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LOWER BOUNDS FOR THE L^2 MINIMAL PERIODIC BLOW-UP SOLUTIONS OF CRITICAL NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We consider the nonlinear Schrödinger equation with critical power $iu_t = -\Delta u - |u|^{4/N} u$, $t \geq 0$ and $x \in \mathbb{T}^N$ (the space-periodic case) in H^1. We consider a blow-up solution with minimal mass. We obtain in this context an optimal lower bound for the blow-up rate (that is, for $|\nabla u(t)|_{L^2}$), and we observe that this lower bound equals the blow-up rate (which is explicitly known) of the minimal blow-up solutions in $\mathbb{R}^N$.

1. Introduction

We consider the critical nonlinear Schrödinger equation (NLS)

$$iu_t = -\Delta u - |u|^{4/N} u, \quad t \geq 0, \ x \in \mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N, \qquad (1)$$
$$u(0) = u_0 \in H^1(\mathbb{T}^N).$$

This equation is a particular case of

$$iu_t = -\Delta u - |u|^{p-1} u, \quad t \geq 0, \ x \in \mathbb{T}^N \qquad (1_p)$$

Let us recall the known results in the case $x \in \mathbb{R}^N$.

Ginibre and Velo proved in [2] that for any $p < \frac{N+2}{N-2}$, equation (1_p) is locally well-posed in $H^1(\mathbb{R}^N)$ in the following sense: for any u_0 in $H^1(\mathbb{R}^N)$ there exist a unique $T > 0$ and a solution $u \in \mathcal{C}([0,T[, H^1)$ such that either $T = +\infty$, or $T < +\infty$ and $\lim_{t\to T} |u(t)|_{H^1} = +\infty$.

The following quantities are conserved:

$$\text{Mass: } \forall t \geq 0, \ \int |u(t)|^2 = \int |u_0|^2 \qquad (2)$$

$$\text{Energy: } \forall t \geq 0, \ E(u(t)) = \frac{1}{2}\int |\nabla u(t)|^2 - \frac{1}{p+1}\int |u|^{p+1} = E(u_0) \qquad (3)$$

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$$\text{Linear Momentum: } \forall t \geq 0,\ Im \int \bar{u}(t)\nabla u(t) = Im \int \bar{u}_0 \nabla u_0 \tag{4}$$

$$\text{Viriel law } \big(p = 1 + \frac{4}{N}\big) : \text{if } \ xu_0(x) \in L^2(\mathbb{R}^N), \text{ then} \tag{5}$$
$$\frac{d^2}{dt^2} \int_{\mathbb{R}^N} |u(t,x)|^2 |x|^2\, dx = 16E(u_0).$$

The exponent $p = 1 + \frac{4}{N}$ is critical in the following sense: it is well-known that if $p < 1 + \frac{4}{N}$, solutions of (1_p) are globally defined (that is $T = +\infty$), whereas blow-up may occur from (5) (i.e. one may have $T < +\infty$) for $p \geq 1 + \frac{4}{N}$. We now focus on the case $p = 1 + \frac{4}{N}$.

There is an additional structure for $p = 1 + \frac{4}{N}$: the Schrödinger equation (1), then has a conformal invariance. If $u(t,x)$ is a solution of (1), then

$$\frac{1}{(T-t)^{N/2}} \exp\Big[-\frac{i|x|^2}{4(T-t)}\Big] u\Big(\frac{1}{T-t}, \frac{x}{T-t}\Big), \quad \text{for } t < T$$

is also a solution of (1).

Consider the unique solution Q of the following equation:

$$v = \Delta v + |v|^{4/N} v \quad (6'), \quad x \in \mathbb{R}^N, \quad Q = Q(|x|), \quad Q > 0. \tag{6}$$

Uniqueness of such a solution Q has been proved by Kwong in [4]. Note that $u(t,x) = e^{it}Q(x)$ is a solution of (1). $Q \neq 0$ is the solution of $(6')$ with minimal mass (that is, if $v \neq 0$ is solution of (6'), then $\int v^2 \geq \int Q^2$).

Weinstein proved in [13] the following Gagliardo-Nirenberg's type inequality: $\forall v \in H^1(\mathbb{R}^N)$, one has

$$\frac{1}{4/N + 2} \int |v|^{4/N+2} \leq \frac{1}{2} \Big(\frac{\int |v|^2}{\int Q^2}\Big)^{2/N} \int |\nabla v|^2. \tag{7}$$

Using the conservation of the mass and of the energy and (7), one can prove that if $\int_{\mathbb{R}^N} |u_0|^2 < \int_{\mathbb{R}^N} Q^2$, then $\int |\nabla u(t)|^2$ is uniformly bounded in time and $u(t)$ is globally defined.

Applying the conformal transformation to the solution $e^{it}Q(x)$ of equation (1), one can find a blow-up solution S of NLS with mass $\int |Q|^2$: $S(t,x) = \frac{1}{(T-t)^{N/2}} e^{-i|x|^2/4(T-t) + i/(T-t)} Q(\frac{x}{T-t})$. Thus $\int_{\mathbb{R}^N} Q^2$ is a critical mass for the blow-up problem in that sense.

Note that the blow-up rate of the minimal blow-up solutions (that is, $u(t)$ is a blow-up solution and $|u(t)|_{L^2} = |Q|_{L^2}$) is explicitly known. Indeed, Merle proved in [7] that if u is such a solution, and up to the invariants of the equation,

$$u(t,x) = S(t,x) = \frac{1}{(T-t)^{N/2}} e^{-i|x|^2/4(T-t)+i/(T-t)} Q\Big(\frac{x}{T-t}\Big),$$

and hence

$$|\nabla u(t)|_{L^2} \sim \frac{|\nabla Q|_{L^2}}{T-t}.$$

Note that for a greater mass ($|u(t)|_{L^2} > |Q|_{L^2}$) there exist blow-up solutions with the lower blow-up rate $|\nabla u(t)|_{L^2} \sim \frac{1}{\sqrt{T-t}} \log|\log(T-t)|$. This fact has been proved for $N = 1$ by Perelman in [12] and has been observed numerically for $N = 2$ by Landman, Papanicolau, C. Sulem and P. L. Sulem in [5]. Also note that the proof of the local well-posedness of (1) in H^1 leads to the following lower bound: $|\nabla u(t)|_{L^2} \geq \frac{c}{\sqrt{T-t}}$ for a solution $u(t)$ of (1) blowing-up at time T.

Finally, Merle proved in [6] that the self-similar blow-up solution $S(t)$ can be extended weakly in L^2 in some sense after the blow-up time, and these solutions are stable with respect to the initial data (and belong to H^1). But a non-uniqueness phenomenon arises at the blow-up time: the extension of the solution is only unique up to an arbitrary phase parameter $e^{i\theta}$ (there is a loss of information). Let us now claim several facts for the space-periodic solutions of (1) ($x \in \mathbb{T}^N$).

Bourgain proved in [1] that the Cauchy problem for equation (1) is locally well-posed in $H^1(\mathbb{T}^N)$ in the same sense as in $H^1(\mathbb{R}^N)$ in dimensions $N = 1, 2$. It is expected to be true in large dimension, and we will assume this fact, but it is an open problem.

