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GLOBAL EXISTENCE AND ASYMPTOTIC STABILITY FOR VISCOELASTIC PROBLEMS *

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Abstract. One considers the damped semilinear viscoelastic wave equation

$$u_{tt} - \Delta u + \alpha u + f(u) + \int_0^t g(t-\tau)\Delta u(\tau)\, d\tau + h(u_t) = 0 \text{ in } \Omega \times (0,\infty),$$

where Ω is any bounded or finite measure domain of $\mathbf{R}^n$, $\alpha \geq 0$ and f, h are power like functions. The existence of global regular and weak solutions is proved by means of the Faedo-Galerkin method and uniform decay rates of the energy are obtained following the perturbed energy method by assuming that g decays exponentially.

1. Introduction

This paper is devoted to the study of the existence and uniform decay rates of solutions $u = u(x,t)$ of the damped semilinear viscoelastic wave equation

$$\begin{aligned} &u_{tt} - \Delta u + \alpha u + f(u) + \int_0^t g(t-\tau)\Delta u(\tau)\, d\tau + h(u_t) = 0 \text{ in } \Omega \times (0,\infty) \\ &u(x,0) = u^0(x); \quad u_t(x,0) = u^1(x) \;\; x \in \Omega, \end{aligned} \tag{$*$}$$

where Ω is any bounded or finite measure domain of $\mathbf{R}^n$. If Ω has a nonempty boundary Γ then it will be assumed regular only if we are interested that $u = 0$ on Γ. Otherwise no regularity is required upon Γ.

For the sake of readability we assume that

$$f(s) = \gamma\, |s|^{\xi}\, s \quad \text{and} \quad h(s) = \beta\, |s|^{\rho}\, s, \tag{1.1}$$

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where γ, ξ, β and ρ are positive constants such that

$$0 < \xi, \rho \leq \frac{2}{n-2} \text{ if } n \geq 3 \quad \text{and } \Omega \text{ is bounded} \tag{1.2}$$

or Ω is unbounded but possesses finite measure,

$$0 < \xi = \rho = \frac{2}{n-2} \text{ if } n \geq 3 \quad \text{and } \Omega \text{ is unbounded,} \tag{1.3}$$

$$\xi, \rho > 0 \text{ if } n = 1, 2 \text{ and } \Omega \text{ is any.} \tag{1.4}$$

Let us consider the Sobolev space $H^1(\Omega)$ endowed with the inner product

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)},$$

and define the subspace of $H^1(\Omega)$, denoted by $H_0^1(\Omega)$, as the closure of $C_0^\infty(\Omega)$ in the strong topology of $H^1(\Omega)$. Then given $v \in H_0^1(\Omega)$ and denoting by $\hat{v}$ the null extension of v to the whole $\mathbf{R}^n$, we have the following useful properties:

$$\hat{v} \in H^1(\mathbf{R}^n), \tag{1.5}$$

$$\frac{\partial \hat{v}}{\partial x_j} = \widehat{\frac{\partial v}{\partial x_j}}, \tag{1.6}$$

$$||v||_{H_0^1(\Omega)} = ||\hat{v}||_{H^1(\mathbf{R}^n)}. \tag{1.7}$$

From (1.3) and (1.4) and according to the Sobolev imbeddings one has

$$H^1(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n) \quad \text{i f } n = 1, \tag{1.8}$$

$$H^1(\mathbf{R}^n) \hookrightarrow L^q(\mathbf{R}^n) \quad \forall q \in [2, \infty) \text{ if } n = 2, \tag{1.9}$$

$$H^1(\mathbf{R}^n) \hookrightarrow L^{\frac{2n}{n-2}}(\mathbf{R}^n) = L^{2(\rho+1)}(\mathbf{R}^n) \quad \text{if } n \geq 3. \tag{1.10}$$

Based on the property (1.7) the imbeddings (1.8)-(1.10) still hold by changing $H^1(\mathbf{R}^n)$ by $H_0^1(\Omega)$ and $\mathbf{R}^n$ by Ω. Indeed, let us prove, for instance, property (1.10). We have, for all $u \in H_0^1(\Omega)$

$$||u||_{L^{\frac{2n}{n-2}}(\Omega)} \leq ||\hat{u}||_{L^{\frac{2n}{n-2}}(\mathbf{R}^n)} \leq C\, ||\hat{u}||_{H^1(\mathbf{R}^n)} = C\, ||u||_{H_0^1(\Omega)}$$

which proves that

$$H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega) = L^{2(\rho+1)}(\Omega) \quad \text{if } n \geq 3. \tag{1.11}$$

By the way, since $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, from (1.11) and interpolation arguments we also deduce that

$$H_0^1(\Omega) \hookrightarrow L^r(\Omega) \quad \forall r \in [2, 2(\rho+1)]. \tag{1.12}$$

Next we make some remarks about early works in asymptotic behaviour of solutions for some problems related to $(*)$. When $g = 0$ and Ω is a bounded domain of $\mathbf{R}^n$ with smooth boundary, problem $(*)$ was studied by several authors. It is worth mentioning a paper by M. Nakao [16] who was the first to obtain the decay rate given in (2.4) (see theorem 2.1 below). Further generalizations have been done by E. Zuazua [18]who obtained analogous decay rates for a large class of hyperbolic problems. In the same direction we can mention the results in A. Haraux [7] and A. Haraux and E. Zuazua [8]. Now, when $\Omega = \mathbf{R}^n$ and $g = 0$ the exponential decay for the semilinear wave equation was obtained by E. Zuazua [17] where a linear damping term is effective at infinity. Also, in the whole space, it is well known that the energy $E(t)$ (see 1.13 below) has the following algebraic decay rate

$$E(t) \leq C(E(0))\,(1+t)^{-(2-n\rho)/\rho} \quad \text{if } 0 < \rho < 2/n$$

which was proved by M. Nakao in [15]. Furthermore, it is known that if $\rho > 2/n$ the energy never decays for small amplitude solutions, which was proved by T. Motai in [13].

On the other hand, when $h(s) = 0$ and Ω is a bounded domain with smooth boundary, J. E. Munoz Rivera [13] and S. Jiang and J. E. Munoz Rivera [9] proved, in the framework of nonlinear viscoelasticity, the exponential decay by assuming that the kernel of the memory decays exponentially. In the context of linear viscoelasticity with nonlocal linearity we can cite the works of C. M. Dafermos [4], C. M. Dafermos and J. A. Nohel [5], among others.

Some related results involving viscoelastic problems with boundary dissipation were obtained, for smooth bounded domains, by J. E. Lagnese [10] and M. M. Cavalcanti et al. [3]. A natural question in this context is about the viscoelastic effects on the boundary. This was considered in M. Aassila et al. [2], where uniform decay rates for the wave equation subject to viscoelastic effects and nonlinear feedback on the boundary were proved by considering bounded domains. At this point it is important to mention the work of M. Aassila [1] and the part of the work of M. Aassila et al. [2] in connection with the asymptotic behaviour for the wave equation in unbounded domains with finite measure. However, in both works [1,2], no decay rates were obtained as in this present paper.

