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BIFURCATION FROM INFINITY IN A CLASS OF NONLOCAL ELLIPTIC PROBLEMS

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1. Introduction

We study the nonlocal problem

$$-\Delta u = \frac{\mu f(u)}{\left[\int_\Omega f(u)dx\right]^p} \ \text{ in } \Omega,\ u>0 \ \text{ in } \Omega,\ \ u=0 \text{ on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded domain in R^N $(N \geq 1)$ with C^2 boundary $\partial\Omega$, p is a constant, μ is a bifurcation parameter, and f is a locally Lipschitz continuous function on $[0,\infty)$ and satisfies

$$f(u) > 0 \text{ for } u \geq 0,\ \overline{\lim}_{u\to\infty}\frac{f(u)}{u} = 0. \tag{1.2}$$

Such problems arise in various situations of practical importance, such as modelling Ohmic heating and plasma physics (see, e.g., [18, 19, 4, 13]), and have attracted considerable attention in recent years; we refer to [3, 4, 13, 14, 18, 19, 20], and the references therein for more details.

Condition (1.2) shows that $f(u)$ is sublinear near infinity. It is well-known that for the corresponding local problem

$$-\Delta u = \lambda f(u) \ \text{ in } \Omega,\ u>0 \text{ in } \Omega,\ u=0 \text{ on } \partial\Omega, \tag{1.3}$$

bifurcation from infinity at a finite λ value is possible only if the nonlinearity $f(u)$ is asymptotically linear or superlinear near infinity. Under condition (1.2), this is not possible; instead, bifurcation from infinity occurs exactly at $\lambda = \infty$, i.e., there exists a sequence of solutions (λ_n, u_n) of (1.3) such that $\lambda_n \to \infty$, $\|u_n\|_\infty \to \infty$; moreover, if (λ_n', u_n') is an arbitrary sequence of solutions to (1.3) with $\|u_n'\|_\infty \to \infty$, then we must have $\lambda_n' \to \infty$.

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In contrast, with proper choices of p, bifurcation from infinity for the nonlocal problem (1.1) with f satisfying (1.2) can occur at both finite and infinite values of μ. Our main interest in this paper is to determine the effect of the value of p on the location of μ where bifurcation from infinity occurs. As is the case for the local problem (1.3) (e.g., [7, 21]), such information is crucial in determining the global structure of the solution set $\{(\mu, u)\}$ of (1.1), and hence the existence, multiplicity and nonexistence of solutions to this nonlocal problem (see Theorem 2.3).

Our first main result deals with nonlinearities satisfying (1.2) and

$$\int_0^\infty f(u)du < \infty.$$

Theorem 1.1. *Suppose f satisfies* (1.2) *and $I_f = \int_0^\infty f(u)du < \infty$, and in the case the space dimension $N \geq 3$, suppose also $\lim_{u\to\infty} uf(u) = 0$. Then bifurcation from infinity for* (1.1) *is determined as follows. When $p = 2$, it occurs exactly at $\mu = 2I_f|\partial\Omega|^2$ ($\mu = 8I_f$ when $N = 1$); when $p > 2$, it occurs exactly at $\mu = 0$; and when $p < 2$, this occurs exactly at $\mu = \infty$.*

For nonlinearities satisfying (1.2) but with $\int_0^\infty f(u)du = \infty$, the behaviour of bifurcation from infinity is more difficult to understand. Subsequently we consider a subclass of them.

Theorem 1.2. *Suppose f satisfies* (1.2) *and*

$$\lim_{u\to\infty} \frac{f(u)}{u^\alpha} = c, \tag{1.4}$$

for some constants α and c, $-1 < \alpha < 1$, $c > 0$. Then bifurcation from infinity for (1.1) *occurs in the following way: When $p\alpha/(\alpha - 1) = 1$, this occurs exactly at $\mu = (c\int_\Omega U^\alpha dx)^p$; when $p\alpha/(\alpha - 1) > 1$, it occurs exactly at $\mu = 0$; when $p\alpha/(\alpha - 1) < 1$, it occurs exactly at $\mu = \infty$.*

Here U denotes the unique positive solution to

$$-\Delta U = cU^\alpha,\ U|_{\partial\Omega} = 0. \tag{1.5}$$

Further details about this equation and its solutions can be found in Remark 3.2 later.

Theorems 1.1 and 1.2 reveal some important differences in the behaviour between integrable and nonintegrable sublinear nonlinearities. This is reflected by the value of the critical number p and the location of bifurcation from infinity at the critical p. More precisely, for integrable nonlinearities, by Theorem 1.1, the critical p is the universal number $p = 2$, and when $p = 2$, the location of bifurcation from infinity, $\mu = 2I_f|\partial\Omega|^2$, depends only on the size of $\partial\Omega$ and the integral I_f; while the particular asymptotic behaviour of $f(u)$ near $u = \infty$ and the shape of Ω do not seem to play a role. In sharp contrast, in Theorem 1.2, for nonintegrable nonlinearities, both the critical value of p, namely $p = (\alpha - 1)/\alpha$, and the location of μ at the critical p, i.e., $\mu = (c \int_\Omega U^\alpha dx)^p$, are determined solely by the asymptotic behaviour of $f(u)$ near $u = \infty$ and the domain Ω.

A borderline case between integrable and nonintegrable nonlinearities is when (1.4) holds with $\alpha = -1$, that is, $\lim_{u\to\infty} uf(u) = c$ for some positive constant c. We will show (Theorem 3.5) that in this case, bifurcation from infinity for (1.1) occurs exactly at $\mu = 0$ when $p \geq 2$, and exactly at $\mu = \infty$ when $p < 2$. Thus at the critical value of p, namely $p = 2$, the location of bifurcation from infinity is the same as when $p < 2$. This is very different to the cases with integrable nonlinearities, and also different from the other nonintegrable cases.

A related situation which is not covered by (1.2) is when f satisfies

$$f(u) > 0 \text{ for } 0 \leq u < a \text{ and } f(u) \leq 0 \text{ for } u \geq a, \tag{1.6}$$

where a is a positive constant. When f satisfies (1.6), the local problem (1.3) has a positive solution u for every $\lambda > 0$ but $0 < u(x) < a$ always holds for $x \in \Omega$. Therefore bifurcation from infinity never occurs in this case. However, it is well-known (see, [2] and [8]) that in this case, any solution of (1.3) converges to a locally uniformly in Ω as $\lambda \to \infty$. Therefore we might say that bifurcation from $u = a$ occurs exactly at $\lambda = \infty$ for (1.3). Our techniques for proving Theorem 1.1 can be used to show that when (1.6) is satisfied, then the conclusions of Theorem 1.1 hold with "bifurcation from infinity" replaced by "bifurcation from $u = a$" and I_f replaced by $J_f = \int_0^a f(u)du$.

