy (MR1) (respectively, (MR3)) for every $s\in\mathbb{R}_+$ and

$$\begin{aligned} c_p &:= \sup_{s\in\mathbb{R}_+} \|\mathcal{D}(\mathcal{A}_s+\mathcal{D})^{-1}\|_{\mathcal{L}(L_p)} < \infty, \\ \text{respectively,}\quad \tilde{c}_p &:= \sup_{s\in\mathbb{R}_+} \|\mathcal{D}'((\mathcal{A}_s+\mathcal{D})^{-1})'\|_{\mathcal{L}(L_{p'})} < \infty. \end{aligned}$$

Here, $\mathcal{A}_s$ is the multiplication operator on $L^p(0,T;X)$ associated with the family $(A(s+t))_{t\in[0,T]}$. If each family $(A(s+t))_{t\in[0,T]}$ satisfies (MR2), it is easy to see that the corresponding evolution families U_s satisfy

$$U_s(t_2+\tau, t_1+\tau) = U_{s+\tau}(t_2,t_1)$$

whenever $s\geq 0$ and $0\leq t_1 < t_2 \leq t_2+\tau\leq T$. Hence, there is an evolution family $(U(t,s))_{t>s\geq 0}$ such that $U(t,s) = U_s(t-s,0)$ whenever $0\leq s<t\leq s+T$.

Theorem 6.3. *Let* $(A(t))_{t\geq 0}$ *and* $(B(t))_{t\geq 0}$ *be two families of closed,* densely defined *linear operators on* X *satisfying* (MR1)–(MR3) *uniformly on* $\mathbb{R}_+$ *for some* $1<p<\infty$. *Assume that the respective evolution families* $(U(t,s))_{t\geq s\geq 0}$ *and* $(V(t,s))_{t\geq s\geq 0}$ *are exponentially bounded and strongly continuous, and* V *has exponential dichotomy on* $[a,\infty)$ *for some* $a\geq 0$. *Assume also that there exists* $\mu\in\bigcap_{t\geq a}(\varrho(A(t))\cap\varrho(B(t)))$ *such that* $R(\mu,A(\cdot)) - R(\mu,B(\cdot))$ *is bounded and strongly measurable on* $[a,\infty)$ *and*

$$\lim_{t\to\infty} \|R(\mu,A(t)) - R(\mu,B(t))\| = 0.$$

Then U has exponential dichotomy on $[b, \infty)$ for some $b \geq 0$.

Proof. As in the proof of Theorem 6.2 we may without loss of generality assume that $a = 0$. Noting that the assumptions imply hypothesis (R) of Section 5, the claim then follows from Theorem 5.7 and Proposition 6.1. □

Remark 6.4. In the context of Theorems 6.2 and 6.3, if V is exponentially stable then U is exponentially stable for $t \geq s \geq a$ for some a (see [37, Theorem 3.3]).

7. Long-time estimates

In this section, $J = \mathbb{R}_+$ will again be unbounded, and we shall consider two evolution families $(U(t,s))_{t>s\geq 0}$ and $(V(t,s))_{t>s\geq 0}$. The results of Sections 3, 4 and 5 provide various estimates for $\|U(s+t,s) - V(s+t,s)\|$ which are valid for $T/2 \leq t \leq T$ and hence for t in compact subsets of $(0,\infty)$. We now want to extend these to estimates for large time.

Proposition 7.1. *Let $(U(t,s))_{t>s\geq 0}$ and $(V(t,s))_{t>s\geq 0}$ be two evolution families on X, and suppose that $\|V(t,s)\| \leq 1$ whenever $t > s \geq 0$. Let $(t_r)_{r=0,1,\dots,m}$ be a finite, strictly increasing sequence such that*

$$\rho := \sum_{r=0}^{m-1} \|U(t_{r+1}, t_r) - V(t_{r+1}, t_r)\| < 1.$$

Then

$$\|U(t_m, t_0) - V(t_m, t_0)\| \leq \frac{\rho}{1-\rho}.$$

Proof. Let $\alpha_r = \|U(t_r, t_0) - V(t_r, t_0)\|$ and $\beta_r = \|U(t_r, t_{r-1}) - V(t_r, t_{r-1})\|$. Then

