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BOUNDED HOLOMORPHIC FUNCTIONAL CALCULUS FOR NON-DIVERGENCE FORM DIFFERENTIAL OPERATORS [1]

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Abstract. Let L be a second-order elliptic partial differential operator of non-divergence form acting on $\mathbf{R^n}$ with bounded coefficients. We show that for each $1 < p_0 < 2, L$ has a bounded H_∞-functional calculus on $L^p(\mathbf{R^n})$ for $p_0 < p < \infty$ if the BMO norm of the coefficients is sufficiently small.

1. Introduction

Let L be a generator of a bounded holomorphic semigroup on a Banach space E. We say that L has a bounded holomorphic functional calculus if the norm of the operator $f(L)$ (see definition (2.4)) is bounded on E by $\|f\|_\infty$ for appropriate bounded holomorphic function f (see definition (2.5)). An example is $f(z) = z^{is}$, s real, then $f(L) = L^{is}$ is the purely imaginary power of L.

When the Banach space E is an L^p space, existence of bounded holomorphic functional calculus implies a number of interesting properties of L. These include square function estimates which are important tools in harmonic analysis (see [15, 5]), and maximal regularity property which is useful in estimates for non-linear differential equations [11].

Holomorphic functional calculus and purely imaginary powers for a non-divergence form differential operator was investigated by many authors, using different methods. Boundedness of these operators was established when certain smoothness conditions are imposed on the coefficients of the differential operators. See for example [17, 6] for the case of the smooth coefficients;

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[16, 2] for Hölder continuous coefficients; [7, 10] for a small L^∞ perturbation of $-\triangle$ on $\mathbf{R^n}$ or uniformly continuous coefficients.

In this paper, we show that for each $1 < p_0 < 2, L$ has a bounded H_∞-functional calculus on $L^p(\mathbf{R^n})$ for $p_0 < p < \infty$ if the BMO norm of the coefficients is sufficiently small. This gives the weakest assumption known so far on the coefficients for L to possess a bounded holomorphic functional calculus.

Our strategy is as follows: we combine the technique of multilinear expansion as in [7, 10] with estimates using wavelet decomposition to study kernel of the operator $f(L)$. We also use certain similar estimates to those on commutators in [3]. Boundedness of $f(L)$ then follows from standard Calderón-Zygmund operator theory.

In Section 2, we give definitions related to holomorphic functional calculus, definitions of wavelets, vaguelettes and their properties. In Section 3, we estimate the bounds on the resolvents of L, and then state our main result (Theorem 3.2). We also state a proposition on the operator norm of the multilinear expansion which implies Theorem 3.2 when BMO norm of the coefficients is sufficiently small. In sections 4 and 5, we prove estimates on kernels of commutators and singular integral operators in the multilinear expansion. We finish the proof of Theorem 3.2 in section 6 by evaluating the operator norm of each term in the multilinear expansion, using the standard T(1) Theorem.

We remark that with some modifications, our method is applicable to a system of differential operators. This should give similar results to ours in the case of systems of elliptic operators.

Throughout, the letter "C" will denote (possibly different) constants that are independent of the essential variables.

2. Preliminaries

In this section we recall some definitions and results on the functional calculus and wavelets.

2.1. **Operators of type ω and holomorphic functional calculus.**

We first review some facts of the functional calculus as introduced by McIntosh [13]; see also [1]. Let $0 \le \omega < \pi$ be given. Then

$$S_\omega := \{z \in \mathbf{C} : |\arg z| \le \omega\} \cup \{0\}$$

denotes the closed sector of angle ω and S^0_ω denotes its interior, while $\dot{S}_\omega := S_\omega \backslash \{0\}$. An operator L on some Banach space E is said to be of type ω if L

is closed and densely defined, $\sigma(L) \subset S_\omega$, and for each $\theta \in (\omega, \pi]$ there exists a constant C_θ such that

$$|\eta| \|(\eta I - L)^{-1}\|_{\mathcal{L}(E)} \leq C_\theta, \qquad \eta \in -\dot{S}_{\pi-\theta}.$$