This research was motivated by an interesting paper by Bebernes and Lacey [3], and Theorems 1.1 and 1.2 above are generalizations of some of the results obtained there. More precisely, under the extra condition that $f(u)$ is decreasing when $N \geq 2$, Theorem 1.1 follows from [3, Theorem 2.2 (and its proof) and Theorem 3.4 part 1], while for the case $N = 1$, Theorem 1.2 is proved in [3, Theorem 3.4 part 2].

On page 942 of [3], it is hinted that the one dimensional results in part 2 of Theorem 3.4 there, for nonlinearities satisfying $\lim_{u\to\infty} f(u)/u^\alpha = c > 0$, $\alpha \geq -1$, can be recovered in high dimensions by analyzing $\int_{\partial\Omega} |\partial u/\partial n| dx$, but no details are available. It seems appropriate to make some comments on what are likely to be obtained by this method, where estimates for $\partial u/\partial n$ with u a solution of (1.3) are obtained through a boundary layer analysis, by constructing suitable upper and lower solutions using solutions of the ordinary differential equation $V'' + f(V) = 0$ on large intervals. For the case that $\int_0^\infty f(u)du < \infty$, this method is used in the proof of Theorem 2.2 in [3]. Without assuming that $f(u)$ is decreasing, this method gives a lower bound for $-\partial u/\partial n|_{\partial\Omega}$ for the *maximal* solution u of (1.3) with large λ. To obtain an upper bound, the additional assumption that $f(u)$ is decreasing is needed (which also implies uniqueness of solutions to (1.3)). Assuming the hint in [3] is to apply this method in an analogous way for the case that $f(u)$ behaves like cu^α near infinity, then the restriction that $\alpha < 1$ seems needed to guarantee the existence of a *maximal* solution of (1.3) (indeed, upper and lower solution techniques do not work in general once $\alpha > 1$), and in which case, again, a lower bound can be found only for the maximal solution. To obtain an upper bound, then the extra restriction that $f(u)$ be decreasing is needed again (which forces $\alpha \leq 0$). In view of our result in Theorem 1.2, it is unclear whether the lower bound and the upper bound established in this way can agree in the limit $\lambda \to \infty$, though it is the case when $\int_0^\infty f(u)du < \infty$, as is shown in the proof of Theorem 2.2 in [3]. If they do not agree in the limit, then an exact location of the value of μ where bifurcation from infinity occurs, at the critical value of p, does not seem obtainable by this method, even under the very restrictive condition that $f(u)$ decreases with u.

Compared with the boundary layer analysis method used in [3], our method for proving Theorem 1.1 is simpler and seems more natural, where the main ingredients are the Pohozaev identity and the comparison principle. Our method for proving Theorem 1.2 is different, and relies on the limiting problem (1.5). Here our condition on the nonlinearity is more restrictive (it must behave like a power function near infinity), but on the other hand, the method extends readily to the case that the Laplace operator Δ is replaced by a general second order uniformly elliptic operator of the form

$$Lu = \sum a_{ij}(x)u_{x_i x_j} + \sum b_i(x)u_{x_i} + c(x)u,$$

and we can also allow the function f to be dependent on x (see Remark 3.6 for more details). In contrast, our method for proving Theorem 1.1 uses special properties of the Laplace operator and the fact that f is independent of x, and it is unclear to us how this theorem can be extended to the above general situation.

Let us end this section by recalling some important facts about the local problem (1.3). These results will be used in the proofs of our main results.

For completeness and convenience of the reader, the proof of these conclusions is included in the last section.

Theorem 1.3. *Suppose f satisfies* (1.2) *or* (1.6). *Then the following conclusions are true.*

(*i*) *For every $\lambda > 0$,* (1.3) *has a maximal and minimal solution, which we denote by U_λ and u_λ, respectively.*

(*ii*) *Both u_λ and U_λ increase with λ, and when* (1.2) *is satisfied, then as $\lambda \to \infty$, u_λ* (*and hence U_λ*) *converges to ∞ locally uniformly in Ω; when* (1.6) *is satisfied, then u_λ and U_λ both converge to a locally uniformly in Ω as $\lambda \to \infty$.*

(*iii*) *If $\tilde{U}_\lambda$ and $\tilde{u}_\lambda$ are the maximal and minimal solutions of* (1.3) *with Ω replaced by $\tilde{\Omega}$ which satisfies $\tilde{\Omega} \supset \Omega$, then $\tilde{U}_\lambda \geq U_\lambda$, $\tilde{u}_\lambda \geq u_\lambda$ on Ω.*

(*iv*) *Let Σ denote the set of all $(\lambda, u) \in [0, \infty) \times C(\overline{\Omega})$ which solve* (1.3). *Then the component Σ_0 of $\overline{\Sigma}$ that contains the point* $(0, 0)$ *is unbounded, and $(\lambda, u_\lambda) \in \Sigma_0$ for each $\lambda > 0$.*

The rest of this paper is organized as follows. Section 2 is mainly devoted to the proof of Theorem 1.1, where some existence and nonexistence results are also presented as direct consequences of Theorem 1.1. We also treat the case that f satisfies (1.6). Our method here is different from that in [3], and is simpler. In section 3, we prove Theorem 1.2 and also study the borderline case that (1.4) is satisfied with $\alpha = -1$. A key step is to obtain an asymptotic limit of the solutions of (1.3) as $\lambda \to \infty$, which is given in Theorem 3.1. In section 4, we prove Theorem 1.3.

2. Integrable nonlinearities

The main purpose of this section is to prove Theorem 1.1. As the case $N = 1$ is already covered by Theorem 3.4 in [3], we assume $N \geq 2$ throughout this section unless otherwise specified.

We start by recalling a useful relation between the nonlocal problem (1.1) and the local one (1.3). Let (μ, u) be a solution to (1.3). Then, evidently, (λ, u) solves (1.3) with

$$\lambda = \mu / [\int_\Omega f(u) dx]^p.$$

Conversely, if (λ, u) solves (1.3), then (μ, u) solves (1.1) with

$$\mu = \lambda [\int_\Omega f(u) dx]^p.$$

Since bifurcation from infinity for (1.3) occurs exactly at $\lambda = \infty$ (see Theorem 1.3), it is now clear that the location of bifurcation from infinity for (1.1) is determined by the limit of $\lambda[\int_\Omega f(u)dx]^p$ as $\lambda \to \infty$ along solutions (λ, u) of (1.3).

For each $\lambda > 0$, let A_λ denote the set of solutions u to (1.3). We have the following result.