$$\begin{aligned} \alpha_{r+1} \leq &\|U(t_{r+1}, t_r) - V(t_{r+1}, t_r)\| \, \|U(t_r, t_0)\| \\ &+ \|V(t_{r+1}, t_r)\| \, \|U(t_r, t_0) - V(t_r, t_0)\| \leq \beta_{r+1}(1+\alpha_r) + \alpha_r. \end{aligned} \tag{7.1}$$

Moreover, $\alpha_1 = \beta_1$. Assume inductively that

$$\alpha_r \leq \sum_{k=1}^{r} \Big(\sum_{j=1}^{r} \beta_j \Big)^k. \tag{7.2}$$

By (7.1),

$$\alpha_{r+1} \leq \beta_{r+1} + \sum_{k=1}^{r} \beta_{r+1} \Big(\sum_{j=1}^{r} \beta_j \Big)^k + \sum_{j=1}^{r} \beta_j + \sum_{k=1}^{r-1} \Big(\sum_{j=1}^{r} \beta_j \Big) \Big(\sum_{j=1}^{r} \beta_j \Big)^k$$

$$\leq \sum_{j=1}^{r+1} \beta_j + \sum_{k=1}^{r} \Big(\sum_{j=1}^{r+1} \beta_j \Big) \Big(\sum_{j=1}^{r} \beta_j \Big)^k \leq \sum_{k=1}^{r+1} \Big(\sum_{j=1}^{r+1} \beta_j \Big)^k .$$

Thus, (7.2) is proved by induction, and the result follows immediately. □

Corollary 7.2. *Let $(A(t))_{t\geq 0}$ and $(B(t))_{t\geq 0}$ be two families of closed linear operators which satisfy the Acquistapace–Terreni or the Kato–Tanabe conditions, and let $(U(t,s))_{t>s\geq 0}$ and $(V(t,s))_{t>s\geq 0}$ be the respective evolution families generated by A and B. Suppose that $\|V(t,s)\| \leq 1$ whenever $t > s \geq 0$ and that at least one of the following conditions is satisfied:*

(a) *Each family $(A(t))$ and $(B(t))$ satisfies either the Kato–Tanabe conditions or the Acquistapace–Terreni conditions and* (AT)′, *and*

$$\sum_{k=0}^{\infty} \sup_{kT\leq \tau \leq (k+1)T} \big\| R(\omega, A(\tau)) - R(\omega, B(\tau)) \big\|^{\eta} < \infty$$

for some $\eta \in (0,1)$;

(b) *Both $(A(t))$ and $(B(t))$ satisfy the Acquistapace–Terreni conditions, and either $\sum_{k=0}^{\infty} \Big(\int_{kT}^{(k+1)T} \big\| R(\omega, A(\tau)) - R(\omega, B(\tau)) \big\| \, d\tau \Big)^{\eta} < \infty$ for some $\eta \in (0, \gamma^2)$; or $\sum_{k=0}^{\infty} \sup_{kT\leq\tau\leq(k+1)T} \big\| R(\omega, A(\tau)) - R(\omega, B(\tau)) \big\|^{\eta} < \infty$ for some $\eta \in (0, \gamma\delta/(1+\gamma\delta))$.*

Then

$$\lim_{s\to\infty} \sup_{t\geq T} \big\| U(s+t,s) - V(s+t,s) \big\| = 0.$$

Proof. Assume that (a) holds. For $a \geq 0$, let

$$G_{\infty}(a) = \sup_{a\leq\tau\leq a+T} \big\| R(\omega, A(\tau)) - R(\omega, B(\tau)) \big\|.$$

By Theorem 3.4 applied to the families $(A(t+a) - \omega)_{t\in[0,T]}$ and $(B(t+a) - \omega)_{t\in[0,T]}$, there is a constant C such that $\|U(t+a,a) - V(t+a,a)\| \leq C G_{\infty}(a)^{\eta}$ whenever $a \geq 0$ and $T/2 \leq t \leq T$.