The three conservation laws (2), (3) and (4) remain true for $u_0 \in H^1(\mathbb{T}^N)$. As in the $\mathbb{R}^N$ case, using the conservation of the mass and of the energy and Gagliardo-Nirenberg's type inequalities, one can prove that if $p < 1 + \frac{4}{N}$, (1_p) is globally well-posed in $H^1(\mathbb{T}^N)$ as soon as this equation is locally well-posed. In what follows, we will always assume $p = 1 + 4/N$ (critical exponent). We shall see that in the space-periodic case, $\int_{\mathbb{R}^N} Q^2$ keeps its role of minimal mass: we will prove that no blow-up can occur for $\int_{\mathbb{T}^N} |u_0|^2 < \int_{\mathbb{R}^N} Q^2$, and there exist blow-up solutions $u(t)$ such that $\int_{\mathbb{T}^N} |u_0|^2 = \int_{\mathbb{R}^N} Q^2$. Existence of such solutions was proved by Ogawa and Tsutsumi in [11] in the case $N = 1$. Indeed, using a perturbation method previously introduced by Merle in [10], they proved the following result (see their Theorem 1.2): There exists a solution $u(t)$ of (1) in $\mathcal{C}([0, T[; H^1(\mathbb{T}))$ for some finite $T > 0$ such that, as $t \to T$, $|\nabla u(t)|_{L^2} = O((T-t)^{-1})$ and $|u(t,x)|^2 \rightharpoonup |Q|^2_{L^2}\, \delta_0$.

The aim of this paper is to study periodic blow-up solutions $u(t)$ with $\int_{\mathbb{T}^N} |u(t)|^2 = \int_{\mathbb{R}^N} Q^2$. Our main result is the following:

Theorem 1. *There exists* $\gamma > 0$ *such that*

$$\forall t \in [0, T[, \quad |\nabla u(t)|_{L^2} \geq \frac{\gamma}{T-t}.$$

Note that the above mentioned result of Ogawa and Tsutsumi for $N = 1$ in [11] implies the optimality of this theorem.

Remark 1.1. Uniqueness of minimal blow-up solutions (up to the invariants of the equation) is unknown, and blow-up solutions of different speed are expected to exist for $\int |u_0|^2 > \int Q^2$.

The paper is organized as follows: Section 2 is devoted to the proof that $u(t)$ concentrates in L^2 around a point $x(t)$ as $t \to T$ with Q as asymptotic profile after rescaling. We also prove that this center of mass $x(t)$ admits a limit x_0 as t approaches the blow-up time T.

In Section 3, we study the localized viriel $g(t)$ of the solution and obtain an upper bound for this quantity. Finally, we will obtain from this bound a lower bound for the blow-up rate $|\nabla u(t)|_{L^2}$ through a minoration of $|\nabla u(t)|_{L^2}^2\, g(t)$.

The main ideas introduced in this paper are the following: using bounds on $u(t)$ far from its concentration point $x(t)$, we obtain *a priori* bounds on $x(t)$ and then we prove the convergence of $x(t)$ (Section 2). Using these bounds again, we obtain a crucial bound for $g''(t)$ (the second order time-derivative of the localized viriel $g(t)$ of $u(t)$) in Section 3. Finally, we apply the result of uniformity of the blow-up proved by Merle in [9] to bound the second order time-derivative of the localized viriel of some solutions of (1) with sub-critical mass (see the appendix).

2. L^2 CONCENTRATION OF THE SOLUTION

2.1. **Preliminary variational results.** As Weinstein in [14] for the $\mathbb{R}^N$-case (his proof was based on the principle of concentration-compactness of Lions), we claim the following result:

Proposition 2.1. *Let* u_n *in* $H^1(\mathbb{T}^N)$ *be a sequence verifying the following hypotheses:*

$\int_{\mathbb{T}^N} |u_n|^2 \leq \int_{\mathbb{R}^N} Q^2$, $\lambda_n^2 = \int_{\mathbb{T}^N} |\nabla u_n|^2 \xrightarrow{n\to\infty} +\infty$, $\exists c_0 \,/\, E(u_n) \leq c_0$.

(*i*) *Then there exist sequences* $x_n \in [-\lambda_n/2, \lambda_n/2]^N$ *and* $e^{i\theta_n} \in \mathbb{S}$ *such that*

$$\tilde{u}_n(x) := \frac{1}{\lambda_n^{N/2}} e^{i\theta_n} u_n\Big(\frac{x - x_n}{\lambda_n}\Big) \rightharpoonup \omega_0^{N/2} Q(\omega_0 x) \text{ as } n \to \infty,$$

where $\omega_0 = \frac{1}{|\nabla Q|_{L^2}}$.

(*ii*) $\int_{\mathbb{T}^N} |u_n|^2 \xrightarrow{n\to\infty} \int_{\mathbb{R}^N} Q^2$.

Corollary 2.1. *Let* $u_0 \in H^1(\mathbb{T}^N)$ *such that* $\int_{\mathbb{T}^N} |u_0|^2 < \int_{\mathbb{R}^N} Q^2$. *Then the solution of* (1) *of initial data* u_0 *is globally defined.*

Proof of Corollary 2.1. Indeed, if $u(t)$ blows up at time T, considering a sequence (t_n) such that $t_n \nearrow T$, we notice that $u_n = u(t_n)$ verifies the hypotheses of proposition 2.1, so a contradiction follows from ii). □

Proof of Proposition 2.1. (ii) is a direct consequence of (i).

(i) Proof follows similar arguments as in $\mathbb{R}^N$. Let u_n in $H^1(\mathbb{T}^N)$ such that

$$\int_{\mathbb{T}^N} |u_n|^2 \leq \int_{\mathbb{R}^N} Q^2, \quad E(u_n) \leq C$$

and $|\nabla u_n|_{L^2} \xrightarrow{n\to\infty} +\infty$. Let us then define

$$\lambda_n = [|\nabla u_n|_{L^2}] \sim |\nabla u_n|_{L^2}$$

($[a]$ stands for the above integer part of the real number a) and then

$$\tilde{u}_n(x) = \frac{1}{\lambda_n^{N/2}} u\left(\frac{x}{\lambda_n}\right).$$

Hence, we have $\tilde{u}_n \in H^1(\mathbb{R}^N/(\lambda_n \mathbb{Z}^N))$,

$$\int_{K_n} |\tilde{u}_n|^2 = \int_{\mathbb{T}^N} |u_n|^2, \quad \int_{K_n} |\nabla \tilde{u}_n|^2 \xrightarrow{n\to\infty} 1$$

(and it remains smaller than 1) and

$$E_{K_n}(\tilde{u}_n) := \frac{1}{2}\int_{K_n} |\nabla \tilde{u}_n|^2 - \frac{1}{4/N+2}\int_{K_n} |\tilde{u}_n|^{4/N+2} = \frac{1}{\lambda_n^2} E(u_n) \leq \frac{c}{\lambda_n^2},$$

where $K_n = \{x \,/\, |x|_\infty \leq \lambda_n/2\}$ stands for the cube of size λ_n centered at 0.

