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A DYNAMICAL APPROACH FOR THE STABILITY OF SECOND ORDER DISSIPATIVE SYSTEMS

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Abstract. We study the stability of nonlinear dissipative dynamical systems. New results concerning the convergence of a solution to a critical point are given in various situations.

1. Introduction

when dealing numerically with the minimization of a function $\phi : H \to \mathbb{R}$, or more generally with the calculation of the critical points of ϕ, one usually uses some process generating a sequence $(u_n)_{n\in\mathbb{N}}$ with properties like: $\lim_{t\to+\infty} \nabla\phi(u_n) = 0$ or, still better, $u_n \to \overline{u}$ as $n \to +\infty$ where $\overline{u}$ is a critical point of ϕ. If the discrete dependence of the sequence $(u_n)_{n\in\mathbb{N}}$ on step n can, at least formally, be turned into the continuous dependence on some parameter t, interpreted as the time, then the discrete process may become a continuous dynamical system with trajectories $t \to u(t)$, and the question now is the asymptotic behavior of $\nabla\phi(u(t))$ or $u(t)$ in relation with the critical points of ϕ. This passage from the discrete to the continuous is best illustrated by the steepest descent method, also known as the gradient method:

$$u_{i+1} - u_i + h\nabla\phi(u_i) = 0, \quad h > 0, \quad x_0 \in H,$$

whose continuous version is

$$u'(t) + \nabla\phi(u(t)) = 0, \quad u(0) = u_0 \in H.$$

A lot of work has been devoted to the continuous gradient equation. Indeed, let $S(t)$ be a strongly continuous semigroup of contractions on a closed convex subset C of a real Hilbert space H and let A be its generator. Bruck [3] introduced a simple condition on A, called demipositivity, that is sufficient to guarantee the existence of the weak limit of $S(t)x$ for each $x \in C$. It

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turns out that several important classes of maximal monotone operators are indeed demipositive. The most interesting classes are the subdifferentials of lower semicontinuous convex functions φ that assume a minimum in H, and single-valued everywhere defined operators A satisfying the condition $(Ax - Ay, x - y) \geq c\|Ax - Ay\|^2$ for some $c > 0$ and for which there exists $x_0 \in H$ such that $Ax_0 = 0$. As a consequence of the more general results in [3] it is shown that the method of steepest descent for convex functions converges weakly. More specifically, we have the following theorem:

Theorem. *Let φ be a lower semicontinuous proper convex function on H that assumes a minimum in H. If $S(t)$ is the semigroup generated by $\partial\varphi$ on $\overline{D(\partial\varphi)}$, then for every $x \in \overline{D(\partial\varphi)}$, $S(t)x$ converges weakly to a minimum point of φ.*

In [5], Jiang, Xu and Roach established necessary and sufficient conditions for the strong convergence of the continuous nonexpansive semigroups generated by accretive operators in Banach spaces and of the discrete steepest descent method for such operators. Quite recently, they presented in [6] analogous theorems for the weak convergence of such semigroups and iterative processes. In this connection, it may also be of interest to note that a conjecture of R.E. Bruck and S. Reich [4] concerning the weak convergence of the discrete steepest descent method for accretive operators has been verified by Jiang, H.K. Xu and Z.-B. Xu [7].

To cope with that problem, one is tempted to introduce a perturbation to the system, acting as a regularization in fact, and write the second order in time continuous problem

$$u''(t) + \lambda u'(t) + \nabla\phi(u(t)) = 0, \tag{1}$$

$$u(0) = u_0, \quad u'(0) = u_1. \tag{2}$$

Let us mention that the first to consider equations (1)-(2) in this context was B.T. Poljak [12]. He studied a two-step discrete algorithm called “heavy ball with friction” method, which may be interpreted as an explicit discretization of (1). The problem (1)-(2) modelizes the motion of a heavy material point $M(t) = (u(t), \phi(u(t)))$ sliding on a profile defined by ϕ, the damping term $\lambda u'(t)$ ($\lambda > 0$), corresponds to a viscous mechanical friction. The dynamical approach to iterative methods in optimization has many advantages: it provides a deep insight on the expected behavior of the method, and sometimes the techniques used in the continuous case can be adapted to obtain results for the discrete algorithm. On the other hand, a continuous dynamical system satisfying nice properties may suggest new iterative methods.