Let $\varepsilon > 0$. By assumption, there exists m such that $\sum_{k=m}^{\infty} G_{\infty}(kT)^{\eta} < \varepsilon/4C(1+\varepsilon)$. Given $s > mT$ and $t \geq T$, choose $s = t_0 < t_1 < \cdots < t_m = s+t$ such that $T/2 \leq t_{r+1} - t_r \leq T$. Each interval $[kT, (k+1)T]$ intersects at most 4 intervals $[t_r, t_r + T]$, so $\sum_{r=0}^{m-1} G_{\infty}(t_r)^{\eta} < \varepsilon/C(1+\varepsilon)$. Hence, $\sum_{r=0}^{m-1} \|U(t_{r+1},t_r) - V(t_{r+1},t_r)\| < \varepsilon/(1+\varepsilon)$. By Proposition 7.1, $\|U(s+t,s) - V(s+t,s)\| < \varepsilon$.

The proof for case (b) is similar, using Theorem 4.7 instead of Theorem 3.4. □

Remarks 7.3. 1. It is easy to see that the conditions (a) and (b) in Corollary 7.2 are independent of T.

2. If the function $\tau \mapsto \big\|R(\omega, A(\tau)) - R(\omega, B(\tau))\big\|$ is decreasing, and $\eta > 0$, then the following are equivalent:

(a) $\sum_{k=0}^{\infty} \sup_{kT \le \tau \le (k+1)T} \big\|R(\omega, A(\tau)) - R(\omega, B(\tau))\big\|^{\eta} < \infty$;

(b) $\sum_{k=0}^{\infty} \Big(\int_{kT}^{(k+1)T} \big\|R(\omega, A(\tau)) - R(\omega, B(\tau))\big\| \, d\tau\Big)^{\eta} < \infty$;

(c) $\int_0^{\infty} \big\|R(\omega, A(\tau)) - R(\omega, B(\tau))\big\|^{\eta} \, d\tau < \infty$.

3. Suppose that $(B(t))$ is autonomous, so that $B(t) = B$ and $V(t,s) = e^{(t-s)B}$. Suppose that V is bounded. Then there is an equivalent norm on X such that V is contractive. Thus, Corollary 7.2 also applies in this case.

Using Theorem 5.7 and Corollary 5.8 instead of Theorems 3.4 and 4.7 in the proof of Corollary 7.2, one obtains the following corollary in the case of maximal regularity.

Corollary 7.4. *Let* $(A(t))_{t\ge 0}$ *and* $(B(t))_{t \ge 0}$ *be two families of closed linear operators satisfying* (MR1)–(MR3) *for some* $1 < p < \infty$ *uniformly on* $\mathbb{R}_+$ *and also satisfying* (MR3) *for some* $1 < q < \infty$ *uniformly on* $\mathbb{R}_+$. *Let* $(U(t,s))_{t>s\ge 0}$ *and* $(V(t,s))_{t>s\ge 0}$ *be the respective evolution families generated by* A *and* B. *Suppose that* $\big\|V(t,s)\big\| \le 1$ *whenever* $t > s \ge 0$ *and that*

$$\sum_{k=0}^{\infty} \sup_{kT \le \tau \le (k+1)T} \|R(\mu, A(\tau)) - R(\mu, B(\tau))\|^r < \infty,$$

where $\mu \in \bigcap_{t \ge 0} (\varrho(A(t)) \cap \varrho(B(t)))$ *and* $r = \max\{\frac{1}{pp'}, \frac{q'-1}{p'q'-p'+q'}\}$. *Then*

$$\lim_{s\to\infty} \sup_{t \ge T} \big\|U(s+t,s) - V(s+t,s)\big\| = 0.$$

Finally in this section, we consider asymptotic behaviour in cases when V is contractive but not exponentially stable, and we show that U has the same asymptotic behaviour as V under the assumptions of Corollary 7.2 or Corollary 7.4.