We remark the above mentioned results are marked by the following features: (a) When $h = 0$, the domain Ω is bounded with smooth boundary $\partial\Omega$; (b) When $g = 0$, the domain Ω is bounded with smooth boundary $\partial\Omega$ or Ω is the whole space $\mathbf{R}^n$ or Ω is an exterior domain with smooth boundary; (c) When $h \neq 0$ and $g \neq 0$, the domain Ω can be unbounded but has finite measure.

The main goal of the present paper is to deal with the semilinear wave equation subject to viscoelastic effects with nonlinear dissipation for general domains with finite measure. To our best knowledge, this natural generalization was not early considered in the literature. Besides, in our analysis and in what concerns the existence results for unbounded domains, we are not using nice properties like spectral resolution of self-adjoint operators, regularity of elliptic problems nor the Poincaré inequality.

The proof of the global existence of solutions to problem $(*)$ is based on the Faedo-Galerkin method. It is interesting to observe that when Ω is a general domain and therefore the well-known Aubin-Lions lemma is not valid, we need to establish a new argument in order to pass to the limit in the nonlinear terms. This is done in Section 3.

To prove the uniform decay rates of the energy

$$E(t) = \frac{1}{2}\int_\Omega \Big\{ |u_t(x,t)|^2 + |\nabla u(x,t)|^2 + \alpha |u(x,t)|^2 + \frac{\gamma}{\xi+2}|u(x,t)|^{\xi+2}\Big\} dx \tag{1.13}$$

we need to define a modified energy function. In fact, a formal computation gives

$$E'(t) = -\beta \int_\Omega |u_t(x,t)|^{\rho+2}\, dx + \int_0^t g(t-\tau)\,(\nabla u(\tau), \nabla u_t(t))\, d\tau, \tag{1.14}$$

which shows that we do not have any information about the sign of $E'(t)$. To overcome this problem we use an argument from Dafermos [4] to define a new energy function $e(t)$ such that $e'(t) \leq 0$ and, of course, $E(t) \leq Ce(t)$ for some positive constant C. This will be discussed in section 4. The proof of the decay for $e(t)$ is based on the perturbed energy method by constructing a suitable Liapunov functional, as in E. Zuazua [17].

Our paper is organized as follows. In section 2 we present some notations and state our main results. In section 3 we prove existence and uniqueness for regular and weak solutions and finally in section 4 we we give the proofs of the uniform decay.

2. Notations and statement of results

We begin by introducing some notations that will be used throughout this work. For the standard $L^p(\Omega)$ space we write

$$(u,v) = \int_\Omega u(x)v(x)\, dx, \quad |u|^2 = \int_\Omega |u(x)|^2\, dx \quad \text{and} \quad ||u||_p^p = \int_\Omega |u(x)|^p\, dx.$$

Next we put

$$\mathcal{H} = \left\{ u \in H_0^1(\Omega); \Delta u \in L^2(\Omega)\right\}. \tag{2.1}$$

Then $\mathcal{H}$ is a Hilbert space endowed with the natural inner product

$$(u,v)_{\mathcal{H}} = (u,v)_{H_0^1(\Omega)} + (\Delta u, \Delta v)\,. \tag{2.2}$$

The precise assumptions on the memory term of $(*)$ are given in the sequel. We assume that $g : \mathbf{R}_+ \to \mathbf{R}_+$ is a bounded C^2 function satisfying

$$1 - \int_0^\infty g(s)ds = l > 0 \tag{$H.1$}$$

and such that there exist positive constants ξ_1, ξ_2 and ξ_3 satisfying

$$-\xi_1 g(t) \le g'(t) \le -\xi_2 g(t) \quad \forall t \ge 0, \tag{H.2}$$

$$0 \le g''(t) \le \xi_3 g(t) \quad \forall t \ge 0. \tag{H.3}$$

Now we are in a position to state our main results.

Theorem 2.1. *Let the initial data* $\left\{u^0, u^1\right\}$ *belong to* $\mathcal{H} \times H_0^1(\Omega)$ *and assume that assumptions* (1.1)-(1.4), $(H.1)$-$(H.3)$ *hold. Then, problem* $(*)$ *possesses a unique regular solution* u *in the class*

$$u \in L^\infty(0,\infty; H_0^1(\Omega)),\ u' \in L^\infty(0,\infty; H_0^1(\Omega)),\ u'' \in L^\infty(0,\infty; L^2(\Omega)). \tag{2.3}$$

Moreover, if $||g||_{L^1(0,\infty)}$ *is sufficiently small and* Ω *has finite measure, the energy* $E(t)$ *has the following decay rates*

$$E(t) \le \left(\varepsilon\theta t + [E(0)]^{-\rho/2}\right)^{-2/\rho}, \quad \forall t \ge 0 \quad \forall \varepsilon \in (0, \varepsilon_0] \tag{2.4}$$

where θ *and* ε_0 *are positive constants.*

Theorem 2.2. *Let the initial data* $\{u^0, u^1\}$ *belong to* $H_0^1(\Omega) \times L^2(\Omega)$ *and assume that same hypotheses of theorem 2.1 hold. Then, problem* $(*)$ *possesses a unique weak solution* u *in the class*

$$u \in C^0([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega)).$$

Besides, assuming that $||g||_{L^1(0,\infty)}$ *is small enough and* Ω *has finite measure, the weak solution has the same decay given in* (2.4).

3. Existence and uniqueness of solutions

In this section we first prove the existence and uniqueness of regular solutions to problem $(*)$. Then, we extend the same result to weak solutions using a density argument.

3.1. **Regular solutions.** Let (ω_ν) be a basis in $\mathcal{H}$ and let us consider V_m the space generated by $\omega_1, \cdots, \omega_m$. Let

$$u_m(t) = \sum_{j=1}^{m} \delta_{jm}(t)\omega_j \tag{3.1}$$

be the solution of the approximate Cauchy problem

$$\begin{aligned}\left(u_m''(t), w\right) + \alpha\left(\nabla u_m, \nabla w\right) + \alpha\left(u_m(t), w\right) + \beta\left(\left|u_m'\right|^\rho u_m'(t), w\right)\\ - \int_0^t g(t-\tau)\left(\nabla u_m(\tau), \nabla w\right) + \gamma(|u_m(t)|^\xi u_m(t), w) = 0,\end{aligned} \tag{3.2}$$

$$u_m(0) = u_{0m} \rightarrow u^0 \text{ in } \mathcal{H}, \quad u_m'(0) = u_{1m} \rightarrow u^1 \text{ in } H_0^1(\Omega). \tag{3.3}$$

We observe that in view of assumptions (1.2)-(1.4) and taking (1.12) into account, the nonlinear terms in (3.2) are well defined, that is, belong to $L^2(\Omega)$. By standard methods in differential equations, we prove the existence of solutions to the approximate problem on some interval $[0, t_m)$ and this solution can be extended to the closed interval $[0, T]$ by using the first estimate below.

