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SCATTERING AND SELF-SIMILAR SOLUTIONS FOR THE NONLINEAR WAVE EQUATION

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Abstract. We study scattering theory and self-similar solutions for the nonlinear wave equation $\Box u + \lambda |u|^{p-1}u = 0$ in two or three space dimensions under the assumption $p_0(n) < p < 1 + 4/(n-1)$, where $p_0(n)$ is the larger root of the equation $(n-1)p^2 - (n+1)p - 2 = 0$. The relation between the theory of scattering and that of self-similar solutions is considered from the point of view of asymptotically free solutions and asymptotically self-similar solutions.

1. Introduction

This paper is concerned with the asymptotic behavior of small solutions to the semilinear wave equation of the form

$$\Box u + F(u) = 0,\ t \in \mathbb{R},\ x \in \mathbb{R}^n. \tag{1.1}$$

Here u is a real-valued unknown function, $\Box = \partial_t^2 - \Delta$ and the nonlinear function F is assumed to be of the form

$$F(u) = \lambda |u|^{p-1}u\ (\lambda \in \mathbb{R},\ p > 1) \tag{1.2}$$

throughout this paper. We develop the study of the long-time behavior from the point of view of the existence of asymptotically free solutions and asymptotically self-similar solutions. More precisely, in the first half of this paper, we consider the scattering theory in two or three space dimensions and prove that if

$$p_0(n) := \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)} < p \le 1 + \frac{4}{n-1} \tag{1.3}$$

and initial data are small in the $X \times Y$-norm (defined in (3.2)–(3.3) below), then the integral equation (3.4) has a unique global solution and the solution

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has the asymptotic states in $X \times Y$ and is asymptotically free in the sense of (3.7). It is also proved that the wave operators are defined in a neighborhood of zero in $X \times Y$. As a result the scattering operator can be defined in a neighborhood of zero in $X \times Y$.

In the second half of this paper we study the existence of self-similar solutions and asymptotically self-similar solutions. It is proved in the case of $n = 2, 3$ that if $p_0(n) < p < 1 + 4/(n-1)$ and small initial data are homogeneous in a certain sense, then the integral equation (5.3) has a self-similar solution. Moreover, we show the existence of asymptotically (in time) self-similar solutions when initial data are asymptotically (in $\mathbb{R}^n$) homogeneous in the sense of (7.7).

Our basic method of proof is a contraction-mapping argument. For the purpose we make use of the $L^{q'}$-L^q estimate of Pecher [25] and the infinitesimal generators of the Lorentz group and the dilation operator. We also generalize some of the inequalities due to Li–Zhou [18] to have variants of the Sobolev embedding (Lemma 2.9) and radius-angular mixed-norm inequalities (Lemma 2.8). For the estimate of the interaction in the integral equations, such generalized Li–Zhou inequalities as in Lemma 2.9 have an advantage over the Hardy–Littlewood–Sobolev inequality in the respect that we can take into account the difference of behaviors of solutions near the characteristic cone and away from it.

Here we shall refer to previous results on scattering theory and the existence of self-similar solutions to (1.1)–(1.2). There are a number of papers concerning scattering theory for small data. Putting

$$u_\delta(t,x) := \delta^{2/(p-1)} u(\delta t, \delta x), \quad (\delta > 0)$$

for a solution u to (1.1)–(1.2), we see that u_δ also satisfies (1.1)–(1.2). Since the transformation $v(x) \mapsto \delta^{2/(p-1)} v(\delta x)$ leaves the $\dot{H}^\theta_2$-norm ($\theta := n/2 - 2/(p-1)$) invariant, we expect that the theory of scattering can be developed in a neighborhood of zero in $\dot{H}^\theta_2 \times \dot{H}^{\theta-1}_2$. Indeed, Lindblad–Sogge and Nakamura–Ozawa have proved that this is the case for $p \geq 1 + 4/(n-1)$ ($n \geq 2$) [19], [24]. For the lower power $p \leq 1 + 4/(n-1)$, in [27], [16] and [38], Pecher, Kubota–Mochizuki and Tsutaya have improved earlier works due to Strauss [35] and Mochizuki–Motai [21]–[23] in two or three space dimensions and filled the gap between $p_0(n)$ and the lower bound for p in [21]–[23] in the framework of classical solutions by the method of John [10]. In this paper we work with a subspace of $C(\mathbb{R}; \dot{H}^{\theta-\varepsilon}_2(\mathbb{R}^n))$ and give a new proof of the scattering for $p_0(n) < p \leq 1 + 4/(n-1)$. It would be worthwhile to mention that in Theorem 3.1 the initial data (f, g) and the asymptotic states $(f_\pm, g_\pm)$

take their values in the same space $X \times Y$. Similarly, in Theorem 4.1, the initial value (f_-, g_-) at $t = -\infty$ and the interacting state $(u(0), \partial_t u(0))$ also take their values in the same space $X \times Y$. Taking into consideration the fact that this property is not satisfied in [16], [27] and [38], the author believes that our space $X \times Y$ is more suitable for the study of scattering than that in [16], [27] and [38].

For the latter subject, Pecher [28] has succeeded in proving the existence of small self-similar solutions and asymptotically self-similar solutions to (1.1)–(1.2) for $n = 3$ and $p > (4 + \sqrt{13})/3 (> p_0(3))$. Ribaud–Youssfi [30] have announced that they have studied the same problem in the general space dimensions $n \geq 2$ and have shown the existence of small self-similar solutions and asymptotically self-similar solutions for

$$p > p_{\mathrm{mm}}(n) := \frac{n^2 + 3n - 2 + \sqrt{(n^2 + 3n - 2)^2 - 8n(n-1)}}{2n(n-1)} (> p_0(n))$$

with an additional and technical restriction on the range of the allowed values of p. The number $p_{\mathrm{mm}}(n)$ appeared in the scattering theory developed by Mochizuki–Motai. By modifying the method of John [10], Pecher [29] has recently refined his earlier result in [28] and shown the existence of self-similar solutions and asymptotically self-similar solutions for $n = 3$ and $p > p_0(3)$. Our proof is inspired by the earlier paper of Pecher [28], and we study the problem in the case $n = 2$ as well as $n = 3$ under the condition $p > p_0(n)$.

One observation on the relation between the scattering theory and self-similar solutions is made below Theorem 7.2 (see Remark (1)). There seems to be some room for further investigation on the relation between these two subjects.

Finally we mention that the number $p_0(n)$, which is the larger root of the equation $(n-1)p^2 - (n+1)p - 2 = 0$, is widely known to be the critical power for the global existence and nonexistence of small solutions to (1.1)–(1.2) (see, e.g., [5], [8], [10], [20], [31], [32], [39]).

We conclude this section by giving the notation used throughout this paper. Following Klainerman [11]–[12], we introduce several partial differential operators as follows: $\partial_0 = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ $L_j = t\partial_j + x_j \partial_t$ $(j = 1, \dots, n)$, $\Omega_{kl} = x_k \partial_l - x_l \partial_k$ $(1 \leq k < l \leq n)$, $L_0 = t\partial_t + x_1 \partial_1 + \cdots + x_n \partial_n$. These operators $\partial_0, \dots, \partial_n, L_1, \dots, L_n, \Omega_{12}, \dots, \Omega_{1n}, \Omega_{23}, \dots, \Omega_{n-1n}$ and L_0 are denoted by $\Gamma_0, \dots, \Gamma_\nu$ in this order, where $\nu = (n^2 + 3n + 2)/2$. On the other hand the standard operators $\partial_0, \partial_1, \dots, \partial_n$ are not so useful for the study of self-similar solutions, and we thus denote partial differential operators

$L_1, \dots, L_n, \Omega_{12}, \dots, \Omega_{1n}, \Omega_{23}, \dots, \Omega_{n-1n}$ and L_0 by $\dot\Gamma_1, \dots, \dot\Gamma_{\nu-n}$ in this order, mainly in Sections 5 and 7. In principle, the notation $\Gamma_0, \dots, \Gamma_\nu$ will be used in the sections where scattering theory is studied. For a multi-index $\alpha = (\alpha_0, \dots, \alpha_\nu)$ we denote $\Gamma_0^{\alpha_0} \cdots \Gamma_\nu^{\alpha_\nu}$ by Γ^α, and for $\alpha = (\alpha_1, \dots, \alpha_{\nu-n})$ we denote $\dot\Gamma_1^{\alpha_1} \cdots \dot\Gamma_{\nu-n}^{\alpha_{\nu-n}}$ by $\dot\Gamma^\alpha$. Moreover we shall need the operator $\omega = \sqrt{-\Delta}$.

It is also necessary to define the norm for $1 \le p, q < \infty$,

$$\begin{aligned} ||v(\cdot)||_{p,q} &:= ||v(r\zeta) r^{(n-1)/p}||_{L^p(\mathbb{R}^+; L^q(S^{n-1}))} \\ &= \Big(\int_0^\infty \Big(\int_{S^{n-1}} |v(r\zeta)|^q dS_\zeta \Big)^{p/q} r^{n-1} dr \Big)^{1/p}, \end{aligned} \tag{1.4}$$

where $r = |x|$ and $\zeta \in S^{n-1}$. If $p = q$, it is obvious that this norm coincides with the usual L^p norm. We also define

$$\begin{cases} ||v(\cdot)||_{p,\infty} = \Big(\displaystyle\int_0^\infty \Big(\sup_{\zeta \in S^{n-1}} |v(r\zeta)| \Big)^p r^{n-1} dr \Big)^{1/p} & \text{for } 1 \le p < \infty, \\ ||v(\cdot)||_{\infty,q} = \displaystyle\sup_{r>0} \Big(\int_{S^{n-1}} |v(r\zeta)|^q dS_\zeta \Big)^{1/q} & \text{for } 1 \le q < \infty. \end{cases} \tag{1.5}$$

These types of norms are effectively utilized for the existence theory of solutions to fully nonlinear wave equations in [17] and [18], and for the existence and scattering theory for semilinear wave equations in [9] and [39]. Let N be a nonnegative integer and Ψ be a characteristic function of a set of $\mathbb{R}^{n+1}$. We define the norms

$$\begin{cases} ||u(t,\cdot)||_{\Gamma,N,p,q,\Psi} := \displaystyle\sum_{|\alpha| \le N} ||\Psi(t,\cdot)\Gamma^\alpha u(t,\cdot)||_{p,q}, \\ ||u(t,\cdot)||_{\dot\Gamma,N,p,q,\Psi} := \displaystyle\sum_{|\alpha| \le N} ||\Psi(t,\cdot)\dot\Gamma^\alpha u(t,\cdot)||_{p,q}. \end{cases} \tag{1.6}$$

When $\Psi \equiv 1$ in (1.6), we omit the sub-index Ψ. When $p = q$, we omit q. When $N = 0$, we omit the sub-indices Γ and N or $\dot\Gamma$ and N. In sum, the notation of the norms in (1.6) is abbreviated to

$$||u(t,\cdot)||_{\Gamma,N,p,q,\Psi} = \begin{cases} ||u(t,\cdot)||_{\Gamma,N,p,q}, & \text{if } \Psi \equiv 1, \\ ||u(t,\cdot)||_{\Gamma,N,p,\Psi}, & \text{if } p = q, \\ ||u(t,\cdot)||_{p,q,\Psi}, & \text{if } N = 0. \end{cases} \tag{1.7}$$

The same rule is also true of the norm $||u(t,\cdot)||_{\dot\Gamma,N,p,q,\Psi}$. Consistently with this rule, we denote the usual L^p norm by $||u(t,\cdot)||_p$.

For $s \in \mathbb{R}$ and $1 \le p \le \infty$ we define the function space

$$L^{s,p} = L^{s,p}(\mathbb{R}^n) = \{\ f \in \mathcal{S}'(\mathbb{R}^n)\ :\ \mathcal{F}^{-1}[(1+|\xi|^2)^{s/2}\hat f] \in L^p(\mathbb{R}^n)\ \} \tag{1.8}$$

equipped with the norm $||f||_{L^{s,p}} := ||\mathcal{F}^{-1}[(1+|\xi|^2)^{s/2}\hat{f}]||_{L^p}$. $\mathcal{S}'$ is the set of all the tempered distributions on $\mathbb{R}^n$, the Fourier transform of f is denoted by $\hat{f}$ or $\mathcal{F}[f]$, and $\mathcal{F}^{-1}$ is the inverse Fourier transform. This norm $||f||_{L^{s,p}}$ should not be confused with the norm $||v||_{p,q}$ defined in (1.4). For a noninteger $s > 0$, $1 \le p < \infty$ and an open set $\Omega \subset \mathbb{R}^n$ we denote by $W^{s,p}(\Omega)$ the space of all the locally integrable functions f on Ω such that

$$||f||_{W^{s,p}(\Omega)} := \Big(\sum_{|\alpha| \le [s]} \|\partial^\alpha f\|^p_{L^p(\Omega)} + \sum_{|\alpha|=[s]} \iint_{\Omega\times\Omega} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|^p}{|x-y|^{n+\sigma p}} dx\, dy\Big)^{\frac{1}{p}} < \infty. \tag{1.9}$$

Here $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, by $[s]$ we mean the largest integer not greater than s and we have defined $\sigma := s - [s]$. Our definition of the fractional-order space $W^{s,p}(\Omega)$ is different from that given in [1] on page 205. But, for suitably nice domains Ω, $W^{s,p}(\Omega)$ defined above coincides with the fractional-order space given in [1, Definition 7.36] on page 205 (see [1, Theorem 7.48]). For an integer $s \ge 0$ and $1 \le p \le \infty$ we define $W^{s,p}(\Omega)$ as the usual Sobolev space of functions whose distributional derivatives up to order s belong to $L^p(\Omega)$. If $\Omega = \mathbb{R}^n$, we usually denote the norms $||f||_{L^p(\mathbb{R}^n)}$ and $||f||_{W^{s,p}(\mathbb{R}^n)}$ as $||f||_{L^p}$ and $||f||_{W^{s,p}}$, respectively. If $1 < p < \infty$ and $\delta > 0$, then we have for every $s \ge 0$

$$L^{s+\delta,p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n) \hookrightarrow L^{s-\delta,p}(\mathbb{R}^n) \tag{1.10}$$

by Theorem 7.63 (g) in [1]. If $p = 2$, then it is well-known that $L^{s,2}(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ for $s \ge 0$.

We shall work with the homogeneous Sobolev space $\dot{H}^s_r = \dot{H}^s_r(\mathbb{R}^n)$ $(s \in \mathbb{R}, 1 \le r < \infty)$. We refer the reader to the appendix of [6, page 502] and [7, page 569] for the definition of the homogeneous Sobolev space. Though the definition of the space $\dot{H}^s_r$ given in [6, page 502] and [7, page 569] allows $r = 1$, we shall work with the case $1 < r < \infty$ throughout this paper. Then we may use the Sobolev embedding $\dot{H}^{s_2}_{r_2} \hookrightarrow \dot{H}^{s_1}_{r_1}$ $(1/r_1 - s_1/n = 1/r_2 - s_2/n, 1 < r_2 < r_1 < \infty)$. Denoting by $\mathcal{Z} = \mathcal{Z}(\mathbb{R}^n)$ the set of all the Schwartz functions $v \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{v}$ and all the derivatives of $\hat{v}$ vanish at $\xi = 0$, we shall also employ the fact that $\mathcal{Z}$ is dense in $\dot{H}^s_r$ $(s \in \mathbb{R}, 1 < r < \infty)$. Furthermore, the duality $\dot{H}^s_r(\mathbb{R}^n)' = \dot{H}^{-s}_{r'}(\mathbb{R}^n)$ $(s \in \mathbb{R}, 1 < r < \infty, 1/r + 1/r' = 1)$ will be also employed. For any interval I and any Banach space B we denote by $C(I; B)$, $BC(I; B)$ and $C_w(I; B)$ the space of continuous, bounded continuous and weakly continuous functions, respectively, from I to B. We denote by $\alpha(p)$ the variable defined by $\alpha(p) = 1/2 - 1/p$. The variables $\beta(p)$

and $\gamma(p)$ are also defined by

$$\alpha(p) = \frac{2}{n+1}\beta(p) = \frac{1}{n-1}\gamma(p). \tag{1.11}$$

The Hölder-conjugate exponent of p is denoted by p': $1/p + 1/p' = 1$.

This paper is organized as follows. The next section is devoted to the proof of several inequalities which we shall effectively use in solving the associated integral equations. In Section 3 we solve the Cauchy problem for small data and prove that all the solutions thereby obtained are asymptotically free as $t \to \pm\infty$. In Section 4 we show the existence of the wave operator and scattering operator. In Section 5 our consideration is turned to the self-similar solutions, and we shall prove another global existence theorem. Section 6 deals with the case where initial data are homogeneous functions as in (6.1). We shall find that they give rise to self-similar solutions. In the final section the existence of asymptotically self-similar solutions is shown.

2. PRELIMINARIES

In this section we prove a number of lemmas which play central roles in the proof of our theorems. Let $\chi_1 = \chi_1(t,x)$ be the characteristic function of the set $\{\,(t,x) \in \mathbb{R}^{n+1} : |x| \le (1+|t|)/2\,\}$ and define $\chi_2 := 1 - \chi_1$. These two functions will be used for the study of the scattering problem. For the study of self-similar solutions we introduce the characteristic function $\chi_3 = \chi_3(t,x)$ of the set $\{\,(t,x) \in (0,\infty) \times \mathbb{R}^n : |x| \le t/2\,\}$ and define $\chi_4 := 1 - \chi_3$. Moreover, let $\Phi_a = \Phi_a(x)$ be the characteristic function of the set $\{\,x \in \mathbb{R}^n : |x| \ge a\,\}$ $(a > 0)$. These characteristic functions will be frequently used throughout this paper.

Lemma 2.1. *For any $s \in \mathbb{R}$, the following commutation relations hold*:

$$[L_j, \omega^s] = s\omega^{s-2}\partial_j\partial_t,\ \ j = 1, \dots, n, \tag{2.1}$$

$$[\Omega_{kl}, \omega^s] = 0,\ \ 1 \le k < l \le n, \tag{2.2}$$

$$[L_0, \omega^s] = -s\omega^s. \tag{2.3}$$

Proof. They all can be easily verified by means of the Fourier transformation. Thus we omit the details. □

Lemma 2.2. *Let $n \ge 1$. The inequality*

$$||\langle |t| - |\cdot|\rangle \partial_a u(t,\cdot)||_2 \le C||u(t,\cdot)||_{\Gamma,1,2}\ (a = 0, \dots, n) \tag{2.4}$$

holds for any u with $\Gamma^\alpha u \in C(\mathbb{R}; L^2(\mathbb{R}^n))$ $(|\alpha| \le 1)$. Here $\langle |t| - |x|\rangle = 1 + |\,|t| - |x|\,|$.

Proof. It is enough to show this inequality for any C^1-function with $\Gamma^\alpha u \in C(\mathbb{R}; L^2(\mathbb{R}^n))$ $(|\alpha| \le 1)$. Then (2.4) is a consequence of the fact that the differential operators ∂_a $(a = 0, \ldots, n)$ can be written as

$$\partial_j = \frac{tL_j + \displaystyle\sum_{k=1}^{n} x_k(x_j\partial_k - x_k\partial_j) - x_j L_0}{t^2 - |x|^2}, \ \partial_t = \frac{tL_0 - \displaystyle\sum_{j=1}^{n} x_j L_j}{t^2 - |x|^2}. \qquad (2.5)$$

Lemma 2.3. (1) *Assume $n \ge 2$, $1/q \ge 1/p - 1/n \ge 0$ and $1 \le p < q < \infty$. Then the following (a) and (b) hold.*

(*a*) *If $\Gamma^\alpha u \in C(\mathbb{R}; L^p(\mathbb{R}^n))$ $(|\alpha| \le 1)$, then the inequality*

$$||u(t,\cdot)||_{q,\chi_1} \le C(1+|t|)^{-n(1/p-1/q)}||u(t,\cdot)||_{\Gamma,1,p} \qquad (2.6)$$

holds.

(*b*) *If $u \in L^1_{loc}((0,\infty)\times\mathbb{R}^n)$ satisfies $t^\delta \dot{\Gamma}^\alpha u \in L^\infty((0,\infty); L^p(\mathbb{R}^n))$ $(|\alpha| \le 1)$ for some $\delta \ge 0$, then the inequality*

$$||u(t,\cdot)||_{q,\chi_3} \le Ct^{-n(1/p-1/q)-\delta} \sum_{|\alpha|\le 1} \operatorname*{ess\,sup}_{t>0} t^\delta ||\dot{\Gamma}^\alpha u(t,\cdot)||_p \qquad (2.7)$$

holds for almost all $t > 0$.

(2) *Assume $n \ge 2$, $n < p < \infty$. Then the following (c) and (d) hold.*

(*c*) *If $\Gamma^\alpha u \in C(\mathbb{R}; L^p(\mathbb{R}^n))$ $(|\alpha| \le 1)$, then the inequality*

$$||u(t,\cdot)||_{\infty,\chi_1} \le C(1+|t|)^{-n/p}||u(t,\cdot)||_{\Gamma,1,p} \qquad (2.8)$$

holds.

(*d*) *If $u \in L^1_{loc}((0,\infty)\times\mathbb{R}^n)$ satisfies $t^\delta \dot{\Gamma}^\alpha u \in L^\infty((0,\infty); L^p(\mathbb{R}^n))$ $(|\alpha| \le 1)$ for some $\delta \ge 0$, then the inequality*

$$||u(t,\cdot)||_{\infty,\chi_3} \le Ct^{-n/p-\delta} \sum_{|\alpha|\le 1} \operatorname*{ess\,sup}_{t>0} t^\delta ||\dot{\Gamma}^\alpha u(t,\cdot)||_p \qquad (2.9)$$

holds for almost all $t > 0$.

Proof. We first prove part (1). As far as (*a*) is concerned, it is enough to show (2.6) for any C^1-function u with $\Gamma^\alpha u \in C(\mathbb{R}; L^p(\mathbb{R}^n))$ $(|\alpha| \le 1)$. Following [12] and [18], we start with the Sobolev embedding on the ball in $\mathbb{R}^n$ (see [18] on page 1215),

$$||v||_{L^q(B_a)} \le Ca^{-n(1/p-1/q)} \sum_{|\alpha|\le 1} a^{|\alpha|}||\partial_x^\alpha v||_{L^p(B_a)}. \qquad (2.10)$$

Here by B_a we mean the ball in $\mathbb{R}^n$ with the center at the origin and the radius $a > 0$. The inequality (2.10) with $a = 1$ is just the Sobolev embedding on the unit ball, and we can obtain (2.10) for general values of a by scaling. Noting that (2.5) yields an inequality

$$
\begin{aligned}
|\partial_j u(t,x)| &\le \frac{|\partial_j u(t,x)| + \sum_{|\alpha|=1} |\dot{\Gamma}^\alpha u(t,x)|}{1+|\,|t|-|x|\,|} \\
&\le \frac{|\partial_j u(t,x)| + \sum_{|\alpha|=1} |\dot{\Gamma}^\alpha u(t,x)|}{(1+|t|)/2}
\end{aligned}
\tag{2.11}
$$

for $|x| \le (1+|t|)/2$ and choosing $a = (1+|t|)/2$, we obtain (2.6) from (2.10).