Theorem 2.1. *Suppose that f satisfies* (1.2) *and $I_f = \int_0^\infty f(u)du < \infty$. Moreover, when $N \geq 3$, assume further that $\lim_{u\to\infty} uf(u) = 0$. Then*

$$\lim_{\lambda\to\infty} \sqrt{\lambda} \int_\Omega f(u)dx = |\partial\Omega|\sqrt{2I_f} \quad \textit{uniformly for } u \in A_\lambda. \tag{2.1}$$

Proof. Integrating (1.3) over Ω, and using integration by parts, we obtain

$$-\int_{\partial\Omega} \big(\partial u/\partial n\big)ds = \lambda \int_\Omega f(u)dx,$$

where n denotes the unit outward normal of $\partial\Omega$. Thus, (2.1) follows if we can prove the following.

$$\lim_{\lambda\to\infty} (-\partial u/\partial n)/\sqrt{\lambda} = \sqrt{2I_f} \text{ uniformly for } x \in \partial\Omega, u \in A_\lambda. \tag{2.2}$$

We will do this in three steps. Firstly we show that (2.2) is true when Ω is a ball, then we show it holds if Ω is an annulus, and finally we show (2.2) holds for a general bounded domain with C^2 boundary.

Step 1. (2.2) holds if Ω is a ball.

Without loss of generality, we may assume that $\Omega = B_R = \{x \in R^N : |x| < R\}$, for some $R > 0$. In this case, by the well-known Gidas-Ni-Nirenberg result, any solution u of (1.3) is radially symmetric: $u(x) = u(r)$, $r = |x|$, and $u'(r) < 0$ for $0 < r \leq R$. Since $u_\lambda \leq u \leq U_\lambda$, and they are equal at $r = R$, where we use the notation in Theorem 1.3 for maximal and minimal solutions of (1.3), the value of $u'(R)$ is between $u'_\lambda(R)$ and $U'_\lambda(R)$. Therefore, it suffices to prove (2.2) for u_λ and U_λ.

Let us use $v(r) = v_\lambda(r)$ to denote either u_λ or U_λ. By the well-known Pohozaev identity ([P]),

$$\lambda \int_{B_R} \big(NF(v) + (1 - N/2)vf(v)\big)dx = (1/2)v'(R)^2 R|\partial B_R|,$$

where $F(v) = \int_0^v f(s)ds$. By Theorem 1.3, as $\lambda \to \infty$, $v(r) = v_\lambda(r) \to \infty$ uniformly for $r \in [0, R - \epsilon]$, for any ϵ, $R > \epsilon > 0$. It follows that

$$\lim_{\lambda\to\infty} \int_{B_{R-\epsilon}} \big(NF(v) + (1 - N/2)vf(v)\big)dx = NI_f|B_{R-\epsilon}|.$$

But the integrand as a function of $\lambda > 0$ and $r \in [0, R]$ is uniformly bounded due to our assumptions on f, hence we can let $\epsilon \to 0$ in the above identity and obtain

$$\lim_{\lambda\to\infty} \int_{B_R} \big(NF(v) + (1 - N/2)vf(v)\big)dx = NI_f|B_R|.$$

It follows that

$$\lim_{\lambda\to\infty} v'(R)^2/\lambda = (2N|B_R|I_f)/(R|\partial B_R|) = 2I_f.$$

Clearly this implies (2.2) and step 1 is complete. Let us remark that it is crucial for our later steps that this limit is independent of the size of the ball.

Step 2. (2.2) holds if Ω is an annulus.

We may assume that $\Omega = B_R \backslash \overline{B}_\rho$, for some $R > \rho > 0$. Let us first observe that u_λ and U_λ are radially symmetric, though it is unknown whether the other solutions of (1.3) are also radially symmetric. This is because, due to the maximality and minimality, and the rotational invariance of the equation, $u_\lambda(x) \le u_\lambda(\Gamma x)$ and $U_\lambda(x) \ge U_\lambda(\Gamma x)$ for any rotation Γ of Ω around its center.

Let us again use $v(r) = v_\lambda(r)$ to denote either u_λ or U_λ, and observe that it is enough to prove (2.2) for such v only. As in step 1 we use Pohozaev's identity and deduce,

$$\lim_{\lambda\to\infty} (1/2)\Big(v'(R)^2R|\partial B_R| - v'(\rho)^2\rho|\partial B_\rho|\Big)/\lambda = \big(|B_R| - |B_\rho|\big)NI_f. \tag{2.3}$$

If we can show $v'(R)^2/\lambda \to 2I_f$ as $\lambda \to \infty$, then it follows from (2.3) that $v'(\rho)^2/\lambda \to 2I_f$ too, and this would conclude step 2. To this end, we consider two auxiliary problems:

$$-\Delta w = \lambda f(w) \quad \text{in } B_R,\ w|_{\partial B_R} = 0$$

and

$$-\Delta z = \lambda f(z) \quad \text{in } B,\ z|_{\partial B} = 0,$$

where B is a ball contained in $B_R \setminus \overline{B}_\rho$ and touches ∂B_R at some point $x_0 \in \partial B_R$.

By Theorem 1.3 we know that our auxiliary problems have positive solutions w_λ and z_λ, respectively. We suppose that w_λ is the maximal solution and z_λ is the minimal solution. Then, again by Theorem 1.3, $v \le w_\lambda$ on $B_R \setminus B_\rho$ and $v \ge z_\lambda$ on B. It follows that

$$\partial z_\lambda/\partial n \ge \partial v/\partial n \ge \partial w_\lambda/\partial n$$

at x_0. Now we use our conclusion obtained in step 1 to w_λ and z_λ, and we obtain from the above inequality that

$$-v'(R)/\sqrt{\lambda} = -\big(\partial v(x_0)/\partial n\big)/\sqrt{\lambda} \to \sqrt{2I_f}$$

as $\lambda \to \infty$, as we wanted. This concludes step 2.

Step 3. (2.2) holds if Ω is an arbitrary bounded domain with C^2 boundary.

Our smoothness assumption on Ω assures that it satisfies a uniform interior and exterior ball condition, i.e., there exists $\rho > 0$ such that each point $x_0 \in \partial\Omega$ can be touched by a ball of radius ρ contained entirely in Ω, and also be touched by a ball of radius ρ lying entirely outside Ω. We now fix such a point x_0 and a ball B_1 inside Ω touching $\partial\Omega$ at x_0, and another ball B_2 outside Ω touching its boundary at x_0. We suppose that both balls are of radius ρ. We then choose a third ball which has the same center as B_2 but has radius greater than the diameter of Ω so that when we move x_0 along $\partial\Omega$ with the three balls being determined as above, then Ω always lies inside the annulus $B_3 \setminus B_2$.