Let $\mathcal{E}$ be a closed, translation-biinvariant subspace of the space $\mathrm{BUC}(\mathbb{R}_+, X)$ of all bounded, uniformly continuous functions from $\mathbb{R}_+$ to X (equipped with the topology of uniform convergence). By definition, this means that, for $f \in \mathrm{BUC}(\mathbb{R}_+, X)$ and $t \ge 0$,

$$S(t)f \in \mathcal{E} \iff f \in \mathcal{E}. \tag{7.3}$$

Here, S denotes the left-shift semigroup on $\mathrm{BUC}(\mathbb{R}_+, X)$, so $S(t)f(s) = f(s+t)$. Examples of translation-biinvariant spaces include the space $C_0(\mathbb{R}_+, X)$ of all continuous functions $f : \mathbb{R}_+ \to X$ which vanish at infinity, and the space $\mathrm{AAP}(\mathbb{R}_+, X)$ of all asymptotically almost periodic functions (see [7, Section 4.7]).

We say that an evolution family $(U(t,s))_{t>s\geq 0}$ belongs to the class $(\mathcal{E})$, and we write $U \in (\mathcal{E})$, if $U(s+T+\cdot, s)x \in \mathcal{E}$ for all $s \geq 0$ and all $x \in X$. By (7.3), this condition is independent of $T > 0$.

Remark 7.5. For an arbitrary bounded evolution family U, the functions $U(s+T+\cdot, s)x$ may not be uniformly continuous. However, if U is a bounded evolution family generated by $(A(t))_{t\geq 0}$ satisfying the Acquistapace–Terreni or the Kato–Tanabe conditions, then, for $s \geq 0$ and $x \in X$, the function $t \mapsto U(s+T+t, s)x$ has bounded derivative $t \mapsto A(s+T+t)U(s+T+t, s+t)U(s+t, s)x$ and is therefore uniformly continuous.

Proposition 7.6. *Let U and V be two bounded evolution families. Suppose that $V \in (\mathcal{E})$ and*

$$\lim_{s\to\infty} \sup_{t\geq T} \big\| U(s+t, s) - V(s+t, s) \big\| = 0.$$

Then $U \in (\mathcal{E})$.

Proof. Let $s \geq 0$ and $x \in X$ be fixed, and let $u(t) = U(s+T+t, s)x$ $(t \geq 0)$. Let $\varepsilon > 0$. There exists $s' \geq s$ such that $\big\|U(s'+T+t, s') - V(s'+T+t, s')\big\| < \varepsilon$ for all $t \geq 0$. Let $v(t) = V(s'+T+t, s')U(s', s)x$. Then $v \in \mathcal{E}$ and $\big\|S(s'-s)u(t) - v(t)\big\| \leq M\varepsilon$ for all $t \geq 0$, where $M = \sup_{t\geq 0}\big\|U(s+t, s)x\big\| < \infty$. There is a function $\tilde{v} \in \mathrm{BUC}(\mathbb{R}_+, X)$ such that $S(s'-s)\tilde{v} = v$ and $\big\|\tilde{v}(t) - u(t)\big\| \leq M\varepsilon$ for all $t \geq 0$. Then $\tilde{v} \in \mathcal{E}$. Since $\mathcal{E}$ is closed, it follows that $u \in \mathcal{E}$, as required. □

Corollary 7.7. *Under the assumptions of Corollary* 7.2 *or Corollary* 7.4, *if $V \in (\mathcal{E})$, then $U \in (\mathcal{E})$.*

8. Discussion and example

8.1. **The Hilbert space case.** We have obtained estimates for the difference of two evolution families, and consequential results on asymptotic behaviour, under the assumption that the generating families of operators satisfy the Acquistapace–Terreni conditions, the Kato–Tanabe conditions or the maximal regularity assumptions. These conditions are mutually independent.

In the case of the Acquistapace–Terreni conditions and the Kato–Tanabe conditions, this is remarked in [2, Theorem 7.9], where other types of generation conditions are also discussed.