Next we prove (b). We choose an arbitrary positive number σ and fix it. When $(t,x) \in (0,\infty) \times \mathbb{R}^n$ satisfies $|x| < t/2$, (2.5) yields an inequality

$$
|\partial_j u(t,x)| \le Ct^{-1} \sum_{|\alpha|=1} |\dot{\Gamma}^\alpha u(t,x)| \tag{2.12}
$$

for any $u \in C^1((0,\infty) \times \mathbb{R}^n)$. Therefore we have by choosing $a = t/2$ in (2.10)

$$
||u(t,\cdot)||_{q,\chi_3} \le Ct^{-n(1/p-1/q)-\delta} \sup_{t>\sigma} t^\delta ||u(t,\cdot)||_{\dot{\Gamma},1,p} \tag{2.13}
$$

for $t > \sigma$ if $u \in C^1((\sigma,\infty) \times \mathbb{R}^n)$ satisfies $t^\delta \dot{\Gamma}^\alpha u \in L^\infty((\sigma,\infty); L^p(\mathbb{R}^n))$ $(|\alpha| \le 1)$. Making use of the standard approximation method and (2.13), we shall show (b). Let us choose a smooth function $\eta = \eta(t,x)$ such that $\operatorname{supp} \eta \subset \{\, (t,x) \in \mathbb{R}^{n+1} : |(t,x)| < 1 \,\}$, $\eta \ge 0$ and $\int_{\mathbb{R}^{n+1}} \eta \, dx \, dt = 1$. Define $\eta_j(t,x) = j^{n+1} \eta(jt, jx)$ for $j = 1, 2, \dots$. For any $u \in L^1_{loc}((0,\infty) \times \mathbb{R}^n)$ and any integer j satisfying $\sigma - 1/j > \sigma/2$, we define $C^\infty((\sigma,\infty) \times \mathbb{R}^n)$-functions $J_j u(t,x)$ by

$$
\begin{aligned}
J_j u(t,x) &= \int_0^\infty \int_{\mathbb{R}^n} \eta_j(t-\tau, x-y) u(\tau,y) \, d\tau \, dy \\
&= \int_{\sigma/2}^\infty \int_{\mathbb{R}^n} \eta_j(t-\tau, x-y) u(\tau,y) \, d\tau \, dy.
\end{aligned}
\tag{2.14}
$$

The second equality in (2.14) is true because $|t-\tau| < 1/j$ by the support condition of η, and thus $\tau > t - 1/j > \sigma - 1/j > \sigma/2$ for $t > \sigma$. By direct computation we find

$$
\begin{aligned}
L_k J_j u(t,x) &= \int_{\sigma/2}^\infty \int_{\mathbb{R}^n} \eta_j(t-\tau, x-y) L_k u(\tau,y) \, d\tau \, dy \\
&+ \int_{\sigma/2}^\infty \int_{\mathbb{R}^n} j^{n+1} (L_k \eta)(j(t-\tau), j(x-y)) u(\tau,y) \, d\tau \, dy, \ k = 1, \dots, n,
\end{aligned}
\tag{2.15}
$$

$$\Omega_{kl}J_ju(t,x) = \int_{\sigma/2}^{\infty}\int_{\mathbb{R}^n} \eta_j(t-\tau,x-y)\Omega_{kl}u(\tau,y)\,d\tau\,dy \tag{2.16}$$
$$+ \int_{\sigma/2}^{\infty}\int_{\mathbb{R}^n} j^{n+1}(\Omega_{kl}\eta)(j(t-\tau),j(x-y))u(\tau,y)\,d\tau\,dy,\ 1 \le k < l \le n,$$
$$L_0J_ju(t,x) = \int_{\sigma/2}^{\infty}\int_{\mathbb{R}^n} \eta_j(t-\tau,x-y)L_0u(\tau,y)\,d\tau\,dy \tag{2.17}$$
$$+ (n+1)\int_{\sigma/2}^{\infty}\int_{\mathbb{R}^n} \eta_j(t-\tau,x-y)u(\tau,y)\,d\tau\,dy$$
$$+ \int_{\sigma/2}^{\infty}\int_{\mathbb{R}^n} j^{n+1}(L_0\eta)(j(t-\tau),j(x-y))u(\tau,y)\,d\tau\,dy.$$

We now show $t^\delta\dot{\Gamma}^\alpha J_ju \in L^\infty(\sigma,\infty;L^p(\mathbb{R}^n))$ ($|\alpha| \le 1$). By the support condition of η we may assume $t < \tau + 1/j$. Then, noting that the condition $\sigma - 1/j > \sigma/2$ is equivalent to $1/j < \sigma/2$ and recalling that $\tau > \sigma/2$ when $t > \sigma$, we find $t < \tau + 1/j < \tau + \sigma/2 < 2\tau$ for $t > \sigma$. We thus have for $t > \sigma$

$$t^\delta||J_ju(t,\cdot)||_p \le 2^\delta\int_{\sigma/2}^{\infty}||\eta_j(t-\tau,\cdot)||_1\tau^\delta||u(\tau,\cdot)||_p\,d\tau \le 2^\delta\operatorname*{ess\,sup}_{\tau>0}\tau^\delta||u(\tau,\cdot)||_p. \tag{2.18}$$

This implies $t^\delta J_ju \in L^\infty(\sigma,\infty;L^p(\mathbb{R}^n))$. Using the identities (2.15)–(2.17), we easily find $t^\delta\dot{\Gamma}^\alpha J_ju \in L^\infty(\sigma,\infty;L^p(\mathbb{R}^n))$ ($|\alpha| = 1$) in the same way as in (2.18). Moreover, we have

$$\sup_{t>\sigma} t^\delta||J_ju(t,\cdot)||_{\dot{\Gamma},1,p} \le C\sum_{|\alpha|\le 1}\operatorname*{ess\,sup}_{t>0} t^\delta||\dot{\Gamma}^\alpha u(t,\cdot)||_p, \tag{2.19}$$

where the constant C is independent of j and σ. Combining (2.19) with (2.13), we now get

$$||J_ju(t,\cdot)||_{q,\chi_3} \le Ct^{-n(1/p-1/q)-\delta}\sum_{|\alpha|\le 1}\operatorname*{ess\,sup}_{t>0} t^\delta||\dot{\Gamma}^\alpha u(t,\cdot)||_p \tag{2.20}$$

for $t > \sigma$. Since the constant C on the right-hand side of (2.20) is independent of j, there exist a subsequence $\{t^{n(\frac{1}{p}-\frac{1}{q})+\delta}\chi_3J_{\tilde{j}}u\} \subset \{t^{n(\frac{1}{p}-\frac{1}{q})+\delta}\chi_3J_ju\}$ and w such that $t^{n(1/p-1/q)+\delta}\chi_3J_{\tilde{j}}u \to w$, weak-$*$ in $L^\infty(\sigma,\infty;L^q(\mathbb{R}^n))$ as $\tilde{j} \to \infty$ and

$$||w||_{L^\infty(\sigma,\infty;L^q(\mathbb{R}^n))} \le \liminf_{\tilde{j}\to\infty} ||t^{n(1/p-1/q)+\delta}\chi_3J_{\tilde{j}}u||_{L^\infty(\sigma,\infty;L^q(\mathbb{R}^n))}. \tag{2.21}$$

On the other hand, we easily see

$$\int_\sigma^\infty\int_{\mathbb{R}^n} \phi(t,x)w(t,x)\,dt\,dx = \int_\sigma^\infty\int_{\mathbb{R}^n} \phi(t,x)t^{n(1/p-1/q)+\delta}\chi_3(t,x)u(t,x)\,dt\,dx \tag{2.22}$$

for any $\phi \in C_0^\infty((\sigma,\infty)\times\mathbb{R}^n)$ because $J_{\tilde{j}}u \to u$ in $L^q(\operatorname{supp}\phi)$ as $\tilde{j}\to\infty$. Noting that $C_0^\infty((\sigma,\infty)\times\mathbb{R}^n)$ is a dense set in $L^1(\sigma,\infty;L^{q'}(\mathbb{R}^n))$ because of $q'<\infty$, we conclude that $w = t^{n(1/p-1/q)+\delta}\chi_3 u$ in $L^\infty(\sigma,\infty;L^q(\mathbb{R}^n))$. Finally, since σ is arbitrary and the constant C in (2.20) is independent of σ, we get (2.7) from (2.20)–(2.21). Thus we have completed the proof of (1).

Next we shall prove (2). Our proof starts with the Sobolev embedding

$$||v||_{L^\infty(B_a)} \le Ca^{-n/p}\sum_{|\alpha|\le 1} a^{|\alpha|}||\partial_x^\alpha v||_{L^p(B_a)}. \tag{2.23}$$

For $a=1$ this is just the Sobolev embedding on the unit ball (see, e.g., [1] on page 97) and a scaling argument gives us (2.23) for the general values of $a>0$. Repeating the same argument as in the proof of (1), we can obtain the inequalities (2.8) and (2.9). Thus we have finished the proof of Lemma 2.3.

Lemma 2.4. *Assume $n\ge 2$, $1\le p<n$ and $1/p \ge 1/q \ge 1/p-1/n$. Then the inequality*

$$||\langle x\rangle^{n(1/p-1/q)}v||_{L^q} \le C(||v||_{L^p} + ||\langle x\rangle\nabla v||_{L^p}) \tag{2.24}$$

holds for any $v\in L^p(\mathbb{R}^n)$ with $\langle x\rangle\nabla v \in L^p(\mathbb{R}^n)$.

Proof. The proof starts with the well-known inequality (see, e.g., [37, Proposition 2.2] on page 4)

$$||v||_{L^q} \le C||\nabla v||_{L^p}, \quad v\in C_0^1(\mathbb{R}^n) \tag{2.25}$$

where $1\le p<n$ and $1/q = 1/p-1/n$. Thus, replacing v with $\langle x\rangle v$, we get

$$||\langle x\rangle v||_{L^q} \le C(||v||_{L^p} + ||\langle x\rangle\nabla v||_{L^p}), \quad v\in C_0^1(\mathbb{R}^n) \tag{2.26}$$

when $1\le p<n$ and $1/q=1/p-1/n$. By interpolation we have (2.24) for $v\in C_0^1(\mathbb{R}^n)$. Finally, by density, (2.24) remains true for any $v\in L^p(\mathbb{R}^n)$ with $\langle x\rangle\nabla v\in L^p(\mathbb{R}^n)$. □

We write $x(\in\mathbb{R}^n)$ as $x=(x',x'')$, $x'=x_1$, $x''=(x_2,\dots,x_n)$.

Lemma 2.5. *Assume $n\ge 2$. (1) If $1<q\le p<\infty$, then the inequality*

$$||v||_{L^p(\mathbb{R}_{x'};L^q(\mathbb{R}^{n-1}_{x''}))} \le C||v||_{L^{s,q}}, \quad s>1/q-1/p \tag{2.27}$$

holds for all $v\in L^{s,q}(\mathbb{R}^n)$.

(2) *If $1 < q < \infty$, then the inequality*

$$||v||_{L^\infty(\mathbb{R}_{x'};L^q(\mathbb{R}^{n-1}_{x''}))} \le C||v||_{L^{s,q}}, \quad s > 1/q \tag{2.28}$$

holds for all $v \in L^{s,q}(\mathbb{R}^n)$.

Proof. Our starting point of the proof is the following trace theorem: If $s > 0$, $1 < p \le q < \infty$, $\chi := s - n/p + (n-1)/q > 0$ and χ is not an integer, then $W^{s,p}(\mathbb{R}^n) \hookrightarrow W^{\chi,q}(\mathbb{R}^{n-1})$ (see [1, Theorem 7.58 (ii)]). In particular, for any $1 < q < \infty$, $0 < \varepsilon < 1$ and $\delta > 0$ we have $L^{1/q+\varepsilon+\delta,q}(\mathbb{R}^n) \hookrightarrow W^{1/q+\varepsilon,q}(\mathbb{R}^n) \hookrightarrow W^{\varepsilon,q}(\mathbb{R}^{n-1}) \hookrightarrow L^q(\mathbb{R}^{n-1})$. Rewriting $\varepsilon + \delta$ as ε, we get

$$||u||_{L^\infty(\mathbb{R}_{x'};L^q(\mathbb{R}^{n-1}_{x''}))} \le C||u||_{L^{1/q+\varepsilon,q}}, \quad 1 < q < \infty, \ \varepsilon > 0. \tag{2.29}$$

This is just the inequality (2.28). Furthermore, using [2, Theorem 5.1.2] and [1, Theorem 7.65], we obtain by the complex interpolation between (2.29) and a trivial inequality $||u||_{L^q(\mathbb{R}_{x'};L^q(\mathbb{R}^{n-1}_{x''}))} \le ||u||_{L^q}$

$$||u||_{L^p(\mathbb{R}_{x'};L^q(\mathbb{R}^{n-1}_{x''}))} \le C||u||_{L^{\theta(1/q+\varepsilon),q}} = C||u||_{L^{1/q-1/p+\varepsilon\theta,q}} \tag{2.30}$$

for $0 < \theta < 1$, $1 < q < \infty$ and $1/p = (1-\theta)/q$. Since ε is an arbitrary positive number, the inequality (2.27) has been proved for $1 < q < p < \infty$. In the case of $p = q$, (2.27) obviously holds. □

Lemma 2.6. *If r, p_1, $p_2 \in (1,\infty)$, q_1, $q_2 \in (1,\infty]$, $1/r = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$ and $s \ge 0$, then the inequality*

$$||fg||_{\dot H^s_r} \le C||f||_{\dot H^s_{p_1}}||g||_{L^{q_1}} + C||f||_{L^{q_2}}||g||_{\dot H^s_{p_2}} \tag{2.31}$$

holds for any f, $g \in \mathcal{S}(\mathbb{R}^n)$ such that $f \in \dot H^s_{p_1}(\mathbb{R}^n)$ and $g \in \dot H^s_{p_2}(\mathbb{R}^n)$.

Proof. For $s = 0$ this is just the Hölder inequality. Christ and Weinstein [4, Proposition 3.3] proved this inequality for $0 < s < 1$ and q_1, $q_2 \in (1,\infty)$ in one dimension, and their argument obviously remains valid for the proof of the higher-dimensional case as well as for $s > 0$ and q_1, $q_2 \in (1,\infty]$. □

Recall that Φ_a ($a > 0$) is the characteristic function of the set $\{\, x \in \mathbb{R}^n : |x| \ge a \,\}$.

Lemma 2.7. *Assume $n \ge 2$.*

(1) *If $1 < q < p < \infty$, $0 < \delta < (n-1)(1/q - 1/p)$ and $1/q - 1/p + \delta \le n/2$, then the inequality*

$$||\Phi_a v||_{p,q} \le Ca^{-(n-1)(1/q-1/p)+\delta}||v||_{\dot H_q^{1/q-1/p+\delta}}, \ v \in \dot H_q^{1/q-1/p+\delta}(\mathbb{R}^n) \tag{2.32}$$

holds. Here the constant C is independent of a.

(2) *If* $1 < q \le p < \infty$, $\delta, \tilde\delta > 0$, $1/q - 1/p + \delta < n/q$ *and* $1/q - 1/p + \delta \le n/2$, *then the inequality*

$$||\Phi_{1/2}|x|^{(n-1)(1/q-1/p)-(\delta+\tilde\delta)}v||_{p,q} \le C||v||_{\dot H_q^{1/q-1/p+\delta}},\ v \in \dot H_q^{1/q-1/p+\delta}(\mathbb{R}^n) \tag{2.33}$$

holds.

(3) *If* $1 < q < \infty$, $1/q < s < n/q$ *and* $s \le n/2$, *then the inequality*

$$||\,|x|^{n/q-s}v\,||_{\infty,q} \le C||v||_{\dot H_q^s},\ v \in \dot H_q^s(\mathbb{R}^n) \tag{2.34}$$

holds.

Remark. The inequality (2.33) is enough for our purpose although it is not a scale-invariant one. Indeed, we shall use it only in Sections 3 and 4.

Proof of Lemma 2.7. By density we have only to show these inequalities for any $v \in \mathcal{Z}(\mathbb{R}^n)$. It is convenient to use Paley–Littlewood dyadic decompositions, which are defined in the following way. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be spherically symmetric with $0 \le \psi \le 1$, $\psi(x) = 1$ for $|x| \le 1$ and $\psi(x) = 0$ for $|x| \ge 2$. Define radial functions $\varphi_0(x) = \psi(x) - \psi(2x)$ and $\varphi_j(x) = \varphi_0(2^{-j}x)$ for $j = \pm1, \pm2, \dots$. Then it is easily checked that $\operatorname{supp}\varphi_j \subset \{\, x \in \mathbb{R}^n : 2^{j-1} \le |x| \le 2^{j+1}\,\}$, $0 \le \varphi_j \le 1$ and

$$\sum_{j=-\infty}^{\infty} \varphi_j(x) = 1 \text{ for } |x| > 0. \tag{2.35}$$

Note that, in this sum, there are at most two nonvanishing terms for every $|x|$. We write polar coordinates as $(r, \theta_1, \dots, \theta_{n-1}) \in (0,\infty) \times (0,\pi) \times \cdots \times (0,\pi) \times (0, 2\pi)$ and the transformation $(r, \theta_1, \dots, \theta_{n-1}) \mapsto x$ defined by $x_1 = r\cos\theta_1$, $x_2 = r\sin\theta_1\cos\theta_2, \dots,$ $x_{n-1} = r\sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1}$, $x_n = r\sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1}$ is denoted by Φ (the reader should not confuse this with the characteristic function Φ_a introduced at the beginning of this section). By Σ_a ($a \in S^{n-1}$) we mean the set $\{\,\zeta \in S^{n-1} : \zeta\cdot a > 1/2\,\}$. When $a = (0, \dots, 0, 1)$, we simply write Σ_a as Σ_+.

Let $w := \varphi_0 v$. We begin with the proof of the inequalities (2.36)–(2.37), from which (2.32)–(2.34) will follow:

$$\Big(\int_0^\infty\Big(\int_{\Sigma_+}|w(r\zeta)|^q dS_\zeta\Big)^{\frac{p}{q}} r^{n-1}dr\Big)^{\frac{1}{p}} \le C||w||_{L^{\frac1q-\frac1p+\delta,q}},\ 1 < q \le p < \infty,\ \delta > 0, \tag{2.36}$$

$$\sup_{r>0}\Big(\int_{\Sigma_+}|w(r\zeta)|^q dS_\zeta\Big)^{1/q} \le C||w||_{L^{s,q}},\ 1 < q < \infty,\ s > 1/q. \tag{2.37}$$

Write the set $\{ x \in \mathbb{R}^n : 1/2 < |x| < 2 \}$ as D and define $D_+ := \{ x \in D : x/|x| \in \Sigma_+ \}$. Then we choose a suitable bounded domain $\Omega_+ (\supset D_+)$ with smooth boundary so that $\Phi \in C^\infty(\bar{\tilde{\Omega}}_+)$, $\Phi^{-1} \in C^\infty(\bar{\Omega}_+)$ $(\tilde{\Omega}_+ := \Phi^{-1}(\Omega_+))$. We simply write $w \circ \Phi(r, \theta_1, \ldots, \theta_{n-1})$ as $\tilde{w}(r, \theta_1, \ldots, \theta_{n-1})$. Since $\partial\tilde{\Omega}_+$ is compact and smooth, we can extend the function $\tilde{w}$ to be defined in the whole space with the help of the extension operator E (see [1], pp. 84–86, 207–208). Then we find

$$\begin{aligned}
&\sup_{r>0} \Big(\int_{\Sigma_+} |w(r\zeta)|^q dS_\zeta \Big)^{1/q} \qquad (2.38)\\
&= \sup_{1/2<r<2} \Big(\int_{\{(\theta_1,\ldots,\theta_{n-1}):\sin\theta_1 \cdots \sin\theta_{n-1} > 1/2\}} |\tilde{w}(r, \theta_1, \ldots, \theta_{n-1})|^q \\
&\qquad\qquad (\sin\theta_1)^{n-2} \cdots \sin\theta_{n-2} d\theta_1 \cdots d\theta_{n-1} \Big)^{1/q} \\
&\le \sup_{r\in\mathbb{R}} ||E\tilde{w}||_{L^q(\mathbb{R}^{n-1})} \le C||E\tilde{w}||_{L^{s,q}} \le C||E\tilde{w}||_{W^{s+\delta,q}(\mathbb{R}^n)} \\
&\le C||\tilde{w}||_{W^{s+\delta,q}(\tilde{\Omega}_+)} \le C||w||_{W^{s+\delta,q}(\Omega_+)} \le C||w||_{W^{s+\delta,q}(\mathbb{R}^n)} \le C||w||_{L^{s+2\delta,q}},
\end{aligned}$$

where the inequality (2.28) was used at the second inequality, and at the third and last inequalities the inclusion (1.10) was employed. At the fifth inequality we also employed the fact that the operator $w \mapsto \tilde{w}$ transforms $W^{s+\delta,q}(\Omega_+)$ boundedly on $W^{s+\delta,q}(\tilde{\Omega}_+)$. Rewriting $s + 2\delta$ as s, we obtain (2.37).

Next we show (2.36). Taking account of the fact that $r^{n-1} \le 2^{n-1}$ on $\operatorname{supp} w \subset \{1/2 \le |x| \le 2\}$, we see that

$$\begin{aligned}
&\Big(\int_0^\infty \Big(\int_{\Sigma_+} |w(r\zeta)|^q dS_\zeta \Big)^{p/q} r^{n-1} dr \Big)^{1/p} \qquad (2.39)\\
&\le C \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}^{n-1}} |E\tilde{w}(r, \theta_1, \ldots, \theta_{n-1})|^q d\theta_1 \cdots d\theta_{n-1} \Big)^{p/q} dr \Big)^{1/p} \\
&\le C||E\tilde{w}||_{L^{1/q-1/p+\delta,q}} \le C||E\tilde{w}||_{W^{1/q-1/p+2\delta,q}(\mathbb{R}^n)} \\
&\le C||\tilde{w}||_{W^{1/q-1/p+2\delta,q}(\tilde{\Omega}_+)} \le C||w||_{W^{1/q-1/p+2\delta,q}(\Omega_+)} \\
&\le C||w||_{W^{1/q-1/p+2\delta,q}(\mathbb{R}^n)} \le C||w||_{L^{1/q-1/p+3\delta,q}}.
\end{aligned}$$

Rewriting $1/q - 1/p + 3\delta$ as $1/q - 1/p + \delta$, we have (2.36).

Here it is remarked that the inequalities (2.36)–(2.37) remain true even if Σ_+ is replaced by Σ_a $(a \in S^{n-1})$. This is obvious because the norms in the spaces $L^{1/q-1/p+\delta,q}(\mathbb{R}^n)$ and $L^{s,q}(\mathbb{R}^n)$ are invariant under the rotation of the variables.

Our next task is to remove φ_0 from the norms $||w||_{L^{1/q-1/p+\delta,q}}$ $(1 < q \le p < \infty,\ \delta > 0,\ 1/q - 1/p + \delta < n/q$ and $1/q - 1/p + \delta \le n/2)$ and $||w||_{L^{s,q}}$ $(1 < q < \infty,\ 1/q < s < n/q$ and $s \le n/2)$. Giving q_1 by $1/q_1 = 1/q - s/n$ and making use of Lemma 2.6, we have

$$\begin{aligned} ||w||_{L^{s,q}} &\le C||w||_{L^q} + C||w||_{\dot{H}^s_q} \qquad (2.40)\\ &\le C||\varphi_0||_{L^{n/s}}||v||_{L^{q_1}} + C||\varphi_0||_{\dot{H}^s_{n/s}}||v||_{L^{q_1}} + C||\varphi_0||_{L^\infty}||v||_{\dot{H}^s_q}\\ &\le C||v||_{\dot{H}^s_q}. \end{aligned}$$

In the first inequality we have applied the Fourier multiplier theorem to the operator defined as $\mathcal{F}^{-1}(1+|\xi|^2)^{s/2}/(1+|\xi|^s)\mathcal{F}$. In the last inequality we have employed the Sobolev embedding $\dot{H}^s_q(\mathbb{R}^n) \hookrightarrow L^{q_1}(\mathbb{R}^n)$. We have also used the Young inequality to bound $||\varphi_0||_{\dot{H}^s_{n/s}}$ as

$$||\varphi_0||_{\dot{H}^s_{n/s}} \le C||\,|\xi|^s\hat{\varphi}_0\,||_{L^{(n/s)'}} \le C, \qquad (2.41)$$

taking the condition $n/s \ge 2$ into account. In a similar way we obtain, by giving q_2 and q_3 as $1/q = 1/q_2 + 1/q_3$ and $1/q_3 = (1/q - 1/p + \delta)/n$,

$$\begin{aligned} &||w||_{L^{1/q-1/p+\delta,q}} \le C||w||_{L^q} + C||w||_{\dot{H}^{1/q-1/p+\delta}_q} \qquad (2.42)\\ &\le C||\varphi_0||_{L^{q_3}}||v||_{L^{q_2}}\\ &+C||\varphi_0||_{\dot{H}^{1/q-1/p+\delta}_{q_3}}||v||_{L^{q_2}} + C||\varphi_0||_{L^\infty}||v||_{\dot{H}^{1/q-1/p+\delta}_q} \le C||v||_{\dot{H}^{1/q-1/p+\delta}_q}. \end{aligned}$$

Since S^{n-1} is covered with finitely many sets, say, $\Sigma_{a_1}, \dots, \Sigma_{a_N}$ for some $a_1, \dots, a_N \in S^{n-1}$, we get from (2.36)–(2.37), (2.40) and (2.42)

$$||\varphi_0 v||_{p,q} \le C||v||_{\dot{H}^{1/q-1/p+\delta}_q}, \qquad (2.43)$$

$$1 < q \le p < \infty,\ \delta > 0,\ 1/q - 1/p + \delta < n/q,\ 1/q - 1/p + \delta \le n/2,$$

$$||\varphi_0 v||_{\infty,q} \le C||v||_{\dot{H}^s_q},\ 1 < q < \infty,\ 1/q < s < n/q,\ s \le n/2. \qquad (2.44)$$

By scaling we have for $j = 0, \pm1, \pm2, \dots$

$$||\varphi_j v||_{p,q} \le C2^{-j\{(n-1)(1/q-1/p)-\delta\}}||v||_{\dot{H}^{1/q-1/p+\delta}_q}, \qquad (2.45)$$

$$||\varphi_j v||_{\infty,q} \le C2^{-j(n/q-s)}||v||_{\dot{H}^s_q}, \qquad (2.46)$$

where the conditions of p, q, s and δ are the same as in (2.43)–(2.44). Now we are ready to derive (2.32)–(2.34). Assuming $1 < q < p < \infty$, $0 < \delta < (n-1)(1/q - 1/p)$ and $1/q - 1/p + \delta \le n/2$, we have from (2.45)

$$||\Phi_1 v||_{p,q} \le ||\sum_{j=0}^{\infty}\varphi_j v||_{p,q} \le \sum_{j=0}^{\infty}||\varphi_j v||_{p,q} \qquad (2.47)$$

$$\leq C||v||_{\dot{H}_q^{1/q-1/p+\delta}} \sum_{j=0}^{\infty} 2^{-j\{(n-1)(1/q-1/p)-\delta\}} \leq C||v||_{\dot{H}_q^{1/q-1/p+\delta}}.$$

For the general values of $a > 0$, (2.32) follows from (2.47) by scaling. As for (2.33), we get immediately from (2.45)

$$\begin{aligned} &||\Phi_{1/2}|x|^{(n-1)(1/q-1/p)-(\delta+\tilde{\delta})}v||_{p,q} \leq ||\sum_{j=-1}^{\infty} \varphi_j |x|^{(n-1)(1/q-1/p)-(\delta+\tilde{\delta})}v||_{p,q} \\ &\leq C||\sum_{j=-1}^{\infty} \varphi_j 2^{j\{(n-1)(1/q-1/p)-(\delta+\tilde{\delta})\}}v||_{p,q} \qquad (2.48) \\ &\leq C\sum_{j=-1}^{\infty} (2^{-\tilde{\delta}})^j ||\varphi_j 2^{j\{(n-1)(1/q-1/p)-\delta\}}v||_{p,q} \\ &\leq C||v||_{\dot{H}_q^{1/q-1/p+\delta}} \sum_{j=-1}^{\infty} (2^{-\tilde{\delta}})^j = C||v||_{\dot{H}_q^{1/q-1/p+\delta}}. \end{aligned}$$

Here we have employed the fact that $\operatorname{supp} \varphi_j \subset \{2^{j-1} \leq |x| \leq 2^{j+1}\}$.