Our aim in this paper is to extend Bruck's theorem and to prove that it is a consequence of the asymptotic convergence of the solution of (1)-(2). We prove that if ϕ is convex and bounded from below then the trajectory $\{u(t),\, t \to +\infty\}$ is minimizing for ϕ, and if the infimum of ϕ on H is attained then $u(t)$ converges weakly towards a minimizer of ϕ. If, furthermore, ϕ is strongly convex then the convergence is strong. In the case when ϕ is a Morse function with precompact trajectories, we establish their asymptotic convergence. These positive results naturally raise the question whether, for any smooth ϕ, the trajectories of system (1)-(2) converge. Thanks to the works of Palis- de Melo [10] and Polácik- Rybakowski [12], we know that there exists a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ for which at least one trajectory does not converge as $t \to +\infty$. So, we give a sufficient condition on $\phi : \mathbb{R}^n \to \mathbb{R}$ under which every trajectory converges to an equilibrium as $t \to +\infty$.

Our paper is organized as follows: in Section 2, we prove the global existence and uniqueness of a solution to (1)-(2). In Section 3, we establish the convergence theorems when ϕ is convex or a Morse function with additional appropriate conditions. Finally, in Section 4, we give a sufficient condition on $\phi : \mathbb{R}^n \to \mathbb{R}$ under which every trajectory $(u(t))$ of (1)-(2) converges to a limit as $t \to +\infty$.

2. Global existence

Let H be a real Hilbert space, $\langle \cdot, \cdot \rangle$ denotes the associated inner product and $|\cdot|$ the corresponding norm. Let us consider a mapping $\phi : H \to \mathbb{R}$ and $\lambda > 0$ a real constant. We are interested in the existence of $u : [0, +\infty) \to H$ solution to (1)-(2). The energy, along every trajectory of (1), is defined by

$$E(t) = \tfrac{1}{2}|u'(t)|^2 + \phi(u(t)).$$

The existence result is:

Theorem 2.1. *Assume that $\phi : H \to \mathbb{R}$ satisfies: ϕ is continuously differentiable on H; ϕ is bounded from below on H; $\nabla\phi$ is Lipschitz continuous on the bounded subsets of H. Then, there exists a unique solution $u \in C^2([0, +\infty), H)$ of* (1)-(2), *furthermore the corresponding energy $E(t)$ is non-increasing and bounded from below, and hence converges to a limit E_∞. Moreover, $u' \in L^\infty(0, +\infty; H) \cap L^2(0, +\infty; H)$.*

Proof of Theorem 2.1. The existence and uniqueness of a local solution for (1)-(2) follows from the Cauchy-Lipschitz theorem. Let $u(t)$ be the corresponding maximal solution defined on $[0, T_{\max})$, to prove that $T_{\max} = +\infty$ we need to show that $u'(t)$ is bounded. Indeed, $E'(t) = -\lambda|u'(t)|^2 \leq 0$, hence $E(t) \le E(0)$ and since ϕ is bounded from below we obtain

$$\sup_{t\in[0,T_{\max})} |u'(t)| < +\infty.$$

Let us prove now that $u' \in L^\infty(0,+\infty;H) \cap L^2(0,+\infty;H)$. Since the energy is non-increasing and ϕ is bounded from below we have

$$\tfrac{1}{2}|u'(t)|^2 \le \tfrac{1}{2}|u_0|^2 + \phi(u_0) - \inf\phi.$$

Hence, $u' \in L^\infty(0,+\infty;H)$. On the other hand

$$\int_0^t |u'(s)|^2\,ds = \frac{1}{\lambda}(E(0) - E(t))$$

and hence

$$\int_0^{+\infty} |u'(s)|^2\,ds = \frac{1}{\lambda}(E(0) - E_\infty), \quad \text{and} \quad u' \in L^2(0,+\infty;H).$$

Theorem 2.2. *In addition to the hypotheses of Theorem* 2.1, *assume that ϕ is coercive, i.e.,* $\lim_{t\to+\infty}\phi(t) = +\infty$, (*or if* $u \in L^\infty(0,+\infty;H)$), *then we have*

$$\lim_{t\to+\infty} u'(t) = 0, \quad \lim_{t\to+\infty} u''(t) = 0, \quad \lim_{t\to+\infty} \nabla\phi(u(t)) = 0, \quad \lim_{t\to+\infty} \phi(u(t)) = E_\infty.$$

Proof. First, we remark that since $\nabla\phi$ is bounded on the bounded subsets of H, then $u'' \in L^\infty(0,+\infty;H)$. Now, the assertion $\lim_{t\to+\infty} u'(t) = 0$ follows from the fact that $u \in L^2(0,+\infty;H)$ and $u'' \in L^\infty(0,+\infty;H)$. Now, if we prove that $\lim_{t\to+\infty} u''(t) = 0$, then by (1) we get $\lim_{t\to+\infty} \nabla\phi(u(t)) = 0$.