The First Estimate: Setting $w = u_m'(t)$ in (3.2), we obtain

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}|u_m'(t)|^2 + \frac{1}{2}|\nabla u_m(t)|^2 + \frac{\alpha}{2}|u_m(t)|^2 + \frac{\gamma}{\xi+2}\|u_m(t)\|_{\xi+2}^{\xi+2}\right\} + \beta\|u_m'(t)\|_{\rho+2}^{\rho+2}\\ &= -g(0)\left|\nabla u_m(t)\right|^2 + \frac{d}{dt}\Big[\int_0^t g(t-\tau)\left(\nabla u_m(\tau), \nabla u_m(t)\right) d\tau\Big]\\ &\quad - \int_0^t g'(t-\tau)\left(\nabla u_m(\tau), \nabla u_m(t)\right) d\tau.\end{aligned} \tag{3.4}$$

Integrating (3.4) over $(0, t)$, taking (3.3) and $(H.2)$ into account and using the inequality $ab \leq \frac{1}{4\eta}a^2 + \eta b^2$, where η is an arbitrary positive number, we deduce

$$\begin{aligned}&\frac{1}{2}|u_m'(t)|^2 + \frac{1}{2}|\nabla u_m(t)|^2 + \frac{\alpha}{2}|u_m(t)|^2 + \frac{\gamma}{\xi+2}\|u_m(t)\|_{\xi+2}^{\xi+2} + \beta\int_0^t \|u_m'(s)\|_{\rho+2}^{\rho+2} ds\\ &\leq k_1 + \frac{1}{2}\int_0^t \left|\nabla u_m(s)\right|^2 ds + \frac{\xi_1^2}{2}\left|\left|g\right|\right|_{L^1(0,\infty)}^2 \int_0^t \left|\nabla u_m(s)\right|^2 ds\\ &\quad + \eta\left|\nabla u_m(t)\right|^2 + \frac{1}{4\eta}\left|\left|g\right|\right|_{L^1(0\infty)}\left|\left|g\right|\right|_{L^\infty(0,\infty)} \int_0^t \left|\nabla u_m(\tau)\right|^2 d\tau,\end{aligned}$$

where k_1 is a positive constant. Employing Gronwall's lemma and choosing $\eta > 0$ sufficiently small, from the last inequality we obtain the first estimate

$$|u_m'(t)|^2+|\nabla u_m(t)|^2+|u_m(t)|^2+\|u_m(t)\|_{\xi+2}^{\xi+2}+\int_0^t \|u_m'(s)\|_{\rho+2}^{\rho+2}ds \leq L_1 \quad (3.5)$$

where L_1 is a positive constant independent of $m \in \mathbf{N}$ and $t \in [0,T]$.

The Second Estimate: First of all, we are going to estimate $u_m''(0)$ in L^2 norm. Considering $w = u_m''(0)$ and $t = 0$ in (3.2), we infer

$$\left|u_m''(0)\right|^2 \leq \left\{|\Delta u_{0m}| + \alpha\,|u_{0m}| + \beta\,||u_{1m}||_{2(\rho+1)}^{(\rho+1)} + \gamma\,||u_{om}||_{2(\xi+1)}^{(\xi+1)}\right\}\left|u_m''(0)\right|. \quad (3.6)$$

This shows, in view of the imbedding (1.12) the convergences in (3.2), that $|u_m''(0)| \leq L_2$, where L_2 is a positive constant independent of $m \in \mathbf{N}$ and $t \in [0,T]$. Now, taking the derivative of (3.2) with respect to t and setting $w = u_m''(t)$, it follows that

$$\begin{aligned}
&\frac{d}{dt}\Big\{\frac{1}{2}|u_m''(t)|^2+\frac{1}{2}|\nabla u_m'(t)|^2+\frac{\alpha}{2}|u_m'(t)|^2\Big\}+\beta(\rho+1)\int_\Omega |u_m'|^\rho (u_m'')^2dx\\
&= -(\xi+1)\gamma\int_\Omega |u_m|^\xi\, u_m' u_m''\,dx + g(0)\frac{d}{dt}\big(\nabla u_m(t),\nabla u_m'(t)\big) - g(0)\left|\nabla u_m'(t)\right|^2\\
&\quad - g'(0)\big(\nabla u_m(t),\nabla u_m'(t)\big) + \frac{d}{dt}\Big\{\int_0^t g'(t-\tau)\big(\nabla u_m(\tau),\nabla u_m'(t)\big)\,d\tau\Big\} \qquad (3.8)\\
&\quad - \int_0^t g''(t-\tau)\big(\nabla u_m(\tau),\nabla u_m'(t)\big)\,d\tau.
\end{aligned}$$

Analysis of $I_1 := \int_\Omega |u_m|^\xi\, u_m' u_m''\,dx$. Since

$$\frac{\xi}{2(\xi+1)}+\frac{1}{2(\xi+1)}+\frac{1}{2}=1,$$

from the generalized Hölder inequality and considering the first estimate into account, we deduce

$$\begin{aligned}
|I_1| &\leq \|u_m(t)\|_{H_0^1(\Omega)}^\xi \|u_m'(t)\|_{H_0^1(\Omega)}|u_m'(t)| \leq k_2\|u_m'(t)\|_{H_0^1(\Omega)}|u_m'(t)|\\
&\leq \frac{k_2}{2}\left\{\left|u_m'(t)\right|^2+\left|\nabla u_m'(t)\right|^2+\left|u_m''(t)\right|^2\right\},
\end{aligned} \quad (3.9)$$

where k_2 is a positive constant.