It remains to show (2.34). Recall that $0 \leq \varphi_j \leq 1$ and that there exist at most two nonvanishing terms in the sum of (2.35). Therefore for every $r > 0$ we can choose j such that $r\zeta \in \operatorname{supp} \varphi_j$ and $\varphi_j(r\zeta) \geq 1/2$ for all $\zeta \in S^{n-1}$. Since $2^{j-1} \leq |x| \leq 2^{j+1}$ on $\operatorname{supp} \varphi_j$, from (2.46) we have the inequality

$$r^{n/q-s}||v(r\cdot)||_{L^q(S^{n-1})} \leq C||v||_{\dot{H}_q^s}, \ 1 < q < \infty, \ \tfrac{1}{q} < s < \tfrac{n}{q}, \ s \leq \tfrac{n}{2}. \qquad (2.49)$$

This immediately leads to (2.34). Thus we have completed the proof. □

Lemma 2.8. *Assume $n \geq 2$, $2 \leq q < p < \infty$ and $0 < \delta < (n-1)(1/q-1/p)$.*

(1) *The inequality*

$$||u(t,\cdot)||_{p,q,\chi_2} \leq C(1+|t|)^{-(n-1)(1/q-1/p)+\delta}||u(t,\cdot)||_{\dot{H}_q^{1/q-1/p+\delta}}, \ t \in \mathbb{R} \quad (2.50)$$

holds for $u \in C(\mathbb{R}; \dot{H}_q^{1/q-1/p+\delta}(\mathbb{R}^n))$.

(2) *The inequality*

$$||u(t,\cdot)||_{p,q,\chi_4} \leq Ct^{-(n-1)(1/q-1/p)+\delta}||u(t,\cdot)||_{\dot{H}_q^{1/q-1/p+\delta}}, \ \text{a.a.}\, t > 0 \qquad (2.51)$$

holds for $u \in L^\infty(0,\infty; \dot{H}_q^{1/q-1/p+\delta}(\mathbb{R}^n))$.

Proof. This lemma is an immediate consequence of part (1) of Lemma 2.7.

Lemma 2.9. *Assume $n \geq 2$, $2 \leq q < \infty$ and $1/q < s < n/q$. Define p by $1/p = 1/q' + s/n$.*

(1) *If* $\chi_1(t,\cdot)h(t,\cdot) \in L^p(\mathbb{R}^n)$ *and* $||h(t,\cdot)||_{1,q',\chi_2} < \infty$ *for* $t \in \mathbb{R}$*, then* $h(t,\cdot) \in \dot{H}_{q'}^{-s}(\mathbb{R}^n)$ *and the inequality*

$$||h(t,\cdot)||_{\dot{H}_{q'}^{-s}} \leq C||h(t,\cdot)||_{p,\chi_1} + C(1+|t|)^{-(n/q-s)}||h(t,\cdot)||_{1,q',\chi_2},\ t \in \mathbb{R} \quad (2.52)$$

holds.

(2) *If* $x_j\chi_1(t,\cdot)h(t,\cdot) \in L^p(\mathbb{R}^n)$ $(j=1,\dots,n)$ *and* $\| |\cdot|^{1-(n/q-s)}h(t,\cdot)\|_{1,q',\chi_2} < \infty$ *for* $t \in \mathbb{R}$*, then* $x_jh(t,\cdot) \in \dot{H}_{q'}^{-s}(\mathbb{R}^n)$ *and the inequality*

$$\begin{aligned} &||x_jh(t,\cdot)||_{\dot{H}_{q'}^{-s}} \\ &\leq C||x_jh(t,\cdot)||_{p,\chi_1} + C||\,|\cdot|^{1-(n/q-s)}h(t,\cdot)\,||_{1,q',\chi_2},\ t \in \mathbb{R},\ j=1,\dots,n \end{aligned} \quad (2.53)$$

holds. If $x_j\chi_3(t,\cdot)h(t,\cdot) \in L^p(\mathbb{R}^n)$ *and* $||\,|\cdot|^{1-(n/q-s)}h(t,\cdot)\,||_{1,q',\chi_4} < \infty$ *for almost all* $t > 0$*, then* $x_jh(t,\cdot) \in \dot{H}_{q'}^{-s}(\mathbb{R}^n)$*. Moreover the same inequality as* (2.53) *holds for almost all* $t > 0$ *if the characteristic functions* χ_1*,* χ_2 *are replaced with* χ_3*,* χ_4*, respectively.*

(3) *If* $\chi_3(t,\cdot)h(t,\cdot) \in L^p(\mathbb{R}^n)$ *and* $||h(t,\cdot)||_{1,q',\chi_4} < \infty$ *for a.a.* $t > 0$*, then* $h(t,\cdot) \in \dot{H}_{q'}^{-s}(\mathbb{R}^n)$ *and the inequality*

$$||h(t,\cdot)||_{\dot{H}_{q'}^{-s}} \leq C||h(t,\cdot)||_{p,\chi_3} + Ct^{-(n/q-s)}||h(t,\cdot)||_{1,q',\chi_4},\ \text{a.a.}\, t > 0 \quad (2.54)$$

holds.

Proof. We begin with the proof of (2.52). We see that

$$\begin{aligned} &||h(t,\cdot)||_{\dot{H}_{q'}^{-s}} = \sup\left|\Big(h(t),v\Big)\right| \quad (2.55)\\ &\leq \sup\left|\Big(h(t)\chi_1(t),v\Big)\right| + \sup\left|\Big(h(t)\chi_2(t),v\Big)\right| \\ &\leq \sup||h(t)||_{p,\chi_1}||v||_{L^{p'}} + \sup||h(t)||_{1,q',\chi_2}||\Phi_{(1+|t|)/2}v||_{\infty,q} \\ &\leq C||h(t)||_{p,\chi_1} + C(1+|t|)^{-(n/q-s)}||h(t)||_{1,q',\chi_2}. \end{aligned}$$

Here by $(\cdot,\cdot)$ we mean the usual L^2 inner product, and the supremum is taken over all $v \in \dot{H}_q^s(\mathbb{R}^n)$ with $||v||_{\dot{H}_q^s} = 1$. In the last inequality we have used the embedding $\dot{H}_q^s(\mathbb{R}^n) \hookrightarrow L^{p'}(\mathbb{R}^n)$ and (2.34). The proof of (2) and (3) is similar; thus we omit it. □

We shall need pointwise (in time) estimates for the free evolution operator. The following is due to Pecher [25], and we shall use it for the study of self-similar solutions.

Lemma 2.10. *Assume* $n \geq 2$*,* $s \in \mathbb{R}$ *and* $2 \leq r < \infty$*. The inequalities*

$$||(\cos\omega t)f||_{\dot{H}_r^s} \leq C|t|^{-\gamma(r)}||f||_{\dot{H}_{r'}^{s+2\beta(r)}}, \quad (2.56)$$

$$||(\omega^{-1}\sin\omega t)g||_{\dot H^s_r} \le C|t|^{-\gamma(r)}||g||_{\dot H^{s+2\beta(r)-1}_{r'}} \tag{2.57}$$

hold for any $f\in \dot H^{s+2\beta(r)}_{r'}(\mathbb{R}^n)$ *and* $g\in \dot H^{s+2\beta(r)-1}_{r'}(\mathbb{R}^n)$.

For $g\in\mathcal{S}'(\mathbb{R}^n)$ we know that $(\omega^{-1}\sin\omega t)g\in C(\mathbb{R};\mathcal{S}'(\mathbb{R}^n))$ and a tempered distribution on $\mathbb{R}^{n+1}$ is defined by

$$\psi\mapsto \langle\!\langle \frac{\sin\omega t}{\omega}g,\psi\rangle\!\rangle := \int_{\mathbb{R}}\langle g,\frac{\sin\omega t}{\omega}\psi(t,\cdot)\rangle\,dt,\quad \psi\in\mathcal{S}(\mathbb{R}^{n+1}).$$

For $f\in\mathcal{S}'(\mathbb{R}^n)$, the corresponding

$$\psi\mapsto \langle\!\langle (\cos\omega t)f,\psi\rangle\!\rangle := \int_{\mathbb{R}}\langle f,(\cos\omega t)\psi(t,\cdot)\rangle\,dt,\quad \psi\in\mathcal{S}(\mathbb{R}^{n+1})$$

also defines a tempered distribution on $\mathbb{R}^{n+1}$.

Lemma 2.11. *For any* $(f,g)\in\mathcal{S}'(\mathbb{R}^n)\times\mathcal{S}'(\mathbb{R}^n)$, $u_0(t):=(\cos\omega t)f+(\omega^{-1}\sin\omega t)g$ *satisfies the following identities in* $\mathcal{S}'(\mathbb{R}^{n+1})$:

$$L_ju_0(t)=-\frac{\sin\omega t}{\omega}\partial_jf-(\sin\omega t)\omega(x_jf)+(\cos\omega t)(x_jg),\ \ j=1,\dots,n, \tag{2.58}$$

$$\Omega_{kl}u_0(t)=(\cos\omega t)(\Omega_{kl}f)+\frac{\sin\omega t}{\omega}\Omega_{kl}g,\ \ 1\le k<l\le n, \tag{2.59}$$

$$L_0u_0(t)=\sum_{j=1}^n(\cos\omega t)(x_j\partial_jf)-(n-1)\frac{\sin\omega t}{\omega}g+\sum_{j=1}^n\frac{\sin\omega t}{\omega}\partial_j(x_jg). \tag{2.60}$$

Proof. We shall show $L_j(\omega^{-1}\sin\omega t)g=(\cos\omega t)(x_jg)$ only, because the others can be verified in a similar way. Observing $(\omega^{-1}\sin\omega t)L_j\psi=\partial_t((\omega^{-1}\sin\omega t)x_j\psi)-x_j(\cos\omega t)\psi$, we have, for any $\psi\in\mathcal{S}(\mathbb{R}^{n+1})$,

$$\begin{aligned}\langle\!\langle\psi,L_j\frac{\sin\omega t}{\omega}g\rangle\!\rangle &= -\int_{\mathbb{R}}\langle\frac{\partial}{\partial t}\Big(\frac{\sin\omega t}{\omega}x_j\psi(t,\cdot)\Big),g\rangle dt+\int_{\mathbb{R}}\langle x_j(\cos\omega t)\psi(t,\cdot),g\rangle dt\\ &= -\int_{\mathbb{R}}\frac{d}{dt}\langle\frac{\sin\omega t}{\omega}x_j\psi(t,\cdot),g\rangle\,dt+\int_{\mathbb{R}}\langle(\cos\omega t)\psi(t,\cdot),x_jg\rangle\,dt\\ &= \langle\!\langle\psi,(\cos\omega t)x_jg\rangle\!\rangle.\end{aligned}\tag{2.61}$$

3. Asymptotically free solutions

First we introduce the sets of functions in which the initial data will be taken. Recall that the nonlinear term F of the equation (1.1) has the form $\lambda|u|^{p-1}u$. We give $\theta:=n/2-2/(p-1)$, and for any $p>p_0(n)$ we fix an arbitrary ε satisfying

$$0 < \varepsilon < \frac{(n-1)p^2 - (n+1)p - 2}{4p(p-1)}. \tag{3.1}$$

Let $\langle x \rangle = \sqrt{1+|x|^2}$. The initial data will be taken in the space $X \times Y$ defined by

$$X = \{ \ f \in \dot{H}_2^{\theta-\varepsilon}(\mathbb{R}^n) \cap \dot{H}_2^{1+\theta-\varepsilon}(\mathbb{R}^n) \ : \tag{3.2}$$
$$||f||_X^2 := ||\omega^{\theta-\varepsilon} f||_{L^2}^2 + ||\langle x \rangle \omega^{1+\theta-\varepsilon} f||_{L^2}^2 < \infty \ \},$$
$$Y = \{ \ g \in \dot{H}_2^{-1+\theta-\varepsilon}(\mathbb{R}^n) \cap \dot{H}_2^{\theta-\varepsilon}(\mathbb{R}^n) \ : \tag{3.3}$$
$$||g||_Y^2 := ||\omega^{-1+\theta-\varepsilon} g||_{L^2}^2 + ||\langle x \rangle \omega^{\theta-\varepsilon} g||_{L^2}^2 < \infty \ \}.$$

$X \times Y$ is the Hilbert space with the norm defined by $||(f,g)||_{X\times Y}^2 := ||f||_X^2 + ||g||_Y^2$. Now we can state the main theorem in this section.

Theorem 3.1. *Suppose that $n = 2,3$ and $p_0(n) < p \le 1 + 4/(n-1)$. There exists a constant $\delta_0 > 0$ depending on n, p, ε and λ with the following property*:

For any $(f,g) \in X \times Y$ with $||(f,g)||_{X\times Y} < \delta_0$ the integral equation

$$u(t) = u_0(t) - \lambda \int_0^t \frac{\sin \omega(t-\tau)}{\omega} |u(\tau)|^{p-1} u(\tau)\, d\tau \tag{3.4}$$

($u_0(t) = (\cos \omega t) f + (\omega^{-1} \sin \omega t) g$) has a unique solution $u(t,x)$ satisfying

$$\Gamma^\alpha u(t), \ \omega^{-1}\partial_t u(t) \in BC(\mathbb{R}; \dot{H}_2^{\theta-\varepsilon}(\mathbb{R}^n)) \quad (|\alpha| \le 1). \tag{3.5}$$

Moreover, there exists a unique pair of functions (f_+, g_+), $(f_-, g_-) \in X \times Y$ satisfying

$$||(f_\pm, g_\pm)||_{X\times Y} \le C_1 ||(f,g)||_{X\times Y} \ \textit{for some constant } C_1 > 0, \tag{3.6}$$
$$||\big(\Gamma^\alpha u(t) - \Gamma^\alpha u_\pm(t), \omega^{-1}\partial_t u(t) - \omega^{-1}\partial_t u_\pm(t)\big)||_{\dot{H}_2^{\theta-\varepsilon} \times \dot{H}_2^{\theta-\varepsilon}} \to 0 \tag{3.7}$$

$(t \to \pm\infty)$ for $|\alpha| \le 1$. Here $u_\pm(t) = (\cos \omega t) f_\pm + (\omega^{-1} \sin \omega t) g_\pm$.

For any $(f,g) \in X \times Y$ let us study the integral equation (3.4) for positive time. This equation can be solved for negative time in the same way. We introduce the set of functions Z_δ ($\delta > 0$) as follows:

$$\begin{aligned} Z_\delta := \Big\{ & u = u(t,x) : \Gamma^\alpha u, \ \omega^{-1}\partial_t u \in C([0,\infty); \dot{H}_2^{\theta-\varepsilon}(\mathbb{R}^n)), \ |\alpha| \le 1, \\ & u(0,x) = f(x), \ \partial_t u(0,x) = g(x), \\ & ||u||_Z = \sum_{|\alpha|\le 1} \sup_{t>0} ||\Gamma^\alpha u(t)||_{\dot{H}_2^{\theta-\varepsilon}} + \sup_{t>0} ||\omega^{-1}\partial_t u(t)||_{\dot{H}_2^{\theta-\varepsilon}} \le \delta \Big\}. \end{aligned}$$

Z_δ is a nonempty, complete metric space with the metric $||u-v||_Z$ (u, $v \in Z_\delta$) if $||(f,g)||_{X\times Y}$ is sufficiently small relative to δ. In what follows we simply denote $||(f,g)||_{X\times Y}$ by Λ. It follows immediately from Lemma 2.11 that $u_0(t) = (\cos\omega t)f + (\omega^{-1}\sin\omega t)g \in Z_{C_2\Lambda}$ for a suitable constant $C_2 > 0$.

In order to prove Theorem 3.1 we define the mapping

$$M : u \mapsto Mu = u_0(t) - I_0[u](t) \ (u \in Z_{2C_2\Lambda}), \tag{3.8}$$

where

$$I_0[u](t) = \lambda \int_0^t \frac{\sin\omega(t-\tau)}{\omega} |u(\tau)|^{p-1} u(\tau)\, d\tau.$$

Our proof is based on the contraction-mapping principle. We shall show

Proposition 3.2. *There exists a constant $\delta_0 > 0$ depending on n, p, ε and λ such that if $\Lambda < \delta_0$, then the mapping M carries $Z_{2C_2\Lambda}$ into itself and satisfies*

$$||Mu - Mv||_Z \le \frac{1}{2} ||u-v||_Z \ (u,\, v \in Z_{2C_2\Lambda}). \tag{3.9}$$

By this proposition the mapping M has a unique fixed point in $Z_{2C_2\Lambda}$ which is a solution to the integral equation (3.4) for positive time. The proof of uniqueness is similar to that of Proposition 3.2. Therefore we devote ourselves to the proof of Proposition 3.2.

Proof of Proposition 3.2. From now on the power p of the nonlinear term of the integral equation (3.4) is always assumed to satisfy $p_0(n) < p \le 1 + 4/(n-1)$. Letting the operators L_j, Ω_{kl} and L_0 act on $I_0[u]$, we have

$$L_j I_0[u](t) = \frac{\sin\omega t}{\omega}[x_j F(f)] + \int_0^t \frac{\sin\omega(t-\tau)}{\omega} L_j F(u(\tau))\, d\tau, \tag{3.10}$$

$$\Omega_{kl} I_0[u](t) = \int_0^t \frac{\sin\omega(t-\tau)}{\omega} \Omega_{kl} F(u(\tau))\, d\tau, \tag{3.11}$$

$$L_0 I_0[u](t) = 2\int_0^t \frac{\sin\omega(t-\tau)}{\omega} F(u(\tau))\, d\tau + \int_0^t \frac{\sin\omega(t-\tau)}{\omega} L_0 F(u(\tau))\, d\tau. \tag{3.12}$$

As for the proof of (3.10)–(3.12), refer to Section 5 below where we shall meet a somewhat complicated situation. The continuity in time of $\Gamma^\alpha I_0[u]$ ($|\alpha| \le 1$) as well as $\omega^{-1}\partial_t I_0[u]$ in $\dot H_2^{\theta-\varepsilon}$ can be easily verified. Therefore we focus our attention on the estimate of them in the $L^\infty(0,\infty;\dot H_2^{\theta-\varepsilon})$-norm. In view of (3.10)–(3.12) and the basic fact that $\partial_t I_0[u] = \int_0^t \cos\omega(t-\tau) F(u(\tau))\, d\tau$ we obtain for $u \in Z_{2C_2\Lambda}$

$$||Mu||_Z \le ||u_0||_Z + ||I_0[u]||_Z \tag{3.13}$$

$$\le C_2\Lambda + \sum_{j=1}^{n} ||x_j F(f)||_{\dot H_2^{-1+\theta-\varepsilon}} + 4\sup_{t>0}\int_0^t ||F(u(\tau))||_{\dot H_2^{-1+\theta-\varepsilon}}\,d\tau$$

$$+\,2\sum_{j=1}^{n}\sup_{t>0}\int_0^t \|\partial_j F(u(\tau))\|_{\dot H_2^{-1+\theta-\varepsilon}} d\tau + \sum_{|\alpha|=1}\sup_{t>0}\int_0^t \|\dot\Gamma^\alpha F(u(\tau))\|_{\dot H_2^{-1+\theta-\varepsilon}} d\tau.$$

Here we have estimated the $\dot H_2^{\theta-\varepsilon}$-norm of $\partial_t I_0[u]$ in the following way:

$$\begin{aligned} ||\partial_t I_0[u]||_{\dot H_2^{\theta-\varepsilon}} &\le \int_0^t ||\cos\omega(t-\tau)\omega^{-2+\theta-\varepsilon}\omega^2 F(u(\tau))||_{L^2}\,d\tau \\ &\le \sum_{j=1}^{n}\int_0^t ||\partial_j F(u(\tau))||_{\dot H_2^{-1+\theta-\varepsilon}}\,d\tau. \end{aligned}$$

It should be remarked that the equality

$$||\Gamma^\alpha F(u(\tau))||_{\dot H_2^{-1+\theta-\varepsilon}} = ||F'(u(\tau))\Gamma^\alpha u(\tau)||_{\dot H_2^{-1+\theta-\varepsilon}}$$

($|\alpha|=1$) is true for $u\in Z_{2C_2\Lambda}$. This can be checked by mollifying a function $u\in Z_{2C_2\Lambda}$ in the x-variables and then taking the limit in the usual manner.

Now we shall show for $|\alpha|\le 1$

$$\begin{aligned} &||F'(u(\tau))\Gamma^\alpha u(\tau)||_{\dot H_2^{-1+\theta-\varepsilon}} \\ &\le C(1+\tau)^{-1-\varepsilon(p-1)}\Big(\sum_{|\beta|\le 1}\sup_{t>0}||\Gamma^\beta u(t)||_{\dot H_2^{\theta-\varepsilon}}\Big)^{p-1}\sup_{t>0}||\Gamma^\alpha u(t)||_{\dot H_2^{\theta-\varepsilon}}. \end{aligned} \tag{3.14}$$

Note that the inequality $1/2 < 1-(\theta-\varepsilon) < n/2$ is true for $p\le 1+4/(n-1)$. Then, for the proof of (3.14), we can make use of (2.52) with $q=2$ and $s = 1-(\theta-\varepsilon)$ to obtain

$$\begin{aligned} &||F'(u(\tau))\Gamma^\alpha u(\tau)||_{\dot H_2^{-1+\theta-\varepsilon}} \\ &\le C||\,|u(\tau)|^{p-1}\Gamma^\alpha u(\tau)\,||_{l,\chi_1} + C(1+\tau)^{-(n/2-1+\theta-\varepsilon)}||\,|u(\tau)|^{p-1}\Gamma^\alpha u(\tau)||_{1,2,\chi_2}, \end{aligned} \tag{3.15}$$

where $1/l = 1/2 + (1-\theta+\varepsilon)/n$. Choosing l_1 and l_2 by $1/l = 1/l_1 + 1/l_2$, $1/l_2 = 1/2-(\theta-\varepsilon)/n$, we estimate the first term on the right-hand side of (3.15) in the case $n=2$

$$\begin{aligned} \cdots &\le C||u(\tau)||^{p-1}_{(p-1)l_1,\chi_1}||\Gamma^\alpha u(\tau)||_{l_2} \\ &\le C(1+\tau)^{n/l_1}||u(\tau)||^{p-1}_{\infty,\chi_1}||\Gamma^\alpha u(\tau)||_{l_2} \\ &\le C(1+\tau)^{-(p-1)n/l_2+n/l_1}||u(\tau)||_{\Gamma,1,l_2}||\Gamma^\alpha u(\tau)||_{l_2} \end{aligned} \tag{3.16}$$

$$\leq C(1+\tau)^{-1-\varepsilon(p-1)}\Big(\sum_{|\beta|\leq 1}\sup_{t>0}||\Gamma^\beta u(t)||_{\dot{H}_2^{\theta-\varepsilon}}\Big)^{p-1}\sup_{t>0}||\Gamma^\alpha u(t)||_{\dot{H}_2^{\theta-\varepsilon}}.$$

Here we have employed (2.8) in the third inequality. We have also used the Sobolev embedding $\dot{H}_2^{\theta-\varepsilon}(\mathbb{R}^n)\hookrightarrow L^{l_2}(\mathbb{R}^n)$. In the case of $n=3$, instead of (2.8) we can employ (2.6) by virtue of the assumption $p\leq 3$ to obtain the estimate

$$\begin{aligned}&||F'(u(\tau))\Gamma^\alpha u(\tau)||_{l,\chi_1} \\ &\leq C(1+\tau)^{-1-\varepsilon(p-1)}\Big(\sum_{|\beta|\leq 1}\sup_{t>0}||\Gamma^\beta u(t)||_{\dot{H}_2^{\theta-\varepsilon}}\Big)^{p-1}\sup_{t>0}||\Gamma^\alpha u(t)||_{\dot{H}_2^{\theta-\varepsilon}}.\end{aligned} \tag{3.17}$$

Next we turn our attention to the second term on the right-hand side of (3.15). Let us choose l_3 and l_4 by $1/2=1/l_3+1/l_4$,

$$\frac{1}{l_4}-\frac{1/l_4-1/p+\varepsilon}{n}=\frac{1}{2}-\frac{\theta-\varepsilon}{n}. \tag{3.18}$$

Observe that the condition $\varepsilon<[(n-1)p^2-(n+1)p-2]/4p(p-1)$ is equivalent to $l_4>2$. Making use of the Sobolev embedding $W^{1,l_4}(S^{n-1})\hookrightarrow L^\infty(S^{n-1})$, we have

$$\begin{aligned}&||\,|u(\tau)|^{p-1}\Gamma^\alpha u(\tau)\,||_{1,2,\chi_2}\leq ||u(\tau)||^{p-1}_{p,(p-1)l_3,\chi_2}||\Gamma^\alpha u(\tau)||_{p,l_4,\chi_2} \\ &\leq C||u(\tau)||^{p-1}_{p,\infty,\chi_2}||\Gamma^\alpha u(\tau)||_{p,l_4,\chi_2} \\ &\leq C\Big(\sum_{|\beta|\leq 1}||\Omega^\beta u(\tau)||_{p,l_4,\chi_2}\Big)^{p-1}||\Gamma^\alpha u(\tau)||_{p,l_4,\chi_2} \\ &\leq C(1+\tau)^{-p(n-1)(1/l_4-1/p)+p\varepsilon}\Big(\sum_{|\beta|\leq 1}||\Omega^\beta u(\tau)||_{\dot{H}_2^{\theta-\varepsilon}}\Big)^{p-1}||\Gamma^\alpha u(\tau)||_{\dot{H}_2^{\theta-\varepsilon}}.\end{aligned} \tag{3.19}$$

The inequality (2.50) and the Sobolev embedding

$$\dot{H}_2^{\theta-\varepsilon}(\mathbb{R}^n)\hookrightarrow \dot{H}_{l_4}^{1/l_4-1/p+\varepsilon}(\mathbb{R}^n)$$

have been used at the last inequality. Observing

$$-(\frac{n}{2}-1+\theta-\varepsilon)-p(n-1)(\frac{1}{l_4}-\frac{1}{p})+p\varepsilon=-1-\varepsilon(p-1), \tag{3.20}$$

we have obtained the desired estimate for the second term on the right-hand side of (3.15).