We claim that $\lim_{t\to+\infty} -H_\phi(u(t))u'(t) = 0$ where H_ϕ is the Hessian of ϕ. Indeed, assume for a moment, that ϕ is C^2 (and hence u is C^3), then

$$u''' + \lambda u'' + H_\phi(u(t))u'(t) = 0.$$

Since $\nabla\phi$ is Lipschitz continuous on the bounded subsets of H, then H_ϕ is bounded on the bounded subsets of H and thanks to $u \in L^\infty(0,+\infty;H)$, $\lim_{t\to+\infty} u'(t) = 0$, our claim follows. Because of

$$(u'')' + \lambda(u'') = -H_\phi(u(t))u'(t) \quad \text{and} \quad \lim_{t\to+\infty} H_\phi(u(t))u'(t) = 0,$$

we deduce, after integration, that $\lim_{t\to+\infty} u''(t) = 0$.

When ϕ is only C^2, the idea is to replace u''' by $v_h(t) = \frac{u'(t+h)-u'(t)}{h}$. From the equation (1), at the points t and $t+h$, we obtain

$$v_h'(t) + \lambda v_h(t) = -\frac{\nabla\phi(u(t+h)) - \nabla\phi(u(t))}{h}.$$

Similar arguments as above permit us to write: $\lim_{t\to+\infty}(\sup_h |v_h(t)|) = 0$. Since, for all $t \geq 0$, $|u''(t)| \leq \sup_{h>0} |v_h(t)|$, we conclude that

$$\lim_{t\to+\infty} u''(t) = 0.$$

Remark 2.3. If the trajectory $u(t)$ was precompact for the norm topology in H, the assertions

$$\lim_{t\to+\infty} u'(t) = 0, \quad \text{and} \quad \lim_{t\to+\infty} \nabla\phi(u(t)) = 0$$

may be deduced from the classical LaSalle's invariance principle. In the next section, we give convergence theorems without compactness hypotheses. Getting rid of the compactness hypotheses in establishing convergence theorems is not a new idea, it was in fact developed by the author in [1].

3. Convergence theorems

In this section, in addition to the hypotheses in Theorem 2.1, we need to make further assumptions of topological nature (precompactness of the trajectory), or geometrical nature (ϕ convex), or differential nature (Morse function), to obtain convergence of the trajectories as $t \to +\infty$.

When $\phi : H \to \mathbb{R}$ is a convex function, the critical points are the global minima of ϕ. Hence $\{x \in H : \phi(x) = \inf \phi\}$ is a closed convex non empty subset of H which may contain an infinite number of elements. In the next theorem, we prove that if $\phi : H \to \mathbb{R}$ is a convex function which is C^1, with $\nabla\phi$ Lipschitz continuous on the bounded sets, ϕ bounded from below and $\{x \in H : \phi(x) = \inf \phi\} \neq \emptyset$, then every trajectory weakly converges to a global minimum of ϕ. More precisely

Theorem 3.1. *In addition to hypotheses of Theorem* 2.1, *assume that* $\{x \in H : \phi(x) = \inf \phi\} \neq \emptyset$*, then for all* $(u_0, u_1) \in H \times H$*, the unique solution of* (1)–(2) *satisfies: there exists* $\overline{u} \in \{x \in H : \phi(x) = \inf \phi\}$ *such that*

$$u(t) \rightharpoonup \overline{u} \text{ weakly in } H \text{ as } t \to +\infty$$

and

$$\lim_{t\to+\infty} \phi(u(t)) = \min \phi.$$

For the proof of Theorem 3.1, we need the two following lemmas:

Lemma 3.2 (Opial [9]). *Let* H *be a Hilbert space and* $u : [0, +\infty) \to H$ *be a function such that there exists* $S(\neq \emptyset) \subset H$ *such that:*

(*i*) $\forall t_n \to +\infty$ *with* $u(t_n) \rightharpoonup \overline{u}$ *weakly in* H*, we have* $\overline{u} \in S$*;*

(*ii*) $\forall z \in S$, $\lim_{t\to+\infty} |u(t) - z|$ *exists.*

Then, $u(t) \rightharpoonup \overline{u} \in S$*, as* $t \to +\infty$*.*

Lemma 3.3. *If* $f \in C^1([0,+\infty),\mathbb{R}_+)$ *satisfies*

$$f'(t) + \lambda f(t) \le g(t)$$

where $\lambda > 0$ *and* $g \in L^1([0,+\infty),\mathbb{R}_+)$*, then*

$$[f]_+ := \max(f,0) \in L^1([0,+\infty),\mathbb{R}),$$

and consequently $\lim_{t\to+\infty} f(t)$ *exists.*

Proof of Lemma 3.3. We have

$$e^{\lambda t} f'(t) + \lambda e^{\lambda t} f(t) \le e^{\lambda t} g(t).$$

Integrating we get

$$[f(t)]_+ \le e^{-\lambda t}[f(0)]_+ + e^{-\lambda t}\int_0^t e^{\lambda\tau} g(\tau)\,d\tau.$$