We then claim that

Step 1 (non-vanishing). (i) $\exists y_n, \exists R_1 > 0, \exists m_1, m_2 > 0$ so that

$$\begin{cases} (\alpha) & \liminf_{n\to\infty} \int_{B(y_n,R_1)} |\tilde{u}_n|^{4/N+2} \geq m_1, \\ (\beta) & \liminf_{n\to\infty} \int_{B(y_n,R_1)} |\tilde{u}_n|^2 \geq m_2, \end{cases}$$

(ii) Moreover, this result is true for any sequence $(\tilde{u}_n)$ such that (for some $c_0 > 0$ and (c_1, c_2), $0 < c_1 < c_2$) : $|\tilde{u}_n|_{L^2} \leq c_0$, $c_1 \leq \int |\nabla \tilde{u}_n|^2 \leq c_2$, $c_1 \leq \int |\tilde{u}_n|^{4/N+2} \leq c_2$ and $\limsup_{n\to\infty} E(\tilde{u}_n) = 0$.

Proof. The proof follows similar arguments as Glangetas and Merle in [3].

Since $E_{K_n}(\tilde{u}_n) \leq \frac{c}{\lambda_n^2}$, we can write

$$\int_{K_n} |\tilde{u}_n|^{4/N+2} = (\frac{2}{N}+1)\int_{K_n} |\nabla \tilde{u}_n|^2 - (\frac{4}{N}+2)E_{K_n}(\tilde{u}_n)$$
$$\geq (\frac{2}{N}+1)\int_{K_n} |\nabla \tilde{u}_n|^2 - (\frac{4}{N}+2)\frac{c}{\lambda_n^2} \xrightarrow{n\to\infty} \frac{2}{N}+1 > 1,$$

hence we have for n great enough

$$\int_{K_n} |\tilde{u}|^{4/N+2} \geq 1.$$

But $\int_{K_n}(|\nabla \tilde{u}_n|^2 + \tilde{u}_n^2) \leq 1 + \int Q^2$, so that we can write

$$\int_{K_n} |\tilde{u}_n|^{4/N+2} \geq 1 \geq \frac{1}{1+\int Q^2}\int_{K_n}(|\nabla \tilde{u}_n|^2 + \tilde{u}_n^2). \tag{8}$$

K_n is the union of λ_n^N disjointed cubes $C(y_i)$ centered at y_i of size 1. Hence (8) can be written

$$\sum_{y_i}\int_{C(y_i)} |\tilde{u}_n|^{4/N+2} \geq \sum_{y_i}\frac{1}{1+\int Q^2}\int_{C(y_i)}(|\nabla \tilde{u}_n|^2 + \tilde{u}_n^2),$$

from which we immediately deduce the existence of y_n such that

$$\int_{C(y_n)} |\tilde{u}_n|^{4/N+2} \geq \frac{1}{1+\int Q^2}\int_{C(y_n)}(|\nabla \tilde{u}_n|^2 + \tilde{u}_n^2).$$

We then have

$$\frac{1}{1+\int Q^2}\int_{C(y_n)}(|\nabla \tilde{u}_n|^2+\tilde{u}_n^2) \leq \int_{C(y_n)} |\tilde{u}_n|^{\frac{4}{N}+2} \leq C_1\Big(\int_{C(y_n)}(|\nabla \tilde{u}_n|^2+\tilde{u}_n^2)\Big)^{\frac{2}{N}+1}$$

by Sobolev embedding. Thus we obtain

$$\int_{C(y_n)}(|\nabla \tilde{u}_n|^2 + \tilde{u}_n^2) \geq C_2 = \Big(\frac{1}{C_1(1+\int Q^2)}\Big)^{N/2}$$

and then

$$\int_{C(y_n)} |\tilde{u}_n|^{4/N+2} \geq m_1 = \frac{C_2}{1+\int Q^2}.$$

This proves (α) with $R_1 = 1+\sqrt{n}/2$ (R_1 such that $C(y_n) \subset B(y_n, R_1)$).

To obtain (β), we shall use Weinstein's inequality. In that purpose, let $\xi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ such that $0 \leq \xi \leq 1$, $\xi \equiv 1$ over $C(y_n)$ and

$$supp(\xi) \subset B(y_n, 1+\sqrt{n}/2) = B(y_n, R_1).$$

We then have

$$\begin{aligned} m_1 &\le \int_{C(y_n)} |\tilde{u}|^{4/N+2} \le \int_{\mathbb{R}^N} |\xi\tilde{u}_n|^{4/N+2} \\ &\le (\frac{2}{N}+1)\Big(\frac{\int_{\mathbb{R}^N}|\xi\tilde{u}_n|^2}{\int Q^2}\Big)^{2/N}\int_{\mathbb{R}^N}|\nabla(\xi\tilde{u}_n)|^2 \\ &\le 2(\frac{2}{N}+1)\Big(\frac{\int_{\mathbb{R}^N}|\xi\tilde{u}_n|^2}{\int Q^2}\Big)^{2/N}\Big(\int_{K_n}|\nabla\tilde{u}_n|^2+|\nabla\xi|_\infty^2\int_{K_n}|\tilde{u}_n|^2\Big) \\ &\le C\Big(\int_{B(y_n,R_1)}|\tilde{u}_n|^2\Big)^{2/N}, \end{aligned}$$

from which we deduce

$$\int_{B(y_n,R_1)}|\tilde{u}_n|^2 \ge m_2 = \Big(\frac{C}{m_1}\Big)^{N/2}.$$

This achieves the proof of Step 1. □

We now prove that, after extraction of a sub-sequence, we obtain the **Step 2 (Compacity result).** By similar arguments as Merle in [8], we claim the following:

Lemma 2.1. *There exist* $y_n \in K_n = [-\lambda_n/2, +\lambda_n/2]^N$, $U \in H^1$, $m_2 > 0$, $R_1 > 0$ *such that*

$$\tilde{u}_n(. - y_n) \rightharpoonup U \text{ in } H^1 \text{ as } n \to \infty, \quad \int_{B(0,R_1)}|U|^2 \ge m_2, \quad E(U) \le 0.$$

Proof. (the constant numbers R_1 and m_2 are given by Step 1). This lemma can be proved by induction over the integer k_0 such that $\int|\tilde{u}_n|^2 < k_0 m_2$.

Remark 2.1. There exist $0 < c_1 < c_2$ such that $c_1 \le \int|\nabla\tilde{u}_n|^2 \le c_2$ and $c_1 \le \int|\tilde{u}_n|^{4/N+2} \le c_2$.

* By what precedes, the lemma is proved for $k_0 = 1$.

* Let us now assume that the lemma is true for some k_0 and let us prove it for $k_0 + 1$.