Our next task is to bound $||x_jF(f)||_{\dot{H}_2^{-1+\theta-\varepsilon}}$, which appears in (3.13). This would be dealt with through the use of Lemma 2.4 at least in the case $n=3$. Instead of Lemma 2.4 we employ (2.33)–(2.34) and the idea of the proof of Lemma 2.9 to treat both the cases $n = 2$ and $n = 3$ together. Recall that Φ_a $(a > 0)$ is the characteristic function of the set $\{x \in \mathbb{R}^n : |x| \geq a\}$. We see by choosing l as in (3.15)

$$\begin{aligned}
&||x_j F(f)||_{\dot{H}_2^{-1+\theta-\varepsilon}} = \sup |(x_j F(f), v)| \qquad (3.21)\\
&\leq \sup |((1-\Phi_{1/2})x_j F(f), v)| + \sup |(\Phi_{1/2} x_j F(f), v)|\\
&\leq \sup ||(1-\Phi_{1/2})x_j F(f)||_{L^l} ||v||_{L^{l'}}\\
&\ + \sup ||\Phi_{1/2}|x|^{1-(n/2-1+\theta-\varepsilon)} F(f)||_{1,2} ||\, |x|^{n/2-(1-\theta+\varepsilon)} v\, ||_{\infty,2}\\
&\leq C||(1-\Phi_{1/2})x_j F(f)||_{L^l} + C||\Phi_{1/2}|x|^{1-(n/2-1+\theta-\varepsilon)} F(f)||_{1,2},
\end{aligned}$$

where the supremum is taken over all $v \in \dot{H}_2^{1-\theta+\varepsilon}(\mathbb{R}^n)$ with the norm one. In the last inequality we have used the Sobolev embedding $\dot{H}_2^{1-\theta+\varepsilon}(\mathbb{R}^n) \hookrightarrow L^{l'}(\mathbb{R}^n)$ and (2.34). To continue the estimate we follow the argument in (3.16)–(3.19). It follows from the standard Sobolev embedding that

$$||(1-\Phi_{1/2})x_j F(f)||_{L^l} \leq C(||f||_{\dot{H}_2^{\theta-\varepsilon}} + ||f||_{\dot{H}_2^{1+\theta-\varepsilon}})^{p-1} ||f||_{\dot{H}_2^{\theta-\varepsilon}}. \qquad (3.22)$$

Moreover, we have by making use of (3.20) and (2.33)

$$\begin{aligned}
&||\Phi_{1/2}|x|^{1-(n/2-1+\theta-\varepsilon)} F(f)||_{1,2} = C||\Phi_{1/2}|x|^{\{1-(n/2-1+\theta-\varepsilon)\}/p} f||_{p,2p}^p \qquad (3.23)\\
&\leq C||\Phi_{1/2}|x|^{\{1-(n/2-1+\theta-\varepsilon)\}/p} f||_{p,\infty}^p\\
&\leq C\Big(\sum_{|\beta|\leq 1} ||\Phi_{1/2}|x|^{(n-1)(1/l_4-1/p)-\varepsilon-(p-1)\varepsilon/p} \Omega^\beta f||_{p,l_4}\Big)^p\\
&\leq C\Big(\sum_{|\beta|\leq 1} ||\Omega^\beta f||_{\dot{H}_{l_4}^{1/l_4-1/p+\varepsilon}}\Big)^p \leq C\Big(||f||_{\dot{H}_2^{\theta-\varepsilon}} + \sum_{1\leq j<k\leq n} ||x_j \partial_k f||_{\dot{H}_2^{\theta-\varepsilon}}\Big)^p\\
&\leq C(||f||_{\dot{H}_2^{\theta-\varepsilon}} + ||\, |x| \omega^{1+\theta-\varepsilon} f||_{L^2})^p.
\end{aligned}$$

Therefore we finally obtain from (3.13), (3.14) and (3.21)–(3.23)

$$\begin{aligned}
||Mu||_Z \leq\ & C_2 \Lambda + C_3(||f||_{\dot{H}_2^{\theta-\varepsilon}} + ||\langle x\rangle \omega^{1+\theta-\varepsilon} f||_{L^2})^p \qquad (3.24)\\
&+ C_4\Big(\sum_{|\beta|\leq 1} \sup_{t>0} ||\Gamma^\beta u(t)||_{\dot{H}_2^{\theta-\varepsilon}}\Big)^{p-1} \sum_{|\alpha|\leq 1} \sup_{t>0} ||\Gamma^\alpha u(t)||_{\dot{H}_2^{\theta-\varepsilon}}
\end{aligned}$$

for suitable constants C_3, $C_4 > 0$. Furthermore, repeating essentially the same argument as in (3.13) and (3.15)–(3.19), we get

$$||Mu - Mv||_Z \leq C_5\Big(\sum_{|\beta|\leq 1} (\sup_{t>0} ||\Gamma^\beta u(t)||_{\dot{H}_2^{\theta-\varepsilon}} + \sup_{t>0} ||\Gamma^\beta v(t)||_{\dot{H}_2^{\theta-\varepsilon}})\Big)^{p-1}$$

$$\times \sum_{|\alpha|\le 1} \sup_{t>0} ||\Gamma^\alpha u(t) - \Gamma^\alpha v(t)||_{\dot H_2^{\theta-\varepsilon}}. \tag{3.25}$$

If Λ is small so that $C_2\Lambda + C_3\Lambda^p + C_4(2C_2\Lambda)^p \le 2C_2\Lambda$ and $C_5(4C_2\Lambda)^{p-1} \le 1/2$ may hold, M turns out to carry $Z_{2C_2\Lambda}$ into itself and satisfy (3.9). Thus we have completed the proof of Proposition 3.2.

Our next task is to show (3.6)–(3.7). It is enough to prove the existence of $(f_+, g_+) \in X \times Y$ satisfying (3.6)–(3.7). Using the solution u obtained above, we define

$$u_+(t) := u_0(t) - \int_0^\infty \frac{\sin\omega(t-\tau)}{\omega} F(u(\tau))\, d\tau. \tag{3.26}$$

Then, as is easily seen, u_+ satisfies $\Box u_+ = 0$ and $u_+, \omega^{-1}\partial_t u_+ \in BC(\mathbb{R}; \dot H_2^{\theta-\varepsilon} \cap \dot H_2^{1+\theta-\varepsilon})$. Define $(f_+, g_+) := (u_+(0), \partial_t u_+(0))$. In order to prove (3.6)–(3.7) we need to verify that the free solution u_+ satisfies $L_j u_+$, $\Omega_{kl}u_+$, $L_0 u_+ \in BC(\mathbb{R}; \dot H_2^{\theta-\varepsilon}(\mathbb{R}^n))$ and

$$\begin{aligned} L_j u_+(t) = L_j u_0(t) - \frac{\sin\omega t}{\omega}[x_j F(f)] \\ - \int_0^\infty \frac{\sin\omega(t-\tau)}{\omega} L_j F(u(\tau))\, d\tau,\ j = 1, \dots, n, \end{aligned} \tag{3.27}$$

$$\Omega_{kl} u_+(t) = \Omega_{kl} u_0(t) - \int_0^\infty \frac{\sin\omega(t-\tau)}{\omega} \Omega_{kl} F(u(\tau))\, d\tau,\ 1 \le k < l \le n, \tag{3.28}$$

$$\begin{aligned} L_0 u_+(t) = L_0 u_0(t) - 2\int_0^\infty \frac{\sin\omega(t-\tau)}{\omega} F(u(\tau))\, d\tau \\ - \int_0^\infty \frac{\sin\omega(t-\tau)}{\omega} L_0 F(u(\tau)) d\tau \end{aligned} \tag{3.29}$$

hold. For a moment we assume (3.27)–(3.29). Then, (3.7) immediately follows from (3.10)–(3.12) and (3.27)–(3.29). In fact we find for $t > 0$ and $|\alpha| \le 1$

$$||\Gamma^\alpha u(t) - \Gamma^\alpha u_+(t)||_{\dot H_2^{\theta-\varepsilon}} \le C \int_t^\infty (1+\tau)^{-1-\varepsilon(p-1)} d\tau \le C(1+t)^{-\varepsilon(p-1)}. \tag{3.30}$$

The same estimate is true for the $\dot H_2^{\theta-\varepsilon}$-norm of $\omega^{-1}\partial_t u - \omega^{-1}\partial_t u_+$. Finally, we need to check $(u_+(0), \partial_t u_+(0)) \in X \times Y$. It is actually possible to show the fact that $(u_+(t), \partial_t u_+(t)) \in X \times Y$ for all $t \in \mathbb{R}$. To check $u_+(t) \in X$ for any $t \in \mathbb{R}$, we employ (2.4) together with a simple inequality $\langle x\rangle \le 1 + |x| \le$

$1+|\,|x|-|t|\,|+|t|$ to find

$$\begin{aligned}&||\langle x\rangle\partial_j\omega^{\theta-\varepsilon}u_+(t)||_2 \qquad (3.31)\\&\le ||(1+|\,|t|-|x|\,|)\partial_j\omega^{\theta-\varepsilon}u_+(t)||_2+|t|\times||\partial_j\omega^{\theta-\varepsilon}u_+(t)||_2\\&\le C||\omega^{\theta-\varepsilon}u_+(t)||_{\Gamma,1,2}+|t|\times||u_+(t)||_{\dot H_2^{1+\theta-\varepsilon}}\\&\le C\Big(\sum_{|\alpha|\le 1}||\Gamma^\alpha u_+(t)||_{\dot H_2^{\theta-\varepsilon}}+||\omega^{-1}\partial_t u_+(t)||_{\dot H_2^{\theta-\varepsilon}}\Big)+|t|\times||u_+(t)||_{\dot H_2^{1+\theta-\varepsilon}},\end{aligned}$$

where Lemma 2.1 has been used in the last inequality. Since the X-norm is equivalent to

$$||\omega^{\theta-\varepsilon}f||_{L^2}+\sum_{j=1}^n||\langle x\rangle\partial_j\omega^{\theta-\varepsilon}f||_{L^2}, \qquad (3.32)$$

we can find that $u_+(t)\in X$ for any $t\in\mathbb{R}$. In the same way as above we also see that $\partial_t u_+(t)\in Y$ for any $t\in\mathbb{R}$.

It remains to show (3.27)–(3.29). We shall prove (3.27) only, because the proofs of the others are easier. For $\sigma>0$ put

$$u_+^\sigma(t):=u_0(t)-\int_0^\sigma\frac{\sin\omega(t-\tau)}{\omega}F(u(\tau))\,d\tau. \qquad (3.33)$$

Since $u_+^\sigma\to u_+$ and $\omega^{-1}\partial_t u_+^\sigma\to\omega^{-1}\partial_t u_+$ in $BC(\mathbb{R};\dot H_2^{\theta-\varepsilon}\cap\dot H_2^{1+\theta-\varepsilon})$ as $\sigma\to+\infty$, it follows that $L_ju_+^\sigma(t)\to L_ju_+(t)$ $(\sigma\to+\infty)$ in $\mathcal{S}'(\mathbb{R}^n)$ for every $t\in\mathbb{R}$. On the other hand, the equality

$$\begin{aligned}L_j\int_0^\sigma\frac{\sin\omega(t-\tau)}{\omega}F(u(\tau))\,d\tau&=-\frac{\sin\omega(t-\sigma)}{\omega}x_jF(u(\sigma))+\frac{\sin\omega t}{\omega}x_jF(f)\\&+\int_0^\sigma\frac{\sin\omega(t-\tau)}{\omega}L_jF(u(\tau))\,d\tau\ \text{ in } BC(\mathbb{R};\dot H_2^{\theta-\varepsilon}(\mathbb{R}^n))\end{aligned}$$

together with the fact that $x_jF(u(\sigma))\to 0$ $(\sigma\to+\infty)$ in $\dot H_2^{-1+\theta-\varepsilon}(\mathbb{R}^n)$ (see (3.34)–(3.35)) implies

$$L_ju_+^\sigma(t)\to L_ju_0(t)-\frac{\sin\omega t}{\omega}x_jF(f)-\int_0^\infty\frac{\sin\omega(t-\tau)}{\omega}L_jF(u(\tau))\,d\tau$$

in $BC(\mathbb{R};\dot H_2^{\theta-\varepsilon}(\mathbb{R}^n))$ as $\sigma\to+\infty$. In view of the fact that $\dot H_2^{\theta-\varepsilon}\hookrightarrow\mathcal{S}'$ we find that the equality (3.27) holds and $L_ju_+\in BC(\mathbb{R};\dot H_2^{\theta-\varepsilon}(\mathbb{R}^n))$. We also know from (3.28)–(3.29) that $\Omega_{kl}u_+$, $L_0u_+\in BC(\mathbb{R};\dot H_2^{\theta-\varepsilon}(\mathbb{R}^n))$.

We must prove $x_jF(u(\sigma)) \to 0$ in $\dot{H}_2^{-1+\theta-\varepsilon}(\mathbb{R}^n)$ as $\sigma \to +\infty$. Employing (2.53), we have

$$||x_jF(u(\sigma))||_{\dot{H}_2^{-(1-\theta+\varepsilon)}} \tag{3.34}$$
$$\leq C||x_jF(u(\sigma))||_{l,\chi_1} + C||\,|x|^{1-(n/2-1+\theta-\varepsilon)}F(u(\sigma))\,||_{1,2,\chi_2},$$

where $1/l = 1/2 + (1-\theta+\varepsilon)/n$ as in (3.15). Proceeding as in (3.16)–(3.17), we can estimate $||x_jF(u(\sigma))||_{l,\chi_1} \leq C(1+\sigma)^{-\varepsilon(p-1)}$. Moreover, for any δ with $0 < \delta < \varepsilon(p-1)/p$ we get in the same way as in (3.23)

$$||\,|x|^{1-(n/2-1+\theta-\varepsilon)}F(u(\sigma))\,||_{1,2,\chi_2} = C||\,|x|^{\{1-(n/2-1+\theta-\varepsilon)\}/p}u(\sigma)\,||^p_{p,2p,\chi_2}$$
$$\leq C||\,|x|^{\{1-(n/2-1+\theta-\varepsilon)\}/p}u(\sigma)\,||^p_{p,\infty,\chi_2} \tag{3.35}$$
$$\leq C\Big(\sum_{|\beta|\leq 1}||\,|x|^{(n-1)(1/l_4-1/p)-\varepsilon-\varepsilon(p-1)/p}\Omega^\beta u(\sigma)||_{p,l_4,\chi_2}\Big)^p$$
$$\leq C(1+\sigma)^{-\varepsilon(p-1)+p\delta}\Big(\sum_{|\beta|\leq 1}||\Phi_{1/2}|x|^{(n-1)(1/l_4-1/p)-\varepsilon-\delta}\Omega^\beta u(\sigma)||_{p,l_4}\Big)^p$$
$$\leq C(1+\sigma)^{-\varepsilon(p-1)+p\delta}\Big(\sum_{|\beta|\leq 1}||\Omega^\beta u(\sigma)||_{\dot{H}_{l_4}^{1/l_4-1/p+\varepsilon}}\Big)^p$$
$$\leq C(1+\sigma)^{-\varepsilon(p-1)+p\delta}\Big(\sum_{|\beta|\leq 1}||\Omega^\beta u(\sigma)||_{\dot{H}_2^{\theta-\varepsilon}}\Big)^p.$$

Therefore we have completed the proof of Theorem 3.1. □

4. Wave operator

The main theorem in this section is concerned with the existence of the wave operator for negative time. In a similar way the wave operator for positive time can be shown to exist. Let ε be the number picked in the previous section, and let X, Y be the same spaces as those defined in (3.2)–(3.3).

Theorem 4.1. *Suppose that $n = 2, 3$ and $p_0(n) < p \leq 1 + 4/(n-1)$. There exists a constant $\delta_1 > 0$ depending on n, p, ε and λ with the following property*:

For any $(f_-, g_-) \in X \times Y$ with $||(f_-, g_-)||_{X\times Y} < \delta_1$ the integral equation

$$u(t) = u_-(t) - \lambda\int_{-\infty}^{t}\frac{\sin\omega(t-\tau)}{\omega}|u(\tau)|^{p-1}u(\tau)\,d\tau \tag{4.1}$$

$(u_-(t) = (\cos\omega t)f_- + (\omega^{-1}\sin\omega t)g_-)$ *has a unique solution* $u(t,x)$ *satisfying*

$$\Gamma^\alpha u(t),\ \omega^{-1}\partial_t u(t) \in BC\big((-\infty,0];\dot H_2^{\theta-\varepsilon}(\mathbb{R}^n)\big), \tag{4.2}$$

$$||\big(\Gamma^\alpha u(t)-\Gamma^\alpha u_-(t), \omega^{-1}\partial_t u(t)-\omega^{-1}\partial_t u_-(t)\big)||_{\dot H_2^{\theta-\varepsilon}\times \dot H_2^{\theta-\varepsilon}} \to 0 \tag{4.3}$$

$(t\to-\infty)$ *for* $|\alpha|\le 1$*. Moreover, this solution satisfies*

$$||(u(0),\partial_t u(0))||_{X\times Y} \le C_6||(f_-,g_-)||_{X\times Y} \tag{4.4}$$

for some constant $C_6>0$*.*

Remark. (1) The integral in (4.1) is interpreted as the improper integral in $\dot H_2^{\theta-\varepsilon}(\mathbb{R}^n)$. (2) Let B_r be an open ball of $X\times Y$ centered at the origin with radius $r>0$. By Theorem 4.1 we know the existence of the wave operator for negative time as a mapping from B_{δ_1} into $B_{C_6\delta_1}$. Moreover, combined with Theorem 3.1, this theorem implies the existence of the scattering operator as a mapping from B_δ into $B_{C_1C_6\delta}$ for any δ with $\delta\le\min(\delta_1,\delta_0/C_6)$.

Proof of Theorem 4.1. We introduce the set V of functions on $(-\infty,0]\times\mathbb{R}^n$ as follows:

$$V := \{u=u(t,x) : \Gamma^\alpha u,\ \omega^{-1}\partial_t u \in C((-\infty,0];\dot H_2^{\theta-\varepsilon}(\mathbb{R}^n)),\ |\alpha|\le 1$$
$$||u||_V = \sum_{|\alpha|\le 1}\sup_{t<0}||\Gamma^\alpha u(t)||_{\dot H_2^{\theta-\varepsilon}} + \sup_{t<0}||\omega^{-1}\partial_t u(t)||_{\dot H_2^{\theta-\varepsilon}} <\infty\}.$$

We denote by V_δ $(\delta>0)$ the set of all $u\in V$ such that $||u||_V\le\delta$. V_δ is complete with respect to the metric $||u-v||_V$ $(u,\,v\in V_\delta)$ for every $\delta>0$.

The strategy of the proof is very similar to that of the proof of Theorem 3.1. Thus we state only the sketch of the proof. Denoting $||(f_-,g_-)||_{X\times Y}$ by Λ_-, we know from Lemma 2.11 that $u_-\in V_{C_2\Lambda_-}$ for the same constant C_2 as in the previous section. Defining the mapping

$$S: u\mapsto u_-(t)-\lambda\int_{-\infty}^t \frac{\sin\omega(t-\tau)}{\omega}|u(\tau)|^{p-1}u(\tau)\,d\tau, \tag{4.5}$$

we next need to prove that there exists a constant $\delta_1>0$ depending on n, p, ε and λ such that if $\Lambda_-<\delta_1$, then S is the contraction on $V_{2C_2\Lambda_-}$. Since (4.1) includes an improper integral, we should be careful in letting the operators L_j, Ω_{kl} and L_0 act on such an integral term. As in (3.27)–(3.29) and (3.33)–(3.35), we can proceed to obtain

$$L_j\int_{-\infty}^t\frac{\sin\omega(t-\tau)}{\omega}F(u(\tau))\,d\tau = \int_{-\infty}^t\frac{\sin\omega(t-\tau)}{\omega}L_jF(u(\tau))\,d\tau, \tag{4.6}$$

$$\Omega_{kl}\int_{-\infty}^t\frac{\sin\omega(t-\tau)}{\omega}F(u(\tau))\,d\tau = \int_{-\infty}^t\frac{\sin\omega(t-\tau)}{\omega}\Omega_{kl}F(u(\tau))\,d\tau, \tag{4.7}$$

$$L_0 \int_{-\infty}^{t} \frac{\sin \omega(t-\tau)}{\omega} F(u(\tau))\, d\tau \tag{4.8}$$
$$= 2\int_{-\infty}^{t} \frac{\sin \omega(t-\tau)}{\omega} F(u(\tau)) d\tau + \int_{-\infty}^{t} \frac{\sin \omega(t-\tau)}{\omega} L_0 F(u(\tau)) d\tau.$$

Then, being on the same lines as in the previous section, we can show the contraction property of the mapping S. The asymptotic property (4.3) is shown as in (3.30). As for (4.4), we employ (2.4) and proceed as in (3.31). □

5. Self-similar solutions

In this section we shall prove another global existence theorem, which allows us to show the existence of self-similar solutions for some homogeneous initial data.

Throughout this section the space dimension n is assumed to be two or three. For any p with $p_0(n) < p < 1 + 4/(n-1)$ we fix an arbitrary ε satisfying

$$0 < \varepsilon < \min\Big(\frac{(n-1)p^2 - (n+1)p - 2}{2p(p-1)}, \frac{2}{p-1} - \frac{n-1}{2}, \frac{2}{p-1} - \frac{n}{p} \Big). \tag{5.1}$$

We also give a number q by

$$\frac{1}{q} - \frac{\frac{1}{q} - \frac{1}{p}}{n} = \frac{1}{2} - \frac{\frac{n}{2} - \frac{2}{p-1}}{n}. \tag{5.2}$$

Observe that the assumption $\varepsilon < [(n-1)p^2 - (n+1)p - 2]/2p(p-1)$ is equivalent to $\varepsilon < \gamma(q)$. This fact will play an important role in our argument below. Let us introduce the set W of functions on $(0,\infty) \times \mathbb{R}^n$ as follows:

$$\begin{aligned} W := \{\, u = u(t,x) \,:\, & u(t) \text{ is Bochner-measurable function such that} \\ & u : (0,\infty) \to \dot H_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n), \\ & \|u\|_W = \sum_{|\alpha|\le 1} \operatorname*{ess\,sup}_{t>0} t^{\varepsilon} \|\dot\Gamma^{\alpha} u(t)\|_{\dot H_q^{1/q-1/p+\varepsilon}} < \infty \,\}. \end{aligned}$$

We denote by W_δ $(\delta > 0)$ the set of all $u \in W$ such that $\|u\|_W \le \delta$. W_δ is complete with respect to the metric $\|u - v\|_W$ $(u, v \in W_\delta)$ for every $\delta > 0$. The global existence theorem in this section is as follows.