Let now w_λ be the maximal solution of (1.3) with Ω replaced by the annulus $B_3 \setminus \overline{B}_2$, and z_λ the minimal solution of (1.3) with Ω replaced by B_1. Then Theorem 1.3 infers that $z_\lambda \le v$ in B_1 and $v \le w_\lambda$ in Ω. In a similar fashion as in step 2, we can now use these inequalities to obtain

$$\partial z_\lambda(x_0)/\partial n \ge \partial v(x_0)/\partial n \ge \partial w_\lambda(x_0)/\partial n.$$

By steps 1 and 2 the first and third terms in the above inequality are of the order $-2I_f\sqrt{\lambda}$ as $\lambda \to \infty$. Hence

$$-\partial v(x_0)/\partial n/\sqrt{\lambda} \to 2I_f \quad \text{as } \lambda \to \infty.$$

Moreover, $\partial z_\lambda(x_0)/\partial n$ and $\partial w_\lambda(x_0)/\partial n$ depend only on the size of the balls, and are independent of the position of the balls. Therefore when we move x_0 along $\partial\Omega$, $\partial v(x_0)/\partial n$ is bounded between two quantities which is independent of x_0 and are of the order $-2I_f\sqrt{\lambda}$ as $\lambda \to \infty$. This shows that $-\big(\partial v/\partial n\big)/\sqrt{\lambda} \to 2I_f$ uniformly on $\partial\Omega$ as $\lambda \to \infty$. This conclude step 3 and thus completes the proof of Theorem 2.1. □

Remark 2.2. An inspection of the above proof shows that instead of requiring

$$I_f = \int_0^\infty f(u)du < \infty \text{ and } \lim_{u\to\infty} uf(u) = 0 \text{ (when } N \ge 3), \tag{2.4}$$

it is enough to just ask

$$\lim_{u\to\infty}[F(u) + (1/N - 1/2)uf(u)] = \Delta_f \in (0, \infty). \tag{2.5}$$

Let us see, however, that if (1.2) is satisfied, then (2.5) and (2.4) are equivalent and we must have $\Delta_f = I_f$ when (2.5) holds. That (2.4) implies (2.5) is trivial. To see that (2.5) implies (2.4) when $N \geq 3$, we first observe that $\Delta_f \leq I_f \leq \infty$, due to $F(u) + (1/N - 1/2)f(u) < F(u)$. It suffices to show that $\Delta_f < I_f$ can never happen. Otherwise, we can find two positive constants ϵ and M such that

$$F(M) \geq \Delta_f + 2\epsilon \tag{2.6}$$

and

$$F(u) + (1/N - 1/2)uf(u) \leq \Delta_f + \epsilon,\ \forall u \geq M. \tag{2.7}$$

From (2.7) we deduce

$$(-1/\beta)[u^{-\beta}F(u)]' \leq (\Delta_f + \epsilon)u^{-\beta-1},\ \ \beta = 2N/(N-2) > 2,\ \forall u \geq M.$$

Integrating this inequality, we obtain

$$(-1/\beta)[u^{-\beta}F(u) - M^{-\beta}F(M)] \leq (-1/\beta)(\Delta_f + \epsilon)(u^{-\beta} - M^{-\beta}),\ \forall u \geq M.$$

Now we use (2.6) and obtain

$$F(u) \geq (\Delta_f + \epsilon) + M^{-\beta}[F(M) - \Delta_f - \epsilon]u^{\beta} \geq \Delta_f + \epsilon + M^{-\beta}\epsilon u^{\beta},\ \forall u \geq M.$$

Since $\beta > 2$, this is impossible in view of our assumption in (1.2) that $\overline{\lim}_{u\to\infty} f(u)/u = 0$.

Proof of Theorem 1.1. This follows easily from Theorem 2.1. Indeed, let (μ, u) be a solution of (1.1) with $u \in A_\lambda$ and $\mu = \lambda[\int_\Omega f(u)dx]^p$, then, by Theorem 2.1,

$$\mu = \lambda^{1-p/2}[\lambda^{1/2}\int_\Omega f(u)dx]^p \to \begin{cases} \infty & \text{if } p < 2 \\ (2I_f)|\partial\Omega|^2 & \text{if } p = 2 \\ 0 & \text{if } p > 2 \end{cases}$$

as $\lambda \to \infty$, uniformly for $u \in A_\lambda$. The conclusions of Theorem 1.1 clearly follow from this. □

Let us now see some simple consequences of Theorem 1.1 on existence and nonexistence of solutions to (1.1), and on the global structure of the solution set

$$\tilde{\Sigma} = \{(\mu, u) \in (0, \infty) \times C(\overline{\Omega}) : (\mu, u) \text{ solves } (1.1)\}.$$

Theorem 2.3. *Let f be as in Theorem 2.1. Then the following holds.*

(i) If $p < 2$, then $\tilde{\Sigma}$ has a component connecting (0,0) and (∞, ∞). In particular, (1.1) *has at least one solution u for each $\mu > 0$.*

(ii) If $p = 2$, then $\tilde{\Sigma}$ has a component connecting $(0,0)$ and $(2I_f|\partial\Omega|^2, \infty)$. In particular, (1.1) *has at least one solution u for each $\mu \in (0, 2I_f|\partial\Omega|^2)$. Moreover, there exists $\mu^* \geq 2I_f|\partial\Omega|^2$ such that* (1.1) *has no solution when $\mu > \mu^*$.*

(iii) If $p > 2$, then $\tilde{\Sigma}$ has a component connecting $(0,0)$ and (0,∞). Moreover, there exists μ_* and μ^*, $0 < \mu_* \leq \mu^* < \infty$, such that (1.1) *has at least two solutions when $\mu \in (0, \mu_*)$, at least one solution when $\mu = \mu_*$, and no solution when $\mu > \mu^*$.*

Proof. (i) In this case, by Theorem 1.1, bifurcation from infinity occurs exactly at $\mu = \infty$. We now look at the connected set of solutions to (1.1) given by

$$\tilde{\Sigma}_0 = \{(\mu, u) : \mu = \lambda[\int_\Omega f(u)dx]^p,\ (\lambda, u) \in \Sigma_0\},$$

where Σ_0 is defined in Theorem 1.3. It is evident that this set connects (0,0) (when $\lambda = 0$) and (∞, ∞) when $\lambda \to \infty$.