Using part (ii) of Step 1, we know there exists a sequence of points $y_{1,n}$ in K_n such that $\int_{B(y_{1,n},R_1)}|\tilde{u}_n|^2 \ge m_2$. Moreover, there exists $\hat{u} \in H^1 = H^1(\mathbb{R}^N)$ such that $\tilde{u}_n(.-y_{1,n}) \rightharpoonup \hat{u}$ in H^1, after extraction of a sub-sequence. We then decompose $\tilde{u}_n$ as follows: $\tilde{u}_n(. - y_{1,n}) = v_{1,n}(.) + v_{2,n}(.)$, where

(α) $v_{2,n}(x) = 0$ if $|x| \le R_n/2$, $v_{1,n}(x) = 0$ if $|x| \ge R_n$, with $R_n \xrightarrow{n\to\infty} \infty$ and $R_n/\lambda_n \xrightarrow{n\to\infty} 0$,

(β) $\displaystyle\int_{R_n/2\le|x|\le R_n}\left(|\nabla\tilde u_n|^2+|\tilde u_n|^{4/N+2}\right)\xrightarrow{n\to\infty}0,$

($\gamma 1$) $\displaystyle\int\left(|v_{1,n}|^2+|v_{2,n}|^2-|\tilde u_n|^2\right)\xrightarrow{n\to\infty}0,$

($\gamma 2$) $\displaystyle\int\left(|\nabla v_{1,n}|^2+|\nabla v_{2,n}|^2-|\nabla\tilde u_n|^2\right)\xrightarrow{n\to\infty}0,$

($\gamma 3$) $\displaystyle\int\left(|v_{1,n}|^{4/N+2}+|v_{2,n}|^{4/N+2}-|\tilde u_n|^{4/N+2}\right)\xrightarrow{n\to\infty}0,$

(δ) so that : $\lim_n E(v_{1,n})+E(v_{2,n})=0.$

Note that, from the convergence of R_n to infinity, $v_{1,n}\to\hat u$ in $L^2_{loc}(\mathbb{R}^N)$. More particularly, $\int|\hat u|^2\ge\int_{B(0,R_1)}|\hat u|^2\ge m_2$, hence, from ($\gamma 1$),

$$\int|v_{2,n}|^2<k_0m_2\qquad\text{for } n \text{ large enough.}$$

Two cases may then occur:

Case 1: $E(\hat u)\le\lim_n E(v_{1,n})\le 0$. The proposition is proved with $U=\hat u$ and $y_n=y_{1,n}$.

Case 2: $E(\hat u)>0$. Then, $P_1=\lim_n E(v_{1,n})>0$, so that from (δ), $E(v_{2,n})\to -P_1$, which implies

$$E(v_{2,n})\le-\frac{P_1}{2}\qquad\text{for } n \text{ large enough.}$$

From the induction hypothesis, there exists $y_{2,n}$ and $U\in H^1$ so that $v_{2,n}(.-y_{2,n})\rightharpoonup U$ in H^1, with

$$\int|U|^2\ge m_2,\quad E(U)\le 0.$$

With the choice $y_n=y_{1,n}+y_{2,n}$, we then have $\tilde u_n(.-y_n)\rightharpoonup U$ in H^1, where U verifies the required properties. This achieves the proof of the lemma. □

We can now achieve the proof of Proposition 2.1. It is sufficient to prove that U can be written $U(x)=e^{i\theta}\omega^{N/2}Q(\omega(x-x_0))$. But from the convergence of u_n to U, we know that $0<m_2\le|U|_{L^2}\le|Q|_{L^2}$ and Step 2 ensures us that $E(U)\le 0$. From the variational characterization of Q, we can then claim that there exist $\omega>0$, $x_0\in\mathbb{R}^N$ and $e^{i\theta}\in\mathbb{S}$ such that $U(x)=e^{i\theta}\omega^{N/2}Q(\omega(x-x_0))$. Hence, we can choose x_n and θ_n to obtain the result of proposition 2.1. □

2.2. **Concentration result.** Let t_n be a sequence such that $t_n\nearrow T$. Let us define $u_n=u(t_n)$, and then $x'_n=\frac{x_n}{\lambda_n}$ where x_n is defined by Proposition

2.1 so that $\frac{1}{\lambda_n^{N/2}} u_n(\frac{\cdot}{\lambda_n} - x_n) \rightharpoonup \omega_0^{N/2} Q(\omega_0 x)$ in H^1, which implies that

$$|u_n(. - x'_n)|^2 \rightharpoonup |Q|_{L^2}^2 \, \delta_0 \text{ in } \mathcal{D}'(\mathbb{T}^N).$$

From this result, it is then classical that we can claim the existence of a continuous function $x(t) \in \mathbb{T}^N$, such that

$$|u(t, x - x(t))|^2 \rightharpoonup |Q|_{L^2}^2 \, \delta_0.$$

Our purpose now is to prove the

Proposition 2.2. *$x(t)$ admits a limit x_0 as t tends to T, and $|u(t,x)|^2 \rightharpoonup |Q|_{L^2}^2 \, \delta_{x_0}$.*

Proof. The proof of Proposition 2.2 is based on the two following mains ideas: we first deduce from Proposition 2.1 bounds on u far from the point of concentration $x(t)$. Using a localized first order momentum (through bounds on its derivative with respect to time) we then obtain an *a priori* bound on $|x(t)|$. We can then prove the convergence of $x(t)$ using again this first order momentum.

Step 1 (Bounds on $u(t)$ far from $x(t)$). We must start bounding the L^2-norm of $\nabla u(t)$ and the $L^{4/N+2}$-norm of $u(t)$ far from the point $x(t)$:

Lemma 2.2. *For all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\forall t \in [0, T[, \int_{\mathbb{T}^N \setminus B(0,\varepsilon)} |\nabla u(t, x - x(t))|^2 \, dx \leq C_\varepsilon. \tag{9}$$

Moreover, $\forall \varepsilon > 0$,

$$\lim_{t \nearrow T} \int_{\mathbb{T}^N \setminus B(0,\varepsilon)} |u(t, x - x(t))|^{4/N+2} \, dx = 0. \tag{10}$$

Proof. These estimates are based on the following result (see Proposition 2.1 in [7] and the remark therein):

Lemma 2.3. *Let $(v_n)_n$ in $H^1(\mathbb{T}^N)$ such that, for some real number c_0,*

$$\begin{cases} E(v_n) \leq c_0, \; |v_n|_{L^2} \leq |Q|_{L^2}, \; |\nabla v_n|_{L^2} \to +\infty, \\ \exists R_0 \text{ such that } \forall n, \displaystyle\int_{|x| \geq R_0} |v_n(x)|^2 \leq \varepsilon, \end{cases}$$

where ε only depends on N. Then there exists $c_1 > 0$ such that for any n,

$$\int_{|x| \geq 4R_0} |\nabla v_n(x)|^2 \, dx \leq c_1.$$

But Proposition 2.1 implies that $v_n = u_n(t_n, . - x(t_n))$ verifies the hypotheses of Lemma 2.3 (for n greater than some n_0) with any real number R_0, and consequently the conclusion holds for any $4R_0 > 0$, which yields that (9) follows directly from Lemma 2.3. (10) is then obtained by interpolation, since $2+4/N < 2^*$, $u(t, x - x(t))$ is bounded in $H^1(\mathbb{T}^N \setminus B(0,\varepsilon))$ and tends to 0 in $L^2(\mathbb{T}^N \setminus B(0,\varepsilon))$. Hence Lemma 2.2 is proved. □

Step 2 (Localized first order momentum). We shall now use this result to prove the convergence of $x(t)$. We need to define a localized first order momentum of u.