Theorem 5.1. *Suppose $n = 2, 3$ and $p_0(n) < p < 1 + 4/(n-1)$. Let δ_2 be any positive constant satisfying both $C_7(2\delta_2)^{p-1} \le 1/2$ and $C_8(4\delta_2)^{p-1} < 1$, where C_7 and C_8 are constants given above* (5.73) *and in* (5.73)*, respectively.*

(1) *If* $(f,g) \in \mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ *is given so that the free solution* $u_0(t) = (\cos\omega t)f + (\omega^{-1}\sin\omega t)g$ *may satisfy* $u_0 \in W_{\delta_2}$, *then the integral equation*

$$u(t) = u_0(t) - \lambda \int_0^t \frac{\sin\omega(t-\tau)}{\omega} |u(\tau)|^{p-1} u(\tau)\, d\tau,\ t > 0 \tag{5.3}$$

has a unique solution in $W_{2\delta_2}$.
(2) *Furthermore, if the free solution satisfies*

$$t^{\varepsilon}\omega^{-1}\partial_t u_0 \in L^{\infty}(0,\infty; \dot{H}_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n))$$

as well as $u_0 \in W_{\delta_2}$, *then*

$$t^{\varepsilon}\omega^{-1}\partial_t u \in L^{\infty}(0,\infty; \dot{H}_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n)), \tag{5.4}$$

$$u - u_0,\ \omega^{-1}\partial_t u - \omega^{-1}\partial_t u_0 \in C([0,\infty); \dot{H}_2^{\theta-\varepsilon}(\mathbb{R}^n))\ (\theta = \frac{n}{2} - \frac{2}{p-1}), \tag{5.5}$$

$$u(t) \to f,\ \partial_t u(t) \to g \text{ in } \mathcal{S}'(\mathbb{R}^n) \text{ as } t \to +0. \tag{5.6}$$

(3) *If, in addition to the assumption in* (1), f *is homogeneous of degree* $-2/(p-1)$ *and* g *is homogeneous of degree* $-1-2/(p-1)$, *then the solution* u *given in* (1) *satisfies the identity* $u(t,x) = \delta^{2/(p-1)}u(\delta t, \delta x)$ *for all* $\delta > 0$.

Proof of Theorem 5.1. In order to prove Theorem 5.1 we define the mapping

$$M : u \mapsto u_0(t) - I_0[u](t)\ (u \in W_{2\delta_2}), \tag{5.7}$$

where we have put

$$I_0[u](t) := \lambda \int_0^t \frac{\sin\omega(t-\tau)}{\omega} |u(\tau)|^{p-1} u(\tau)\, d\tau$$

as before. This integral is interpreted as the Bochner integral in

$$\dot{H}_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n).$$

The contraction-mapping principle is employed again. We shall show that M carries $W_{2\delta_2}$ into itself and it is the contraction mapping on $W_{2\delta_2}$.

We start with showing that M carries $W_{2\delta_2}$ into itself. Since it is assumed that $u_0 \in W_{\delta_2}$, our task is to check $I_0[u] \in W_{\delta_2}$. For this aim we shall devote ourselves to showing for $u \in W$

$$\operatorname*{ess\,sup}_{t>0} t^{\varepsilon} ||\dot{\Gamma}^{\alpha} I_0[u](t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \le C||u||_W^p,\ |\alpha| \le 1. \tag{5.8}$$

Here we make a remark. Since the homogeneous Sobolev space

$$\dot{H}_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n)$$

is separable, $I_0[u]$ turns out to be Bochner measurable once

$$I_0[u] \in C_w((0,\infty); \dot{H}_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n))$$

has been shown. Employing the argument in (5.56)–(5.57) and (5.26)–(5.30) below (see also (5.74)), we can show $I_0[u] \in C([0,\infty); \dot{H}_2^{\theta-\varepsilon}(\mathbb{R}^n))$. Thereby $I_0[u] \in C([0,\infty); \mathcal{S}'(\mathbb{R}^n))$ because $\dot{H}_2^{\theta-\varepsilon}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ holds now. The estimate (5.8) with $\dot{\Gamma}^\alpha = 1$ immediately implies that

$$I_0[u] \in C_w((0,\infty); \dot{H}_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n))$$

because $\mathcal{S}(\mathbb{R}^n)$ is dense in $\dot{H}_{q'}^{-(1/q-1/p+\varepsilon)}(\mathbb{R}^n)$.

(5.8) with $\dot{\Gamma}^\alpha = L_j$ is the most troublesome to show. We shall therefore focus our attention on this case for a while. Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be any nonnegative, radial function such that $\operatorname{supp} \eta \subset \{\, x \in \mathbb{R}^n : |x| < 2 \,\}$ and $\eta(x) = 1$ for $|x| \le 3/2$ (the reader should not confuse this function with another one, $\eta = \eta(t,x)$, introduced in the proof of Lemma 2.3). For $\tau > 0$ and $x \in \mathbb{R}^n$, we define $\eta_1(\tau,x) := \eta(x/\tau)$ and $\eta_2(\tau,x) := 1 - \eta_1(\tau,x)$. Since our argument below shows that, for any $t > 0$, $\omega^{-1} \sin\omega(t-\tau)[\eta_j(\tau)F(u(\tau))]$ is Bochner integrable in $\dot{H}_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n)$ over $(0,t)$ and

$$\int_0^t \Big|\Big| \frac{\sin\omega(t-\tau)}{\omega}[\eta_j(\tau)F(u(\tau))] \Big|\Big|_{\dot{H}_q^{1/q-1/p+\varepsilon}} d\tau \le C t^{-\varepsilon} ||u||_W^p, \;\; j=1,2, \tag{5.9}$$

we find for any $T > 0$

$$\int_0^t \frac{\sin\omega(t-\tau)}{\omega}[\eta_j(\tau)F(u(\tau))]\, d\tau \in L^\infty(T,\infty; \dot{H}_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n)) \subset \mathcal{D}'((0,\infty)\times\mathbb{R}^n). \tag{5.10}$$

For a technical reason, we divide $I_0[u]$ into two parts:

$$I_0[u](t) = \sum_{j=1}^2 \int_0^t \frac{\sin\omega(t-\tau)}{\omega}[\eta_j(\tau)F(u(\tau))]\, d\tau =: \sum_{j=1}^2 I_j[u](t). \tag{5.11}$$

We shall start with calculating $L_j I_2[u]$. For any $\psi \in C_0^\infty((0,\infty)\times\mathbb{R}^n)$

$$\langle\!\langle L_j I_2[u], \psi \rangle\!\rangle = -\langle\!\langle I_2[u], t\partial_j\psi \rangle\!\rangle - \langle\!\langle I_2[u], x_j\partial_t\psi \rangle\!\rangle. \tag{5.12}$$

Furthermore,

$$\begin{aligned} \langle\!\langle I_2[u], x_j\partial_t\psi \rangle\!\rangle &= \int_0^\infty \langle x_j\partial_t\psi(t), \int_0^t \frac{\sin\omega(t-\tau)}{\omega}[\eta_2(\tau)F(u(\tau))]\, d\tau \rangle\, dt \\ &= \int_0^\infty \int_0^t \langle x_j\partial_t\psi(t), \frac{\sin\omega(t-\tau)}{\omega}[\eta_2(\tau)F(u(\tau))] \rangle\, d\tau\, dt \end{aligned} \tag{5.13}$$

$$= \int_0^\infty \int_0^t \langle \eta_2(\tau)F(u(\tau)), \frac{\sin\omega(t-\tau)}{\omega}[x_j\partial_t\psi(t)]\rangle\, d\tau\, dt$$
$$= \int_0^\infty \int_0^t \langle \eta_2(\tau)F(u(\tau)), \partial_t\Big(\frac{\sin\omega(t-\tau)}{\omega}[x_j\psi(t)]\Big)\rangle\, d\tau\, dt$$
$$- \int_0^\infty \int_0^t \langle \eta_2(\tau)F(u(\tau)), \cos\omega(t-\tau)[x_j\psi(t)]\rangle\, d\tau\, dt.$$

Observe that $\int_0^t \langle \eta_2(\tau)F(u(\tau)), \omega^{-1}\sin\omega(t-\tau)[x_j\psi(t)]\rangle\, d\tau \in C^1([0,\infty))$ and

$$\frac{d}{dt}\int_0^t \langle \eta_2(\tau)F(u(\tau)), \frac{\sin\omega(t-\tau)}{\omega}[x_j\psi(t)]\rangle\, d\tau = \int_0^t \langle \eta_2(\tau)F(u(\tau)), \partial_t\Big(\frac{\sin\omega(t-\tau)}{\omega}[x_j\psi(t)]\Big)\rangle\, d\tau. \tag{5.14}$$

Noting $\cos\omega(t-\tau)[x_j\psi(t)] = x_j\cos\omega(t-\tau)\psi(t) + (t-\tau)\omega^{-1}\sin\omega(t-\tau)\partial_j\psi(t)$, we obtain from (5.12)–(5.14)

$$\begin{aligned}\langle\langle L_jI_2[u],\psi\rangle\rangle &= \int_0^\infty\int_0^t \langle \eta_2(\tau)F(u(\tau)), x_j\cos\omega(t-\tau)\psi(t)\rangle\, d\tau\, dt \\ &\quad - \int_0^\infty\int_0^t \langle \eta_2(\tau)F(u(\tau)), \tau\frac{\sin\omega(t-\tau)}{\omega}\partial_j\psi(t)\rangle\, d\tau\, dt.\end{aligned} \tag{5.15}$$

It is shown below that both $\cos\omega(t-\tau)[x_j\eta_2(\tau)F(u(\tau))]$ and $\tau\omega^{-1}\sin\omega(t-\tau)\partial_j(\eta_2(\tau)F(u(\tau)))$ are Bochner integrable in $\dot H_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n)$ over the interval $(0,t)$. Thus, it follows from (5.15) that

$$\begin{aligned}\langle\langle L_jI_2[u],\psi\rangle\rangle &= \langle\langle \int_0^t \cos\omega(t-\tau)[x_j\eta_2(\tau)F(u(\tau))]\, d\tau, \psi\rangle\rangle \\ &\quad + \langle\langle \int_0^t \frac{\sin\omega(t-\tau)}{\omega}\tau\partial_j(\eta_2(\tau)F(u(\tau)))\, d\tau, \psi\rangle\rangle\end{aligned} \tag{5.16}$$

for all $\psi \in C_0^\infty((0,\infty)\times\mathbb{R}^n)$.

Next we calculate $L_jI_1[u]$. Since it will be shown later that, for $\dot\Gamma^\alpha = 1$ or L_j, the estimate

$$||\dot\Gamma^\alpha(\eta_1(\tau)F(u(\tau)))||_{\dot H_{q'}^{-1+1/q-1/p+\varepsilon+2\beta(q)}} \le C\tau^{-1-\varepsilon+\gamma(q)}$$

holds, we find that $\eta_1F(u)$ and $L_j(\eta_1F(u))$ belong to

$$L^\infty(T,\infty;\dot H_{q'}^{-1+1/q-1/p+\varepsilon+2\beta(q)}(\mathbb{R}^n))$$

for any $T > 0$, and thereby they belong to $\mathcal{D}'((0,\infty)\times\mathbb{R}^n)$. Fix an arbitrary $t > 0$. For any $\varphi \in C_0^\infty(\mathbb{R}^n)$ and any $\chi = \chi(\tau) \in C_0^\infty(0,\infty)$ with $\operatorname{supp}\chi \subset (0,t)$, we see that

$$\begin{aligned}
&\langle\!\langle \partial_\tau x_j(\eta_1 F(u)), \chi\frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle\!\rangle \\
&= \langle\!\langle L_j(\eta_1 F(u)), \chi\frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle\!\rangle + \langle\!\langle \eta_1 F(u), \tau\chi\frac{\sin\omega(t-\tau)}{\omega}\partial_j\varphi\rangle\!\rangle \\
&= \int_0^t \chi(\tau)\langle L_j(\eta_1(\tau)F(u(\tau))), \frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle\, d\tau \\
&+ \int_0^t \tau\chi(\tau)\langle \eta_1(\tau)F(u(\tau)), \frac{\sin\omega(t-\tau)}{\omega}\partial_j\varphi\rangle\, d\tau.
\end{aligned} \tag{5.17}$$

On the other hand, by the definition of the weak derivative, we have

$$\begin{aligned}
&\langle\!\langle \partial_\tau x_j(\eta_1 F(u)), \chi\frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle\!\rangle \\
&= -\int_0^t \chi'(\tau)\langle \eta_1(\tau)F(u(\tau)), x_j\frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle\, d\tau \\
&+ \int_0^t \chi(\tau)\langle \eta_1(\tau)F(u(\tau)), x_j\cos\omega(t-\tau)\varphi\rangle\, d\tau.
\end{aligned} \tag{5.18}$$

Fixing an arbitrary $t > 0$, we see that (5.17) and (5.18) yield the equality

$$\begin{aligned}
&\int_0^t \chi(\tau)\langle L_j(\eta_1(\tau)F(u(\tau))), \frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle\, d\tau \\
&+ \int_0^t \tau\chi(\tau)\langle \eta_1(\tau)F(u(\tau)), \frac{\sin\omega(t-\tau)}{\omega}\partial_j\varphi\rangle\, d\tau \\
&= -\int_0^t \chi'(\tau)\langle \eta_1(\tau)F(u(\tau)), x_j\frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle\, d\tau \\
&+ \int_0^t \chi(\tau)\langle \eta_1(\tau)F(u(\tau)), x_j\cos\omega(t-\tau)\varphi\rangle\, d\tau
\end{aligned} \tag{5.19}$$

for any $\chi \in C_0^\infty(0,t)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$. Note that the identity

$$\begin{aligned}
&\frac{d}{d\tau}\langle \eta_1(\tau)F(u(\tau)), x_j\frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle \\
&= \langle L_j(\eta_1(\tau)F(u(\tau))), \frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle \\
&+ \tau\langle \eta_1(\tau)F(u(\tau)), \frac{\sin\omega(t-\tau)}{\omega}\partial_j\varphi\rangle - \langle \eta_1(\tau)F(u(\tau)), x_j\cos\omega(t-\tau)\varphi\rangle
\end{aligned} \tag{5.20}$$

holds in $\mathcal{D}'(0,t)$ by (5.19). Here the derivative in the τ-variable on the left-hand side is understood in the weak sense. Later it will be shown that the three terms on the right-hand side of (5.20) as well as

$$\langle \eta_1(\tau)F(u(\tau)), x_j\omega^{-1}\sin\omega(t-\tau)\varphi\rangle$$

itself belong to $L^1(0,t)$ (the cut-off function η_1 plays a role here). Therefore we find $\langle \eta_1 F(u(\tau)), x_j\omega^{-1}\sin\omega(t-\tau)\varphi\rangle \in W^{1,1}(0,t)$ and

$$\begin{aligned}
&\int_0^t \langle L_j(\eta_1(\tau)F(u(\tau))), \frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle\, d\tau \\
&-\int_0^t \frac{d}{d\tau}\langle \eta_1(\tau)F(u(\tau)), x_j\frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle\, d\tau \\
&=\int_0^t \langle \eta_1(\tau)F(u(\tau)), x_j\cos\omega(t-\tau)\varphi\rangle\, d\tau \\
&-\int_0^t \tau\langle \eta_1(\tau)F(u(\tau)), \frac{\sin\omega(t-\tau)}{\omega}\partial_j\varphi\rangle\, d\tau.
\end{aligned} \tag{5.21}$$

Even if φ is replaced with any $\psi=\psi(t,x)\in C_0^\infty((0,\infty)\times\mathbb{R}^n)$, (5.21) obviously remains true. Since $I_1[u]$ satisfies the equality

$$\begin{aligned}
\langle\!\langle L_jI_1[u],\psi\rangle\!\rangle &= \int_0^\infty\int_0^t \langle \eta_1(\tau)F(u(\tau)), x_j\cos\omega(t-\tau)\psi(t)\rangle\, d\tau\, dt \\
&-\int_0^\infty\int_0^t \langle \eta_1(\tau)F(u(\tau)), \tau\frac{\sin\omega(t-\tau)}{\omega}\partial_j\psi(t)\rangle\, d\tau\, dt
\end{aligned} \tag{5.22}$$

as $I_2[u]$ satisfies (5.15), we conclude by (5.21)–(5.22) that the identity

$$\begin{aligned}
\langle\!\langle L_jI_1[u],\psi\rangle\!\rangle &= \int_0^\infty\int_0^t \langle L_j(\eta_1(\tau)F(u(\tau))), \frac{\sin\omega(t-\tau)}{\omega}\psi(t)\rangle\, d\tau\, dt \\
&-\int_0^\infty\int_0^t \frac{d}{d\tau}\langle \eta_1(\tau)F(u(\tau)), x_j\frac{\sin\omega(t-\tau)}{\omega}\psi(t)\rangle\, d\tau\, dt
\end{aligned} \tag{5.23}$$

holds for $\psi\in C_0^\infty((0,\infty)\times\mathbb{R}^n)$. It is remarked again that the τ-derivative in the second term on the right-hand side of (5.23) is understood in the weak sense.

Our next step is to estimate $L_jI_1[u]$ and $L_jI_2[u]$ by making use of the formulae (5.16) and (5.23). We begin with estimating $L_jI_1[u]$. Our first task is to verify

$$\langle \eta_1(\tau)F(u(\tau)), x_j\omega^{-1}\sin\omega(t-\tau)\varphi\rangle \in W^{1,1}(0,t)\ (\varphi\in C_0^\infty(\mathbb{R}^n)).$$

It follows from Lemma 2.10 that

$$
\begin{aligned}
&|\langle \eta_1(\tau)F(u(\tau)), x_j \frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle| \\
&\le C|t-\tau|^{-\gamma(q)}||x_j\eta_1(\tau)F(u(\tau))||_{\dot H_{q'}^{-1+1/q-1/p+\varepsilon+2\beta(q)}}||\varphi||_{\dot H_{q'}^{-(1/q-1/p+\varepsilon)}}.
\end{aligned}
\tag{5.24}
$$

Observe that the assumption $\varepsilon < 2/(p-1)-(n-1)/2$ in (5.1) is equivalent to the inequality $1/q < 1-(1/q-1/p+\varepsilon)-2\beta(q)$. Furthermore, the inequality $1-(1/q-1/p+\varepsilon)-2\beta(q) < n/q$ obviously holds because it is equivalent to $1/p-(n-1)/2 < \varepsilon$ and the inequality $1/p-(n-1)/2 < 0$ holds now. Hence we can apply Lemma 2.9 (2) to proceed:

$$
\begin{aligned}
&||x_j\eta_1(\tau)F(u(\tau))||_{\dot H_{q'}^{-(1-(1/q-1/p+\varepsilon)-2\beta(q))}} \le C||x_j\eta_1(\tau)F(u(\tau))||_{l,\chi_3} \\
&\quad + C||\,|x|^{1-n/q+1-(1/q-1/p+\varepsilon)-2\beta(q)}\eta_1(\tau)F(u(\tau))\,||_{1,q',\chi_4} \\
&\le C\tau||F(u(\tau))||_{l,\chi_3} + C\tau^{1-n/q+1-(1/q-1/p+\varepsilon)-2\beta(q)}||F(u(\tau))||_{1,q',\chi_4},
\end{aligned}
\tag{5.25}
$$

where $1/l = 1/q' + (1-(1/q-1/p+\varepsilon)-2\beta(q))/n$. We have also used the fact that $\tau/2 \le |x| \le 2\tau$ for $x \in \operatorname{supp}\eta_1(\tau)\chi_4(\tau)$. Define l_1 and l_2 as $1/l = 1/l_1 + 1/l_2$, where $1/l_2 = 1/q-(1/q-1/p+\varepsilon)/n$. When $n=2$, we obtain by using (2.9) with $p = l_2$

$$
\begin{aligned}
&||\,|u(\tau)|^{p-1}u(\tau)\,||_{l,\chi_3} \le ||u(\tau)||^{p-1}_{(p-1)l_1,\chi_3}||u(\tau)||_{l_2} \\
&\le C\tau^{n/l_1}||u(\tau)||^{p-1}_{\infty,\chi_3}||u(\tau)||_{\dot H_q^{1/q-1/p+\varepsilon}} \le C\tau^{-1-\varepsilon+\gamma(q)} \\
&\times\Big(\sum_{|\alpha|\le 1}\operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||\dot\Gamma^{\alpha}u(t)||_{\dot H_q^{1/q-1/p+\varepsilon}}\Big)^{p-1}\operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||u(t)||_{\dot H_q^{1/q-1/p+\varepsilon}}.
\end{aligned}
\tag{5.26}
$$

Here we have used $-n(p-1)/l_2 + n/l_1 - \varepsilon p = -1-\varepsilon+\gamma(q)$. On the other hand, we can use (2.7) instead of (2.9) to obtain in the case of $n=3$

$$
\begin{aligned}
&||\,|u(\tau)|^{p-1}u(\tau)\,||_{l,\chi_3} \le ||u(\tau)||^{p-1}_{(p-1)l_1,\chi_3}||u(\tau)||_{l_2} \le C\tau^{-1-\varepsilon+\gamma(q)} \\
&\times\Big(\sum_{|\alpha|\le 1}\operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||\dot\Gamma^{\alpha}u(t)||_{\dot H_q^{1/q-1/p+\varepsilon}}\Big)^{p-1}\operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||u(t)||_{\dot H_q^{1/q-1/p+\varepsilon}}.
\end{aligned}
\tag{5.27}
$$

In order to estimate $||F(u(\tau))||_{1,q',\chi_4}$, which is on the last line of (5.25), we proceed by using the Hölder inequality:

$$
||\,|u(\tau)|^{p-1}u(\tau)\,||_{1,q',\chi_4} \le ||u(\tau)||^{p-1}_{p,(p-1)q_0,\chi_4}||u(\tau)||_{p,q,\chi_4},
\tag{5.28}
$$

where $q_0 = q/(q-2)$. Since the condition $\varepsilon < (n-1)(1/q-1/p)$ is equivalent to $\varepsilon < 2/(p-1)-n/p$, which is assumed in (5.1), we can employ (2.51) to obtain

$$||u(\tau)||_{p,q,\chi_4} \le C\tau^{-(n-1)(1/q-1/p)} \operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||u(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \tag{5.29}$$

for almost all $\tau > 0$. Furthermore, since the condition $1 > (n-1)/q$ is equivalent to the inequality $p^2 - 2p - 1 > 0$, which is satisfied for $p > p_0(n)$ $(n = 2, 3)$, we may use the Sobolev embedding on the unit sphere and then (2.51) to get

$$\begin{aligned} ||u(\tau)||^{p-1}_{p,(p-1)q_0,\chi_4} &\le C||u(\tau)||^{p-1}_{p,\infty,\chi_4} \le C\sum_{|\beta|\le 1} ||\Omega^{\beta}u(\tau)||^{p-1}_{p,q,\chi_4} \\ &\le C\tau^{-(p-1)(n-1)(1/q-1/p)}\Big(\sum_{|\beta|\le 1} \operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||\Omega^{\beta}u(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}}\Big)^{p-1} \end{aligned} \tag{5.30}$$

for almost all $\tau > 0$. Observe $-n/q + 1 - (1/q - 1/p + \varepsilon) - 2\beta(q) - p(n-1)(1/q - 1/p) = -1 - \varepsilon + \gamma(q)$. Then we finally obtain from (5.24)–(5.30)

$$\begin{aligned} &|\langle \eta_1(\tau)F(u(\tau)), x_j\frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle| \\ &\le C|t-\tau|^{-\gamma(q)}\tau^{-\varepsilon+\gamma(q)}||u||^p_W||\varphi||_{\dot{H}_{q'}^{-(1/q-1/p+\varepsilon)}}. \end{aligned} \tag{5.31}$$

Since $\gamma(q) = 1/p-(2/(p-1)-(n-1)/2) < 1/p-\varepsilon < 1$ (the first inequality is true by virtue of (5.1)), we find $\langle \eta_1(\tau)F(u(\tau)), x_j\omega^{-1}\sin\omega(t-\tau)\varphi\rangle \in L^1(0,t)$ for any fixed $t > 0$.