Now, thanks to Fubini's theorem we have

$$\begin{aligned}\int_0^{+\infty}\Big(\int_0^t e^{-\lambda(t-\tau)} g(\tau)\,d\tau\Big)\,dt &= \int_0^{+\infty}\int_\tau^{+\infty} e^{-\lambda(t-\tau)} g(\tau)\,dt d\tau \\ &= \frac{1}{\lambda}\int_0^{+\infty} g(\tau)\,d\tau < +\infty.\end{aligned}$$

Proof of Theorem 3.1. We apply Opial's lemma with $S = \{x \in H : \nabla\phi(x) = 0\}$. Because of Theorem 2.2 we know that $\lim_{t\to+\infty} \nabla\phi(u(t)) = 0$. If $u(t_n) \rightharpoonup z$ weakly, then noticing that

$$\phi(s) \ge \phi(u(t_n)) + \langle \nabla\phi(u(t_n)), s - u(t_n)\rangle \ \ \forall s \in H$$

we can pass to the lower limit to obtain

$$\forall s \in H, \quad \phi(s) \ge \phi(z),$$

that is, $z \in \{x \in H : \nabla\phi(x) = 0\} = S$.

Now, set $f(t) = \frac{1}{2}|u(t) - z|^2$, then we have

$$\begin{aligned}f''(t) + \lambda f'(t) &= |u'(t)|^2 + \langle u(t) - z, u''(t) + \lambda u'(t)\rangle \\ &= |u'(t)|^2 - \langle u(t) - z, \nabla\phi(u(t))\rangle.\end{aligned}$$

Since $z \in S$ and $\nabla\phi(z) = 0$, and by the monotonicity of $\nabla\phi$:

$$\langle u(t) - z, \nabla\phi(u(t))\rangle = \langle u(t) - z, \nabla\phi(u(t)) - \nabla\phi(z)\rangle \ge 0$$

which yields

$$f''(t) + \lambda f'(t) \le |u'(t)|^2.$$

Thanks to Lemma 3.3, we have $[f'(t)]_+ \in L^1(0,+\infty;\mathbb{R})$. To end the proof of Theorem 3.1, we have to show that $\lim_{t\to+\infty}\phi(u(t)) = \min\phi$. By the convexity inequality, we have

$$\phi(s) \geq \phi(u(t)) + \langle \nabla\phi(u(t)), s - u(t)\rangle.$$

Since $u(t) \rightharpoonup \overline{u}$ weakly in H and $\nabla\phi(u(t)) \to 0$ strongly in H, we deduce that

$$\phi(s) \geq \limsup_{t\to+\infty} \phi(u(t)) \geq \liminf_{t\to+\infty} \phi(u(t)) \geq \phi(\overline{u}).$$

Hence,

$$\phi(\overline{u}) = \min\phi = \lim_{t\to+\infty}\phi(u(t)). \qquad \square$$

It is worth completing Theorem 3.1 by a strong convergence theorem. We say that $\phi : H \to \mathbb{R}$ is strongly convex if for any $R > 0$ there exists $\beta_R : \mathbb{R}_+ \to \mathbb{R}_+$ with $\beta_R(t_n) \to 0 \Rightarrow t_n \to 0$, such that for any x, $y \in H$ with $|x| < R$, $|y| < R$ we have

$$\langle \nabla\phi(x) - \nabla\phi(y), x - y\rangle \geq \beta_R(|x - y|).$$

Theorem 3.4. *In addition to the hypotheses of Theorem* 3.1, *assume that* ϕ *is strongly convex, then each trajectory* $u(t)$ *of* (1)–(2) *is norm convergent as* $t \to +\infty$ *to the unique global minimizer* $\overline{u}$ *of* ϕ.

Proof. Let $u(t)$ be a trajectory of (1)–(2), we know that it is bounded: $\exists R > 0$, $\forall t \in [0,+\infty)$, $|u(t)| \leq R$. Since ϕ is strongly convex it has a unique minimizer $\overline{u} = \operatorname{argmin}\phi$ and we have

$$\langle \nabla\phi(\overline{u}) - \nabla\phi(u(t)), \overline{u} - u(t)\rangle \geq \beta_R(|u(t) - \overline{u}|).$$

Since $\nabla\phi(\overline{u}) = 0$, we deduce that

$$\beta_R(|u(t) - \overline{u}|) \leq \langle u''(t) + \lambda u'(t), \overline{u} - u(t)\rangle.$$

Because of Theorem 2.2, we know that $\lim_{t\to+\infty} u'(t) = \lim_{t\to+\infty} u''(t) = 0$ and since $u(t)$ is bounded, it follows that $\lim_{t\to+\infty}\beta_R(|u(t) - \overline{u}|) = 0$ and hence $u(t) \to \overline{u}$ strongly as $t \to +\infty$.