In this purpose, let us define a function $\psi \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$, $\mathbb{Z}^N$-periodic, and such that $\psi(x) = x$ on a neighborhood of 0.

Let $\tilde{\psi} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ be a function verifying the following hypotheses:

$\tilde{\psi}$ is odd and 1-periodic, $\tilde{\psi}(x) = x$ over $[-1/100, 1/100]$, $\tilde{\psi} = 0$ over $[1/2 - 1/100, 1/2]$, $\tilde{\psi}' \leq 1$, $\int_0^{1/2} \tilde{\psi} = 0$ and there exists $0 < a < 1/2$ such that $\tilde{\psi} \geq 0$ on $[0, a]$ and $\tilde{\psi} \leq 0$ on $[a, 1/2]$.

One then defines $\tilde{\varphi}(x) = 2\int_0^x \tilde{\psi}(t)\, dt$, and then

$$\begin{aligned} \varphi(x_1, \ldots, x_N) &= \tilde{\varphi}(x_1) + \cdots + \tilde{\varphi}(x_N) \\ \psi(x_1, \ldots, x_N) &= (\tilde{\psi}(x_1), \ldots, \tilde{\psi}(x_N)) = \frac{1}{2}\nabla\varphi. \end{aligned} \tag{11}$$

Note that this function φ belongs to $\mathcal{C}^\infty(\mathbb{T}^N, \mathbb{R})$, that $\varphi \geq 0$ over $\mathbb{T}^N$, and that $\varphi(x) = |x|^2$, $\psi(x) = x$ over $[-1/100, 1/100]^N \supset B = B_{1/100} = B(0, 1/100)$.

We now study the time evolution of

$$f_{t_0}(t) = \int_{\mathbb{T}^N} |u(t,x)|^2 \psi(x + x(t_0))$$

(on a neighborhood of T^-), t_0 being a time close enough to T. Up to a translation, we assume in a first time that $x(t_0) = 0$.

In the $\mathbb{R}^N$ case, we would have set $\psi(x) = x$ and hence $f_{t_0}(t)$ would be the first order momentum of $u(t)$. But we need to define such a $\mathbb{Z}^N$-periodic function ψ in order to perform integrations by parts. In the $\mathbb{R}^N$ case, $\frac{d}{dt}\int_{\mathbb{R}^N} |u|^2 x$ is a constant vector, but this result no more holds in the $\mathbb{T}^N$ case for this localized momentum $f_{t_0}(t)$. Nevertheless, using Step 1, we obtain a uniform bound for $f'_{t_0}(t)$.

i] On the one hand, there exists M_0 such that

$$|x(t)| \leq \frac{1}{400} \Rightarrow |f'_{t_0}(t)| \leq M_0.$$

Indeed, after computations, we obtain

$$f'_{t_0}(t) = 2Im \int_{\mathbb{T}^N} \bar{u}\nabla u + 2Im \sum_k \Big(\int_{\mathbb{T}^N\setminus B} \bar{u}(\tilde{\psi}'(x_k) - 1)\frac{\partial u}{\partial x_k}\Big)\overrightarrow{e_k},$$

where $(\overrightarrow{e_1}, \dots, \overrightarrow{e_N})$ is the canonical bases of $\mathbb{R}^N$.

But from (4), $\forall t$, $Im \int_{\mathbb{T}^N} \bar{u}(t) = Im \int \bar{u}_0 \nabla u_0$. We deduce from this conservation law that

$$f'_{t_0}(t) = 2Im \int_{\mathbb{T}^N} \bar{u}_0\nabla u_0 + 2Im \sum_{k=1}^{N} \Big(\int_{\mathbb{T}^N\setminus B} \bar{u}(\tilde{\psi}'(x_k) - 1)\frac{\partial u}{\partial x_k}\Big)\overrightarrow{e_k},$$

and hence

$$|f'_{t_0}(t)| \leq 2\Big|\int \bar{u}_0\nabla u_0\Big| + 2\Big(|\tilde{\psi}'|_{L^\infty} + 1\Big)\,|\bar{u}|_{L^2}\,|\nabla u|_{L^2(\mathbb{T}^N\setminus B_{1/100})}\,. \tag{12}$$

From (12) (and (2)), we deduce that $f'_{t_0}(t)$ is bounded as soon as ∇u is bounded in $L^2(\mathbb{T}^N \setminus B)$, which occurs as soon as $x(t) \in B_{1/400}$. Moreover, it clearly appears that the bound M_0 of $|f'_{t_0}(t)|$ only depends on u_0 (but not on the choice of t_0).

ii] On the other hand, we can now obtain the following *a priori* bound: for t_0 close enough to T, one has

$$\forall t \geq t_0,\ |x(t)| \leq \frac{1}{400}.$$

Working by contradiction, assume there exists $t_1 \in]t_0, T[$ such that $|x(t)| = \frac{1}{400}$, and $|x(t)| < \frac{1}{400}$ for $t_0 \leq t < t_1$.

Let us remark that

$$f_{t_0}(t_0) = \int |u(t, x - x(t))|^2 \psi(x)\, dx$$

tends to $|Q|^2_{L^2}\,\psi(0) = \overrightarrow{0}$ as $t \nearrow T$. Hence there exists a function $\delta_0(t_0)$ such that $\delta_0(t_0) \to 0$ as $t_0 \to T$, and

$$\forall t \geq t_0,\ |x(t)| \leq \frac{1}{400} \Rightarrow |f_{t_0}(t) - |Q|^2_{L^2}\,x(t)| \leq \delta_0.$$

We then have

$$\begin{aligned} |Q|^2_{L^2}\,|x(t_1)| &\leq |f_{t_0}(t)| + \delta_0(t_0) \leq |f_{t_0}(t_0)| + \sup_{[t_0,t_1]} |f'_{t_0}|(t_1 - t_0) + \delta_0(t_0) \\ &\leq M_0(T - t_0) + 2\delta_0(t_0) \to 0 \text{ as } t_0 \to T. \end{aligned}$$

Hence, if we fix t_0 close enough to T so that $M_0(T-t_0)+2\delta_0(t_0) \le \frac{|Q|_{L^2}^2}{500}$, we will obtain $\frac{|Q|_{L^2}^2}{400} = |Q|_{L^2}^2 |x(t_1)| \le \frac{|Q|_{L^2}^2}{500}$, which is a contradiction, and the *a priori* bound is proved.

So, we have proved that $|x(t)|$ remains smaller than $1/400$ and hence $f'_{t_0}(t)$ is bounded, so we deduce that $f_{t_0}(t)$ admits a limit as $t \nearrow T$, and we shall note this limit $-|Q|_{L^2}^2 x_0$.