Using the formula (5.20), we next show that the weak derivative in the τ-variable of $\langle \eta_1(\tau)F(u(\tau)), x_j\omega^{-1}\sin\omega(t-\tau)\varphi\rangle$ also belongs to $L^1(0,t)$. We shall first show

$$||L_j\big(\eta_1(\tau)F(u(\tau))\big)||_{\dot{H}_{q'}^{-(1-(1/q-1/p+\varepsilon)-2\beta(q))}} \le C\tau^{-1-\varepsilon+\gamma(q)}||u||^p_W,\ \text{a.a.}\,\tau > 0, \tag{5.32}$$

which together with Lemma 2.10 will immediately imply

$$\langle L_j\big(\eta_1(\tau)F(u(\tau))\big), \omega^{-1}\sin\omega(t-\tau)\varphi\rangle \in L^1(0,t).$$

Fix an arbitrary $\sigma > 0$. For any integer k satisfying $\sigma - 1/k > \sigma/2$, we define $C^{\infty}\big((\sigma,\infty)\times\mathbb{R}^n\big)$-functions $u_k(t,x) := J_k u(t,x)$ as in (2.14). Proceeding as in (5.25), we have

$$\begin{aligned} &||L_j\big(\eta_1(\tau)F(u_k(\tau))\big)||_{\dot{H}_{q'}^{-(1-(1/q-1/p+\varepsilon)-2\beta(q))}} \le C||L_j\big(\eta_1(\tau)F(u_k(\tau))\big)||_{l,\chi_3} \\ &+ C\tau^{-n/q+1-(1/q-1/p+\varepsilon)-2\beta(q)}||L_j\big(\eta_1(\tau)F(u_k(\tau))\big)||_{1,q',\chi_4}. \end{aligned} \tag{5.33}$$

Note that $|L_j\eta_1(\tau,x)| \le C$ for $|x| \le \tau/2$. Hence it is easy to prove

$$\begin{aligned}
&||L_j(\eta_1(\tau)F(u_k(\tau)))||_{l,\chi_3} \qquad (5.34)\\
&\le C\tau^{-1-\varepsilon+\gamma(q)}\Big(\sum_{|\alpha|\le 1}\sup_{t>\sigma} t^{\varepsilon}||\dot\Gamma^{\alpha}u_k(t)||_{\dot H_q^{1/q-1/p+\varepsilon}}\Big)^{p-1}\\
&\times\Big(\sup_{t>\sigma} t^{\varepsilon}||u_k(t)||_{\dot H_q^{1/q-1/p+\varepsilon}} + \sup_{t>\sigma} t^{\varepsilon}||L_j u_k(t)||_{\dot H_q^{1/q-1/p+\varepsilon}}\Big),\ \tau\ge\sigma
\end{aligned}$$

in almost the same way as in (5.26)–(5.27). Indeed, we have only to use (2.13) instead of (2.7) in the case of $n = 3$, and to use the inequality

$$||u_k(t,\cdot)||_{\infty,\chi_3} \le Ct^{-n/p-\delta}\sup_{t>\sigma} t^{\delta}||u_k(t,\cdot)||_{\dot\Gamma,1,p},\ n<p<\infty \qquad (5.35)$$

instead of (2.9) in the case of $n = 2$. The proof of (5.35) is similar to that of (2.13), and it is therefore left to the reader. Note $\omega^{1/q-1/p+\varepsilon}J_k u = J_k\omega^{1/q-1/p+\varepsilon}u$. In the same way as in (2.18) we get

$$\sup_{t>\sigma} t^{\varepsilon}||u_k(t)||_{\dot H_q^{1/q-1/p+\varepsilon}} \le 2^{\varepsilon}\operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||u(t)||_{\dot H_q^{1/q-1/p+\varepsilon}}. \qquad (5.36)$$

Furthermore, since it follows from (2.15) that

$$\begin{aligned}
\omega^{1/q-1/p+\varepsilon}L_j u_k(t,x) &= \int_{\sigma/2}^{\infty}\int_{\mathbf{R}^n}\eta_k(t-\tau,x-y)(\omega^{1/q-1/p+\varepsilon}L_j u)(\tau,y)\,d\tau\,dy\\
&+\int_{\sigma/2}^{\infty}\int_{\mathbf{R}^n}k^{n+1}(L_j\eta)(k(t-\tau),k(x-y))(\omega^{1/q-1/p+\varepsilon}u)(\tau,y)\,d\tau\,dy, \qquad (5.37)
\end{aligned}$$

we also obtain as in (2.18)

$$\begin{aligned}
&\sup_{t>\sigma} t^{\varepsilon}||L_j u_k(t)||_{\dot H_q^{1/q-1/p+\varepsilon}} \qquad (5.38)\\
&\le C(\operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||L_j u(t)||_{\dot H_q^{1/q-1/p+\varepsilon}} + \operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||u(t)||_{\dot H_q^{1/q-1/p+\varepsilon}}).
\end{aligned}$$

For $\dot\Gamma^{\alpha} = \Omega_{kl}$ or L_0, the inequality

$$\begin{aligned}
&\sup_{t>\sigma} t^{\varepsilon}||\dot\Gamma^{\alpha}u_k(t)||_{\dot H_q^{1/q-1/p+\varepsilon}} \qquad (5.39)\\
&\le C(\operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||\dot\Gamma^{\alpha}u(t)||_{\dot H_q^{1/q-1/p+\varepsilon}} + \operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||u(t)||_{\dot H_q^{1/q-1/p+\varepsilon}})
\end{aligned}$$

also holds. It should be noted here that the constants C on the right-hand side of (5.38) and (5.39) are independent of k and σ. Combining (5.34) with (5.36) and (5.38)–(5.39), we have for $\tau > \sigma$

$$||L_j(\eta_1(\tau)F(u_k(\tau)))||_{l,\chi_3} \qquad (5.40)$$

$$\leq C\tau^{-1-\varepsilon+\gamma(q)}\Big(\sum_{|\alpha|\leq 1}\operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||\dot{\Gamma}^{\alpha}u(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}}\Big)^{p-1}$$
$$\times(\operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||u(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}}+\operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||L_j u(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}}).$$

We turn our attention to the estimate for $||L_j(\eta_1(\tau)F(u_k(\tau)))||_{1,q',\chi_4}$. Since the inequality $|L_j\eta_1(\tau,x)|\leq C$ holds for $\tau/2\leq|x|\leq 2\tau$, we get

$$||L_j(\eta_1(\tau)F(u_k(\tau)))||_{1,q',\chi_4} \tag{5.41}$$
$$\leq C\tau^{-p(n-1)(1/q-1/p)}\Big(\sum_{|\beta|\leq 1}\sup_{t>\sigma} t^{\varepsilon}||\Omega^{\beta}u_k(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}}\Big)^{p-1}$$
$$\times(\sup_{t>\sigma} t^{\varepsilon}||u_k(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}}+\sup_{t>\sigma} t^{\varepsilon}||L_j u_k(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}})$$
$$\leq C\tau^{-p(n-1)(1/q-1/p)}\Big(\sum_{|\beta|\leq 1}\operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||\Omega^{\beta}u(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}}\Big)^{p-1}$$
$$\times(\operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||u(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}}+\operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||L_j u(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}}),\ \tau\geq\sigma$$

by repeating the same argument as in (5.28)–(5.30) and (5.36)–(5.39). Combining (5.33) with (5.40)–(5.41), we therefore obtain

$$||L_j(\eta_1(\tau)F(u_k(\tau)))||_{\dot{H}_{q'}^{-(1-(1/q-1/p+\varepsilon)-2\beta(q))}}\leq C\tau^{-1-\varepsilon+\gamma(q)}||u||_W^{p-1} \tag{5.42}$$
$$\times(\operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||u(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}}+\operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||L_j u(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}})$$

for $\tau\geq\sigma$. Since the constant C is independent of k, there exist a subsequence $\{u_{\tilde{k}}\}\subset\{u_k\}$ and w_j such that $\tau^{1+\varepsilon-\gamma(q)}L_j(\eta_1(\tau)F(u_{\tilde{k}}(\tau)))\to w_j$, weak-$*$ in $L^{\infty}(\sigma,\infty;\dot{H}_{q'}^{-(1-(1/q-1/p+\varepsilon)-2\beta(q))}(\mathbb{R}^n))$ as $\tilde{k}\to\infty$ and

$$||w_j||_{L^{\infty}(\sigma,\infty;\dot{H}_{q'}^{-(1-(1/q-1/p+\varepsilon)-2\beta(q))})} \tag{5.43}$$
$$\leq\liminf_{\tilde{k}\to\infty}||\tau^{1+\varepsilon-\gamma(q)}L_j(\eta_1F(u_{\tilde{k}}))||_{L^{\infty}(\sigma,\infty;\dot{H}_{q'}^{-(1-(1/q-1/p+\varepsilon)-2\beta(q))})}.$$

We show $w_j=\tau^{1+\varepsilon-\gamma(q)}L_j(\eta_1F(u))$. It is first noted that the inequality

$$||F(u_{\tilde{k}}(\tau))-F(u(\tau))||_{\dot{H}_{q'}^{-(1-(1/q-1/p+\varepsilon)-2\beta(q))}} \tag{5.44}$$
$$\leq C\tau^{-1+\gamma(q)}||u_{\tilde{k}}(\tau)-u(\tau)||_{\dot{H}_q^{1/q-1/p+\varepsilon}}\Big(\sum_{|\alpha|\leq 1}\operatorname*{ess\,sup}_{t>0} t^{\varepsilon}||\dot{\Gamma}^{\alpha}u(t)||_{\dot{H}_q^{\frac{1}{q}-1/p+\varepsilon}}\Big)^{p-1}$$

holds for almost all $\tau>\sigma$. In order to check (5.44) we have only to repeat essentially the same argument as in (5.25)–(5.30), (2.18) and (5.36). Then, for any $\psi \in C_0^\infty((\sigma,\infty)\times\mathbb{R}^n)$, we have

$$
\begin{aligned}
&|\langle\!\langle \tau^{1+\varepsilon-\gamma(q)} L_j(\eta_1 F(u_{\tilde{k}}) - \eta_1 F(u)), \psi\rangle\!\rangle| \\
&\le \int_\sigma^\infty ||F(u_{\tilde{k}}(\tau)) - F(u(\tau))||_{\dot{H}_{q'}^{-(1-(1/q-1/p+\varepsilon)-2\beta(q))}} \\
&\quad \times ||\eta_1(\tau) L_j(\tau^{1+\varepsilon-\gamma(q)}\psi(\tau))||_{\dot{H}_q^{1-(1/q-1/p+\varepsilon)-2\beta(q)}}\, d\tau \\
&\le C||u||_W^{p-1} \int_\sigma^\infty ||u_{\tilde{k}}(\tau) - u(\tau)||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \\
&\quad \times \tau^{-1+\gamma(q)} ||\eta_1(\tau) L_j(\tau^{1+\varepsilon-\gamma(q)}\psi(\tau))||_{\dot{H}_q^{1-(1/q-1/p+\varepsilon)-2\beta(q)}}\, d\tau \\
&\le C||u||_W^{p-1} ||u_{\tilde{k}} - u||_{L^q(I;\dot{H}_q^{1/q-1/p+\varepsilon})},
\end{aligned} \tag{5.45}
$$

where $I(\subset [\sigma,\infty))$ is any compact interval such that

$$\operatorname{supp} ||\eta_1(\tau) L_j(\tau^{1+\varepsilon-\gamma(q)}\psi(\tau))||_{\dot{H}_q^{1-(1/q-1/p+\varepsilon)-2\beta(q)}} \subset I.$$

Since $\omega^{1/q-1/p+\varepsilon} u_{\tilde{k}} = J_{\tilde{k}} \omega^{1/q-1/p+\varepsilon} u \to \omega^{1/q-1/p+\varepsilon} u$ in $L^q(I\times\mathbb{R}^n)$ as $\tilde{k}\to\infty$, we finally find by (5.45) that $w_j = \tau^{1+\varepsilon-\gamma(q)} L_j(\eta_1 F(u))$. Moreover, it follows from (5.42)–(5.43) that

$$\tau^{1+\varepsilon-\gamma(q)} ||L_j(\eta_1(\tau) F(u(\tau)))||_{\dot{H}_{q'}^{-(1-(1/q-1/p+\varepsilon)-2\beta(q))}} \le C||u||_W^p \tag{5.46}$$

for almost all $\tau > \sigma$. The constant C on the right-hand side above is independent of σ. Since $\sigma(>0)$ is arbitrary, we have finished the proof of (5.32). We then get

$$
\begin{aligned}
&|\langle L_j(\eta_1(\tau) F(u(\tau))), \omega^{-1}\sin\omega(t-\tau)\varphi\rangle| \\
&\le C|t-\tau|^{-\gamma(q)} \tau^{-1-\varepsilon+\gamma(q)} ||u||_W^p ||\varphi||_{\dot{H}_{q'}^{-(1/q-1/p+\varepsilon)}}.
\end{aligned} \tag{5.47}
$$

As we observed below (5.31), the inequality $\gamma(q) < 1$ holds now. Furthermore, the inequality $-1-\varepsilon+\gamma(q) > -1$ is also true because $\varepsilon < \gamma(q)$ (see below (5.2)). Thus (5.47) implies $\langle L_j(\eta_1(\tau)F(u(\tau))), \omega^{-1}\sin\omega(t-\tau)\varphi\rangle \in L^1(0,t)$. In order to finish the proof of $\langle \eta_1(\tau)F(u(\tau)), x_j\omega^{-1}\sin\omega(t-\tau)\varphi\rangle \in W^{1,1}(0,t)$ we need to verify that the last two terms on the right-hand side of (5.20) belong to $L^1(0,t)$. This can be checked by repeating essentially the same argument as before. Hence we have completed the proof of $\langle \eta_1(\tau)F(u(\tau)), x_j\omega^{-1}\sin\omega(t-\tau)\varphi\rangle \in W^{1,1}(0,t)$ ($\varphi \in C_0^\infty(\mathbb{R}^n)$).

We return to the formula (5.23). Fix an arbitrary $t>0$. Then the function

$$\langle \eta_1(\tau)F(u(\tau)), x_j \frac{\sin\omega(t-\tau)}{\omega}\psi(t)\rangle \in W^{1,1}(0,t)$$

is almost everywhere equal to a function $G(\tau) \in C([0,t])$ and

$$\int_0^t \frac{d}{d\tau}\langle \eta_1(\tau)F(u(\tau)), x_j \frac{\sin\omega(t-\tau)}{\omega}\psi(t)\rangle\, d\tau = [G(\tau)]_{\tau=0}^{\tau=t} \tag{5.48}$$

holds. Hence the absolute value of the second term on the right-hand side of (5.23) is estimated as

$$\begin{aligned}\cdots &\le 2\int_0^\infty \max_{0\le\tau\le t}|G(\tau)|\,dt \\ &= 2\int_0^\infty \operatorname*{ess\,sup}_{0<\tau<t}|\langle \eta_1(\tau)F(u(\tau)), x_j\frac{\sin\omega(t-\tau)}{\omega}\psi(t)\rangle|\,dt.\end{aligned} \tag{5.49}$$

Our next step is to prove

$$\operatorname*{ess\,sup}_{0<\tau<t}|\langle \eta_1(\tau)F(u(\tau)), x_j\frac{\sin\omega(t-\tau)}{\omega}\psi(t)\rangle| \le Ct^{-\varepsilon}||u||_W^p||\psi(t)||_{\dot H_{q'}^{-(1/q-1/p+\varepsilon)}}, \tag{5.50}$$

which allows us to obtain the desired estimate

$$\begin{aligned}&|\int_0^\infty t^\varepsilon \int_0^t \frac{d}{d\tau}\langle \eta_1(\tau)F(u(\tau)), x_j\frac{\sin\omega(t-\tau)}{\omega}\psi(t)\rangle\,d\tau\,dt| \\ &\le C||u||_W^p||\psi||_{L^1(0,\infty;\dot H_{q'}^{-(1/q-1/p+\varepsilon)})}.\end{aligned} \tag{5.51}$$

In order to prove (5.50) we divide the proof into two cases: $0<\tau<t/2$ and $t/2<\tau<t$. We begin with the former. Choose r and s so that $\gamma(r)=\varepsilon$ and $\dot H_{q'}^{-(1/q-1/p+\varepsilon)}(\mathbb{R}^n) \hookrightarrow \dot H_{r'}^{-1+s+2\beta(r)}(\mathbb{R}^n)$ may hold. Observing that the inequality $1/r<s<n/r$ is satisfied, we proceed:

$$\begin{aligned}&|\langle \eta_1(\tau)F(u(\tau)), x_j\frac{\sin\omega(t-\tau)}{\omega}\psi(t)\rangle| \\ &\le C|t-\tau|^{-\gamma(r)}||x_j\eta_1(\tau)F(u(\tau))||_{\dot H_{r'}^{-s}}||\psi(t)||_{\dot H_{r'}^{-1+s+2\beta(r)}} \\ &\le Ct^{-\varepsilon}||x_j\eta_1(\tau)F(u(\tau))||_{\dot H_{r'}^{-s}}||\psi(t)||_{\dot H_{q'}^{-(1/q-1/p+\varepsilon)}}.\end{aligned} \tag{5.52}$$

Giving l as $1/l = 1/r' + s/n$, we can obtain

$$\begin{aligned}&||x_j\eta_1(\tau)F(u(\tau))||_{\dot H_{r'}^{-s}} \\ &\le C||x_j\eta_1(\tau)F(u(\tau))||_{l,\chi_3} + C||\,|x|^{1-(n/r-s)}\eta_1(\tau)F(u(\tau))\,||_{1,r',\chi_4} \le C||u||_W^p\end{aligned} \tag{5.53}$$

as in (5.25)–(5.30). Combining (5.52) with (5.53), we have gotten

$$|\langle \eta_1(\tau)F(u(\tau)), x_j\frac{\sin\omega(t-\tau)}{\omega}\psi(t)\rangle| \le Ct^{-\varepsilon}||u||_W^p||\psi(t)||_{\dot H_{q'}^{-(1/q-1/p+\varepsilon)}} \tag{5.54}$$

for almost all $\tau \in (0, t/2)$.

We next consider the opposite case $t/2 < \tau < t$. Proceeding as

$$
\begin{aligned}
&|\langle \eta_1(\tau)F(u(\tau)), x_j \frac{\sin\omega(t-\tau)}{\omega}\psi(t)\rangle| \qquad (5.55)\\
&\le || \frac{\sin\omega(t-\tau)}{\omega}[x_j\eta_1(\tau)F(u(\tau))]||_{\dot H_q^{1/q-1/p+\varepsilon}} ||\psi(t)||_{\dot H_{q'}^{-(1/q-1/p+\varepsilon)}}\\
&\le C||x_j\eta_1(\tau)F(u(\tau))||_{\dot H_2^{-1+\theta+\varepsilon}} ||\psi(t)||_{\dot H_{q'}^{-(1/q-1/p+\varepsilon)}} \ \Big(\theta = \frac{n}{2} - \frac{2}{p-1}\Big)
\end{aligned}
$$

by the Sobolev embedding $\dot H_2^{\theta}(\mathbb{R}^n) \hookrightarrow \dot H_q^{1/q-1/p}(\mathbb{R}^n)$, we may use Lemma 2.9 (2) to obtain

$$
\begin{aligned}
&||x_j\eta_1(\tau)F(u(\tau))||_{\dot H_2^{-(1-\theta-\varepsilon)}}\\
&\le C\tau||F(u(\tau))||_{l,\chi_3} + C\tau^{1-n/2+(1-\theta-\varepsilon)}||F(u(\tau))||_{1,2,\chi_4}, \qquad (5.56)
\end{aligned}
$$

where $1/l = 1/2 + (1-\theta-\varepsilon)/n$. Moreover, taking account of the condition $t/2 < \tau < t$, we see that the right-hand side of (5.56) is estimated as

$$\cdots \le Ct^{-\varepsilon}||u||_W^p \qquad (5.57)$$

in essentially the same way as in (5.26)–(5.30). We thus have shown that the inequality

$$|\langle \eta_1(\tau)F(u(\tau)), x_j\omega^{-1}\sin\omega(t-\tau)\psi(t)\rangle| \le Ct^{-\varepsilon}||u||_W^p||\psi(t)||_{\dot H_{q'}^{-(1/q-1/p+\varepsilon)}}$$

holds for almost all $\tau \in (t/2, t)$, which together with (5.54) leads to (5.50)–(5.51). Therefore the estimate for the second term on the right-hand side of (5.23) has been completed.

As for the first term on the right-hand side of (5.23) we repeat the argument in the proof of (5.47) to obtain

$$
\begin{aligned}
&|\int_0^\infty t^\varepsilon \int_0^t \langle L_j(\eta_1(\tau)F(u(\tau))), \frac{\sin\omega(t-\tau)}{\omega}\psi(t)\rangle\, d\tau\, dt| \qquad (5.58)\\
&\le C\int_0^\infty t^\varepsilon\int_0^t |t-\tau|^{-\gamma(q)}\tau^{-1-\varepsilon+\gamma(q)}\, d\tau ||\psi(t)||_{\dot H_{q'}^{-(1/q-1/p+\varepsilon)}}\, dt ||u||_W^p\\
&\le C||u||_W^p||\psi||_{L^1(0,\infty;\dot H_{q'}^{-(1/q-1/p+\varepsilon)})}.
\end{aligned}
$$

Here we have employed a simple inequality

$$\int_0^t |t-\tau|^{-\gamma(q)}\tau^{-1-\varepsilon+\gamma(q)}\, d\tau \le Ct^{-\varepsilon}, \qquad (5.59)$$

the proof of which is left to the reader. By (5.51) and (5.58) it has been shown that $|\langle\langle t^\varepsilon L_j I_1[u], \psi\rangle\rangle| \le C||u||_W^p ||\psi||_{L^1(0,\infty;\dot H_{q'}^{-(1/q-1/p+\varepsilon)})}$, and hence

$$\operatorname*{ess\,sup}_{t>0} t^\varepsilon ||L_j I_1[u](t)||_{\dot H_q^{1/q-1/p+\varepsilon}} \le C||u||_W^p \tag{5.60}$$

by the duality argument.

To finish the proof of (5.8) for the case $\dot\Gamma^\alpha = L_j$ our next task is to estimate $t^\varepsilon L_j I_2[u]$. It follows from the formula (5.16) that

$$\|L_j I_2[u](t)\|_{\dot H_q^{1/q-1/p+\varepsilon}} \le C\int_0^t |t-\tau|^{-\gamma(q)} ||x_j\eta_2(\tau)F(u(\tau))||_{\dot H_{q'}^{1/q-1/p+\varepsilon+2\beta(q)}} d\tau$$
$$+ C\int_0^t |t-\tau|^{-\gamma(q)}\tau ||\eta_2(\tau)F(u(\tau))||_{\dot H_{q'}^{1/q-1/p+\varepsilon+2\beta(q)}} d\tau. \tag{5.61}$$

We shall show

$$||x_j\eta_2(\tau)F(u(\tau))||_{\dot H_{q'}^{1/q-1/p+\varepsilon+2\beta(q)}} \le C\tau^{-1-\varepsilon+\gamma(q)}||u||_W^p, \text{ a.a. } \tau>0. \tag{5.62}$$

Let σ be an arbitrary positive number. For any integer k satisfying $\sigma - 1/k > \sigma/2$, define $C^\infty((\sigma,\infty)\times\mathbb{R}^n)$-functions $u_k(t,x) := J_k u(t,x)$ as before. We have by (2.54)

$$\begin{aligned} &||x_j\eta_2(\tau)F(u_k(\tau))||_{\dot H_{q'}^{1/q-1/p+\varepsilon+2\beta(q)}} \\ &\le C\sum_{a=1}^n ||\partial_a(x_j\eta_2(\tau)F(u_k(\tau)))||_{\dot H_{q'}^{-1+1/q-1/p+\varepsilon+2\beta(q)}} \\ &\le C\tau^{-n/q+1-(1/q-1/p+\varepsilon)-2\beta(q)}\sum_{a=1}^n ||\partial_a(x_j\eta_2(\tau)F(u_k(\tau)))||_{1,q',\chi_4} \end{aligned} \tag{5.63}$$

for $\tau \ge \sigma$. Here we have used the fact that $\eta_2 = 0$ for $|x| \le \tau/2$. Moreover, since the first derivative $\partial_j v$ of a smooth function v can be written in terms of $L_j v$, $\Omega_{jl} v$ and $L_0 v$ (see (2.5)), we have

$$\begin{aligned} &||\partial_a(x_j\eta_2(\tau)F(u_k(\tau)))||_{1,q',\chi_4} \le C||F(u_k(\tau))||_{1,q',\chi_4} \\ &+ C||x_j\eta_2(\tau)\frac{|u_k(\tau)|^{p-1}}{|x|^2-\tau^2}(\tau L_a + \sum_{l=1}^n x_l(x_a\partial_l - x_l\partial_a) - x_a L_0)u_k(\tau)||_{1,q',\chi_4} \\ &\le C\sum_{|\alpha|\le 1} ||\,|u_k(\tau)|^{p-1}\dot\Gamma^\alpha u_k(\tau)\,||_{1,q',\chi_4} \le C\tau^{-p(n-1)(1/q-1/p)}||u||_W^p \end{aligned} \tag{5.64}$$

for $\tau \ge \sigma$. In the second inequality we have used the fact $|x|^2 - \tau^2 \ge |x|(|x|+\tau)/3$ on the $\operatorname{supp}\eta_2$, and to get the last inequality we have repeated the same argument as in (5.28)–(5.30) and (5.36)–(5.39). Thus we have obtained the inequality

$$||x_j\eta_2(\tau)F(u_k(\tau))||_{\dot H^{1/q-1/p+\varepsilon+2\beta(q)}_{q'}} \le C\tau^{-1-\varepsilon+\gamma(q)}||u||_W^p,\ \tau\ge\sigma. \qquad (5.65)$$

Since the constant C on the right-hand side above is independent of k and σ, we can carry out such an argument as in (5.42)–(5.46), and as a result we obtain (5.62). In the same way it is possible to show

$$\tau||\eta_2(\tau)F(u(\tau))||_{\dot H^{1/q-1/p+\varepsilon+2\beta(q)}_{q'}} \le C\tau^{-1-\varepsilon+\gamma(q)}||u||_W^p$$

for almost all $\tau>0$. Hence we have

$$\begin{aligned}||L_jI_2[u](t)||_{\dot H^{1/q-1/p+\varepsilon}_q} &\le C\int_0^t|t-\tau|^{-\gamma(q)}\tau^{-1-\varepsilon+\gamma(q)}\,d\tau||u||_W^p \qquad (5.66)\\ &\le Ct^{-\varepsilon}||u||_W^p,\quad t>0,\end{aligned}$$

which together with (5.60) completes the proof of (5.8) in the case of $\dot\Gamma^\alpha = L_j$. We next prove the inequality (5.8) in the case $\dot\Gamma^\alpha=L_0$. Our first task is to calculate $L_0I_0[u]$. For any $\psi\in C_0^\infty((0,\infty)\times\mathbb{R}^n)$ we have

$$\begin{aligned}&\langle\!\langle L_0\int_0^t\frac{\sin\omega(t-\tau)}{\omega}F(u(\tau))\,d\tau,\psi\rangle\!\rangle \qquad (5.67)\\ &=-\int_0^\infty\int_0^t\langle F(u(\tau)),\partial_t\Big(\frac{\sin\omega(t-\tau)}{\omega}t\psi(t)\Big)\rangle\,d\tau\,dt\\ &+\int_0^\infty\int_0^t\langle\sum_{j=1}^n x_j\partial_jF(u(\tau)),\frac{\sin\omega(t-\tau)}{\omega}\psi(t)\rangle\,d\tau\,dt\\ &+\int_0^\infty\int_0^t\tau\langle F(u(\tau)),\cos\omega(t-\tau)\psi(t)\rangle\,d\tau\,dt\\ &+\int_0^\infty\int_0^t\langle F(u(\tau)),\frac{\sin\omega(t-\tau)}{\omega}\psi(t)\rangle\,d\tau\,dt.\end{aligned}$$

For the same reason as in (5.14) the first term on the right-hand side above vanishes.