Remark 3.5. The results presented above could be generalized to cover the equation

$$u'' + \Lambda u' + \nabla\phi(u) = 0$$

where $\Lambda : H \to H$ is a bounded self-adjoint linear operator, which we assume to be elliptic: there is $\alpha > 0$ such that for any $x \in H$, $\langle \Lambda x, x\rangle \geq \alpha|x|^2$.

Now, to prove how we can deduce Bruck's theorem from Theorem 3.1. Let $\varepsilon > 0$ be a small real number and consider the problem

$$\varepsilon u'' + u' + \nabla\phi(u) = 0.$$

In the next theorem, we prove that when ϕ is convex, the convergence of the trajectories as $\varepsilon \to 0$ holds for the uniform convergence on $[0, +\infty)$, that is, for a smooth ϕ the theorem of Bruck [3] is obtained as a consequence of Theorem 3.1. More precisely:

Theorem 3.6. *Let $\phi : H \to \mathbb{R}$ be a C^1, convex, bounded from below function with $\nabla\phi$ Lipschitz continuous on bounded sets. Then for any $(u_0, u_1) \in H \times H$ and for any $\varepsilon > 0$, the solution u_ε of*

$$\varepsilon u_\varepsilon'' + u_\varepsilon' + \nabla\phi(u_\varepsilon) = 0, \qquad u_\varepsilon(0) = u_0, \;\; u_\varepsilon'(0) = u_1$$

converges uniformly on $[0, +\infty)$ to u the unique solution of

$$u' + \nabla\phi(u) = 0, \qquad u(0) = u_0$$

and we have

$$\|u_\varepsilon - u\|_{L^\infty(0,+\infty;H)} \le c\sqrt{\varepsilon},$$

where $c > 0$ is a positive constant.

If in addition, $\{x \in H : \phi(x) = \inf\phi\} \neq \emptyset$, then we have

$$u(\infty) =: \textit{weak} - \lim_{t\to+\infty} u(t) \;\; \textit{exists} \quad (\textit{Bruck's theorem}), \tag{3.1}$$

$$\textit{weak} - \lim_{t\to+\infty} u_\varepsilon(t) \to u(\infty) \;\; \textit{strongly in} \;\; H \;\; \textit{as} \; \varepsilon \to 0, \tag{3.2}$$

$$u_\varepsilon' \to u' \quad \textit{in} \quad L^2(0, +\infty; H) \;\; \textit{as} \; \varepsilon \to 0. \tag{3.3}$$

Proof. By the monotonicity of $\nabla\phi$ we have

$$\langle \nabla\phi(u_\varepsilon) - \nabla\phi(u), u_\varepsilon - u\rangle \ge 0,$$

or equivalently

$$\tfrac{1}{2}\frac{d}{dt}|u_\varepsilon - u|^2 + \varepsilon\langle u_\varepsilon'', u_\varepsilon - u\rangle \ge 0.$$

Integration from 0 to t of the above inequality, and then integration by parts yield

$$\tfrac{1}{2}|u_\varepsilon(t) - u(t)|^2 + \varepsilon\langle u_\varepsilon(t) - u(t), u_\varepsilon'(t)\rangle \le \varepsilon \int_0^t \langle u_\varepsilon'(s) - u'(s), u_\varepsilon'(s)\rangle\, ds. \tag{3.4}$$

Since

$$\frac{\varepsilon}{2}|u_\varepsilon'(t)|^2 + \phi(u_\varepsilon(t)) + \int_0^t |u_\varepsilon'(s)|^2\, ds = \frac{\varepsilon}{2}|u_1|^2 + \phi(u_0)$$

we deduce

$$|\varepsilon u_\varepsilon'(t)| \le c\sqrt{\varepsilon} \quad \forall t \ge 0,\ \forall \varepsilon \in (0,1], \quad \sup_{0<\varepsilon\le 1}\Big(\int_0^{+\infty} |u_\varepsilon'(t)|^2\,dt\Big) \le c.$$