We now claim that $x(t)$ converges to this point x_0

$$\begin{aligned} f_{t_0}(t) + |Q|_{L^2}^2 x(t) &= \int |u(t,x)|^2(\psi(x)+x(t))\,dx \\ &= \int |u(t,x-x(t))|^2(\psi(x-x(t))+x(t))\,dx. \end{aligned}$$

But for $|x| \le \frac{3}{400}$, $|x-x(t)| \le \frac{1}{100}$ and hence $\psi(x-x(t))+x(t) = x = \psi(x)$, so that

$$\begin{aligned} |f_{t_0}(t) + |Q|_{L^2}^2 x(t)| \le \Big| \int_{B_{3/400}} |u(t,x-x(t))|^2 \psi(x) \Big| \\ + (\frac{1}{400} + |\psi|_\infty) \int_{\mathbb{T}^N \setminus B_{3/400}} |u(t,x-x(t))|^2 \to 0 \text{ as } t \to T. \end{aligned}$$

From this result and from the convergence of $f_{t_0}(t)$ to $-|Q|_{L^2}^2 x_0$, we deduce immediately the convergence of $x(t)$ to x_0, hence the proof of Proposition 2.2 is achieved. □

We now use a new space-translation so that $x_0 = 0$, that is to say that we assume from now on that u concentrates around the point 0, which also means that $|u(t,x)|^2 \rightharpoonup |Q|_{L^2}^2 \delta_0$ as $t \to T$.

3. Lower bound of the blow-up rate

3.1. **Reduction of the proof of Theorem 1.** We define the localized viriel g by

$$g(t) = \int_{\mathbb{T}^N} |u(t,x)|^2 \varphi(x)\,dx, \tag{13}$$

where $\varphi(x) = |x|^2$ close to zero, see equations (11).

Let us give a link between the result for the localized viriel $g(t)$ (see the further corollary 3.1) and $|\nabla u(t)|_{L^2}$.

Lemma 3.1. *Assume there exists $K > 0$ such that $g(t) \leq K(T-t)^2$. Then there exists $\gamma > 0$ such that*

$$|\nabla u(t)|_{L^2} \geq \frac{\gamma}{T-t}.$$

Proof. ψ being defined by (11), we study the quantity $\int |u(t)|^2 div(\psi)$ as follows: On the one hand, we know that

$$\int |u(t)|^2 div(\psi) =< |u(t)|^2, div(\psi) > \to |Q|_{L^2}^2 \, div(\psi)(0) = N \, |Q|_{L^2}^2 \text{ as } t \to T.$$

On the other hand, we obtain (with an integration by parts and the use of Cauchy-Schwarz's inequality):

$$\int |u(t)|^2 div(\psi) = -\int_{\mathbb{T}^N} \psi.\nabla(|u(t)|^2) = -2Re \int_{\mathbb{T}^N} \bar{u}(t)\nabla u(t).\psi$$
$$\leq 2\,|\nabla u(t)|_{L^2} \left(\int_{\mathbb{T}^N} |u(t)|^2|\psi|^2\right)^{1/2}.$$

We note that with $\check{\varphi}(x) = |\psi(x)|^2 = \tilde{\psi}(x_1)^2 + \cdots + \tilde{\psi}(x_N)^2$ and $\check{\psi}(s) = \tilde{\psi}'(s)\tilde{\psi}(s)$, we have

$$\tfrac{1}{2}\nabla\check{\varphi} = (\check{\psi}(x_1), \ldots, \check{\psi}(x_N)).$$

$\check{\psi}$ and $\check{\varphi}$ verify the same conditions as ψ and φ, so if we define

$$\check{g}(t) = \int_{\mathbb{T}^N} |u(t)|^2 \check{\varphi},$$

we assume there exists a constant K such that $\forall t \in [0,T)$, $\check{g}(t) \leq K(T-t)^2$.

Thus we have obtained that

$$2\,|\nabla u(t)|_{L^2} \sqrt{K}(T-t) \geq 2\,|\nabla u(t)|_{L^2} \sqrt{\check{g}(t)} \geq \int |u(t)|^2 div(\psi) \to N \int Q^2$$

as $t \to T$, and hence

$$|\nabla u(t)|_{L^2} \geq \frac{\gamma}{T-t},$$

for some constant $\gamma > 0$ ($\gamma < \frac{N \int Q^2}{2\sqrt{K}}$), and the proof of Lemma 3.1 is achieved. □

Theorem 1 will now be implied directly by Corollary 3.1.

3.2. **The localized viriel identity.** In this section, we will prove a conservation law (that we call the localized viriel identity) which will later enable us to get the bound of the viriel announced in Section 3.1. We claim that

Proposition 3.1. $\exists K > 0$ *such that* $\forall 0 \leq t < T$, $g''(t) \leq 2K$.

Proof. After computations, one obtains that the two first order derivatives of this quantity with respect to time are given by the following $\forall t \in [0, T[$,

$$g'(t) = 4Im \int_{\mathbb{T}^N} \bar{u} \nabla u . \psi, \tag{14}$$

$$\begin{aligned} g''(t) = 16E(u_0) - \int_{\mathbb{T}^N} |u|^2 \Delta^2 \varphi + \frac{8}{1+2/N} \int_{\mathbb{T}^N} |u|^{4/N+2} \Big(1 - \frac{\Delta \varphi}{2N}\Big) \\ - 8 \int_{\mathbb{T}^N} \sum_k \Big| \frac{\partial u}{\partial x_k} \Big|^2 (1 - \tilde{\psi}'(x_k)). \end{aligned} \tag{15}$$

Since $\tilde{\psi}'(x_k) = 1$ and $\Delta \varphi = 2N$ over $B_{1/100}$, the two last integrals

$$\int_{\mathbb{T}^N} |u|^{4/N+2} (1 - \frac{\Delta \varphi}{2N}) \quad \text{and} \quad \int_{\mathbb{T}^N} \sum_k \Big| \frac{\partial u}{\partial x_k} \Big|^2 (1 - \tilde{\psi}'(x_k))$$

are respectively equal to

$$\int_{\mathbb{T}^N \setminus B} |u|^{4/N+2} (1 - \frac{\Delta \varphi}{2N}) \quad \text{and} \quad \int_{\mathbb{T}^N \setminus B} \sum_k \Big| \frac{\partial u}{\partial x_k} \Big|^2 (1 - \tilde{\psi}'(x_k)),$$

and hence they are bounded from the results of the former section.

Moreover, we clearly have

$$\Big| \int_{\mathbb{T}^N} |u|^2 \Delta^2 \varphi \Big| \leq \big| \Delta^2 \varphi \big|_{L^\infty} \int_{\mathbb{T}^N} |u|^2 = c$$

and in fact

$$\int_{\mathbb{T}^N} |u|^2 \Delta^2 \varphi \to < |Q|^2_{L^2} \, \delta_0, \Delta^2 \varphi >= 0.$$

So we deduce from (15) the result of Proposition 3.1. □

3.3. **Behavior of the viriel around the blow-up time.** As announced in Section 3.1, we will use the result of Proposition 3.1 to obtain an upper bound of the localized viriel $g(t)$.