Hence integrating (3.8) over $(0,t)$ and taking $(H.2)$, $(H.3)$, (3.3), (3.7) and (3.9) under consideration, we conclude

$$\frac{1}{2}\left|u_m''(t)\right|^2+\frac{1}{2}\left|\nabla u_m'(t)\right|^2+\frac{\alpha}{2}\left|u_m'(t)\right|^2+\beta(\rho+1)\int_0^t\int_\Omega\left|u_m'\right|^\rho\left(u_m''\right)^2\,dx\,ds$$

$$\le k_3+\frac{g(0)^2}{4\eta}\left|\nabla u_m(t)\right|^2+2\eta\left|\nabla u_m'(t)\right|^2+k_4\int_0^t\left|\nabla u_m(s)\right|^2ds$$

$$+\frac{(\xi+1)\gamma(k_2+1)}{2}\int_0^t\left\{\left|u_m'(s)\right|^2+\left|\nabla u_m'(s)\right|^2+\left|u_m''(s)\right|^2\right\}ds$$

where k_3 is a positive constant and

$$k_4=\frac{\xi_3^2}{2}\left|\left|g\right|\right|_{L^1(0,\infty)}^2+\frac{\xi_1^2}{2}\left|\left|g\right|\right|_{L^1(0,\infty)}\left|\left|g\right|\right|_{L^\infty(0,\infty)}.$$

Choosing $\eta>0$ small enough, making use of the first estimate and employing Gronwall's lemma, we obtain the second estimate

$$\left|u_m''(t)\right|^2+\left|\nabla u_m'(t)\right|^2+\left|u_m'(t)\right|^2+\int_0^t\int_\Omega\left|u_m'\right|^\rho\left(u_m''\right)^2dxds\le L_3 \tag{3.10}$$

where L_3 is a positive constant independent of $m\in\mathbf{N}$ and $t\in[0,T]$.

3.2. **Analysis of the nonlinear terms.** From the above estimates and considering (1.12) we infer

$$\|\,|u_m'|^\rho u_m'\|_{L^2(0,T;L^2(\Omega))}^2=\int_0^T\int_\Omega\left|u_m'\right|^{2(\rho+1)}dxdt\le C \tag{3.11}$$

where C is a positive constant independent of $m\in\mathbf{N}$. Consequently, we can extract a subsequence (u_μ) of (u_m) such that there exits $\chi\in L^2(0,T;L^2(\Omega))$ which verifies

$$|u_\mu'|^\rho u_\mu'\rightharpoonup\chi\quad\text{weakly in }L^2(0,T;L^2(\Omega)). \tag{3.12}$$

Also, we can deduce that

$$u_\mu'\rightharpoonup u'\quad\text{weak-star in }\ L^\infty(0,T;H_0^1(\Omega)). \tag{3.13}$$

Defining

$$B_i=\left\{x\in\mathbf{R}^n;\left|\left|x\right|\right|_{\mathbf{R}^n}\le i\right\},\ \ i\in\mathbf{N},\quad\Omega\cap B_i=\Omega_i$$

one has $\Omega\subset\bigcup_{i\in\mathbf{N}}\Omega_i$, and according to (3.11), we have for all $i\in A\subset\mathbf{N}$,

$$\||u_\mu'|^\rho u_\mu'\|_{L^2(0,T;L^2(\Omega_i))}\le C. \tag{3.14}$$

On the other hand, from the a priori estimates, we deduce that there exists a subsequence of $\{u_\mu\}$, which we still denote by $\{u_\mu\}$, such that

$$\{u'_\mu\} \text{ is bounded in } L^2(0,T;H_0^1(\Omega)),$$
$$\{u''_\mu\} \text{ is bounded in } L^2(0,T;L^2(\Omega)).$$

Considering that the imbedding $H^1(\Omega_i) \hookrightarrow L^2(\Omega_i)$ is compact, we can extract a subsequence $\{u_{\mu,i}\}$ of $\{u_\mu\}$ such that, in view of (3.13) and making use of Aubin-Lions theorem, we have

$$u'_{\mu,i} \to u' \quad \text{strongly in } L^2(0,T;L^2(\Omega_i)), \tag{3.15}$$

for all $i \in A$. Then,

$$u'_{\mu,i} \to u' \quad \text{a.e. in } \Omega_i \times (0,T)$$

and consequently

$$|u'_{\mu,i}|^\rho u'_{\mu,i} \to |u'|^\rho u' \quad \text{a.e. in } \Omega_i \times (0,T). \tag{3.16}$$

From (3.14) and (3.16) and thanks to Lions lemma, we deduce

$$|u'_{\mu,i}|^\rho u'_{\mu,i} \rightharpoonup |u'|^\rho u' \quad \text{weakly in } L^2(0,T;L^2(\Omega_i)), \tag{3.17}$$

for all $i \in A$. Combining (3.12) and (3.17) we infer that $\chi = |u'|^\rho u'$ a.e. in Ω. Analogously we prove that

$$|u_{\mu,i}|^\xi u_{\mu,i} \rightharpoonup |u|^\xi u \quad \text{weakly in } L^2(0,T;L^2(\Omega)).$$

So, using standard arguments we can pass to the limit in (3.2) in order to obtain

$$u_{tt}-\Delta u+\alpha u+\gamma|u|^\xi u+\int_0^t g(t-\tau)\Delta u(\tau)\,d\tau+\beta|u'|^\rho u' = 0 \text{ in } L^2(0,\infty;L^2(\Omega)). \tag{3.18}$$

3.3. **Uniqueness.** Let u_1 and u_2 be two smooth solutions to problem $(*)$. Then, $z = u_1 - u_2$ verifies

$$\begin{aligned}
&\left(z''(t),w\right)+(\nabla z(t),\nabla w)+\alpha\,(z(t),w)+\beta\left(\left|u'_1(t)\right|^\rho u'_1(t),w\right)\\
&\quad-\beta\left(\left|u'_2(t)\right|^\rho u'_2(t),w\right)-\int_0^t g(t-\tau)\left(\nabla z(\tau),\nabla z'(t)\right)d\tau \qquad (3.19)\\
&\quad=\gamma\left(|u_2(t)|^\xi u_2(t),w\right)-\gamma\left(|u_1(t)|^\xi u_1(t),w\right),\quad \forall\, w\in H_0^1(\Omega).
\end{aligned}$$

Substituting $w = z'(t)$ in (3.19), we see that

$$\frac{d}{dt}\left\{\frac{1}{2}\left|z'(t)\right|^2+\frac{1}{2}\left|\nabla z(t)\right|^2+\frac{\alpha}{2}\left|z(t)\right|^2\right\}$$

$$\leq \int_0^t g(t-\tau)\,(\nabla z(\tau), \nabla w)\,d\tau + C(\gamma,\xi)\int_\Omega \left(|u_2|^\xi + |u_1|^\xi\right)|z|\,|z'|dx.$$

Since

$$\int_0^t g(t-\tau)\left(\nabla z(\tau), \nabla z'(t)\right)d\tau = -g(0)\,|\nabla z(t)|^2$$
$$-\int_0^t g'(t-\tau)\,(\nabla z(\tau), \nabla z(t))\,d\tau \;+\frac{d}{dt}\Big(\int_0^t g(t-\tau)(\nabla z(\tau), \nabla z(t))d\tau\Big),$$

then, taking the assumption $(H.2)$ and the first estimate into account, we obtain

$$\frac{d}{dt}\Big\{\frac{1}{2}\left|z'(t)\right|^2 + \frac{1}{2}\,|\nabla z(t)|^2 + \frac{\alpha}{2}\,|z(t)|^2\Big\}$$
$$\leq \frac{\xi_2^2}{2}\,|\nabla z(t)|^2 + \frac{1}{2}\,||g||_{L^1(0,\infty)}\int_0^t g(t-\tau)\,|\nabla z(\tau)|^2\,d\tau$$
$$+\frac{d}{dt}\Big(\int_0^t g(t-\tau)\,(\nabla z(\tau), \nabla z(t))\,d\tau\Big)$$
$$+\,C_1(\gamma,\xi)\Big[\frac{1}{2}\,|\nabla z(t)|^2 + \frac{1}{2}\,|z(t)|^2\Big] + \frac{1}{2}\left|z'(t)\right|^2.$$