If $\mu \in (0, \pi]$, then

$$H_\infty(S^0_\mu) := \{f : S^0_\mu \to \mathbf{C}; f \text{ is holomorphic and } ||f||_{H_\infty} < \infty\}, \tag{2.1}$$

where $||f||_{H_\infty} := \sup\{|f(z)| : z \in S^0_\mu\}$. In addition, we define

$$\Psi(S^0_\mu) := \{g \in H_\infty(S^0_\mu) : \exists s > 0, \exists c \geq 0 : |g(z)| \leq c \frac{|z|^s}{1 + |z|^{2s}}\}. \tag{2.2}$$

Let $\omega < \theta < \mu$ and let Γ be the oriented contour given by

$$\gamma(t) := \begin{cases} -te^{-i\theta}, & \text{for } \ t < 0, \\ te^{i\theta}, & \text{for } \ t \geq 0, \end{cases}$$

and $\dot{\Gamma} = \Gamma \backslash \{0\}$. If L is of type ω and $g \in \Psi(S^0_\mu)$, we define $g(L) \in L(E)$ by

$$g(L) := -\frac{1}{2\pi i} \int_\Gamma (\eta I - L)^{-1} g(\eta) d\eta. \tag{2.3}$$

If, in addition, L is one-one and has dense range and if $f \in H_\infty(S^0_\mu)$, then

$$f(L) := [h(L)]^{-1} (fh)(L), \tag{2.4}$$

where $h(z) := z(1+z)^{-2}$. It can be shown that $f(L)$ is a well-defined linear operator in E and that this definition is consistent with definition (2.3) for $f \in \Psi(S^0_\mu)$. The definition of $f(L)$ can even extended to encompass unbounded holomorphic functions; see [13] for details.

Given $N \geq 1$ and $\mu \in (0, \pi]$, we say that an operator L has a bounded H_∞ calculus if L is of type $\omega \in [0, \mu)$, if L is one-one and has dense range, and if $f(L) \in \mathcal{L}(E)$ with

$$||f(L)||_{\mathcal{L}(E)} \leq N ||f||_{H_\infty}, \qquad f \in H_\infty(S^0_\mu), \tag{2.5}$$

for some constant $N \geq 1$. We then denote the class of operators L of type ω which satisfy (2.5) by $\mathcal{H}_\infty(E; N, \mu)$. Let us add the following useful observation, which is based on the Convergence Lemma of McIntosh ([13]). It shows that in order to prove (2.5) we can restrict our attention to function $g \in \Psi(S^0_\mu)$. This has the advantage that we can deal with absolutely convergence Dunford-Taylor integral; see the definition (2.3).

Lemma 2.1. *Let L be of type ω and assume in addition that L is one-one with dense range. Then there exists $k \geq 1$ such that the following statement holds: if*

$$||g(L)||_{\mathcal{L}(E)} \leq N||g||_{H_\infty}, \qquad g \in \Psi(S^0_\mu) \tag{2.6}$$

for some $N \geq 1$, then $L \in \mathcal{H}_\infty(E; kN, \mu)$.

Proof. We refer to [10] Lemma 2.1; see also [13, 1].

2.2. **Wavelets.** Now we recall a multiresolution analysis in $L^2(\mathbf{R^n})$ of regularity $r_0 \geq 3$, denoted by $\{V_j\}_{j\in\mathbf{Z}}$, with the associated orthonormal wavelet basis $(\psi_\lambda)_{\lambda\in\mathbf{I}}$. In this notation, $\mathbf{I}$ is the set of all λ's given by $\lambda = (k+\frac{\epsilon}{2})2^{-j}$, where $j \in \mathbf{Z}, k \in \mathbf{Z^n}$ and $\epsilon \in \{0,1\}^n\backslash\{0,0\}$. For each $\lambda \in \mathbf{I}$, we have

$$\psi_\lambda(x) = 2^{jn/2}\psi_{\frac{\epsilon}{2}}(2^jx-k), \quad x \in \mathbf{R^n}.$$

The set $\{\psi_\lambda\}_{\lambda\in\mathbf{I}}$ is an orthonormal basis of $L^2(\mathbf{R^n})$ and unconditional basis of $W^{s,p}(\mathbf{R^n})$ when $1 < p < \infty$ and $|s| \leq 3$. We assume that the $2^n - 1$ functions $\psi_{\frac{\epsilon}{2}}(x)$ are compactly supported, say $\mathrm{supp}\psi_{\frac{\epsilon}{2}}(x) \subseteq [0, 2^M]^n$ for some integer M, and $\int_{\mathbf{R^n}} \psi_{\frac{\epsilon}{2}}(x)x^\alpha dx = 0$ for any multi-index α with $|\alpha| \leq 2$. Next, we state the definition of vaguelette, see Chapter 8 in [14].