We shall start proving the

Proposition 3.2. $g(T) = g'(T) = 0$, *i.e., the functions* g *and* g' *can be considered as continuous up to time* T *with these equalities.*

Proof. We shall proceed through the two following steps

i] $\lim_{t\to T^-} g(t) = 0$ since

$$g(t) =< \underbrace{|u(t)|^2}_{\to |Q|^2_{L^2}\delta_0}, \varphi > \to |Q|^2_{L^2}\, \varphi(0) = 0.$$

Hence one can define g by continuity at time T, with $g(T) = 0$.

ii] To prove that $\lim_{t\to T^-} g'(t) = 0$, we follow the method of Merle in [7], p. 446.

For any integer $n \geq 1$, we consider the solution u_n of (1) associated to the initial data $u_{0,n} = \left(1 - \frac{1}{n}\right) u_0$.

Since $|u_{0,n}|_{L^2} < |u_0|_{L^2} = |Q|_{L^2}$, we know that u_n is globally defined in time. Hence, in particular, we can define the localized viriel g_n of u_n :

$$\forall t \in \mathbb{R},\ g_n(t) := \int_{\mathbb{T}^N} |u_n(t,x)|^2 \varphi(x)\, dx \geq 0. \tag{16}$$

Using continuity arguments as n tends to infinity, we deduce from this non-negativity that $\lim_{t\to T^-} g'(t) = 0$. Indeed, we can start proving the

Lemma 3.2. *g_n'' is bounded independently from n, i.e., there exist $K_1, \alpha > 0$ such that*

$$\forall n \in \mathbb{N},\ \forall t \in [0, T+\alpha], \quad g_n''(t) \leq 2K_1.$$

Proof. See appendix A.

Since g'' is bounded, $g'(t) = g'(0) + \int_0^t g''(s)\, ds$ has a limit (which will be $g'(T)$) as $t \to T$.

Since $g(T) = 0$ and $g(t) \geq 0$, $\forall t < T$, it is obvious that $g'(T) \leq 0$. Our purpose to achieve ii] is to prove that $g'(T) = 0$, i.e., that $g'(T) \geq 0$.

From Lemma 3.2, we can write:

$$\forall n \geq 0,\ \forall 0 < \delta_0 < \alpha,\ 0 \leq g_n(T+\delta_0) \leq g_n(T-\delta_0) + 2g_n'(T-\delta_0)\delta_0 + 4K_1\delta_0^2.$$

By continuity of the solution of (1) with respect to the initial data, we can pass to the limit as n tends to ∞ and hence obtain $\forall 0 < \delta_0 < \alpha$,

$$\begin{array}{lcl} 0 & \leq & g(T-\delta_0) + 2g'(T-\delta_0)\delta_0 + 4K_1\delta_0^2 \\ 0 & \leq & -\delta_0 g'(T) + K\delta_0^2 + 2g'(T-\delta_0)\delta_0 + 4K_1\delta_0^2 \\ 0 & \leq & 2g'(T-\delta_0) - g'(T) + (K+4K_1)\delta_0, \end{array}$$

and passing to the limit as $\delta_0 \searrow 0$ we obtain $g'(T) \geq 0$. Hence we have proved that $g'(T) = 0$. □

From the fact that $g(T) = 0 = g'(T)$ and from Proposition 3.1, we obtain the following:

Corollary 3.1. *There exists a positive constant K such that $\forall 0 \leq t < T$,*

$$0 \leq g(t) \leq K(T-t)^2.$$

We can also deduce from $g(T) = g'(T) = 0$ the following

Corollary 3.2. *Let $u_0 \in H^1(\mathbb{T}^N)$ such that the solution u of (1) blows-up in finite time, and $\int_{\mathbb{T}^N} |u_0|^2 = \int_{\mathbb{R}^N} Q^2$. Then $E(u_0) \geq 0$.*

Proof. In (15), we have $\tilde{\psi}' \leq 1$ and $\Delta\varphi \in L^\infty$, hence

$$g''(t) \leq 16E(u_0) - \int |u(t,x)|^2 \Delta^2\varphi(x)\,dx + c\int_{\mathbb{T}^N \setminus B_{1/100}} |u|^{2+4/N},$$

$$\int |u|^2 \Delta^2 \varphi \to |Q|^2_{L^2}\, \Delta^2\varphi(0) = 0 \quad \text{as } t \to T,$$

and

$$\int_{\mathbb{T}^N \setminus B} |u|^{2+4/N} \to 0$$

(see Lemma 2.2) which implies that

$$\limsup_{t \to T} g''(t) \leq 16E(u_0).$$

But from $g(T) = g'(T) = 0$ and the nonnegativity of $g(t)$, it is obvious that $\limsup_T g'' \geq 0$. We immediately deduce from these inequalities that the energy of any blow-up solution of (1) of minimal mass has a nonnegative energy, that is, $E(u_0) \geq 0$. □

Remark 3.1. Note that in the $\mathbb{T}^N$ case, there exist at any mass elements in H^1 of negative energy: if we consider a constant function α, we have $E(\alpha) = -\frac{\alpha^{4/N+2}}{4/N+2}$. Hence we cannot get in the space-periodic case the blow-up criterion of the $\mathbb{R}^N$ case ($E(u_0) < 0$ implies blow-up if $xu_0 \in L^2(\mathbb{R}^N)$, see (5)).

Appendix A. Stability of the bound of g''

This appendix is devoted to the proof of Lemma 3.2. Note that a sufficient condition to obtain such a result is that the quantity $|\nabla u_n|_{L^2(\mathbb{T}^N \setminus B_{1/100})}$ is bounded (see the proof of Proposition 3.1).

Step 1. We first claim

Lemma A.1. *For all $\gamma > 0$, there exist two real numbers $\delta > 0$ and $C_0 > 0$, an integer n_0 and a sequence of continuous functions $x_n(t)$ (defined over $[T-\delta, T+\delta]$, $\forall n \geq n_0$) such that $\forall n \geq n_0$, $\forall\ t \in [T-\delta, T+\delta]$,*

$$\left\{ \begin{array}{l} \left| \int_{\mathbb{T}^N} |u_n(t, x - x_n(t))|^2 \psi(x)\, dx \right| \leq \dfrac{1}{600}, \\ \int_{\mathbb{T}^N \setminus B_{1/600}} |\nabla u_n(t, x - x_n(t))|^2\, dx \leq C_0, \\ \int_{\mathbb{T}^N \setminus B_{1/300}} |u_n(t, x - x_n(t))|^2\, dx \leq \gamma. \end{array} \right.$$

Proof. Thanks to the alternative contained in the theorem of Merle [9], p. 480 and the fact that u blows up at time T, we claim the

Lemma A.2 (Uniform blow-up)**.**

$$\begin{array}{ll} \forall A > 0, & \exists \alpha(A) > 0,\ \exists m_0(A) \in \mathbb{N} \textit{ such that} \\ & \forall n \geq m_0,\ \forall s \in [T-\alpha, T+\alpha],\ |\nabla u_n(s)|_{L^2} \geq A \end{array} \tag{17}$$