Integrating last inequality over $(0,t)$, we deduce

$$\frac{1}{2}\left|z'(t)\right|^2 + \frac{1}{2}\,|\nabla z(t)|^2 + \frac{\alpha}{2}\,|z(t)|^2 \leq \frac{\xi_2^2}{2}\int_0^t |\nabla z(s)|^2\,ds$$
$$+\frac{1}{2}\|g\|^2_{L^1(0,\infty)}\int_0^t |\nabla z(s)|^2 ds + C_1(\gamma,\xi)\int_0^t \big[\frac{1}{2}|\nabla z(s)|^2 + \frac{1}{2}|z(s)|^2\big]ds \quad (3.20)$$
$$+\,\frac{1}{2}\int_0^t \left|z'(s)\right|^2 ds + \int_0^t g(t-\tau)\,(\nabla z(\tau), \nabla z(t))\,d\tau.$$

Now, since for an arbitrary $\eta > 0$, we have

$$\int_0^t g(t-\tau)\,(\nabla z(\tau), \nabla z(t))\,d\tau$$
$$\leq \eta\,|\nabla z(t)|^2 + \frac{1}{4\eta}\,||g||_{L^1(0,\infty)}\,||g||_{L^\infty(0,\infty)}\int_0^t |\nabla z(\tau)|^2\,d\tau,$$

from (3.20) and the Gronwall's lemma we conclude that $|z'(t)| = |\nabla z(t)| = |z(t)| = 0$. This finishes the proof. □

3.4. **Weak solutions.** Let us consider $\{u^0, u^1\} \in H_0^1(\Omega) \times L^2(\Omega)$. Then, by density, there exists $\left\{u_\mu^0, u_\mu^1\right\} \subset \mathcal{H} \times H_0^1(\Omega)$ such that

$$u_\mu^0 \to u^0 \quad \text{in } H_0^1(\Omega), \tag{3.21}$$

$$u_\mu^1 \to u^1 \quad \text{in } L^2(\Omega). \tag{3.22}$$

Therefore, for each $\mu \in \mathbf{N}$, there exists u_μ, smooth solution of problem $(*)$ verifying

$$\begin{aligned} & u_\mu'' - \Delta u_\mu + \alpha u_\mu + \gamma |u_\mu|^\xi u_\mu + \int_0^t g(t-\tau)\Delta u_\mu(\tau)\,d\tau + \beta |u_\mu'|^\rho u_\mu' = 0 \\ & u_\mu(0) = u_\mu^0, \;\; u_\mu'(0) = u_\mu^1. \end{aligned} \tag{3.23}$$

Repeating the same arguments used in the first estimate, we obtain

$$\left|u_\mu'(t)\right|^2 + |\nabla u_\mu(t)|^2 + |u_\mu(t)|^2 + ||u_\mu(t)||_{\xi+2}^{\xi+2} + \int_0^t \left|\left|u_\mu'(s)\right|\right|_{\rho+2}^{\rho+2} ds \le C, \tag{3.24}$$

where C is a positive constant independent of $\mu \in \mathbf{N}$ and $t \in [0,T]$.

Let $z_{\mu,\sigma} = u_\mu - u_\sigma$ with $\mu, \sigma \in \mathbf{N}$, where u_μ and u_σ are regular solutions of (3.23). Then following the same steps already used in the uniqueness of regular solutions and taking the convergences (3.21) and (3.22) into account, we deduce that there exists u such that

$$u_\mu \to u \quad \text{strongly in } C^0([0,T]; H_0^1(\Omega)), \tag{3.25}$$

$$u_\mu' \to u' \quad \text{strongly in } C^0([0,T]; L^2(\Omega)). \tag{3.26}$$

From (3.24)-(3.26) and the related convergences we are able to pass to the limit to obtain

$$u_{tt} - \Delta u + \alpha u + \gamma\, |u|^\xi\, u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + \beta\, \left|u'\right|^\rho u' = 0$$

in $L^{\frac{\rho+2}{\rho+1}}(0,\infty; H^{-1}(\Omega) + L^{\frac{\rho+2}{\rho+1}}(\Omega))$. The uniqueness of weak solutions requires a regularization procedure and can be obtained using the standard method of Visik-Ladyzhenkaya, cf. J.L. Lions and E. Magenes [11], Chapter 3, Section 8.2.2.

4. Uniform decay

In this section we obtain the uniform decay of the energy $E(t)$ for regular solutions, since the same occurs for weak solutions using standard density arguments.

From (3.18) we have

$$E'(t) = -\beta \left|\left|u'(t)\right|\right|_{\rho+2}^{\rho+2} + \int_0^t g(t-\tau) \left(\nabla u(\tau), \nabla u'(t)\right) d\tau \tag{4.1}$$

where $E(t)$ is defined in (1.13). A direct computation shows that

$$\begin{aligned} &\int_0^t g(t-\tau) \left(\nabla u(\tau), \nabla u'(t)\right) d\tau = \frac{1}{2} \left(g' \diamond \nabla u\right)(t) - \frac{1}{2} \left(g \diamond \nabla u\right)'(t) \\ &+ \frac{d}{dt}\left\{ \frac{1}{2}\Big(\int_0^t g(s)\, ds\Big) \left|\nabla u(t)\right|^2 \right\} - \frac{1}{2} g(t) \left|\nabla u(t)\right|^2 \end{aligned} \tag{4.2}$$

where

$$(g \diamond y)(t) = \int_0^t g(t-s) \left|y(t) - y(s)\right|^2 ds.$$

Defining the modified energy by

$$\begin{aligned} e(t) = \frac{1}{2} \left|u'(t)\right|^2 + \frac{1}{2}\Big(1 - \int_0^t g(s)\, ds\Big) \left|\nabla u(t)\right|^2 + \frac{\alpha}{2} \left|u(t)\right|^2 \\ + \frac{\gamma}{\xi+2} \left|\left|u(t)\right|\right|_{\xi+2}^{\xi+2} + \frac{1}{2} \left(g \diamond \nabla u\right)(t), \end{aligned} \tag{4.3}$$

we obtain from (4.1) and (4.2) that

$$e'(t) = -\beta \left|\left|u'(t)\right|\right|_{\rho+2}^{\rho+2} + \frac{1}{2} \left(g' \diamond \nabla u\right)(t) - \frac{1}{2} g(t) \left|\nabla u(t)\right|^2. \tag{4.4}$$

We observe that taking the assumption $(H.1)$ into account, we deduce that $e(t) \geq 0$. Now, from (4.4) and considering the assumption $(H.2)$ on the kernel g, we have that $e'(t) \leq 0$. Moreover, since

$$E(t) \leq \left(l^{-1} + 2\right) e(t), \tag{4.5}$$

the decay of $E(t)$ is a consequence of the decay of $e(t)$.