Definition 2.2. A family of functions $f_{j,k}(x), j \in \mathbf{Z}, k \in \mathbf{Z^n}$ is called a family of vaguelettes if there exist real numbers $C > 0$, and $0 < \alpha \leq \beta \leq 1$ such that

(*i*) $|f_{j,k}(x)| \leq C2^{nj/2}\dfrac{1}{(1+|2^jx-k|)^{n+\alpha}}, \ \forall x \in \mathbf{R^n}$;

(*ii*) $|f_{j,k}(x) - f_{j,k}(x+h)| \leq C\dfrac{2^{(n/2+\beta)j}|h|^\beta}{(1+|2^jx-k|)^{n+\alpha}}, \ \forall x \in \mathbf{R^n}$ and $|h| \leq 2^{-j}$;

(*iii*) $\displaystyle\int_{\mathbf{R^n}} f_{j,k}(x)dx = 0.$

If $(\theta_\lambda)_{\lambda\in\mathbf{I}}$ is a family of vaguelettes, we define operator T on the space of finite linear combinations of wavelets by the formula: $T\psi_\lambda = \theta_\lambda$, $\lambda \in \mathbf{I}$. We will often encounter such an operator. Here is a summary of its main properties.

Proposition 2.3. *The previously defined operator T extends to a Calderón-Zygmund operator. It is continuos on L^p when $1 < p < \infty$. Moreover, we have $\|T\|_p \leq CC_p$, where C satisfies (i), (ii) in Definition 2.2.*

Proof. We refer to Theorem 1 in [3] or Chapter 8 in [14].

3. Functional calculus for non-divergence form differential operators

We aim to show that a non-divergence operator L has a bounded holomorphic calculus, provided that the BMO norm of the coefficients is sufficiently small. The operator L is defined by

$$L := L(D) := -\sum_{i,j=1}^{n} a_{ij}(x)D_{ij}$$

acting on $W^{2,p}(\mathbf{R^n})$. We use the notations $D_i = \partial/\partial x_i$, $D_{ij} = \partial^2/\partial x_i \partial x_j$ for $1 \le i,j \le n$. Assume that $A(x) = ((a_{ij}(x))_{1\le i,j\le n}$ is an $n \times n$ matrix of complex, L^∞ coefficients, defined on $\mathbf{R^n}$, and satisfying the ellipticity condition

$$\exists \delta > 0, \quad \forall \xi \in \mathcal{C}^{\mathbf{n}}, \qquad \mathrm{Re} \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\bar{\xi}_j \ge \delta |\xi|^2. \tag{3.1}$$

The ellipticity constant of A is the largest δ which occurs in the above definition, denoted by $\delta(A)$. Throughout this paper, the notation ω always means

$$\omega = \sup_{x\in\mathbf{R^n}} \{|\arg\{\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\bar{\xi}_j\}|; \xi \in \mathcal{C}^{\mathbf{n}}\} < \frac{\pi}{2}. \tag{3.2}$$

Now we recall the definition of $BMO(\mathbf{R^n})$ and $VMO(\mathbf{R^n})$. We say that $f(x)$ belongs to $BMO(\mathbf{R^n})$ if

$$||f||_* = \sup_{B\subset\mathbf{R^n}} \frac{1}{|B|}\int_B |f(x) - f_B|dx < \infty,$$

where B ranges in the class of the balls in $\mathbf{R^n}$. Here f_B is the average $|B|^{-1}\int_B f(x)dx$. For $f \in BMO(\mathbf{R^n})$ and $r > 0$ we set

$$\eta(r) = \sup_{x\in\mathbf{R^n}, \rho\le r} \frac{1}{|B_\rho|}\int_{B_\rho} |f(x) - f_{B_\rho}|dx.$$

We say that $f(x) \in VMO(\mathbf{R^n})$ if $\lim_{r\to 0^+}\eta(r) = 0$.