We are going to deduce from lemma A.2 the existence of functions $x_n(t)$ and of constants $C_0 > 0$, $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $\forall\ t \in [T-\delta, T+\delta]$,

$$\left\{ \begin{array}{l} \left| \int_{\mathbb{T}^N} |u_n(t, x - x_n(t))|^2 \psi(x)\, dx \right| \leq \dfrac{1}{600}, \\ \int_{\mathbb{T}^N \setminus B_{1/1200}} |\nabla u_n(t, x - x_n(t))|^2\, dx \leq C_0, \\ \int_{\mathbb{T}^N \setminus B_{1/600}} |u_n(t, x - x_n(t))|^2\, dx \leq \gamma. \end{array} \right. \tag{18}$$

To prove this, we work by contradiction, and in that purpose we assume $\forall C > 0$, $\forall n_0 \in \mathbb{N}$, $\forall \delta > 0$, $\exists n \geq n_0$, $\exists t \in [T-\delta, T+\delta]$ such that $\forall\ y \in \mathbb{T}^N$,

$$\left| \int_{\mathbb{T}^N} |u_n(t, x - y)|^2 \psi(x)\, dx \right| > \frac{1}{600}, \text{ or } \int_{\mathbb{T}^N \setminus B_{1/1200}} |\nabla u_n(t, x - y)|^2\, dx > C,$$
$$\text{or } \int_{\mathbb{T}^N \setminus B_{1/600}} |u_n(t, x - y)|^2\, dx > \gamma.$$

We then choose $C = M \in \mathbb{N}$, and then in $(17) :\ A = M$, hence we obtain the existence of $\alpha(M)$ and $m_0(M)$ such that $\forall n \geq m_0(M)$, $\forall s \in [T - \alpha(M), T + \alpha(M)]$, $|\nabla u_n(s)|_{L^2} \geq M$. We then choose $n_0 = m_0(M)$ and $\delta = \alpha(M)$.

Hence, we obtain the existence of sequences $n_M \in \mathbb{N}$ and $t_M \in \mathbb{R}$ such that $\forall M \in \mathbb{N}$, $|\nabla u_{n_M}(t_M)|^2_{L^2} \geq M$ and $\forall\, y$,

$$\Big| \int_{\mathbb{T}^N} |u_{n_M}(t_M, x-y)|^2 \psi(x)\, dx \Big| > \frac{1}{600}, \text{ or}$$
$$\int_{\mathbb{T}^N \setminus B(y,1/1200)} |\nabla u_{n_M}(t_M, x)|^2\, dx > M, \text{ or} \tag{19}$$
$$\int_{\mathbb{T}^N \setminus B_{1/600}} |u_{n_M}(t_M, x-y)|^2\, dx > \gamma.$$

Let us then define $v_M = u_{n_M}(t_M) \in H^1(\mathbb{T}^N)$. We obtain that $|v_M|_{L^2} \leq |Q|_{L^2}$ (from $|v_M|_{L^2} = |u_{0,n_M}|_{L^2}$), $\exists c_0 \;/\; E(v_M) \leq c_0$ (since $E(v_M) = E(u_{0,n_M})$), $|\nabla v_M|_{L^2} \to +\infty$ as $M \to \infty$.

From Proposition 2.1 and the sequel, we then know that there exists a sequence of points $z_M \in \mathbb{T}^N$ such that
(i) $|v_M(. - z_M)|^2 \rightharpoonup |Q|^2_{L^2}\, \delta_0$ as $M \to \infty$, which implies:

$$\int_{\mathbb{T}^N \setminus B(z_M,1/600)} |v_M|^2 \xrightarrow{M\to\infty} 0, \quad \int_{\mathbb{T}^N} |v_M(x - z_M)|^2 \psi(x)\, dx \xrightarrow{M\to\infty} 0.$$

(ii) $\exists C_{(1/1200)}$ such that $\int_{\mathbb{T}^N \setminus B(z_M,1/1200)} |\nabla v_M(x)|^2\, dx \leq C_{(1/1200)}$, which is a contradiction with (19) for the choice $y = z_M$ (for M great enough).

Conclusion. We have proved the existence of the sequence of functions $x_n(t)$ with the desired properties (see (18)).

Moreover, the functions $x_n(t)$ can be considered as continuous by standard arguments, hence the proof of Lemma A.1 is achieved. □

Step 2. We now claim the

Lemma A.3. *With the former notations,*

$$\forall n \geq n_0,\ \forall t \in [T - \delta, T + \delta],\ |x_n(t)| < \frac{2}{300}.$$

Proof. It is similar to the fact that $|x(t)| \leq \frac{1}{400}$ in the proof of Proposition 2.2, once obtained the following results.

We need to define a localized first order momentum of $u_n(t)$ by

$$f_n(t) := \int_{\mathbb{T}^N} |u_n(t,x)|^2 \psi(x)\, dx \in \mathbb{R}^N.$$

Since $|\nabla u(t)|_{L^2(B_{1/600})} \to +\infty$ as $t \nearrow T$, we can assume that

$$|\nabla u(T - \delta)|^2_{L^2(B_{1/600})} > C_0,$$

where C_0 is given by Lemma A.1, and it follows that we can assume n_0 to be great enough so that

$$\forall n \geq n_0,\ |\nabla u_n(T-\delta)|^2_{L^2(B_{1/600})} > C_0 \tag{20}$$

Assume that $|x_n(T-\delta)| > \frac{1}{300}$ for some $n \geq n_0$. Then $B_{1/600} \subset \mathbb{T}^N \setminus B(x_n(T-\delta), 1/600)$ and hence (see Lemma A.1) :

$$\int_{B_{1/600}} |\nabla u_n(T-\delta)|^2\,dx \leq C_0,$$

which contradicts (20). This proves that $|x_n(T-\delta)| \leq \frac{1}{300}$, $\forall n \geq n_0$. We finally remark that $\forall t \in [T-\delta, T+\delta]$,

$$|x_n(t)| \leq \frac{5}{600} \Rightarrow \int_{\mathbb{T}^N \setminus B_{1/100}} |\nabla u_n(t,x)|^2 \leq C_0 \Rightarrow |f_n'(t)| \leq M,$$

for some constant $M > 0$.

We will moreover assume that $2M\delta \leq \frac{1}{600}$.

From the fact that $|u_{0,n}|_{L^2} \to |Q|_{L^2}$, we can also assume that

$$\forall n \geq n_0,\ |Q|^2_{L^2} - |u_{0,n}|^2_{L^2} \leq \frac{|Q|^2_{L^2}}{100}.$$

Finally, $f_{t_0}(t) \to 0$ as $t \to T$, so we can assume that δ verifies $|f_{t_0}(T-\delta)| < \frac{1}{600}$. Since $f_n(T-\delta) \xrightarrow{n\to\infty} f_{t_0}(T-\delta)$, we can assume n_0 to be great enough so that

$$\forall n \geq n_0,\ |f_n(T-\delta)| \leq \frac{1}{600}.$$

Lemma A.3 then follows by similar arguments as the *a priori* estimates (on $x(t)$) in the proof of Proposition 2.2. □

But we know that

$$|x_n(t)| \leq \frac{5}{600} \Rightarrow \int_{\mathbb{T}^N \setminus B_{1/100}} |\nabla u_n(t,x)|^2 \leq C_0 \Rightarrow |g_n''(t)| \leq K_1,$$

for some constant $K_1 > 0$. Hence, the proof of Lemma 3.2 is achieved. □

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