Let us define the Liapunov functional

$$\psi(t) = [e(t)]^{\rho/2} \left(u'(t), u(t)\right). \tag{4.6}$$

Taking the derivative of $\psi(t)$ with respect to t, substituting

$$u'' = \Delta u - \alpha u - \gamma \left|u(t)\right|^{\xi} u - \beta \left|u'\right|^{\rho} u' - \int_0^t g(t-\tau) \Delta u(\tau)\, d\tau$$

in the obtained expression, it follows that

$$\psi'(t) = \frac{\rho}{2}\left[e(t)\right]^{\frac{\rho-2}{2}} e'(t)\left(u'(t),u(t)\right)$$
$$+\left[e(t)\right]^{\rho/2}\Big\{\left|u'(t)\right|^2 - \left|\nabla u(t)\right|^2 - \alpha\left|u(t)\right|^2 - \beta\left(\left|u'(t)\right|^{\rho}u'(t),u(t)\right) \tag{4.7}$$
$$-\gamma\left(\left|u(t)\right|^{\xi}u(t),u(t)\right) + \int_0^t g(t-\tau)\left(\nabla u(\tau),\nabla u(t)\right)d\tau\Big\}.$$

On the other hand, there exists $C > 0$ such that

$$\left|\left(u'(t),u(t)\right)\right| \le Ce(t) \le Ce(0)$$

which implies that

$$-\frac{\rho}{2}\left[e(t)\right]^{\frac{\rho-2}{2}}\left(u'(t),u(t)\right) \le \frac{C\rho}{2}\left[e(0)\right]^{\rho/2}.$$

However, since $-e'(t) \ge 0$, the last inequality gives

$$\frac{\rho}{2}\left[e(t)\right]^{\frac{\rho-2}{2}}\left(u'(t),u(t)\right)e'(t) \le -C_1 e'(t), \tag{4.8}$$

where $C_1 = \frac{\rho C}{2}\left[e(0)\right]^{\rho/2}$. Then, combining (4.7) and (4.8), we infer

$$\psi'(t) \le -C_1 e'(t) + \left[e(t)\right]^{\rho/2}\Big\{\left|u'(t)\right|^2 - \left|\nabla u(t)\right|^2 - \alpha\left|u(t)\right|^2 \tag{4.9}$$
$$-\beta\left(\left|u'(t)\right|^{\rho}u'(t),u(t)\right) - \gamma\left|\left|u(t)\right|\right|_{\xi+2}^{\xi+2} + \int_0^t g(t-\tau)\left(\nabla u(\tau),\nabla u(t)\right)d\tau\Big\}.$$

Now, making use of the Young's inequality $ab \le C(\eta)a^p + \eta b^q$, $\frac{1}{p}+\frac{1}{q}=1$, noting that $\frac{1}{\frac{\rho+2}{\rho+1}}+\frac{1}{\rho+2}=1$ and considering the imersion $H_0^1(\Omega)\hookrightarrow L^{\rho+2}(\Omega)$ (see 1.12), we infer

$$\begin{aligned}\beta\left(\left|u'(t)\right|^{\rho}u'(t),u(t)\right) &\le \beta\left|\left|u'(t)\right|\right|_{\rho+2}^{\rho+1}\left|\left|u(t)\right|\right|_{\rho+2}\\ &\le C(\eta)\left|\left|u'(t)\right|\right|_{\rho+2}^{\rho+2} + \eta\left|\nabla u(t)\right|^{\rho+2} + \eta\left|u(t)\right|^{\rho+2}.\end{aligned} \tag{4.10}$$

Then, from (4.9) and (4.10), we obtain

$$\psi'(t) \le -C_1 e'(t) + [e(t)]^{\frac{\rho}{2}}|u'(t)|^2 + [e(t)]^{\frac{\rho}{2}}\big\{-|\nabla u(t)|^2 + \eta|\nabla u(t)|^{\rho+2}\big\}$$
$$+\eta\left|u(t)\right|^{\rho+2} - \alpha\left[e(t)\right]^{\rho/2}\left|u(t)\right|^2 + C(\eta)\left[e(t)\right]^{\rho/2}\left|\left|u'(t)\right|\right|_{\rho+2}^{\rho+2} \tag{4.11}$$
$$-\gamma\left[e(t)\right]^{\rho/2}\left|\left|u(t)\right|\right|_{\xi+2}^{\xi+2} + \left[e(t)\right]^{\rho/2}\int_0^t g(t-\tau)\left(\nabla u(\tau),\nabla u(t)\right)d\tau.$$

We observe that in view of (1.13) and (4.5), we get

$$|\nabla u(t)|^{\rho+2} \le 2^{\rho/2}(l^{-1}+2)^{\rho/2}[e(t)]^{\rho/2}|\nabla u(t)|^2 \le m[e(0)]^{\rho/2}|\nabla u(t)|^2 \tag{4.12}$$

where $m = 2^{\rho/2}(l^{-1}+2)^{\rho/2}$. Analogously one has

$$|u(t)|^{\rho+2} \leq (\frac{2}{\alpha})^{\rho/2} [e(0)]^{\rho/2} |u(t)|^2 . \tag{4.13}$$

Thus, from (4.11)-(4.13), we get

$$\begin{aligned} \psi'(t) \leq & -C_1 e'(t) + [e(t)]^{\frac{\rho}{2}} |u'(t)|^2 - [e(t)]^{\frac{\rho}{2}} \{1 - \eta m [e(0)]^{\rho/2}\} |\nabla u(t)|^2 \\ & - [e(t)]^{\rho/2} \left\{ \alpha - \eta (\frac{2}{\alpha})^{\rho/2} [e(0)]^{\rho/2} \right\} |u(t)|^2 \\ & + C(\eta) [e(0)]^{\rho/2} \left| |u'(t)| \right|_{\rho+2}^{\rho+2} - \gamma [e(t)]^{\rho/2} ||u(t)||_{\xi+2}^{\xi+2} \\ & + [e(t)]^{\rho/2} \int_0^t g(t-\tau) (\nabla u(\tau), \nabla u(t)) \, d\tau. \end{aligned} \tag{4.14}$$