Maximal regularity can occur under very weak regularity conditions on the operators $(A(t))$ (weaker than (AT) or (KT)); we refer to the results in

[17, Théorème 2, p. 620], [20], [6, Theorem III.4.10.10] and [35, Theorem 2.5] for details. Conversely, the Acquistapace–Terreni or the Kato–Tanabe conditions do not imply maximal regularity, in general. This occurs even in the autonomous case; i.e., there exists an operator A which generates a holomorphic C_0-semigroup and which does not have maximal regularity (see [14], [26] and [28] for concrete and abstract examples).

However, we want to point out that such an example cannot be found in a Hilbert space X. In the autonomous case this is a simple consequence of (SC) and Plancherel's theorem (see [18]). In the nonautonomous case, by [24, Theorem 3.1, Theorem 3.2], the Acquistapace–Terreni conditions always imply L^p-maximal regularity for every $1 < p < \infty$ when the underlying space is a Hilbert space. It is not too difficult to check that the Acquistapace–Terreni conditions on $\mathbb{R}_+$ even imply (MR1) uniformly on $\mathbb{R}_+$. Moreover, one can verify by duality arguments that (AT)$'$ implies (MR3) uniformly on $\mathbb{R}_+$. Hence, in a Hilbert space, assuming the Acquistapace–Terreni conditions and (AT)$'$, the estimates in Theorem 3.4 *and* in Corollary 5.8 both hold. The exponent r in Corollary 5.8 is always less than $\frac{1}{2}$, but it may be arbitrarily close to $\frac{1}{2}$ when p is large and q is close to 1. Thus, Theorem 3.4 gives a sharper estimate than Corollary 5.8 whenever both are applicable. The exponent η of G_∞ in Theorem 4.7 (2) is always less than $\frac{1}{3}$, so Corollary 5.8 gives a sharper estimate than Theorem 4.7 when the L^p-maximal regularity conditions hold for all $1 < p < \infty$. In general (UMD)-spaces, the Acquistapace–Terreni conditions still imply L^p-maximal regularity, if the boundedness condition in (SC) is replaced by the stronger *R-boundedness*; we refer to [38, Satz 4.2.6] for this result, and to [12], [33] or [41] for the concept of R-boundedness.

8.2. **Parabolic partial differential equations.** In the last part of this article we want to apply our results to the following parabolic partial differential equation:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) + A(t,x,D)u(t,x) = 0, & (t,x) \in (0,\infty) \times \Omega, \\ B(t,x,D)u(t,x) = 0, & (t,x) \in (0,\infty) \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{8.1}$$

We make the following assumptions:

$$\Omega \subset \mathbb{R}^N \text{ is a bounded domain, } \partial\Omega \in C^2, \tag{8.2}$$

$$A(t,x,D)u(x) := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} a_{ij}(t,x) \frac{\partial}{\partial x_j} u(x) + a(t,x)u(x) \text{ in } \Omega, \tag{8.3}$$

$$B(t,x,D)u(x) = \sum_{i,j=1}^{N} a_{ij}(t,x)\nu_i(x)\tfrac{\partial}{\partial x_j}u(x) + b(t,x)u(x) \text{ in } \partial\Omega, \tag{8.4}$$

$$a_{ij} \in C(\mathbb{R}_+; C^1(\bar{\Omega})) \cap C^\mu(\mathbb{R}_+; C(\bar{\Omega})), \tag{8.5}$$

$$a_{ij}(t,x) = a_{ji}(t,x), \tag{8.6}$$

$$\sum_{i,j=1}^{N} a_{ij}(t,x)\xi_i\xi_j \geq \delta(t)|\xi|^2 \quad \text{for all } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^N, \tag{8.7}$$

$$a \in C^\mu(\mathbb{R}_+; C(\bar{\Omega})), \tag{8.8}$$

$$b \in C(\mathbb{R}_+; C^1(\partial\Omega)) \cap C^\mu(\mathbb{R}_+; C(\partial\Omega)) \text{ is positive, and} \tag{8.9}$$

$$u_0 \in L^p(\Omega) \text{ and } 1 < p < \infty. \tag{8.10}$$

Here, the vector $\nu(x) := (\nu_i(x))$ is the outer normal vector at the boundary point $x \in \partial\Omega$, $\delta \in L^\infty_{loc}(\mathbb{R}_+)$ is positive and nonzero almost everywhere, and $\mu \in (\frac{1}{2}, 1]$.