(ii) Now, $\tilde{\Sigma}_0$ connects (0,0) and $(2I_f|\partial\Omega|^2, \infty)$. To see that there exists some $\mu^* \geq 2I_f|\partial\Omega|^2$ such that (1.1) has no positive solution when $\mu > \mu^*$, we first observe that by Theorem 2.1, there exists $\Lambda > 0$ such that

$$\lambda[\int_\Omega f(u)dx]^p \leq 2I_f|\partial\Omega|^2 + 1 \quad \text{for every } \lambda \geq \Lambda,\ u \in A_\lambda.$$

On the other hand,

$$\lambda[\int_\Omega f(u)dx]^p \leq \Lambda(|\Omega|M)^p,\ \ \forall\lambda \in [0, \Lambda],$$

where $M = \max_{0 \leq u \leq \|U_\Lambda\|_\infty} f(u)$. Therefore, we can choose

$$\mu^* = \max\{\Lambda(|\Omega|M)^p, 2I_f|\partial\Omega|^2 + 1\}.$$

(iii) We define $\mu_* = \sup\{\mu : (\mu, u) \in \tilde{\Sigma}_0\}$, which is achieved due to the local compactness of the solution set. The existence of μ^* can be argued as in case (ii) above, with $2I_f|\partial\Omega|^2$ replaced by 0 now. □

Remark 2.4. The conclusions in Theorem 2.3 can be considerably improved if we know that the solution set Σ of the local problem (1.3) is a smooth curve, which is the case if Ω is a ball, or when $f(u)$ is decreasing in u.

Let us now consider the case that (1.6) is satisfied by f.

Theorem 2.5. *If f satisfies* (1.6), *then all the conclusions of Theorems 1.1 and 2.3 hold except that I_f should be replaced by $J_f = \int_0^a f(u)du$ and "bifurcation from infinity" should be replaced by "bifurcation from $u = a$".*

Proof. This follows exactly as in the case that (1.2) is satisfied except that now any $u \in A_\lambda$ converges to a locally uniformly in Ω as $\lambda \to \infty$. As a result, $F(u) \to J_f$ and $uf(u) \to 0$ locally uniformly in Ω as $\lambda \to \infty$, $u \in A_\lambda$. It follows that (2.2) holds with I_f replaced by J_f if $N \geq 2$. The case $N = 1$ follows from an obvious modification of the proof of Theorem 3.4 in [3]. □

3. Nonintegrable nonlinearities

In this section, we consider nonlinearities f which satisfy (1.2) but with $\int_0^\infty f(u)du = \infty$. As will become clear in this section, this case is more difficult to handle. Therefore we consider a typical subclass of these functions, that is those satisfying (1.4), namely, for some constants α and c, $-1 < \alpha < 1$, $c > 0$,

$$\lim_{u\to\infty} f(u)/u^\alpha = c.$$

Let us recall that, for each $\lambda > 0$, A_λ denotes the set of solutions u to (1.3).

Theorem 3.1. *Suppose f satisfies* (1.2) *and* (1.4) *except that here $\alpha \leq -1$ is also allowed. Then*

$$\lim_{\lambda\to\infty} \lambda^{1/(\alpha-1)} u(x) = U(x) \tag{3.1}$$

locally uniformly for $x \in \Omega$, and uniformly for $u \in A_\lambda$, where U is the unique positive solution of

$$-\Delta U = cU^\alpha, \quad U|_{\partial\Omega} = 0. \tag{3.2}$$

Remark 3.2. It is well-known that equation (3.2) has a unique positive solution when $-\infty < \alpha < 1$, and $U \in C^2(\overline{\Omega})$ when $0 \leq \alpha < 1$, $U \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ when $-1 < \alpha < 0$, $U \in C(\overline{\Omega}) \cap C^2(\Omega)$ when $\alpha \leq -1$. We refer to [5, 15, 16] and the references therein for more details. When $\alpha = 0$, Theorem 3.1 follows from [8, Theorem 1]; see also the remarks at the end of section 1 of [8] for the case $0 < \alpha < 1$.

Proof of Theorem 3.1. From our assumptions on f, we can find constants m and M, $0 < m < M$, such that

$$m(1+u)^\alpha \leq f(u) \leq M(1+u)^\alpha, \quad \forall u \geq 0.$$

The function $u \to (1+u)^\alpha$ is decreasing when $\alpha \leq 0$, and when $0 < \alpha < 1$, the function $u \to (1+u)^\alpha/u$ is decreasing. It follows from this observation and Theorem 1.3 that, for every $\lambda > 0$, the following problems

$$-\Delta v = \lambda m(1+v)^\alpha, \ \ v|_{\partial\Omega} = 0 \tag{3.3}$$

and

$$-\Delta w = \lambda M(1+w)^\alpha, \ \ w|_{\partial\Omega} = 0 \tag{3.4}$$

have unique positive solutions, which we denote by v_λ and w_λ, respectively. Clearly, u is an upper solution to (3.3) and a lower solution to (3.4). From the proof of Theorem 1.3, one knows that (3.3) has lower solutions smaller than u and (3.4) has upper solutions bigger than u. It follows that

$$v_\lambda \le u \le w_\lambda, \ \ \forall x \in \Omega, \ \forall u \in A_\lambda. \tag{3.5}$$

Let $V_\lambda = \lambda^{1/(\alpha-1)}(1+v_\lambda)$ and $W_\lambda = \lambda^{1/(\alpha-1)}(1+w_\lambda)$. One readily sees that V_λ satisfies

$$-\Delta Z = \beta Z^\alpha, \ Z|_{\partial\Omega} = \gamma \tag{3.6}$$

with $\beta = m$ and $\gamma = \lambda^{1/(\alpha-1)}$, while W_λ satisfies this equation with $\beta = M$ and $\gamma = \lambda^{1/(\alpha-1)}$. Let us observe that when $\alpha < 0$, by [5, Theorem 1.1] and its proof (mainly Lemma 1.3 there), (3.6) has a unique positive solution Z_γ^β for each $\beta > 0$ and $\gamma \ge 0$, and it decreases with γ, moreover, Z_γ^β varies continuously with γ and converges to Z_0^β in $C(\overline{\Omega})$ as $\gamma \to 0+$. When $0 \le \alpha < 1$, these conclusions remain true with a much simpler proof (uniqueness follows from the concavity of the function $Z \to \beta Z^\alpha$, and existence follows from a simple upper and lower solution argument, while continuity with respect to γ follows from a regularity and compactness consideration). It follows that $V_\lambda \to Z_0^m$ and $W_\lambda \to Z_0^M$ as $\lambda \to \infty$ in $C(\overline{\Omega})$. From (3.5), we find

$$V_\lambda - \lambda^{1/(\alpha-1)} \le \lambda^{1/(\alpha-1)} u \le W_\lambda, \ \forall x \in \Omega, \ \forall \lambda > 0, \ \forall u \in A_\lambda. \tag{3.7}$$

Using these inequalities, we see that for any sequence of positive numbers $\lambda_n \to \infty$, any $u_n \in A_{\lambda_n}$, and any subdomain K of Ω satisfying $\overline{K} \subset \Omega$, we can find a positive constant $\delta = \delta(K)$ such that

$$\delta \le V_0 - \lambda_n^{1/(\alpha-1)} \le U_n = \lambda_n^{1/(\alpha-1)} u_n \le W_{\lambda_n} \le \delta^{-1}, \ \ \forall x \in K, n \text{ large}.$$