Choosing $\eta \leq \min\{1/2m[e(0)]^{\rho/2}, \alpha/2(\frac{2}{\alpha})^{\rho/2}[e(0)]^{\rho/2}\}$, we deduce from (4.14)

$$\begin{aligned} \psi'(t) \leq & -C_1 e'(t) + [e(t)]^{\rho/2} \left| u'(t) \right|^2 - \frac{1}{2} [e(t)]^{\rho/2} |\nabla u(t)|^2 - \frac{\alpha}{2} [e(t)]^{\rho/2} |u(t)|^2 \\ & - \gamma [e(t)]^{\rho/2} ||u(t)||_{\xi+2}^{\xi+2} + C [e(0)]^{\rho/2} \left| |u'(t)| \right|_{\rho+2}^{\rho+2} \\ & + [e(t)]^{\rho/2} \int_0^t g(t-\tau) (\nabla u(\tau), \nabla u(t)) \, d\tau. \end{aligned} \tag{4.15}$$

Now, taking the assumption $(H.2)$ and (4.4) under consideration, we obtain from (4.15)

$$\begin{aligned} \psi'(t) \leq & -(C_1 + C_2) e'(t) - \frac{1}{2} [e(t)]^{\rho/2} |\nabla u(t)|^2 + [e(t)]^{\rho/2} \left| u'(t) \right|^2 \\ & - \frac{\alpha}{2} [e(t)]^{\rho/2} |u(t)|^2 - \frac{\gamma}{\xi+2} [e(t)]^{\rho/2} ||u(t)||_{\xi+2}^{\xi+2} \\ & + [e(t)]^{\rho/2} \int_0^t g(t-\tau) (\nabla u(\tau), \nabla u(t)) \, d\tau, \end{aligned} \tag{4.16}$$

where $C_2 = C[e(0)]^{\rho/2}\beta^{-1}$. Defining the perturbed energy by

$$e_\varepsilon(t) = (1 + \varepsilon(C_1 + C_2))\, e(t) + \varepsilon \psi(t), \quad \varepsilon > 0 \tag{4.17}$$

then, it is easy to verify that there exists $L_1 > 0$ such that

$$|e_\varepsilon(t) - e(t)| \leq \varepsilon L_1 e(t).$$

Thus, considering $\varepsilon \in (0, 1/2L_1]$, we have $\frac{1}{2} e(t) \leq e_\varepsilon(t) \leq 2e(t)$ and consequently

$$2^{-\frac{\rho+2}{2}} [e(t)]^{\frac{\rho+2}{2}} \leq [e_\varepsilon(t)]^{\frac{\rho+2}{2}} \leq 2^{\frac{\rho+2}{2}} [e(t)]^{\frac{\rho+2}{2}} ; \; \varepsilon \in (0, 1/2L_1]. \tag{4.18}$$

Getting the derivative of (4.17) with respect to t taking (4.4) and (4.16) into account, we deduce

$$\begin{aligned} e_{\varepsilon}'(t) \le& -\beta \left|\left|u'(t)\right|\right|_{\rho+2}^{\rho+2} - \frac{\xi_2}{2}\left(g \diamond \nabla u\right)(t) + \varepsilon \left[e(t)\right]^{\rho/2} \left|u'(t)\right|^2 \\ &- \frac{\varepsilon}{2}\left[e(t)\right]^{\rho/2} \left|\nabla u(t)\right|^2 - \frac{\alpha\varepsilon}{2}\left[e(t)\right]^{\rho/2}\left|u(t)\right|^2 - \frac{\varepsilon\gamma}{\xi+2}\left[e(t)\right]^{\rho/2} \left|\left|u(t)\right|\right|_{\xi+2}^{\xi+2} \\ &+ \varepsilon \left[e(t)\right]^{\rho/2} \int_0^t g(t-\tau)\left(\nabla u(\tau), \nabla u(t)\right) d\tau. \end{aligned} \tag{4.19}$$

Having in mind that

$$\begin{aligned} \left|\nabla u(t)\right|^2 = 2e(t) - \left|u'(t)\right|^2 + \Big(\int_0^t g(s)\,ds\Big)\left|\nabla u(t)\right|^2 \\ - \alpha\left|u(t)\right|^2 - \left(g\diamond \nabla u\right)(t) - \frac{2\gamma}{\xi+2}\left|\left|u(t)\right|\right|_{\xi+2}^{\xi+2} \end{aligned}$$

we can rewrite (4.19) as

$$\begin{aligned} e_{\varepsilon}'(t) \le& -\beta \left|\left|u'(t)\right|\right|_{\rho+2}^{\rho+2} - \frac{\xi_2}{2}\left(g \diamond \nabla u\right)(t) + \frac{3\varepsilon}{2}\left[e(t)\right]^{\rho/2}\left|u'(t)\right|^2 \\ &- \varepsilon\left[e(t)\right]^{\frac{\rho+2}{2}} - \frac{\varepsilon}{2}\Big(\int_0^t g(s)\,ds\Big)\left[e(t)\right]^{\rho/2}\left|\nabla u(t)\right|^2 \\ &+ \varepsilon\left[e(t)\right]^{\rho/2}\int_0^t g(t-\tau)\left(\nabla u(\tau), \nabla u(t)\right) d\tau + \frac{\varepsilon}{2}\left[e(t)\right]^{\rho/2}\left(g\diamond\nabla u\right)(t). \end{aligned} \tag{4.20}$$

Since Ω possesses finite measure there exists $C = C(meas(\Omega))$ such that

$$\left|u'(t)\right|^{\rho+2} \le C\left|\left|u'(t)\right|\right|_{\rho+2}^{\rho+2}. \tag{4.21}$$

Then, from (4.20) and (4.21) we deduce

$$\begin{aligned} e_{\varepsilon}'(t) \le& -\frac{\beta}{C}\left|u'(t)\right|^{\rho+2} - \frac{\xi_2}{2}\left(g\diamond\nabla u\right)(t) + \frac{3\varepsilon}{2}\left[e(t)\right]^{\rho/2}\left|u'(t)\right|^2 - \varepsilon\left[e(t)\right]^{\frac{\rho+2}{2}} \\ &+ \varepsilon\left[e(t)\right]^{\rho/2}\int_0^t g(t-\tau)\left(\nabla u(\tau),\nabla u(t)\right)d\tau + \frac{\varepsilon}{2}\left[e(t)\right]^{\rho/2}\left(g\diamond\nabla u\right)(t). \end{aligned} \tag{4.22}$$