It has been shown in [42, Theorem 4.1] that under the above conditions, the family of operators on $L^p(\Omega)$ $(1 < p < \infty)$ defined by

$$D(A(t)) := \{u \in W^{2,p}(\Omega) : B(t,\cdot,D)u = 0 \text{ on } \partial\Omega\}, \tag{8.11}$$

$$A(t)u := A(t,\cdot,D)u \tag{8.12}$$

satisfies the Acquistapace–Terreni conditions and the condition (AT)$'$ (the latter by self-adjointness of $A(t)$ in $L^2(\Omega)$). Actually, in [42], the proof is only given for Neumann boundary conditions (i.e., $b = 0$), but the proof for arbitrary positive b, i.e., for Robin boundary conditions, is similar.

There are many results on the asymptotic behaviour of equation (8.1) in the case that the coefficients A and B are independent of t or at least periodic in time ([9], [16], [22], [29] and [30]). We mention here two propositions about the periodic case. By the Rellich–Kondrachov Theorem [4, Theorem 6.2], [31, Lemma, p.62], the first one (which may be compared with [7, Proposition 5.4.7]) applies in particular to the problem (8.1) in the case of periodic coefficients (see Theorem 8.3 below).

Proposition 8.1. *Let $(A(t))_{t\geq 0}$ be a T-periodic family of operators on a Banach space X. Assume that $A(t)$ has compact resolvent for every $t \geq 0$ and that the Acquistapace–Terreni or the Kato–Tanabe conditions hold. Assume further that the evolution family $(U(t,s))_{t>s\geq 0}$ generated by $(A(t))_{t\geq 0}$ is bounded. Then every orbit $U(s+\cdot,s)x$ $(s \in \mathbb{R}_+, x \in \overline{D(A(s))})$ is asymptotically almost periodic.*

Proof. Since $A(t)$ has compact resolvent, property (EF4) implies that $U(t,s)$ is compact whenever $t > s$.

Define the monodromy operators $V(s) := U(s+T, s)$ $(s \in \mathbb{R}_+)$. Since $V(s)^n = U(s+nT, s)$, the monodromy operators are power-bounded. This and the compactness of $V(s)$ imply that for every $x \in X$ the orbit $\{V(s)^n x : n \in \mathbb{N}\}$ is relatively compact in X, and it follows easily that the sequence $(V(s)^n x)_{n\in\mathbb{N}}$ is asymptotically almost periodic in the sense that its orbit under the left-shift operator is relatively compact in $l^\infty(\mathbb{N}, X)$. The result now follows from [9, Proposition 2.11]. □

The property of exponential dichotomy in the case of nonautonomous *periodic* problems also has a very simple characterization; see, for example, [30, Proposition 6.3.3] in the case of constant domains, and [36, Theorem 14] for the general case (see also [15, Theorem V.2.1]).

Proposition 8.2. *In addition to the assumptions* (8.2)–(8.10)*, assume that the coefficients* a_{ij}*,* a *and* b *are* T*-periodic in time. Let* $(U(t,s))_{t\geq s\geq 0}$ *be the associated evolution family. Then* U *has exponential dichotomy on* $\mathbb{R}_+$ *if and only if the intersection of the spectrum of monodromy operator* $U(T,0)$ *with the unit circle is empty.*

In applications it may happen that the coefficients are not time-independent or time-periodic. However, if the coefficients are asymptotically T-periodic in time in the sense that there exists a T-periodic function $\tilde{a}_{ij} \in C^{\tilde{\mu}}(\mathbb{R}_+; C(\bar{\Omega}))$ such that

$$\lim_{t\to\infty} \|a_{ij}(t,\cdot) - \tilde{a}_{ij}(t,\cdot)\|_\infty = 0,$$

and similarly for the coefficients a and b, then we may apply our results to this case. For example, we obtain the following application.