Let $p \in (1, \infty)$. We have the following lemma.

Lemma 3.1. *Let $\theta \in (\omega, \pi]$ be given, and let the operator L be defined as above. Then there exist positive constants ϵ_0 and C such that $-\dot{S}_{\pi-\theta} \subset \rho(L)$ and*

$$|\eta|\|(\eta I - L)^{-1}\|_{\mathcal{L}(L_p)} \le C_\theta, \quad \eta \in -\dot{S}_{\pi-\theta}, \tag{3.3}$$

provided $||A||_* = \sup_{1\le\alpha,\beta\le n} ||a_{\alpha\beta}||_* < \epsilon_0$. *Moreover,* L *is one-to-one and has dense range.*

Proof. We first prove (3.3). Let $\eta \in -\dot{S}_{\pi-\theta}$ be given. For any $\lambda \in \mathbf{I}$, we define an elliptic operator

$$L_\lambda := -\sum_{\alpha,\beta=1}^{n} m_\lambda(a_{\alpha\beta}) D_{\alpha\beta}, \tag{3.4}$$

and let $l_\lambda(\xi) = \sum_{\alpha,\beta=1}^{n} m_\lambda(a_{\alpha\beta})\xi_\alpha\xi_\beta$, where $m_\lambda(a_{\alpha\beta})$ is the mean value of $a_{\alpha\beta}$ on Q_λ, the unique dyadic cube containing λ and with sides of length 2^{-j}, *i.e.* $m_\lambda(a_{\alpha\beta}) = |Q_\lambda|^{-1}\int_{Q_\lambda} a_{\alpha\beta}(x)dx$. For any $\lambda \in \mathbf{I}$, let $\theta_{\eta,\lambda}(x)$ be a function solving the equation:

$$\big(1-\eta^{-1}L_\lambda\big)\theta_{\eta,\lambda}(x) = \psi_\lambda(x). \tag{3.5}$$

Define

$$Q_\eta\psi_\lambda(x) = \theta_{\eta,\lambda}(x) \qquad \text{and} \qquad (1-\eta^{-1}L)Q_\eta = Id - \overline{[R_\eta, A]}, \tag{3.6}$$

where the kernels $Q_\eta(x,y)$, $\overline{[R_\eta, A]}(x,y)$ of the operators Q_η and $\overline{[R_\eta, A]}$ are given by

$$Q_\eta(x,y) = \sum_\lambda \theta_{\eta,\lambda}(x)\psi_\lambda(y),$$

and

$$\overline{[R_\eta, A]}(x,y) = -\sum_{\alpha,\beta=1}^{n}\sum_\lambda \big[a_{\alpha\beta}(x) - m_\lambda(a_{\alpha\beta})\big]\eta^{-1}D_{\alpha\beta}\theta_{\eta,\lambda}(x)\psi_\lambda(y).$$

For each $p > 1$, from (3.6) we know that a right inverse of $(1-\eta^{-1}L)$ exists if the norm $||\overline{[R_\eta, A]}||_{(p,p)}$ is sufficiently small. Let $\tilde{Q}_\eta$ be the operator defined by

$$\tilde{Q}_\eta\psi_\lambda(x) = (1+4^j\eta^{-1})\theta_{\eta,\lambda}(x).$$

It can be proved that the operator $\tilde{Q}_\eta$ is an isomorphism of L^p onto itself, and then Q_η is an isomorphism of L^p onto $W^{2,p}$ for $1 < p < \infty$. See Chapter 6 in [14]. So, the left inverse of $(1-\eta^{-1}L)$ exists, and is the same as the right one. As in the proof of Proposition 3.1 in [10], we have

$$|\xi|^{|\gamma|}|\partial^\gamma(1-\eta^{-1}l_\lambda(\xi))^{-1}| \le C_\gamma(1+|\eta|^{-1}|\xi|^2)^{-1},$$