The function $U_n = \lambda_n^{1/(\alpha-1)} u_n$ also satisfies

$$-\Delta U_n = \frac{f(u_n)}{u_n^\alpha} U_n^\alpha, \ \ U_n|_{\partial\Omega} = 0. \tag{3.8}$$

Since $u_n \to \infty$ uniformly on K, $f(u_n)/u_n^\alpha \to c$ as $n \to \infty$. Thus the right hand side of the first identity in (3.8) has an L^∞ bound on K independent of n. By standard L^p theory for elliptic operators, we conclude that a subsequence of U_n, say $U_{n^{(1)}}$, converges in $C(K)$ to some function $U^{(1)}$ which satisfies

$$-\Delta U^{(1)} = c(U^{(1)})^\alpha \tag{3.9}$$

on K. Let us also note that, by (3.7),

$$Z_0^m \leq U^{(1)} \leq Z_0^M \tag{3.10}$$

on K. We now choose an increasing sequence of subdomains $K = K_1 \subset K_2 \subset \ldots \subset\subset \Omega$ with $\cup K_n = \Omega$. Applying our above argument to K_2 we can find from the subsequence $U_{n^{(1)}}$ a further subsequence, say $U_{n^{(2)}}$, which converges in $C(K_2)$ to some $U^{(2)}$ that satisfies (3.9) and (3.10) on K_2. Evidently $U^{(2)} = U^{(1)}$ on K_1. Continue in this fashion for each member K_n in the sequence of subdomains, and we finally obtain, by a standard diagonal procedure, a subsequence of U_n, say U_{n_k}, which converges in $C_{loc}(\Omega)$ to a function U satisfying (3.9) and (3.10) on Ω. But (3.10) implies that $\lim_{x\to\partial\Omega} U(x) = 0$. Hence U must be the unique positive solution of (3.2), and therefore, the whole original sequence U_n converges to U in $C_{loc}(\Omega)$. As λ_n is an arbitrary sequence converging to ∞, and also $u_n \in A_{\lambda_n}$ is arbitrary, we conclude that for any compact subset K of Ω, as $\lambda \to \infty$, $\lambda^{1/(\alpha-1)}u$ converges in $C(K)$ to U, uniformly for $u \in A_\lambda$. This finishes our proof. □

Theorem 3.3. *Suppose that f satisfies* (1.2) *and* (1.4). *Then*

$$\lim_{\lambda\to\infty} \lambda^{\alpha/(\alpha-1)} \int_\Omega f(u)dx = c\int_\Omega U^\alpha dx < \infty, \tag{3.11}$$

uniformly in $u \in A_\lambda$, where U is the unique positive solution to (3.2).

Proof. On any compact subset K of Ω, by Theorem 3.1, we have

$$\lambda^{\alpha/(\alpha-1)} f(u(x)) = \frac{f(u(x))}{u(x)^\alpha}[\lambda^{1/(\alpha-1)}u(x)]^\alpha \to cU(x)^\alpha$$

as $\lambda \to \infty$, uniformly for $x \in K$ and $u \in A_\lambda$. It follows that

$$\lambda^{\alpha/(\alpha-1)} \int_K f(u)dx \to c\int_K U^\alpha dx. \tag{3.12}$$

Using the notations in the proof of Theorem 3.1 we have

$$\lambda^{\alpha/(\alpha-1)} f(u) \leq \lambda^{\alpha/(\alpha-1)} M(1+u)^\alpha$$

$$\leq \begin{cases} MW_\lambda^\alpha \leq M(Z_1^M)^\alpha, & \forall\lambda > 1, 0 \leq \alpha < 1, \\ MV_\lambda^\alpha \leq M(Z_0^m)^\alpha, & \forall\lambda > 0, -1 < \alpha < 0. \end{cases}$$

If $\alpha \geq 0$, then both Z_1^M and U are in $C^2(\overline{\Omega})$, and the above inequality implies that

$$\int_{\Omega\setminus K} \lambda^{\alpha/(\alpha-1)} f(u)dx \leq M \int_{\Omega\setminus K} (Z_1^M)^\alpha dx \to 0$$

as $|\Omega \setminus K| \to 0$, uniformly for $u \in A_\lambda$ and $\lambda > 1$. It follows that when we let $K \to \Omega$ in the sense that $|\Omega \setminus K| \to 0$ in (3.12), then we obtain (3.11).

If $-1 < \alpha < 0$, then we must be careful as U^α and $(Z_0^m)^\alpha$ become unbounded near $\partial\Omega$. However, in this case, we can find a positive constant δ such that $U^\alpha \geq \delta$ and $(Z_0^m)^\alpha \geq \delta$ in Ω. It follows that $U \geq \psi$, $Z_0^m \geq \psi$ in Ω, where $-\Delta\psi = \delta$, $\psi|_{\partial\Omega} = 0$. Since $\partial\psi/\partial n < 0$ on $\partial\Omega$, we can find some positive constant ξ such that $\psi(x) \geq \xi d(x, \partial\Omega)$. Therefore, using $\alpha > -1$, we obtain

$$\int_\Omega U^\alpha dx \leq \int_\Omega \psi^\alpha dx < \infty,$$

and as $|\Omega \setminus K| \to 0$,

$$\int_{\Omega\setminus K} U^\alpha dx \leq \int_{\Omega\setminus K} \psi^\alpha dx \to 0,$$

$$\int_{\Omega\setminus K} \lambda^{\alpha/(\alpha-1)} f(u)dx \leq M \int_{\Omega\setminus K} (Z_0^m)^\alpha dx \leq M \int_{\Omega\setminus K} \psi^\alpha dx \to 0,$$

uniformly for $\lambda > 0$ and $u \in A_\lambda$. Now we can obtain (3.11) by letting $K \to \Omega$ in (3.12) as before. This completes the proof. □

Remark 3.4. Let us note that $\int_\Omega U^\alpha dx = \infty$ when $\alpha \leq -1$, which implies that our method here is difficult to use for the integrable nonlinearities covered by section 2. The divergence of the integral is a consequence of Theorem 2.1 of [16], which states

$$c_1[D - \ln d(x)]^{1/2} d(x) \leq U(x) \leq c_2[D - \ln d(x)]^{1/2} d(x),$$

$\forall x \in \Omega$, when $\alpha = -1$, and

$$c_1 d(x)^{2/(1-\alpha)} \leq U(x) \leq c_2 d(x)^{2/(1-\alpha)},\ \forall x \in \Omega, \quad \text{when } \alpha < -1,$$

where c_1, c_2 and D are positive constants, $d(x) = d(x, \partial\Omega)$.