Because $u' \in L^2(0,+\infty;H)$ we deduce that

$$\int_0^t \langle u_\varepsilon'(s) - u'(s), u_\varepsilon'(s)\rangle ds \le \int_0^t |u_\varepsilon'(t)|^2 dt + \Big(\int_0^t |u_\varepsilon'(t)|^2 dt\Big)^{\frac12}\Big(\int_0^t |u'|^2\,dt\Big)^{\frac12}$$

$$\int_0^t \langle u_\varepsilon'(s) - u'(s), u_\varepsilon'(s)\rangle\,ds \le c$$

for all $t > 0$ and $\varepsilon > 0$. Finally, from the above estimates we deduce

$$\tfrac12 |u_\varepsilon(t) - u(t)|^2 \le c\sqrt{\varepsilon}|u_\varepsilon(t) - u(t)| + c\varepsilon, \qquad 0 < \varepsilon \le 1$$

and consequently

$$\|u_\varepsilon - u\|_{L^\infty(0,+\infty;H)} \le c\sqrt{\varepsilon}. \tag{3.5}$$

Now, we turn to the proof of (3.1). For any $v \in H$, $\varepsilon > 0$, and $0 < s \le t < +\infty$, we have

$$\langle u(t) - u(s), v\rangle = \langle u(t) - u_\varepsilon(t), v\rangle + \langle u_\varepsilon(t) - u_\varepsilon(s), v\rangle + \langle u_\varepsilon(s) - u(s), v\rangle.$$

Thanks to (3.5) we have

$$|\langle u(t) - u(s), v\rangle| \le 2c\sqrt{\varepsilon}|v| + \langle u_\varepsilon(t) - u_\varepsilon(s), v\rangle$$

and the weak-limit of $u_\varepsilon(t)$ exists as $t \to +\infty$ by Theorem 3.1, hence

$$\limsup_{s,t\to+\infty} |\langle u(t) - u(s), v\rangle| \le 2c\sqrt{\varepsilon}|v|, \quad \forall \varepsilon > 0.$$

Thus, for any $v \in H$, $\lim_{t\to+\infty}\langle u(t), v\rangle$ exists and owing to the uniform boundedness principle the weak-$\lim_{t\to+\infty} u(t) := u(\infty)$ exists.

To prove (3.2), we have from (3.5)

$$|u_\varepsilon(t) - u(t)| \le c\sqrt{\varepsilon}.$$

As $t \to +\infty$, $u_\varepsilon(t) - u(t) \rightharpoonup u_\varepsilon(\infty) - u(\infty)$ weakly, and by the lower semicontinuity for the weak topology of the norm in H, we deduce that

$$|u_\varepsilon(\infty) - u(\infty)| \le c\sqrt{\varepsilon}$$

which proves the norm convergence of $u_\varepsilon(\infty)$ to $u(\infty)$ as $\varepsilon \to 0$, and (3.2).

From the energy estimate and letting $t \to +\infty$, we have

$$\phi(u_\varepsilon(\infty)) + \int_0^{+\infty} |u_\varepsilon'(s)|^2\,ds \le \frac{\varepsilon}{2}|u_1|^2 + \phi(u_0),$$

and hence

$$\limsup_{\varepsilon\to 0}\Big(\phi(u_\varepsilon(\infty))+\int_0^{+\infty}|u_\varepsilon'(s)|^2\,ds\Big)\le \phi(u_0). \tag{3.6}$$

From the convexity inequality

$$\phi(u(\infty))\ge\phi(u(t))+\langle\nabla\phi(u(t)),u(\infty)-u(t)\rangle$$

and since $\nabla\phi(u(t))\to 0$, we easily deduce that

$$\lim_{t\to+\infty}\phi(u(t))=\phi(u(\infty)). \tag{3.7}$$

Passing to the limit in the energy identity

$$\phi(u(t))+\int_0^t|u'(s)|^2\,ds=\phi(u_0)$$

we get

$$\phi(u_0)=\phi(u(\infty))+\int_0^{+\infty}|u'(s)|^2\,ds.$$

From (3.6)-(3.7), we obtain

$$\limsup_{\varepsilon\to 0}\Big(\phi(u_\varepsilon(\infty))+\int_0^{+\infty}|u_\varepsilon'(s)|^2\,ds\Big)\le\phi(u(\infty))+\int_0^{+\infty}|u'(s)|^2\,ds.$$

Because $\phi(u_\varepsilon(\infty))=\phi(u(\infty))=\min\phi$, we get

$$\limsup_{\varepsilon\to 0}\int_0^{+\infty}|u_\varepsilon'(s)|^2\,ds\le\int_0^{+\infty}|u'(s)|^2\,ds.$$

Since $u_\varepsilon'\rightharpoonup u'$ weakly in $L^2(0,+\infty;H)$, we conclude the strong convergence in $L^2(0,+\infty;H)$. □

Definition 3.7. $\phi: H\to\mathbb{R}$ is called a Morse function if it is C^2 and its Hessian $H_\phi(\overline{u})$ possesses a continuous inverse at every critical point $\overline{u}$.