Fixing an arbitrary $t>0$, we next show that the identity

$$\begin{aligned}&\langle\sum_{j=1}^n x_j\partial_jF(u(\tau)),\frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle+\tau\langle F(u(\tau)),\cos\omega(t-\tau)\varphi\rangle \qquad (5.68)\\ &=\langle L_0F(u(\tau)),\frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle+\langle F(u(\tau)),\frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle\end{aligned}$$

$$- \frac{d}{d\tau}\langle \tau F(u(\tau)), \frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle$$

holds in $\mathcal{D}'(0,t)$ for any $\varphi \in C_0^\infty(\mathbb{R}^n)$. Indeed, for any $\chi = \chi(\tau) \in C_0^\infty(0,\infty)$ with $\operatorname{supp}\chi \subset (0,t)$ and any $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have by the definition of the weak derivative

$$\begin{aligned} &\langle\!\langle \tau\partial_\tau F(u), \chi\frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle\!\rangle = -\langle\!\langle F(u), \partial_\tau\Bigl(\tau\chi\frac{\sin\omega(t-\tau)}{\omega}\varphi\Bigr)\rangle\!\rangle \qquad (5.69)\\ &= -\int_0^t \chi(\tau)\langle F(u(\tau)), \frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle\,d\tau\\ &-\int_0^t \chi'(\tau)\langle F(u(\tau)), \tau\frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle\,d\tau\\ &+\int_0^t \chi(\tau)\langle F(u(\tau)), \tau\cos\omega(t-\tau)\varphi\rangle\,d\tau. \end{aligned}$$

On the other hand, we obviously have

$$\begin{aligned} &\langle\!\langle \tau\partial_\tau F(u), \chi\frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle\!\rangle = \int_0^t \chi(\tau)\langle L_0F(u(\tau)), \frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle\,d\tau\\ &-\int_0^t \chi(\tau)\langle \sum_{j=1}^n x_j\partial_jF(u(\tau)), \frac{\sin\omega(t-\tau)}{\omega}\varphi\rangle\,d\tau. \qquad (5.70) \end{aligned}$$

Now (5.68) follows immediately from (5.69)–(5.70). Note that the identity (5.68) remains valid with φ replaced with $\psi(t,x) \in C_0^\infty((0,\infty)\times\mathbb{R}^n)$. Since the identity (5.68) also implies $\langle \tau F(u(\tau)), \omega^{-1}\sin\omega(t-\tau)\psi(t)\rangle \in W^{1,1}(0,t)$ (the proof of which is left to the reader), we can obtain from (5.67)–(5.68)

$$\begin{aligned} &\langle\!\langle L_0\int_0^t \frac{\sin\omega(t-\tau)}{\omega}F(u(\tau))\,d\tau, \psi\rangle\!\rangle \qquad (5.71)\\ &= \int_0^\infty\int_0^t \langle L_0F(u(\tau)), \frac{\sin\omega(t-\tau)}{\omega}\psi(t)\rangle\,d\tau\,dt\\ &+2\int_0^\infty\int_0^t \langle F(u(\tau)), \frac{\sin\omega(t-\tau)}{\omega}\psi(t)\rangle\,d\tau\,dt\\ &-\int_0^\infty\int_0^t \frac{d}{d\tau}\langle \tau F(u(\tau)), \frac{\sin\omega(t-\tau)}{\omega}\psi(t)\rangle\,d\tau\,dt \end{aligned}$$

for $\psi \in C_0^\infty((0,\infty)\times\mathbb{R}^n)$. The τ-derivative in the last term is understood in the weak sense. Proceeding as we did for the estimate of $L_jI_0[u]$, we then obtain

$$|\langle\langle t^\varepsilon L_0 \int_0^t \frac{\sin\omega(t-\tau)}{\omega} F(u(\tau))\,d\tau, \psi\rangle\rangle| \le C||u||_W^p ||\psi||_{L^1(0,\infty;\dot H_{q'}^{-(1/q-1/p+\varepsilon)})}, \tag{5.72}$$

from which the inequality (5.8) for $\dot\Gamma^\alpha = L_0$ follows.

It remains to show (5.8) for $\dot\Gamma^\alpha = 1$ or Ω_{kl}. For $\dot\Gamma^\alpha = 1$, the proof of (5.8) has been essentially finished in (5.24)–(5.30). The proof of (5.8) for $\dot\Gamma^\alpha = \Omega_{kl}$ is much easier than that of (5.8) for $\dot\Gamma^\alpha = L_j$ or L_0, because the operator ∂_t is not included in Ω_{kl}. We omit the details of the proof of (5.8) for $\dot\Gamma^\alpha = 1$ or Ω_{kl}. Therefore it has been shown that $||I_0[u]||_W \le C_7||u||_W^p$ for some $C_7 > 0$. Since δ_2 is small so that $C_7(2\delta_2)^{p-1} \le 1/2$ may hold, the mapping M defined by (5.7) turns out to carry $W_{2\delta_2}$ into itself.

Next we need to prove that M is the contraction mapping on $W_{2\delta_2}$. Repeating essentially the same argument as we have done above, we can show that the mapping M satisfies

$$||Mu - Mv||_W \le C_8(||u||_W + ||v||_W)^{p-1}||u-v||_W \tag{5.73}$$

for some $C_8 > 0$. Since δ_2 is small so that $C_8(4\delta_2)^{p-1} < 1$ may hold, M is the contraction mapping on $W_{2\delta_2}$. In $W_{2\delta_2}$ the mapping M has a unique fixed point u, and obviously it is a unique solution to (5.3) in $W_{2\delta_2}$. It remains to show (5.4)–(5.6). Taking into account the assumption $t^\varepsilon\omega^{-1}\partial_t u_0 \in L^\infty(0,\infty;\dot H_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n))$ and being on the same line as in the proof of $t^\varepsilon u \in L^\infty(0,\infty;\dot H_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n))$, we can show (5.4). In order to check (5.5) we notice that the estimate

$$||F(u(\tau))||_{\dot H_2^{-1+\theta-\varepsilon}} \le C\tau^{-1+\varepsilon}||u||_W^p \tag{5.74}$$

follows for the same reason as in (5.56)–(5.57) and (5.26)–(5.30). The continuity property (5.5) is an immediate consequence of (5.74). It remains to verify (5.6). First we find from (5.74) that

$$u(t) - u_0(t),\ \omega^{-1}\partial_t u(t) - \omega^{-1}\partial_t u_0(t) \to 0 \text{ in } \dot H_2^{\theta-\varepsilon}(\mathbb{R}^n) \tag{5.75}$$

as $t \to +0$. Since $u_0(t) \to f$ and $\partial_t u_0(t) \to g$ in $\mathcal{S}'(\mathbb{R}^n)$ as $t \to 0$, (5.6) immediately follows in view of the fact that $\dot H_2^s(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ for $s = \theta - \varepsilon$ and $-1+\theta-\varepsilon$.

Our final task is to prove part (3) of Theorem 5.1. Observe that the norm in the space W is invariant under the transformation

$$u(t,x) \mapsto \delta^{2/(p-1)}u(\delta t, \delta x) \quad (\delta > 0).$$

Then, defining $u_\delta := \delta^{2/(p-1)}u(\delta t, \delta x)$ $(\delta > 0)$ for the solution u given in part (1) of Theorem 5.1, we see that u_δ is also a solution to (5.3) in $W_{2\delta_2}$ with the same initial data (f, g) as u has. However, uniqueness in $W_{2\delta_2}$ implies that the solution u given in (1) satisfies the identity $u \equiv u_\delta$ for all $\delta > 0$. Thus we have finished the proof of Theorem 5.1. □

6. Free solutions with homogeneous initial data

Let W be the Banach space defined in the previous section. By $P_l(x)$ and $Q_m(x)$ $(l, m = 0, 1, \dots)$ we denote any homogeneous polynomials of degree l and m, respectively. The purpose of this section is to show that free solutions belong to W when initial data (f, g) have the form

$$\begin{cases} f(x) = \displaystyle\sum_{l=0}^{N_1} P_l(x)|x|^{-2/(p-1)-l}, \\ g(x) = \displaystyle\sum_{m=0}^{N_2} Q_m(x)|x|^{-1-2/(p-1)-m}. \end{cases} \tag{6.1}$$

Here N_1 and N_2 are any nonnegative integers. This is an immediate consequence of the following theorem.

Theorem 6.1. *Let $n = 2, 3$ and $1 < r < \infty$. Suppose that $0 < d < n$, $s \in \mathbb{R}$ and $(n-1)/2 < d + s < n - 1$. Suppose further that*

$$\big(d + s - \frac{n-1}{2}\big)r < 1, \tag{6.2}$$

$$(d + s)r > n. \tag{6.3}$$

If $P(x)$ is a homogeneous polynomial of degree $l \geq 0$, then for $v(x) := \varepsilon_1 P(x)|x|^{-d-l}$ and $w(x) := \varepsilon_1 P(x)|x|^{-1-d-l}$ $(\varepsilon_1 > 0)$ the estimates

$$\sup_{t>0} t^{d+s-n/r}||(\cos\omega t)v||_{\dot H^s_r} \leq C\varepsilon_1, \tag{6.4}$$

$$\sup_{t>0} t^{d+s-n/r}||\frac{\sin\omega t}{\omega}w||_{\dot H^s_r} \leq C\varepsilon_1 \tag{6.5}$$

hold.

Since this theorem has been essentially shown by Pecher [28] for the case of $n = 3$, we shall treat the two-space-dimensional case. We are on the same line as in [28]. First some estimates are shown when initial data are homogeneous and spherically symmetric. By virtue of the decomposition of polynomials as in (6.34) and the boundedness of a certain singular integral operator, the assumption of spherical symmetry is removed and we can obtain (6.4)–(6.5).

Before entering into the proof, we remark that (6.4) and (6.5) immediately follow from the scaling argument once we have shown $||(\cos\omega)v||_{\dot{H}^s_r} \le C\varepsilon_1$ and $||(\omega^{-1}\sin\omega)w||_{\dot{H}^s_r} \le C\varepsilon_1$. This means that our task is reduced to the estimate at $t = 1$. Taking this remark into account, we start our consideration with the next lemma.

Lemma 6.2. *Let $n = 2$. Suppose that $1 \le \Theta < \infty$, $1/2 < \rho < 1$ and*

$$\big(\rho - \frac{1}{2}\big)\Theta < 1 \text{ and } \rho\Theta > 2. \tag{6.6}$$

For $\phi(x) := |x|^{-\rho}$ and $\psi(x) := |x|^{-1-\rho}$, both $(\cos\omega)\phi$ and $(\omega^{-1}\sin\omega)\psi$ belong to $L^\Theta(\mathbb{R}^2)$.

Proof of Lemma 6.2. Our starting point is the classical formula of a solution to a free-wave equation:

$$u_0(t) = \frac{\partial}{\partial t}\Big(\frac{t}{2\pi}\int_{|y|<1}\frac{\phi(x+ty)}{\sqrt{1-|y|^2}}\,dy\Big) + \frac{t}{2\pi}\int_{|y|<1}\frac{\psi(x+ty)}{\sqrt{1-|y|^2}}\,dy. \tag{6.7}$$

It is enough to show $(\cos\omega)\phi \in L^\Theta(\mathbb{R}^2)$ only, because the proof of $(\omega^{-1}\sin\omega)\psi \in L^\Theta(\mathbb{R}^2)$ is similar. Our task is to verify that J_1 and J_2 defined as

$$J_1(x) := \frac{1}{2\pi}\int_{|y|<1}\frac{dy}{\sqrt{1-|y|^2}|x+y|^\rho}, \tag{6.8}$$

$$J_2(x) := \frac{1}{2\pi}\int_{|y|<1}\frac{dy}{\sqrt{1-|y|^2}|x+y|^{1+\rho}} \tag{6.9}$$

have enough pointwise decay property to belong to $L^\Theta(\mathbb{R}^2)$. The observation by Kubota [15] is very helpful to this end. By the standard calculation (see [15] on page 129) we see that J_1 and J_2 are rewritten as $(r := |x|)$

$$J_1(x) = \frac{2}{\pi}\int_0^1\frac{1}{\sqrt{1-\eta^2}}\int_{|r-\eta|}^{r+\eta} z^{1-\rho}\frac{\eta}{\sqrt{[z^2-(r-\eta)^2][(r+\eta)^2-z^2]}}\,dz\,d\eta, \tag{6.10}$$

$$J_2(x) = \frac{2}{\pi}\int_0^1\frac{1}{\sqrt{1-\eta^2}}\int_{|r-\eta|}^{r+\eta} z^{-\rho}\frac{\eta}{\sqrt{[z^2-(r-\eta)^2][(r+\eta)^2-z^2]}}\,dz\,d\eta. \tag{6.11}$$

In what follows we simply denote $[\big(z^2-(r-\eta)^2\big)\big((r+\eta)^2-z^2\big)]^{-1/2}$ by $h(z,\eta,r)$. First we shall prove $J_1 \in L^\Theta(\mathbb{R}^2)$. Following Kubota [15], we invert the order of the $z\eta$-integral in the following way. When $r \le 1$, the condition

$$0 \le \eta \le 1 \text{ and } |\eta - r| \le z \le \eta + r$$

is equivalent to

$$0 \le z \le 1 + r \text{ and } |z - r| \le \eta \le \min(1, z + r).$$

Therefore J_1 is divided into two parts:

$$J_1(x) = \frac{2}{\pi} \int_{1-r}^{1+r} z^{1-\rho} \, dz \int_{|z-r|}^{1} \frac{\eta}{\sqrt{1-\eta^2}} h(z, \eta, r) \, d\eta \tag{6.12}$$
$$+ \frac{2}{\pi} \int_{0}^{1-r} z^{1-\rho} \, dz \int_{|z-r|}^{z+r} \frac{\eta}{\sqrt{1-\eta^2}} h(z, \eta, r) \, d\eta \text{ for } r \le 1.$$

Next suppose $r \ge 1$. In this case the condition

$$0 \le \eta \le 1 \text{ and } |\eta - r| \le z \le \eta + r$$

is equivalent to

$$r - 1 \le z \le r + 1 \text{ and } |z - r| \le \eta \le 1.$$

Therefore we see that

$$J_1(x) = \frac{2}{\pi} \int_{r-1}^{r+1} z^{1-\rho} \, dz \int_{|z-r|}^{1} \frac{\eta}{\sqrt{1-\eta^2}} h(z, \eta, r) \, d\eta \text{ for } r \ge 1. \tag{6.13}$$

Setting

$$I_1(x) = \frac{2}{\pi} \int_{|r-1|}^{r+1} z^{1-\rho} \, dz \int_{|z-r|}^{1} \frac{\eta}{\sqrt{1-\eta^2}} h(z, \eta, r) \, d\eta, \tag{6.14}$$

$$I_2(x) = \frac{2}{\pi} \int_{0}^{1-r} z^{1-\rho} \, dz \int_{|z-r|}^{z+r} \frac{\eta}{\sqrt{1-\eta^2}} h(z, \eta, r) \, d\eta, \tag{6.15}$$

we find

$$J_1(x) = I_1(x) + I_2(x) \text{ for } |x| \le 1, \tag{6.16}$$

$$J_1(x) = I_1(x) \text{ for } |x| \ge 1. \tag{6.17}$$

Before entering into the estimate for I_1 and I_2, we give the following preliminary result, which is useful in estimating the η-integrals in (6.14) and (6.15) (see [15] on page 131):

$$\int_a^b \frac{\eta}{\sqrt{\eta^2 - a^2}\sqrt{b^2 - \eta^2}} \, d\eta = \frac{\pi}{2} \text{ for any } 0 \le a < b. \tag{6.18}$$

We start the estimate for I_1. Observing $(z + r)^2 - \eta^2 \ge (z + r)(z + r - 1)$ for $|z - r| \le \eta \le 1$ and noting the identity $h(z, \eta, r) = h(\eta, z, r)$, we have by

(6.18)

$$\int_{|z-r|}^{1} \frac{\eta}{\sqrt{1-\eta^2}} h(z,\eta,r)\, d\eta \tag{6.19}$$
$$\leq \frac{1}{\sqrt{(z+r)(z+r-1)}} \int_{|z-r|}^{1} \frac{\eta}{\sqrt{(1-\eta^2)(\eta^2-(z-r)^2)}}\, d\eta$$
$$= \frac{\pi/2}{\sqrt{(z+r)(z+r-1)}}.$$

When $r \geq 2$, we get by using $z^{-\rho} \leq (r-1)^{-\rho}$ and $z/\sqrt{(z+r)(z+r-1)} \leq 1$

$$I_1(x) \leq 2(r-1)^{-\rho} \leq Cr^{-\rho}. \tag{6.20}$$

For $1 < r \leq 2$, we estimate $z^{1/2-\rho} \leq (r-1)^{1/2-\rho}$ by virtue of the assumption $\rho > 1/2$ to obtain

$$I_1(x) \leq (r-1)^{-(\rho-1/2)} \int_{r-1}^{r+1} \frac{dz}{\sqrt{z+r}} \leq C(r-1)^{-(\rho-1/2)}. \tag{6.21}$$

For $0 < r < 1$ we easily get

$$I_1(x) \leq (1-r)^{-(\rho-1/2)} \int_{1-r}^{1+r} \frac{dz}{\sqrt{z-1+r}} \leq C(1-r)^{-(\rho-1/2)}. \tag{6.22}$$

As for $I_2(x)$, it follows from (6.18) that

$$\int_{|z-r|}^{z+r} \frac{\eta}{\sqrt{1-\eta^2}} h(z,\eta,r)\, d\eta \leq \frac{\pi/2}{\sqrt{1-(z+r)^2}}. \tag{6.23}$$

Thus, in view of the assumption $\rho < 1$, I_2 is estimated as

$$I_2(x) \leq \int_0^{1-r} \frac{z^{1-\rho}}{\sqrt{1-(z+r)^2}}\, dz \tag{6.24}$$
$$\leq (1-r)^{1-\rho} \int_0^{1-r} \frac{dz}{\sqrt{1-r-z}} \leq C(1-r)^{3/2-\rho} \leq C \text{ for } 0 < r < 1.$$

It therefore follows from (6.20)–(6.22) and (6.24) that

$$J_1(x) \leq \begin{cases} Cr^{-\rho} & \text{for } r \geq 2 \\ C|r-1|^{-(\rho-1/2)} & \text{for } 0 \leq r \leq 2,\ r \neq 1. \end{cases} \tag{6.25}$$

Let us next consider the pointwise estimate for J_2. Repeating the same argument as in (6.12)–(6.17) and defining

$$I_3(x) := \frac{2}{\pi} \int_{|r-1|}^{r+1} z^{-\rho}\, dz \int_{|z-r|}^{1} \frac{\eta}{\sqrt{1-\eta^2}} h(z,\eta,r)\, d\eta, \tag{6.26}$$

$$I_4(x) := \frac{2}{\pi}\int_0^{1-r} z^{-\rho}\,dz \int_{|z-r|}^{z+r} \frac{\eta}{\sqrt{1-\eta^2}} h(z,\eta,r)\,d\eta, \tag{6.27}$$

we find that $J_2(x)$ is also rewritten as

$$J_2(x) = I_3(x) + I_4(x) \text{ for } |x| \le 1, \tag{6.28}$$

$$J_2(x) = I_3(x) \text{ for } |x| \ge 1. \tag{6.29}$$

Estimating the η-integral in (6.26) as we did in (6.19), we find for $r \ge 2$

$$I_3(x) \le \int_{r-1}^{r+1} \frac{dz}{z^\rho \sqrt{(z+r)(z+r-1)}} \le Cr^{-\rho-1}. \tag{6.30}$$

For $0 \le r \le 2$, $r \ne 1$, we utilize the inequality $z + r \ge |r-1| + r \ge 1$ to get

$$\begin{aligned} I_3(x) &\le \int_{|r-1|}^{r+1} \frac{dz}{z^\rho\sqrt{z+r-1}} = \int_{|r-1|}^{r+1} z^{-\rho}(2\sqrt{z+r-1})'\,dz \\ &= \Big[2z^{-\rho}\sqrt{z+r-1}\Big]_{|r-1|}^{r+1} + 2\rho\int_{|r-1|}^{r+1} z^{-\rho-1}\sqrt{z+r-1}\,dz \\ &\le C + C\int_{|r-1|}^{r+1} z^{-\rho-1/2}\,dz \le C + C|r-1|^{-(\rho-1/2)}. \end{aligned} \tag{6.31}$$

In the last inequality the assumption $\rho > 1/2$ has been taken into account.