Proof of Theorem 1.2. Let (μ, u) be a solution to (1.1) with $u \in A_\lambda$ and $\mu = \lambda[\int_\Omega f(u)dx]^p$. By Theorem 3.3, we have, as $\lambda \to \infty$,

$$\mu = \lambda^{1-p\alpha/(\alpha-1)} \Big[\lambda^{\alpha/(\alpha-1)} \int_\Omega f(u)dx\Big]^p$$

$$\to \begin{cases} 0, & \text{if } p\alpha/(\alpha-1) > 1, \\ \left(c\int_\Omega U^\alpha dx\right)^p, & \text{if } p\alpha/(\alpha-1) = 1, \\ \infty, & \text{if } p\alpha/(\alpha-1) < 1, \end{cases}$$

uniformly in $u \in A_\lambda$. Theorem 1.2 thus follows. □

Our next result deals with the borderline case that (1.4) is satisfied with $\alpha = -1$.

Theorem 3.5. *Suppose f satisfies* (1.2) *and* $\lim_{u\to\infty} uf(u) = c > 0$. *Then bifurcation from infinity for* (1.1) *occurs exactly at $\mu = 0$ when $p \geq 2$, and it occurs exactly at $\mu = \infty$ when $p < 2$.*

Proof. Using the notations in the proof of Theorem 3.1 with $\alpha = -1$, we have, for $u \in A_\lambda$,

$$\lambda^{1/2} f(u) \geq \lambda^{1/2} m(1+u)^{-1} \geq mW_\lambda^{-1} = m(Z^M_{\lambda^{-1/2}})^{-1}.$$

In the proof of Theorem 3.1, we have shown that Z^β_γ converges to Z^β_0 uniformly on $\overline{\Omega}$ as $\gamma \to 0$ (which is positive in Ω). Therefore, for any compact subset K of Ω,

$$\underline{\lim}_{\lambda\to\infty} \int_\Omega \lambda^{1/2} f(u)dx \geq \lim_{\lambda\to\infty} \int_K m(Z^M_{\lambda^{-1/2}})^{-1}dx = m\int_K (Z^M_0)^{-1}dx,$$

uniformly for $u \in A_\lambda$. By Remark 3.4, we know that $\int_\Omega (Z^M_0)^{-1}dx = \infty$. Hence $\sup_{K\subset\subset\Omega} \int_K (Z^M_0)^{-1}dx = \infty$. It follows that

$$\lim_{\lambda\to\infty} \int_\Omega \lambda^{1/2} f(u)dx = \infty, \tag{3.13}$$

uniformly for $u \in A_\lambda$.

On the other hand, for any $\eta \in (-1, 0)$ and $u \in A_\lambda$,

$$\lambda^{\eta/(\eta-1)} f(u) \leq \lambda^{\eta/(\eta-1)} M(1+u)^{-1} \leq \lambda^{\eta/(\eta-1)} M(1+u)^\eta.$$

We now follow the argument in the proof of Theorem 3.1 and use the notation there with $\alpha = \eta$. We deduce

$$\lambda^{\eta/(\eta-1)} f(u) \leq M(Z^m_0)^\eta,\ \forall\lambda > 0,\ \forall u \in A_\lambda.$$

Since Z^m_0 satisfies (3.6) with $\beta = m$ and $\alpha = \eta > -1$, it follows that

$$\lambda^{\eta/(\eta-1)} \int_\Omega f(u)dx \leq M\int_\Omega (Z^m_0)^\eta dx < \infty,\ \forall\lambda > 0,\ \forall u \in A_\lambda. \tag{3.14}$$

Suppose now $p \leq 2$. Then from (3.13) we obtain

$$\mu = \lambda^{1-p/2}[\lambda^{1/2}\int_\Omega f(u)dx]^p \to \infty, \tag{3.15}$$

as $\lambda \to \infty$, uniformly for $u \in A_\lambda$.

If $p > 2$, then we can find $\eta > -1$ such that $p\eta/(\eta-1) > 1$. Thus, by (3.14),

$$\mu = \lambda^{1-p\eta/(\eta-1)}[\lambda^{\eta/(\eta-1)} \int_\Omega f(u)dx]^p \to 0, \tag{3.16}$$

as $\lambda \to \infty$, uniformly for $u \in A_\lambda$.

The conclusions of Theorem 3.5 clearly follow from (3.15) and (3.16). □

Some existence and nonexistence results follow immediately from Theorem 1.2 and Theorem 3.5, in the fashion that Theorem 2.3 follows from Theorem 1.1. We leave the details to the interested reader.

Remark 3.6. All our results in this section hold true if we replace the Laplace operator Δ by a general second order uniformly elliptic operator of the form $Lu = \Sigma a_{ij}(x)u_{x_ix_j} + \Sigma b_i(x)u_{x_i} + c(x)u$, with smooth enough coefficients. Moreover, we can also replace $f(u)$ by $f(x,u)$ with $f(x,u)$ satisfying (1.2) uniformly for $x \in \Omega$, and instead of (1.4), $f(x,u)$ satisfies

$$\lim_{u\to\infty} f(x,u)/u^\alpha = k(x)$$

uniformly for $x \in \Omega$, where $k(x)$ is continuous and positive on $\overline{\Omega}$. Then the function U in (3.1) and (3.11) should be replaced by the unique positive solution of

$$-LU = k(x)U^\alpha, \;\; U|_{\partial\Omega} = 0.$$

We can also allow $k(x)$ to vanish and then our theory here can also cover cases where $\alpha \le -1$, for example, when $k(x)/d(x,\partial\Omega)^\beta$ is uniformly bounded on Ω, with $\alpha + \beta > -1$. We refer to [5, 15, 16] for results needed for this new limiting problem.

4. The local problem

We consider the local problem (1.3) and prove Theorem 1.3 in this section. As can be seen from the following proof, the results of Theorem 1.3 hold if Δ is replaced by L as in Remark 3.5, and $f(u)$ is replaced by $f(x,u)$ with $f(x,u)$ satisfying (1.2) or (1.6) uniformly for $x \in \Omega$.

Proof of Theorem 1.3. We only prove for the case that (1.2) is satisfied. When (1.6) is satisfied, the proof is similar and easier; moreover, by [Da3, Theorem 2], if we assume further that f is C^1 and $f'(u) \le 0$ for $u \le a$ but close to a, then $u_\lambda = U_\lambda$ for large λ (see also [2]).

(i) This follows from a standard upper and lower solution argument. Let $\lambda > 0$ be fixed.