$(\eta,\xi) \in -\dot{S}_{\pi-\theta}\times\mathbf{R^n}$, $\gamma \in \mathbf{N^n}$. It is easy to verify that $(1-\eta^{-1}L_\lambda)^{-1}$ is a Fourier multiplier operator with symbol $(1-\eta^{-1}l_\lambda(\xi))^{-1}$. From the standard multiplier result and classical Calderón-Zygmund operator theory we know

that when $1 < p < \infty$:
(i) the operator Q_η is a bounded operator on L^p;
(ii) the functions $(\eta^{-1} D_{\alpha\beta}\theta_{\eta,\lambda}(x))_{\lambda\in\mathbf{I}}$ form a family of vaguelettes. The operator $\overline{[R_\eta, A]}$ can also be proved to be a bounded operator on L^p such that

$$||\overline{[R_\eta, A]}(f)||_p \le C_p ||A||_* ||f||_p, \quad 1 < p < \infty,$$

where C_p is a constant independent of η. This (ii) is proved in Theorem 4′ in [3] or Proposition 2.3 in [12]. It follows from (3.6), (i) and (ii) above that

$$\|\eta(\eta I - L)^{-1}(f)\|_p = ||Q_\eta(I - \overline{[R_\eta, A]})^{-1}(f)||_p \le \sum_{m=0}^{\infty} ||Q_\eta \overline{[R_\eta, A]}^m (f)||_p$$
$$\le C \sum_{m=0}^{\infty} C_p^m ||A||_*^m ||f||_p \le C||f||_p,$$

whenever $\|A\|_* \le \min(\epsilon_0, (2C_p)^{-1})$ for $1 < p < \infty$. Hence $(\eta I - L)$ is one-to-one with closed range and (3.3) is proved.

That L is one-to-one, we leave it to the reader.

The following is the main theorem of this paper.

Theorem 3.2. *Let $\mu \in (\omega, \pi]$ be given, and the operator L defined as above. Then there are constants ϵ_0 and $N \ge 1$ such that $L \in \mathcal{H}_\infty(L_p; N, \mu)$, provided $||A||_* = \sup_{1\le i,j\le n} ||a_{ij}||_* < \epsilon_0$.*

Proof. We fix $\theta \in (\omega, \mu)$. Let $\sigma(t) = e^{i\theta}$ if $t < 0$ and $\sigma(t) = e^{-i\theta}$ if $t \ge 0$.

As in (3.5) and (3.6), for any $\lambda \in \mathbf{I}$, let $\theta_{t,\lambda}(x)$ be a function solving the equation:

$$\big(1 - \sigma(t)t^2 L_\lambda\big)\theta_{t,\lambda}(x) = \psi_\lambda(x). \tag{3.7}$$

Define

$$Q_t\psi_\lambda(x) = \theta_{t,\lambda}(x) \quad \text{and} \quad (1 - \sigma(t)t^2 L)Q_t = Id - \overline{[R_t, A]}, \tag{3.8}$$

whose kernels are given by

$$Q_t(x, y) = \sum_\lambda \big(1 + \sigma(t)t^2 \sum_{u,v=1}^{n} m_\lambda(a_{uv})D_{uv}\big)^{-1} \psi_\lambda(x)\psi_\lambda(y)$$

and

$$\overline{[R_t, A]}(x, y) \quad = \quad -\sigma(t)t^2 \sum_{\alpha,\beta=1}^{n} \sum_\lambda \big[a_{\alpha\beta}(x) - m_\lambda(a_{\alpha\beta})\big]$$

$$\times D_{\alpha\beta}\big(1+\sigma(t)t^2\sum_{u,v=1}^{n} m_\lambda(a_{uv})D_{uv}\big)^{-1}\psi_\lambda(x)\psi_\lambda(y).$$

If $\alpha, \beta \in \{1, 2, \ldots, n\}$, the functions $(t^2 D_{\alpha\beta}\theta_{t,\lambda})_{\lambda\in\mathbf{I}}$ form a family of vaguelettes, and there exists a constant C independent of t such that

$$||\overline{[R_t, A]}(f)||_p \leq C||A||_*||f||_p, \qquad 1 < p < \infty. \tag{3.9}$$