But, since $\frac{1}{\frac{\rho+2}{\rho}} + \frac{2}{\rho+2} = 1$ the Hölder inequality yields

$$\begin{aligned} \left[e(t)\right]^{\rho/2}\left|u'(t)\right|^2 &\le \frac{\rho}{\rho+2}\Big(\mu\left[e(t)\right]^{\rho/2}\Big)^{\frac{\rho+2}{\rho}} + \frac{2}{\rho+2}\Big(\frac{1}{\mu}\left|u'(t)\right|^2\Big)^{\frac{\rho+2}{2}} \\ &\le \mu^{\frac{\rho+2}{\rho}}\left[e(t)\right]^{\frac{\rho+2}{2}} + \frac{1}{\mu^{\frac{\rho+2}{2}}}\left|u'(t)\right|^{\rho+2} \end{aligned} \tag{4.23}$$

where μ is an arbitrary positive constant. Then, combining (4.22) and (4.23), we deduce

$$e_\varepsilon'(t) \le -\Big(\frac{\beta}{C} - \frac{3\varepsilon}{2}\frac{1}{\mu^{\frac{\rho+2}{2}}}\Big)|u'(t)|^{\rho+2} - \Big(\varepsilon[1 - \frac{3}{2}\mu^{\frac{\rho+2}{\rho}}]\Big)[e(t)]^{\frac{\rho+2}{2}} \tag{4.24}$$
$$- \Big(\frac{\xi_2}{2} - \frac{\varepsilon}{2}[e(0)]^{\rho/2}\Big)\Big(g \diamond \nabla u\Big)(t) + \varepsilon[e(t)]^{\rho/2}\int_0^t g(t-\tau)(\nabla u(\tau), \nabla u(t))d\tau.$$

Analysis of $I_2 := \int_0^t g(t-\tau)\,(\nabla u(\tau), \nabla u(t))\,d\tau$. We have

$$|I_2| \le \int_0^t g(t-\tau)\,|\nabla u(t)|\,\{|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|\} \tag{4.25}$$
$$\le \eta|\nabla u(t)|^2 + \frac{1}{4\eta}\Big(\int_0^t g(t-\tau)|\nabla u(\tau) - \nabla u(t)|d\tau\Big)^2 + \Big(\int_0^t g(s)\,ds\Big)|\nabla u(t)|^2$$
$$\le 2\eta l^{-1}e(t) + \frac{1}{4\eta}\,||g||_{L^1(0,\infty)}\,(g \diamond \nabla u)\,(t) + 2\,||g||_{L^1(0,\infty)}\,l^{-1}e(t).$$

From (4.24) and (4.25) we deduce

$$e_\varepsilon'(t) \le -\Big(\frac{\beta}{C} - \frac{3\varepsilon}{2}\frac{1}{\mu^{\frac{\rho+2}{2}}}\Big)\,\big|u'(t)\big|^{\rho+2}$$
$$- \Big[1 - \Big(\frac{3\mu^{\frac{\rho+2}{\rho}}}{2} + 2\eta l^{-1} + 2l^{-1}\,||g||_{L^1(0,\infty)}\Big)\Big]\varepsilon\,[e(t)]^{\frac{\rho+2}{2}} \tag{4.26}$$
$$- \Big\{\frac{\xi_2}{2} - \varepsilon\,[e(0)]^{\rho/2}\Big(\frac{1}{2} + \frac{1}{4\eta}\,||g||_{L^1(0,\infty)}\Big)\Big\}\,(g \diamond \nabla u)\,(t).$$

Choosing μ, η and $||g||_{L^1(0,\infty)}$ sufficiently small so that

$$\theta = 1 - \Big(\frac{3\mu^{\frac{\rho+2}{\rho}}}{2} + 2\eta l^{-1} + 2l^{-1}\,||g||_{L^1(0,\infty)}\Big) > 0,$$

and choosing ε sufficiently small in order to have

$$\frac{\beta}{C} - \frac{3\varepsilon}{2}\frac{1}{\mu^{\frac{\rho+2}{2}}} > 0 \ \text{ and } \ \frac{\xi_2}{2} - \varepsilon\,[e(0)]^{\rho/2}\Big(\frac{1}{2} + \frac{1}{4\eta}\,||g||_{L^1(0,\infty)}\Big) > 0,$$

from (4.26) we arrive at

$$e_\varepsilon'(t) \le -\varepsilon\theta\,[e(t)]^{\frac{\rho+2}{2}}. \tag{4.27}$$

Combining (4.18) and (4.27) we infer that

$$e_\varepsilon'(t) \le -\frac{N}{2^{\frac{\rho+2}{2}}}\,[e_\varepsilon(t)]^{\frac{\rho+2}{2}},$$

where $N = \varepsilon\theta$. Therefore

$$e_\varepsilon'(t)\,[e_\varepsilon(t)]^{-\frac{\rho+2}{2}} \leq -\frac{N}{2^{\frac{\rho+2}{2}}}. \tag{4.28}$$

But since

$$\frac{d}{dt}\,[e_\varepsilon(t)]^{-\rho/2} = -\frac{\rho}{2}\,[e_\varepsilon(t)]^{-\frac{\rho+2}{2}}\,e_\varepsilon'(t),$$

from (4.28) it holds that

$$\frac{d}{dt}\,[e_\varepsilon(t)]^{-\rho/2} \geq \frac{\rho N}{2^{\frac{\rho+4}{2}}}.$$

Integrating the above inequality over $(0,t)$, it follows that

$$[e_\varepsilon(t)]^{-\rho/2} \geq [e_\varepsilon(0)]^{-\rho/2} + \frac{\rho N}{2^{\frac{\rho+4}{2}}}t. \tag{4.29}$$

Finally, from (4.29) we deduce that

$$e_\varepsilon(t) \leq \left\{ [e_\varepsilon(0)]^{-\rho/2} + \frac{\rho N}{2^{\frac{\rho+4}{2}}}t\right\}^{-2/\rho} \leq \left\{2^{\rho/2}\,[e(0)]^{-\rho/2} + \frac{\rho N}{2^{\frac{\rho+4}{2}}}t\right\}^{-2/\rho},$$

which implies

$$e(t) \leq \left\{ [e(0)]^{-\rho/2} + \frac{\rho N}{2^{\rho+2}}t\right\}^{-2/\rho}.$$

The proof now is completed. □

Remarks. (a) We note that when Ω is any and $\rho = 0$, that is, when we consider the linear dissipation, the exponential decay rates are also obtained, namely, we can find positive constants C, θ and ε_0 such that

$$E(t) \leq C\exp(-\varepsilon\theta t) \quad \forall t \geq 0,\ \ \forall \varepsilon \in (0, \varepsilon_0].$$

(b) We also observe that we can repeat all the considerations used above in order to extend our results for a general class of damped problems

$$u_{tt} - (\Delta)^m\,u + \alpha u + f(u) + \int_0^t g(t-\tau)\,(\Delta)^m\,u(\tau)\,d\tau + h(u_t) = 0 \quad \text{in } \Omega\times(0,\infty)$$

$$u(0) = u^0 \in H_0^m(\Omega); \quad u_t(x) = u^1 \in L^2(\Omega),$$

replacing, of course, the assumptions (1.2) and (1.3) for suitable ones according to Sobolev imersions related to $m \in \mathbf{N}$.

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