Since $f(u)/u \to 0$ as $u \to \infty$, for any given $\epsilon > 0$, we can find $L_\epsilon > 0$ such that $0 < f(u)/u < \epsilon$ for $u \geq L_\epsilon$. Thus

$$f(u) \leq M_\epsilon + \epsilon u, \quad M_\epsilon = \max_{0 \leq u \leq L_\epsilon} f(u), \quad \forall u \geq 0.$$

Let ϕ be the unique solution to

$$-\Delta \phi = 1, \quad \phi|_{\partial\Omega} = 0.$$

Then $\phi > 0$ in Ω and for all large α,

$$-\Delta(\alpha\phi) = \alpha \geq \lambda(M_\epsilon + \epsilon\|\phi\|_\infty \alpha) \geq \lambda f(\alpha\phi)$$

if we fix ϵ such that $\lambda\epsilon\|\phi\|_\infty < 1$. This shows that $\alpha\phi$ is an upper solution to (1.3) for all large α, say $\alpha \geq \alpha_0 = \alpha_0(\lambda)$. Clearly 0 is a lower solution to (1.3). Thus (1.3) has at least one positive solution.

Let us note that any positive solution u of (1.3) satisfies $u \leq \alpha_0\phi$. This follows from a device known as Serrin's sweeping principle (see [25]), which in our situation here can be stated as follows.

Serrin's sweeping principle. Let w_α be a family of upper solutions to (1.3) which varies continuously with the parameter α for α belonging to some finite or infinite interval I, and none of them is a solution to (1.3). Suppose that u is a positive solution of (1.3) and $u \leq w_{\alpha_0}$ for some $\alpha_0 \in I$. Then $u \leq w_\alpha$ for all $\alpha \in I$. An analogue for lower solutions also hold.

Choose $M > 0$ large such that $\lambda f(u) + Mu$ is increasing in u on the interval $[0, \alpha_0\|\phi\|_\infty]$, and consider the iterations

$$-\Delta u_n + Mu_n = \lambda f(u_{n-1}) + Mu_{n-1}, \quad u_n|_{\partial\Omega} = 0, \ u_0 = \alpha_0\phi.$$

$$-\Delta v_n + Mv_n = \lambda f(v_{n-1}) + Mv_{n-1}, \quad v_n|_{\partial\Omega} = 0, \ v_0 = 0.$$

One easily checks that as $n \to \infty$, u_n decreases to the maximal positive solution of (1.3) and v_n increases to the minimal positive solution of (1.3). This proves (i).

(ii) Since $f(u) > 0$, one easily sees that for each $n \geq 1$, the functions u_n and v_n defined in the iteration procedures in (i) increase with λ. Hence their limits, U_λ and u_λ increase with λ.

Given any $\alpha > 0$, let $m_\alpha = \min_{x \in \overline{\Omega}, 0 \leq s \leq \alpha} f(s\phi(x))$, where ϕ is defined as in (i) above. Then $m_\alpha > 0$ and whenever $\lambda > \alpha/m_\alpha$, we have

$$-\Delta(s\phi) = s \leq \lambda f(s\phi), \quad \forall s \in (0, \alpha], \ \forall x \in \Omega.$$

Therefore $s\phi$ is a lower solution to (1.3) for every $0 < s \leq \alpha$. By Serrin's sweeping principle, we deduce $u_\lambda \geq \alpha\phi$ whenever $\lambda > \alpha/m_\alpha$. This implies that $u_\lambda \to \infty$ as $\lambda \to \infty$ locally uniformly in Ω.

(iii) Let $\tilde{\phi}$ denote ϕ when Ω is replaced by $\tilde{\Omega}$. Then clearly $\tilde{\phi} \geq \phi$ on Ω. If we use $\tilde{u}_n$ and $\tilde{v}_n$ to denote the iteration sequences as in (i) when Ω is replaced by $\tilde{\Omega}$, it is easily seen that $u_n \leq \tilde{u}_n$ and $v_n \leq \tilde{v}_n$ on Ω. The conclusion in this part of Theorem 1.3 now follows immediately.

(iv) The first conclusion follows from Theorem 3.7 of [24] (see also [6] and [1]). That $(\lambda, u_\lambda) \in \Sigma_0$ follows from Du [9, Theorem 2.5] except that a more restrictive condition on $f(u)$ is required there, namely, f is C^1 and $f'(u)$ is bounded from below on $[0, \infty)$. However, an inspection of the proof of Theorem 4.1 in [9] shows that f is locally Lipschitz is enough. For convenience of the reader, let us briefly recall this proof for our special case here.

Fix $\lambda_0 > 0$ and consider the order interval $K = [0, u_{\lambda_0}] = \{u \in E = C(\overline{\Omega}) : 0 \leq u \leq u_{\lambda_0}\}$. This is a closed convex set in E. Let $M > 0$ be a constant such that $\lambda f(u) + Mu$ is increasing in the interval $[0, \|u_{\lambda_0}\|_\infty + 1]$. Then equation (1.3) is equivalent to the abstract equation $u = A(\lambda, u)$ where $A(\lambda, u) = (-\Delta + M)^{-1}(\lambda f(u) + Mu)$. It is easily checked that A maps $[0, \lambda_0] \times K$ into K and is completely continuous. When $\lambda = 0$, $A(0, u) = u$ has a unique solution in K, that is $u = 0$. When $\lambda = \lambda_0$, $A(\lambda_0, u) = u$ also has a unique solution in K, that is $u = u_{\lambda_0}$, due to our choice of K. Moreover, it is easy to show that the fixed point index of $A(0, \cdot)$ on the convex set K at $u = 0$ is 1. Now it follows from a general topological result (see, Nussbaum [22, Lemma 3.4] and also [9]) that the points $(0, 0)$ and $(\lambda_0, u_{\lambda_0})$ belong to a component of the solution set $\{(\lambda, u) \in [0, \lambda_0] \times K : u = A(\lambda, u)\}$. In particular, $(\lambda_0, u_{\lambda_0}) \in \Sigma_0$. $\square$

Remark 4.1. The extra information that $(\lambda, u_\lambda) \in \Sigma_0$ is useful. For example, if we can only prove (2.1) to hold for the minimal solution $u = u_\lambda$, then all the existence results in Theorem 2.3 are still valid, because we still have some control of the continuum $\tilde{\Sigma}_0$ through (2.1) for u_λ. On the other hand, if we only know (2.1) to hold for the maximal solution $u = U_\lambda$, then we do not have control of $\tilde{\Sigma}_0$ anymore, as it is unclear whether (λ, U_λ) belongs to Σ_0. Indeed, there are counterexamples with nonnegative nonlinearities where the maximal positive solutions and the minimal positive solutions belong to different solution branches (see, e.g.,[17]), and for positive nonlinearities, Dancer [7] has counterexamples where the solution set is not connected. As a result, we cannot conclude that the existence results in Theorem 2.3 still hold. This shows that the proof of the existence results in Theorem 2.2 part 1 of [3] contains a gap.

Remark 4.2. The fact that the minimal solutions belong to the global bifurcation branch holds in many situations. We refer to [12], [10] and [11] for discussions and applications of this fact in systems of equations.

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