Lemma 3.1 shows that $\dot\Gamma \subset \rho(L)$, and that the spectrum of L lies to the right of $\dot\Gamma$. Here, Γ denotes the contour introduced in section 2. If $F \in \Psi(S^0_\mu)$, then $F(L) \in \mathcal{L}(L^p)$ is well defined, thanks to (2.3) and Lemma 3.1. It follows from Lemma 3.1 and (3.8) that we obtain the following representation formula:

$$\begin{aligned} F(L)(f)(x) : &= -\frac{1}{2\pi i}\int_\Gamma (\lambda I - L)^{-1}F(\lambda)d\lambda(f)(x) \qquad (3.10)\\ &= -\frac{1}{\pi i}\sum_{m=0}^{\infty}\int_{-\infty}^{\infty} Q_t\overline{[R_t, A]}^m F(\tau(t))\frac{dt}{t}(f)(x), \end{aligned}$$

where $\tau(t) = t^{-2}\sigma(-t)$. In order to prove that L has a bounded H_∞-functional calculus, it suffices to establish (2.6), since we have proved that L is one-one and has dense range. Let $F \in \Psi(S^0_\mu)$ be fixed and set

$$T_m := \int_{-\infty}^{\infty} Q_t\overline{[R_t, A]}^m F(\tau(t))\frac{dt}{t}, \quad m = 0, 1, 2, \cdots. \tag{3.11}$$

The main estimate in the proof of Theorem 3.2 is the following proposition.

Proposition 3.3. *Suppose that* $1 < p < \infty$. *There exist constants* C, C_p *such that*
(i) $||T_0||_{\mathcal{L}(L^p)} \leq C$ *when* $m = 0$*; and*
(ii) when $m \geq 1$,

$$\|T_m\|_{\mathcal{L}(L^p)} \leq \begin{cases} C^{m+1}||A||_*^{\frac{m}{n+2}}||F||_{H_\infty}, & 2 \leq p < \infty,\\ C_p^{m+1}||A||_*^{\frac{m}{n+2}}||F||_{H_\infty}, & 1 < p < 2.\end{cases}$$

We will postpone the proof of Proposition 3.3 and finish the proof of Theorem 3.2. It follows immediately from Proposition 3.3 that

$$\|F(L)\|_{\mathcal{L}(L^p)} \leq \begin{cases} C||F||_{H_\infty}, & 2 \leq p < \infty,\\ C_p||F||_{H_\infty}, & 1 < p < 2.\end{cases}$$

for all $F \in \Psi(S^0_\mu)$, whenever $\|A\|_* \leq \min(\epsilon_0, (2C_p)^{-n-2})$ for $1 < p < 2$; and whenever $\|A\|_* \leq \min(\epsilon_0, (2C)^{-n-2})$ for $2 \leq p < \infty$. Thanks to Lemma 2.1, the proof of Theorem 3.2 is now complete. What remains of this paper is to prove Proposition 3.3 using several lemmas in Sections 4,5 and 6.

As a corollary of Theorem 3.2, we can show the existence of a bounded H_∞-calculus of the operator L with VMO coefficients, whose proof is obtained by using the method of localization and approximation introduced in [2], Section 3. See also [10], Section 6. We state the result but omit its detailed proof.

Corollary 3.4. *Let $\mu \in (\omega, \pi]$ be given, and the operator L defined as above. We assume that the coefficients $(a_{ij}(x))_{1\le i,j\le n}$ of L belong to $VMO(\mathbf{R^n}) \cap L^\infty(\mathbf{R^n})$. Then there exist constants $s > 0$ and $N \ge 1$ such that*

$$sI + L \in \mathcal{H}_\infty(L_p; N, \mu), \qquad 1 < p < \infty.$$

4. The study of $\overline{[\mathbf{R_t}, \mathbf{A}]}$

In this section we first study the operator $\overline{[R_t, A]}$ defined as in (3.8). We have the following lemma.