It remains to derive the pointwise estimate for $I_4(x)$. We estimate the η-integral in (6.27) as in (6.23) to have

$$\begin{aligned} I_4(x) &\le \int_0^{1-r} \frac{dz}{z^\rho\sqrt{1-(z+r)^2}} \le \int_0^{1-r} \frac{dz}{z^\rho\sqrt{1-r-z}} \\ &= \int_0^{(1-r)/2} \frac{dz}{z^\rho\sqrt{1-r-z}} + \int_{(1-r)/2}^{1-r} \frac{dz}{z^\rho\sqrt{1-r-z}} \\ &\le \sqrt{\tfrac{2}{1-r}}\int_0^{(1-r)/2} z^{-\rho}\,dz + \left(\tfrac{1-r}{2}\right)^{-\rho}\int_{(1-r)/2}^{1-r} \frac{dz}{\sqrt{1-r-z}} \le C(1-r)^{-(\rho-1/2)}. \end{aligned} \tag{6.32}$$

In the last inequality the assumption $\rho < 1$ has been taken into account. Hence we have obtained the pointwise estimate for J_2 as follows:

$$J_2(x) \le \begin{cases} Cr^{-\rho-1} & \text{for } r \ge 2 \\ C|r-1|^{-(\rho-1/2)} & \text{for } 0 \le r \le 2,\ r \ne 1. \end{cases} \tag{6.33}$$

It is now obvious that the estimates (6.25) and (6.33) ensure $(\cos\omega)\phi \in L^\Theta(\mathbb{R}^2)$ under condition (6.6). □

Proof of Theorem 6.1. We may assume $n = 2$ throughout the proof. Let us prove (6.4) first. What we need to show is the estimate $||(\cos\omega)v||_{\dot H^s_r} \le$

$C\varepsilon_1$. Following Pecher's proof for $n=3$ in [28], we start our consideration with the decomposition of a homogeneous polynomial $P(x)$ of degree $l \geq 0$ by virtue of [34, Theorem 2.1 in Chapter 4]:

$$P(x) = P_0(x) + |x|^2 P_1(x) + \cdots + |x|^{2m} P_m(x). \tag{6.34}$$

Here $P_j(x)$ is a homogeneous and harmonic polynomial of degree $l-2j$ ($j = 0, \ldots, m$). When $l - 2j = 0$, P_j is a constant, and we have by taking the assumptions $d < 2$ and $d + s < 1$ (thus $d + s < 2$) into account

$$\begin{aligned} &\omega^s(\cos\omega)(\varepsilon_1|x|^{2j}P_j(x)|x|^{-d-l}) = C\varepsilon_1\omega^s(\cos\omega)|x|^{-d} \\ &= C\varepsilon_1\mathcal{F}^{-1}[|\xi|^s(\cos|\xi|)|\xi|^{-2+d}] = C\varepsilon_1(\cos\omega)|x|^{-(d+s)}. \end{aligned} \tag{6.35}$$

Since $d+s$ satisfies (6.2)–(6.3), we can employ Lemma 6.2 with $\rho = d+s$ to obtain

$$||\cos\omega(\varepsilon_1|x|^{2j}P_j(x)|x|^{-d-l})||_{\dot{H}^s_r} \leq C\varepsilon_1||(\cos\omega)|x|^{-(d+s)}||_{L^r} \leq C\varepsilon_1. \tag{6.36}$$

Next suppose $l-2j \geq 1$. Note that, by virtue of [34, Theorem 4.5 in Chapter 4], we have

$$\mathcal{F}\Big[\frac{P_j(x)}{|x|^{2+(l-2j)}}\Big](\xi) = \frac{CP_j(\xi)}{|\xi|^{l-2j}}. \tag{6.37}$$

Because of the assumption $0 < d < 2$ it is also possible to use [34, Theorem 4.1 in Chapter 4], and we have

$$\begin{aligned} &\omega^s(\cos\omega)(\varepsilon_1|x|^{2j}P_j(x)|x|^{-d-l}) = \varepsilon_1\mathcal{F}^{-1}\Big[|\xi|^s(\cos|\xi|)\mathcal{F}\Big[\frac{P_j(x)}{|x|^{d+l-2j}}\Big](\xi)\Big] \\ &= C\varepsilon_1\mathcal{F}^{-1}\Big[|\xi|^s(\cos|\xi|)\frac{P_j(\xi)}{|\xi|^{2-d+l-2j}}\Big] \\ &= C\varepsilon_1\mathcal{F}^{-1}\Big[\frac{\cos|\xi|}{|\xi|^{2-d-s}}\Big] * \mathcal{F}^{-1}\Big[\frac{P_j(\xi)}{|\xi|^{l-2j}}\Big] = C\varepsilon_1\Big((\cos\omega)\frac{1}{|x|^{d+s}}\Big) * \Big(\frac{P_j(x)}{|x|^{2+l-2j}}\Big). \end{aligned} \tag{6.38}$$

Since P_j is harmonic and homogeneous of degree $l-2j \geq 1$, we see

$$\int_{S^1} P_j(x)\,dS = 2\pi P_j(0) = 0.$$

Taking account of the fact that $P_j(x)/|x|^{l-2j}$ is homogeneous of degree zero, we can apply [34, Theorem 3.1 in Chapter 6] to get the boundedness of the singular integral operator with a kernel $P_j(x)/|x|^{2+l-2j}$. Because of the assumption $1 < r < \infty$ we thus have

$$||\cos\omega(\varepsilon_1|x|^{2j}P_j(x)|x|^{-d-l})||_{\dot{H}^s_r} \leq C\varepsilon_1||\cos\omega|x|^{-(d+s)}||_{L^r} \leq C\varepsilon_1. \tag{6.39}$$

In the last inequality we have made use of Lemma 6.2 with $\rho = d + s$. Combining (6.34) with (6.36) and (6.39), we have finished the proof of the inequality $||(\cos\omega)v||_{\dot{H}^s_r} \le C\varepsilon_1$ for $v(x) = \varepsilon_1 P(x)|x|^{-d-l}$. The proof of (6.5) is similar. Indeed, taking the inequality $d + s + 1 < n$ into account, we can repeat essentially the same argument as above. Thus we have completed the proof of Theorem 6.1. □

As an immediate consequence, we get the following:

Corollary 6.3. *Suppose $n = 2, 3$ and $p_0(n) < p < 1 + 4/(n-1)$. Suppose further that q and ε are the numbers given at the beginning of Section 5. If (f, g) is a pair of functions given in* (6.1)*, the free solution $u_0(t)$ with initial data $(u_0(0), \partial_t u_0(0)) = \varepsilon_1(f, g)$ $(\varepsilon_1 > 0)$ belongs to W, and it satisfies $||u_0||_W \le C\varepsilon_1$ and $t^\varepsilon\omega^{-1}\partial_t u_0 \in L^\infty(0, \infty; \dot{H}_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n))$.*

Proof. We start with the proof of the estimate $||u_0||_W \le C\varepsilon_1$. Noting (2.58)–(2.60) and choosing $d = 2/(p-1)$, $s = 1/q - 1/p + \varepsilon$ and $r = q$, we apply Theorem 6.1. It is enough to check (6.2)–(6.3) because the other conditions are obviously satisfied. The condition (6.2) turns out to be equivalent to

$$\varepsilon < \frac{(n-1)p^2 - (n+1)p - 2}{2p(p-1)},$$

which is assumed in (5.1). On the other hand, the condition (6.3) is equivalent to $\varepsilon > (n-1)/q + 1/p - 2/(p-1)$. This is satisfied because (5.2) implies $(n-1)/q + 1/p - 2/(p-1) = 0$. Thus we have checked the conditions (6.2) and (6.3), and Theorem 6.1 immediately yields the estimate $||u_0||_W \le C\varepsilon_1$.

Next let us prove that $u_0 \in W$. It is enough to show that $u_0(t) : (0, \infty) \to \dot{H}_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n)$ is Bochner measurable. Since $u_0 \in C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^n))$, the estimate

$$||u_0(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \le C\varepsilon_1 t^{-\varepsilon}$$

immediately implies $u_0 \in C_w((0, \infty); \dot{H}_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n))$ (see the remark given just below (5.8)). Noting that the space $\dot{H}_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n)$ is separable, we have shown that $u_0(t) : (0, \infty) \to \dot{H}_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n)$ is Bochner measurable.

Finally we shall check $t^\varepsilon\omega^{-1}\partial_t u_0 \in L^\infty(0, \infty; \dot{H}_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n))$. Our task is to show

$$\sup_{t>0} t^\varepsilon ||(\sin\omega t)f||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \le C, \tag{6.40}$$

$$\sup_{t>0} t^\varepsilon ||\frac{\cos\omega t}{\omega} g||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \le C \tag{6.41}$$

when f and g are functions given in (6.1). Since the inequality $1+2/(p-1) < n$ is true for $p > p_0(n)$, we may employ Theorem 6.1 with $d = 1 + 2/(p-1)$ as well as $s = -1 + 1/q - 1/p + \varepsilon$ (< 0) to show (6.41). As for (6.40), it immediately follows because the inequality

$$||(\sin \omega t)f||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \leq C \sum_{j=1}^{n} ||\frac{\sin \omega t}{\omega} \partial_j f||_{\dot{H}_q^{1/q-1/p+\varepsilon}}$$

holds and $\partial_j f$ has the same form as the function g has in (6.1). Thus we have finished the proof of Corollary 6.3. □

7. Asymptotically self-similar solutions

The aim of the last section is to show the existence of a class of solutions which asymptotically behave like a self-similar solution. One of the main results in this section is

Theorem 7.1. *Suppose $n = 2, 3$ and $p_0(n) < p < 1 + 4/(n-1)$. Let δ_3 be any positive number satisfying $C_9(4\delta_3)^{p-1} < 1$ as well as $C_7(2\delta_3)^{p-1} \leq 1/2$ and $C_8(4\delta_3)^{p-1} < 1$, where C_7 and C_8 are the same constants as those in Section 5 and C_9 is the constant appearing in* (7.4) *below. Suppose further that $(f, g), (\tilde{f}, \tilde{g}) \in \mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ are given so that the corresponding free solutions $u_0(t) = (\cos \omega t)f + (\omega^{-1} \sin \omega t)g$ and $\tilde{u}_0(t) = (\cos \omega t)\tilde{f} + (\omega^{-1} \sin \omega t)\tilde{g}$ may satisfy $u_0, \tilde{u}_0 \in W_{\delta_3}$. If the difference between the free solutions has the stronger time-decay property*

$$\operatorname*{ess\,sup}_{t>0} (1+t)^{\varepsilon_0} t^{\varepsilon} ||u_0(t) - \tilde{u}_0(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}} < \infty \tag{7.1}$$

for a suitable $\varepsilon_0 > 0$ with $\varepsilon + \varepsilon_0 < \gamma(q)$, then the corresponding solutions $u, \tilde{u} \in W_{2\delta_3}$ to the integral equation (5.3) *given by Theorem* 5.1 *also satisfy*

$$\operatorname*{ess\,sup}_{t>0} (1+t)^{\varepsilon_0} t^{\varepsilon} ||u(t) - \tilde{u}(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}} < \infty. \tag{7.2}$$

Remark. If, in addition to the assumptions in the above theorem, the initial data (f, g) satisfy the homogeneity condition in part (3) of Theorem 5.1, then the solution u is self-similar and

$$t^{\varepsilon} ||u(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}} = \text{constant} \quad \text{for a.a. } t > 0. \tag{7.3}$$

It is easy to observe (7.3). Indeed, since u is self-similar and thus satisfies the identity $u(t, x) \equiv \delta^{2/(p-1)} u(\delta t, \delta x)$ $(\delta > 0)$, we have only to choose, e.g., $\delta = 1/t$. Theorem 7.1 implies that $\tilde{u}$ behaves like a self-similar solution u for large time because ε_0 is strictly positive.

Proof of Theorem 7.1. We first observe that a simple inequality

$$(1+t)/(1+\tau) \le 2t/\tau$$

holds for $0 \le \tau \le t$. Indeed, if $t \le 1$, then

$$(1+t)/(1+\tau) \le 2 \le 2t/\tau.$$

When $t \ge 1$, we easily see $(1+t)/(1+\tau) \le 2t/\tau$. Suppose that T is an arbitrary positive number. Repeating essentially the same argument as in the proof of Theorem 5.1 and taking the inequality $(1+t)^{\varepsilon_0} \le [2t(1+\tau)/\tau]^{\varepsilon_0}$ into account, we have for almost all $t \in (0,T)$

$$\begin{aligned}
&(1+t)^{\varepsilon_0} t^{\varepsilon} ||u(t)-\tilde{u}(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \le (1+t)^{\varepsilon_0} t^{\varepsilon} ||u_0(t)-\tilde{u}_0(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \\
&+ C(4\delta_3)^{p-1} t^{\varepsilon+\varepsilon_0} \int_0^t |t-\tau|^{-\gamma(q)} \tau^{-1+\gamma(q)-(\varepsilon+\varepsilon_0)} (1+\tau)^{\varepsilon_0} \tau^{\varepsilon} \\
&\times ||u(\tau)-\tilde{u}(\tau)||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \, d\tau \\
&\le \operatorname*{ess\,sup}_{t>0} (1+t)^{\varepsilon_0} t^{\varepsilon} ||u_0(t)-\tilde{u}_0(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \\
&+ C_9 (4\delta_3)^{p-1} \operatorname*{ess\,sup}_{0<t<T} (1+t)^{\varepsilon_0} t^{\varepsilon} ||u(t)-\tilde{u}(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}}.
\end{aligned} \tag{7.4}$$

Since T is arbitrary and δ_3 is small so that $C_9(4\delta_3)^{p-1} < 1$ may hold, (7.2) follows immediately from (7.1) and (7.4). □

Following Cazenave–Weissler [3] and Pecher [28], we shall give an example of data (f,g), $(\tilde{f},\tilde{g})$ satisfying the assumptions in Theorem 7.1. Since we are interested in the existence of asymptotically self-similar solutions, let us consider the case where the solution u with data (f,g) is self-similar.

Theorem 7.2. *Suppose that $n = 2, 3$ and $p_0(n) < p < 1 + 4/(n-1)$. Suppose further that f and g are functions of the form*

$$\begin{cases} f(x) = \varepsilon_1 \displaystyle\sum_{l=0}^{N_1} P_l(x)|x|^{-2/(p-1)-l} & (\varepsilon_1 > 0), \\ g(x) = \varepsilon_2 \displaystyle\sum_{m=0}^{N_2} Q_m(x)|x|^{-1-2/(p-1)-m} & (\varepsilon_2 > 0) \end{cases} \tag{7.5}$$

where P_l and Q_m are any homogeneous polynomials of degree l and m, respectively, and N_1, $N_2 (\ge 0)$ are any integers. Let $\psi_j, \tilde{\psi}_j$ $(j = 1, 2)$ be compactly supported functions on $\mathbb{R}^n$ such that ψ_1, $\tilde{\psi}_1 \in C^2$, ψ_2, $\tilde{\psi}_2 \in C^1$ and they are identically equal to 1 in a neighborhood of zero. Define $\eta_j := 1 - \psi_j$. Let v

and w *be tempered distributions on* $\mathbb{R}^n$ *satisfying*

$$v,\, x_j\omega v,\, \omega^{-1}w,\, x_jw \in \dot{H}_2^{\theta+\varepsilon}(\mathbb{R}^n) \cap \dot{H}_{q'}^{1/q-1/p+\varepsilon+2\beta(q)}(\mathbb{R}^n) \tag{7.6}$$

for $j = 1, \dots, n$. *If* $\tilde{f}$ *and* $\tilde{g}$ *have the form*

$$\begin{cases} \tilde{f} = \eta_1 f + \varepsilon_3 v + \varepsilon_5 \sum_{l=0}^{\tilde{N}_1} \tilde{\psi}_1 \tilde{P}_l |x|^{-2/(p-1)-l} \\ \tilde{g} = \eta_2 g + \varepsilon_4 w + \varepsilon_6 \sum_{m=0}^{\tilde{N}_2} \tilde{\psi}_2 \tilde{Q}_m |x|^{-1-2/(p-1)-m} \end{cases} \tag{7.7}$$

$(\varepsilon_j > 0,\ j = 3, \dots, 6)$ *where* $\tilde{P}_l$ *and* $\tilde{Q}_m$ *are any homogeneous polynomials of degree* l *and* m, *respectively and* $\tilde{N}_1$, $\tilde{N}_2 (\geq 0)$ *are any integers, then the free solution* $\tilde{u}_0(t) = (\cos \omega t)\tilde{f} + (\omega^{-1} \sin \omega t)\tilde{g}$ *belongs to* W *and satisfies for* $|\alpha| \leq 1$

$$\sup_{t>0} t^{\varepsilon} ||\dot{\Gamma}^{\alpha} (\cos \omega t) \tilde{f}||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \leq C\varepsilon_1 + C\varepsilon_3 + C\varepsilon_5, \tag{7.8}$$

$$\sup_{t>0} t^{\varepsilon} ||\dot{\Gamma}^{\alpha} \frac{\sin \omega t}{\omega} \tilde{g}||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \leq C\varepsilon_2 + C\varepsilon_4 + C\varepsilon_6. \tag{7.9}$$

Moreover, the difference between $u_0(t) := (\cos \omega t) f + (\omega^{-1} \sin \omega t) g$ *and* $\tilde{u}_0(t)$ *satisfies*

$$\sup_{t>0} (1+t)^{\gamma(q)-\varepsilon} t^{\varepsilon} ||u_0(t) - \tilde{u}_0(t)||_{\dot{H}_q^{1/q-1/p+\varepsilon}} < \infty. \tag{7.10}$$

Remark. (1) Let us consider the case where $v = w = 0$ as well as $\tilde{P}_l = \tilde{Q}_m = 0$ in (7.7). Then $\tilde{f}$ is smooth and satisfies $\tilde{f}(x) = O(|x|^{-2/(p-1)})$ for large $|x|$. Due to the lack of sufficient rate of decay it seems that $\tilde{f}$ fails to be in $\dot{H}_2^{\theta-\varepsilon}(\mathbb{R}^n)$ no matter how small $\varepsilon (> 0)$ is, though such a function as $\langle x \rangle^{-2/(p-1)-\delta}$ $(\delta > 0)$ belongs to $\dot{H}_2^{\theta-\varepsilon}(\mathbb{R}^n)$ (in fact, to X) for sufficiently small $\varepsilon > 0$. For the same reason $\tilde{g}$ seems to miss being in $\dot{H}_2^{-1+\theta-\varepsilon}(\mathbb{R}^n)$ however small $\varepsilon (> 0)$ is. Therefore the scattering theory developed in Sections 3 and 4 is not applicable to the above-mentioned case. Theorems 7.1 and 7.2 complement Theorem 3.1 and tell us one way of characterizing the asymptotic behavior of the solution $\tilde{u}$ when the data $(\tilde{f}, \tilde{g})$ have the form in

(7.7). (2) If we consider the data $(\tilde{f}, \tilde{g})$ of the form

$$\begin{cases} \tilde{f} = f + \varepsilon_3 v + \varepsilon_5 \sum_{l=0}^{\tilde{N}_1} \tilde{\psi}_1 \tilde{P}_l |x|^{-2/(p-1)-l}, \\ \tilde{g} = g + \varepsilon_4 w + \varepsilon_6 \sum_{m=0}^{\tilde{N}_2} \tilde{\psi}_2 \tilde{Q}_m |x|^{-1-2/(p-1)-m} \end{cases}$$

instead of (7.7), we can show the same as (7.8)–(7.10). This, together with Theorem 7.1, means the stability of self-similar solutions.

Proof of Theorem 7.2. We start with the proof of (7.8)–(7.9). It is enough to show (7.9) only, because the proof of (7.8) is similar. To begin with, we estimate $(\omega^{-1}\sin\omega t)(\eta_2 g)$. It follows immediately from the embedding $\dot{H}_2^{\theta+\varepsilon}(\mathbb{R}^n) \hookrightarrow \dot{H}_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n)$ and the formulae (2.58)–(2.60) that

$$\sup_{t>0} ||\dot{\Gamma}^\alpha \frac{\sin\omega t}{\omega}(\eta_2 Q_m |x|^{-1-2/(p-1)-m})||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \le C \ (|\alpha| \le 1) \tag{7.11}$$

once we have shown

$$\omega^{-1}(\eta_2 Q_m |x|^{-1-2/(p-1)-m}), \ x_j \eta_2 Q_m |x|^{-1-2/(p-1)-m} \in \dot{H}_2^{\theta+\varepsilon}(\mathbb{R}^n). \tag{7.12}$$

Moreover, thanks to $\varepsilon < \gamma(q)$, Lemma 2.10 together with Corollary 6.3 yields the estimate

$$\begin{aligned} &\sup_{t>1} t^\varepsilon ||\dot{\Gamma}^\alpha \frac{\sin\omega t}{\omega}(\eta_2 Q_m |x|^{-1-2/(p-1)-m})||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \\ &\le \sup_{t>1} t^\varepsilon ||\dot{\Gamma}^\alpha \frac{\sin\omega t}{\omega}(Q_m |x|^{-1-2/(p-1)-m})||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \\ &+ \sup_{t>1} t^\varepsilon ||\dot{\Gamma}^\alpha \frac{\sin\omega t}{\omega}(\psi_2 Q_m |x|^{-1-2/(p-1)-m})||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \le C \ (|\alpha| \le 1) \end{aligned} \tag{7.13}$$

once

$$\omega^{-1}(\psi_2 Q_m |x|^{-1-\frac{2}{p-1}-m}), x_j \psi_2 Q_m |x|^{-1-\frac{2}{p-1}-m} \in \dot{H}_{q'}^{1/q-1/p+\varepsilon+2\beta(q)}(\mathbb{R}^n) \tag{7.14}$$

has been shown.

Let us first show (7.12). Choosing r_0 as $1/2 - (\theta+\varepsilon)/n = 1/r_0 - 1/n$ and noting $r_0 > 1$, we get by the embedding $\dot{H}_{r_0}^1(\mathbb{R}^n) \hookrightarrow \dot{H}_2^{\theta+\varepsilon}(\mathbb{R}^n)$

$$\begin{aligned} &||\omega^{-1}(\eta_2 Q_m |x|^{-1-2/(p-1)-m})||_{\dot{H}_2^{\theta+\varepsilon}} \le C||\omega^{-1}(\eta_2 Q_m |x|^{-1-2/(p-1)-m})||_{\dot{H}_{r_0}^1} \\ &= C||\eta_2 Q_m |x|^{-1-2/(p-1)-m}||_{L^{r_0}} \le C. \end{aligned} \tag{7.15}$$

The last inequality is due to the fact that the function $\langle x\rangle^{-1-2/(p-1)}$ belongs to $L^{r_0}(\mathbb{R}^n)$ thanks to the assumption $\varepsilon > 0$. We can check

$$x_j\eta_2 Q_m|x|^{-1-2/(p-1)-m} \in \dot{H}_2^{\theta+\varepsilon}(\mathbb{R}^n)$$

in a similar way. We next prove (7.14). As we have observed before (see just below (5.24)), the inequality $1/q - 1/p + \varepsilon + 2\beta(q) < 1 - 1/q < 1$ holds now. Giving r_1 by $1/q' - (1/q - 1/p + \varepsilon + 2\beta(q))/n = 1/r_1 - 1/n$ and noting $r_1 > 1$, we find by the embedding $\dot{H}^1_{r_1}(\mathbb{R}^n) \hookrightarrow \dot{H}_{q'}^{1/q-1/p+\varepsilon+2\beta(q)}(\mathbb{R}^n)$ that

$$\begin{aligned} &||\omega^{-1}(\psi_2 Q_m|x|^{-1-2/(p-1)-m})||_{\dot{H}_{q'}^{1/q-1/p+\varepsilon+2\beta(q)}} \\ &\le C||\psi_2 Q_m|x|^{-1-2/(p-1)-m}||_{L^{r_1}} \le C. \end{aligned} \tag{7.16}$$

The last inequality is due to the fact that the function $|x|^{-1-2/(p-1)}$ belongs to L^{r_1} $(|x| < 1)$ thanks to the assumption $\varepsilon < [(n-1)p^2 - (n+1)p - 2]/2p(p-1)$. In a similar way we see $x_j\psi_2 Q_m|x|^{-1-2/(p-1)-m} \in \dot{H}_{q'}^{1/q-1/p+\varepsilon+2\beta(q)}(\mathbb{R}^n)$. Thus we have completed the proof of the estimate for $(\omega^{-1}\sin\omega t)(\eta_2 g)$.

As for the estimate for $(\omega^{-1}\sin\omega t)w$, we easily get

$$\sup_{t>0}(1+t)^{\gamma(q)}||\dot{\Gamma}^\alpha \frac{\sin\omega t}{\omega} w||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \le C \tag{7.17}$$

from the formulae (2.58)–(2.60) and the assumption (7.6).

Let us next show for $|\alpha| \le 1$

$$\sup_{t>0}(1+t)^{\gamma(q)-\varepsilon}t^\varepsilon||\dot{\Gamma}^\alpha \frac{\sin\omega t}{\omega}(\tilde{\psi}_2\tilde{Q}_m|x|^{-1-2/(p-1)-m})||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \le C. \tag{7.18}$$

For $0 < t < 1$ we get by virtue of Corollary 6.3 and (7.12)

$$\begin{aligned} &||\dot{\Gamma}^\alpha \frac{\sin\omega t}{\omega}(\tilde{\psi}_2\tilde{Q}_m|x|^{-1-2/(p-1)-m})||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \\ &\le ||\dot{\Gamma}^\alpha \frac{\sin\omega t}{\omega}(\tilde{Q}_m|x|^{-1-2/(p-1)-m})||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \\ &+ ||\dot{\Gamma}^\alpha \frac{\sin\omega t}{\omega}((1-\tilde{\psi}_2)\tilde{Q}_m|x|^{-1-2/(p-1)-m})||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \le Ct^{-\varepsilon} + C. \end{aligned} \tag{7.19}$$

For $t \ge 1$ (in fact $t > 0$) Lemma 2.10 yields

$$||\dot{\Gamma}^\alpha \frac{\sin\omega t}{\omega}(\tilde{\psi}_2\tilde{Q}_m|x|^{-1-2/(p-1)-m})||_{\dot{H}_q^{1/q-1/p+\varepsilon}} \le Ct^{-\gamma(q)} \tag{7.20}$$

by virtue of (7.14). Then (7.18) immediately follows from (7.19)–(7.20). Thus we have finished the proof of (7.9).

Now it is obvious that $\tilde{u}_0(t) : (0,\infty) \to \dot{H}_q^{1/q-1/p+\varepsilon}(\mathbb{R}^n)$ is Bochner measurable. Thus $\tilde{u}_0$ belongs to W.

Finally, repeating the argument in (7.18)–(7.20) and taking (7.17) into account, we easily obtain (7.10). Therefore we have completed the proof of Theorem 7.2. □

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