Theorem 8.3. *Consider the parabolic partial differential equation* (8.1) *together with the assumptions* (8.2)–(8.10) *on* $L^2(\Omega)$*. Assume that there exist* T*-periodic functions* $\tilde{a}_{ij} \in C(\mathbb{R}_+; C^1(\bar{\Omega})) \cap C^{\tilde{\mu}}(\mathbb{R}_+; C(\bar{\Omega}))$*,* $\tilde{a} \in C^{\tilde{\mu}}(\mathbb{R}_+; C(\bar{\Omega}))$ *and* $\tilde{b} \in C(\mathbb{R}_+; C^1(\partial\Omega)) \cap C^{\tilde{\mu}}(\mathbb{R}_+; C(\partial\Omega))$ *(where* $\tilde{\mu} \in (\frac{1}{2}, 1]$*) such that for some* $\eta \in (0,1)$

$$\tilde{a} \geq 0, \quad \tilde{b} \geq 0 \tag{8.13}$$

$$\tilde{a}_{ij} \text{ satisfies (8.6) and (8.7),} \tag{8.14}$$

$$\sum_{k\geq 0} \|a_{ij} - \tilde{a}_{ij}\|^\eta_{L^\infty((kT,(k+1)T)\times\Omega)} < \infty, \tag{8.15}$$

$$\sum_{k\geq 0} \|a - \tilde{a}\|^\eta_{L^\infty((kT,(k+1)T)\times\Omega)} < \infty, \tag{8.16}$$

$$\sum_{k\geq 0} \|b-\tilde{b}\|^{\eta}_{L^\infty((kT,(k+1)T)\times\partial\Omega)} < \infty. \tag{8.17}$$

Then the solution of (8.1) *is asymptotically almost periodic with values in* $L^2(\Omega)$.

Proof. If in (8.1) we replace the coefficients a_{ij}, a and b by the coefficients $\tilde{a}_{ij}$, $\tilde{a}$ and $\tilde{b}$, then the resulting parabolic partial differential equation leads to a T-periodic, nonautonomous Cauchy problem of the form (1.1) when we define the associated family of elliptic operators $(\tilde{A}(t))$ on $L^2(\Omega)$ as in (8.11) and (8.12). By (8.13) and (8.14), every $\tilde{A}(t)$ is dissipative, and the family $(\tilde{A}(t))$ satisfies the Acquistapace–Terreni conditions as well as $(\mathrm{AT})'$ on $L^2(\Omega)$ (see [42]). The corresponding evolution family $\tilde{U}$ is contractive. Moreover, by the Rellich–Kondrachov theorem [4, Theorem 6.2], [31, Lemma, p. 62], the operators $\tilde{A}(t)$ have compact resolvent. Thus, we may apply Proposition 8.1, and we obtain that every orbit of $\tilde{U}$ is asymptotically almost periodic.

Next, the conditions (8.15)–(8.17) together with [42, Section 4] imply that condition (a) of Corollary 7.2 is satisfied for the families $(A(t))$ and $(\tilde{A}(t))$. Hence, by Corollary 7.2 and Corollary 7.7, the evolution family U associated with problem (8.1) is asymptotically almost periodic in $L^2(\Omega)$. □

Remarks 8.4. 1. Actually the condition (a) of Corollary 7.2 is satisfied under weaker regularity convergence conditions than (8.15)–(8.17), and we have not presented the most general case. Weaker regularity in time can already be found in [42, Section 4]. The regularity conditions in the space variable (as well as the regularity of Ω) can be relaxed if one uses a variational formulation of the problem (8.1) and the operators $A(t)$, and then Gaussian estimates ([8, Theorems 4.8 and 5.3] and [32]) and a variant of the proof of [42, Theorem 4.1]. However, we will not go into details.
2. Problem (8.1) has also been studied assuming Kato–Tanabe conditions on the coefficients; we refer to [5], [27] and [40, Chapter 5].

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