Lemma 4.1. *Let $0 < t < \infty$. The distribution kernel $\overline{[R_t, A]}(x, y)$ of the operator $\overline{[R_t, A]}$ admits a decomposition*

$$\begin{aligned} \overline{[R_t, A]}(x, y) &= \rho_t(x, y) + \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(x) \sqcap_t^{\alpha\beta} (x, y) - \overline{\sqcap}_t(x, y) \qquad (4.1)\\ &\stackrel{\text{def}}{=} \rho_t(x, y) + \sqcap_t(x, y), \end{aligned}$$

where the distribution kernel $\rho_t(x, y)$ is supported in $\{(x, y) \in \mathbf{R^n} \times \mathbf{R^n} : |x - y| \le t\}$, and $\sqcap_t^{\alpha\beta}(x, y)$ satisfies

$$| \sqcap_t^{\alpha\beta} (x, y)| \le Ct^{-n}(1 + \frac{|x - y|}{t})^{-n-2}, \qquad (4.2)$$

and

$$| \nabla_x \sqcap_t^{\alpha\beta}(x, y)| + | \nabla_y \sqcap_t^{\alpha\beta}(x, y)| \le Ct^{-n-1}(1 + \frac{|x - y|}{t})^{-n-3}. \qquad (4.3)$$

The same estimates (4.2) *and* (4.3) *hold for the distribution kernel $\overline{\sqcap}_t(x, y)$.*

Proof. We consider $G_{t,\lambda}(x)$, the fundamental solution of the operator

$$P_{t,\lambda}(D) = -(\sigma^{-1}(t)t^{-2} + \sum_{\alpha,\beta=1}^{n} m_\lambda(a_{\alpha\beta})D_{\alpha\beta}).$$

For any $\alpha, \beta \in \{1, 2, \cdots, n\}$, denote $\theta_{t,\lambda}^{\alpha\beta}(x) = D_{\alpha\beta} G_{t,\lambda} * \psi_\lambda(x)$. We have

$$\overline{[R_t, A]}(x,y) = \sum_{\alpha,\beta=1}^{n} \sum_{\lambda} \big[a_{\alpha\beta}(x) - m_\lambda(a_{\alpha\beta})\big] \theta_{t,\lambda}^{\alpha\beta}(x)\psi_\lambda(y)$$
$$\stackrel{\text{def}}{=} \sum_{\alpha,\beta=1}^{n} \sum_{\lambda} \big[a_{\alpha\beta}(x) - m_\lambda(a_{\alpha\beta})\big] t^{-n} \int_{\mathbf{R^n}} G_\lambda^{\alpha\beta}(\frac{x-w}{t})\psi_\lambda(w)dw\psi_\lambda(y).$$

By the ellipticity condition (3.1) we have

$$|\partial_\xi^\gamma(\hat{G}_\lambda^{\alpha\beta})(\xi)| \leq C_\gamma |\xi|^{-\gamma} \quad \text{for all } \gamma \in \mathbf{N^n}.$$

Then, from standard classical harmonic analysis we have

$$|\partial_x^\gamma(G_\lambda^{\alpha\beta})(x)| \leq C_\gamma^{'} |x|^{-n-\gamma} \quad \text{for all } \gamma \in \mathbf{N^n}.$$

See, for example, Proposition 2 in Chapter 6 ([18]).

Similarly to the proof of Lemma 5.1 in [10], for any $k \in \mathbf{N}$, the functions $(G_\lambda^{\alpha\beta}(x))_{\lambda\in\mathbf{I}}$ satisfy $\int_{\mathbf{R^n}} G_\lambda^{\alpha\beta}(x)x^\gamma dx = 0$ for any multi-index γ with $|\gamma| \leq 1$ and

$$|G_\lambda^{\alpha\beta}(x)| + |\bigtriangledown G_\lambda^{\alpha\beta}(x)| \leq C_k |x|^{-k}, \qquad |x| \geq 1.$$

Let $2^{-j_0-1} \leq t < 2^{-j_0}$ for some $j_0 \in \mathbf{Z}$, and denote $\Theta_{j_0} = j_0 + M + 2 + \log\sqrt{n}$. Set $G_\lambda^{\alpha\beta}(x) = G_{1,\lambda}^{\alpha\beta}(x) + G_{2,\lambda}^{\alpha\beta}(x)$, where $G_{1,\lambda}^{\alpha\beta}(x) = 1$ if $|x| \leq 1/3$; $G_{1,\lambda}^{\alpha\beta}(x) = 0$ if $x \geq 1/2$ and $\int_{\mathbf{R^n}} G_{1,\lambda}^{\alpha\beta}(x)x^\gamma dx = 0$ for any multi-index $|\gamma| \leq 1$. We define