For Morse functions we have the following convergence theorem:

Theorem 3.8. *Let $\phi: H\to\mathbb{R}$ be a Morse function with $\nabla\phi$ Lipschitz continuous on bounded sets. For $v:=(u_0,u_1)\in H\times H$, let u_v be the solution of*

$$u''(t)+\lambda u'(t)+\nabla\phi(u(t))=0,\quad u(0)=u_0,\;\; u'(0)=u_1.$$

For any v such that u_v is precompact, then $u_v(t)$ converges to a critical point of ϕ as $t\to+\infty$.

Proof. Let ω_v be the ω-limit set of the trajectory u_v defined by

$$\omega_v=\{y\in H:\;\;\exists(t_n)_{n\in\mathbb{N}},\; t_n\to+\infty\;\text{ and }\; u(t_n)\to y\;\text{ as }\; n\to+\infty\}.$$

We know that $\omega_v \subset \{x \in H : \nabla\phi(x) = 0\}$ and since ϕ is a Morse function, all the elements of $\{x \in H : \nabla\phi(x) = 0\}$ are isolated. But, as ω_v is a connected set and ω_v is a connected set contained in a set whose elements are all isolated, this means that $\omega_v = \{\overline{u}\}$ a singleton. Consequently, the trajectory u_v which is contained in a compact set and has a unique limit point, necessarily converges to this unique element $\overline{u} \in \{x \in H : \nabla\phi(x) = 0\}$.

4. Further convergence results

In Section 3, we established several convergence theorems in various situations. These positive results naturally raise the question whether, for any smooth ϕ, the trajectories of system (1)–(2) converge. The simpler first order system $u' + \nabla\phi(u) = 0$ has been studied earlier in the literature: if $\phi : \mathbb{R} \to \mathbb{R}$, any solution of $u' + \nabla\phi(u) = 0$ tends to an equilibrium point. As soon as $\phi : \mathbb{R}^n \to \mathbb{R}$ with $n \geq 2$ this becomes false in general even if $n = 2$ and $\phi \in C^\infty(\mathbb{R}^2, \mathbb{R})$, a counterexample due to Palis-de Melo [10] shows that the convergence can fail. By modifying and refining their construction, Polácik and Rybakowski [11] showed that there exists a function $H : \mathbb{R}^2 \to \mathbb{R}$ such that for every function $Z : \mathbb{R}^2 \to \mathbb{R}$ satisfying a certain condition and every metric on $\mathbb{R}^2$ (defined by some symmetric positive definite matrix valued function R on $\mathbb{R}^2$) the corresponding gradient equation

$$\xi' = v(\xi), \quad \xi \in \mathbb{R}^2,$$

where $v(\xi) = -(R(\xi))^{-1}\nabla(H + Z)(\xi)$, admits a bounded solution whose ω-limit set is diffeomorphic to the unit circle S^1.

In this section, we assume that $\phi : \mathbb{R}^n \to \mathbb{R}$ and we establish a sufficient condition under which every trajectory $u(t)$ of

$$u'' + u' + \nabla\phi(u) = 0, \quad u(0) = u_0, \;\; u'(0) = u_1,$$

converges to a limit as $t \to +\infty$. More precisely, we have

Theorem 4.1. *Assume that $\phi : \mathbb{R}^n \to \mathbb{R}$ is analytic, and let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ be a solution to (1)–(2). Then there exists $\overline{u} \in \{x \in \mathbb{R}^n : \nabla\phi(x) = 0\}$ such that*

$$\lim_{t\to+\infty} \left(\|u'(t)\| + \|u(t) - \overline{u}\|\right) = 0.$$

For the proof of Theorem 4.1, we need the following useful lemma:

Lemma 4.2. [8, Page 92] *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be an analytic function in a neighborhood of a point $\overline{u} \in \mathbb{R}^n$. Then there exists $\sigma > 0$ and $0 < \theta < \frac{1}{2}$ such that*

$$\|\nabla\phi(u)\| \geq |\phi(u) - \phi(\overline{u})|^{1-\theta}, \qquad \forall u \in \mathbb{R}^n \;\; \|u - \overline{u}\| < \sigma.$$

Proof of Theorem 4.1. By Theorem 2.2, we know that $\lim_{t\to+\infty}\|u'(t)\| = 0$. Let $v := (u_0, u_1)$ and u the solution of

$$u'' + u' + \nabla\phi(u) = 0, \quad u(0) = u_0, \;\; u'(0) = u_1,$$

then, it is well known that ω_v is a non-empty compact connected set and $\omega_v \subset \{x \in \mathbb{R}^n : \; \nabla\phi(x) = 0\}$. Let $a \in \omega_v$, then up to the change of variable $u = a + w$ and of function $\psi(w) = \phi(a+w) - \phi(a)$, we have $\nabla\psi(w) = \nabla\phi(u)$, and then $0 \in \omega_v$.

Let $\varepsilon > 0$ a positive real number and define

$$E_\varepsilon(t) := \tfrac{1}{2}\|u'(t)\|^2 + (1+\varepsilon)\phi(u(t)) + \varepsilon\langle\nabla\phi(u(t)), u'(t)\rangle.$$

A simple computation shows that

$$E'_\varepsilon(t) = -\|u'(t)\|^2 - \varepsilon\|\nabla\phi(u(t))\|^2 + \varepsilon\langle\nabla^2\phi(u(t))u'(t), u'(t)\rangle.$$

If $\varepsilon > 0$ is such that $\varepsilon\|\nabla^2\phi(u)\|_\infty < \frac{1}{2}$, then we get $E'_\varepsilon(t) \le 0$, and since $0 \in \omega_v$ we deduce that $E_\varepsilon(t) \to 0$ as $t \to +\infty$, $E_\varepsilon(t) \ge 0$ for all $t \ge 0$ and

$$-E'_\varepsilon(t) \ge \frac{\varepsilon}{2}\left(\|u'(t)\|^2 + \|\nabla\phi(u(t))\|^2\right). \tag{4.1}$$

Thanks to the Cauchy-Schwarz and Young inequalities we have

$$\begin{aligned}(E_\varepsilon(t))^{1-\theta} &\le c\left(\|u'\|^{2(1-\theta)} + |\phi(u)|^{1-\theta} + \|\nabla\phi(u)\|^{1-\theta}\|u'\|^{1-\theta}\right)\\ &\le c\left(\|u'\|^{2(1-\theta)} + |\phi(u)|^{1-\theta} + \|\nabla\phi(u)\| + \|u'\|^{\frac{1-\theta}{\theta}}\right),\end{aligned} \tag{4.2}$$

where $c > 0$ is a positive constant, and θ is the one given in Lemma 4.2.

Since $0 \in \omega_v$, there exists a sequence $t_n \to +\infty$ such that $u(t_n) \to 0$ as $n \to +\infty$. Hence, there exists $N > 0$ such that for $n \ge N$ we have

$$\|u(t_n)\| < \frac{\sigma}{2}, \qquad \frac{4c}{\theta\varepsilon}\left(E_\varepsilon(t_n)\right)^\theta < \frac{\sigma}{2}, \tag{4.3}$$

$$\|u'(t)\| \le 1 \quad \forall t \ge t_N. \tag{4.4}$$

Define $\overline{t} := \sup\{t \ge t_N : \|u(s)\| < \sigma \text{ on } [t_N, t]\}$. We claim that $\overline{t} = \infty$. Indeed, if $\overline{t} < \infty$, then by Lemma 4.2, (4.2) and (4.4) we have for all $t \in (t_N, \overline{t})$

$$(E_\varepsilon(t))^{1-\theta} \le 2c\left(\|u'\| + \|\nabla\phi(u)\|\right). \tag{4.5}$$

If there exists $t_0 \in \mathbb{R}_+$ such that $E_\varepsilon(t_0) = 0$, then $E_\varepsilon(t) = 0$ for all $t \ge t_0$ and thanks to (4.1) we deduce the existence of a stationary solution. Otherwise, because of (4.1), (4.5) we have

$$-\frac{d}{dt}\left(E_\varepsilon(t)\right)^\theta \ge \frac{\theta\varepsilon}{4c}\left(\|u'\| + \|\nabla\phi(u)\|\right) \quad \text{for all} \quad t \in (t_N, \overline{t}),$$

which yields by integration over $(t_N, \bar{t})$

$$\int_{t_N}^{\bar{t}} \|u'\| \, dt \leq \frac{4c}{\theta\varepsilon} \left(E_\varepsilon(t_N)\right)^\theta . \tag{4.6}$$

Now, by (4.3), (4.6) and since

$$\|u(\bar{t})\| \leq \int_{t_N}^{\bar{t}} \|u'\| \, dt + \|u(t_N)\|$$

we get $\|u(\bar{t})\| < \sigma$ which contradicts $\bar{t} = \sup\{t \geq t_N : \|u(s)\| < \sigma \text{ on } [t_N, t]\}$. Thus, (4.6) becomes

$$\int_{t_N}^{+\infty} \|u\| \, dt \leq \frac{4c}{\theta\varepsilon} \left(E_\varepsilon(t_N)\right)^\theta < \infty,$$

which implies the existence of the limit for $u(t)$ as $t \to +\infty$.

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