$$\rho_t(x,y) = \sum_{\alpha,\beta=1}^{n} \sum_{\lambda:\lambda(j)>\Theta_{j_0}} \big[a_{\alpha\beta}(x) - m_\lambda(a_{\alpha\beta})\big] t^{-n} \int_{\mathbf{R^n}} G_{1,\lambda}^{\alpha\beta}(\frac{x-w}{t})\psi_\lambda(w)dw\psi_\lambda(y), \tag{4.4}$$

$$\sqcap_t^{\alpha\beta}(x,y) = \sum_{\lambda:\lambda(j)\leq\Theta_{j_0}} t^{-n} \int_{\mathbf{R^n}} G_{1,\lambda}^{\alpha\beta}(\frac{x-w}{t})\psi_\lambda(w)dw\psi_\lambda(y) \tag{4.5}$$
$$+ \sum_{\lambda} t^{-n} \int_{\mathbf{R^n}} G_{2,\lambda}^{\alpha\beta}(\frac{x-w}{t})\psi_\lambda(w)dw\psi_\lambda(y)$$
$$\stackrel{\text{def}}{=} \sqcap_{1,t}^{\alpha\beta}(x,y) + \sqcap_{2,t}^{\alpha\beta}(x,y)$$

$$\overline{\sqcap}_t(x,y) = \sum_{\alpha,\beta=1}^{n} \sum_{\lambda:\lambda(j)\leq\Theta_{j_0}} m_\lambda(a_{\alpha\beta}) t^{-n} \int_{\mathbf{R^n}} G_{1,\lambda}^{\alpha\beta}(\frac{x-w}{t})\psi_\lambda(w)dw\psi_\lambda(y) \tag{4.6}$$

$$+\sum_{\alpha,\beta=1}^{n}\sum_{\lambda} m_\lambda(a_{\alpha\beta})t^{-n}\int_{\mathbf{R^n}} G_{2,\lambda}^{\alpha\beta}(\frac{x-w}{t})\psi_\lambda(w)dw\psi_\lambda(y).$$

Then, by (4.4)-(4.6) we obtain the decomposition (4.1):

$$\overline{[R_t,A]}(x,y)=\rho_t(x,y)+\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(x)\sqcap_t^{\alpha\beta}(x,y)-\overline{\sqcap}_t(x,y).$$

We first consider the kernel $\rho_t(x,y)$. For any $j>\Theta_{j_0}=j_0+M+2+\log\sqrt{n}$, the properties of functions $G_{1,\lambda}^{\alpha\beta}(x)$ and wavelets $\psi_{\frac{\epsilon}{2}}(x)$ with support $\mathrm{supp}\psi_{\frac{\epsilon}{2}}(x)\subset[0,2^M)^n$ show that the variables x,w,y in the expression in $\rho_t(x,y)$ satisfy

$$|w-y|\le 2^{M-j}\sqrt{n}\le 2^{-j_0-2}<t/4 \qquad \text{and} \quad |x-w|\le t/2.$$

This implies

$$|x-y|\le|x-w|+|w-y|\le 3t/4<t, \qquad \text{if } \rho_t(x,y)\ne 0.$$

Hence, the distribution kernel $\rho_t(x,y)$ is supported in the set $\{(x,y)\in\mathbf{R^n}\times\mathbf{R^n}:|x-y|\le t\}$.

Now we study the kernel $\sqcap_{1,t}^{\alpha\beta}(x,y)$ when $|x-y|\le t$. Since $\int_{\mathbf{R^n}} G_{1,\lambda}^{\alpha\beta}(x)dx=0$, we have

$$\begin{aligned}|\sqcap_{1,t}^{\alpha\beta}(x,y)|&\le\sum_{j\le\Theta_{j_0}}2^{jn}|\sum_{k,\epsilon}t^{-n}\int_{\mathbf{R^n}}G_{1,\lambda}^{\alpha\beta}(\frac{x-w}{t})\\&\times(\psi_{\frac{\epsilon}{2}}(2^jw-k)-\psi_{\frac{\epsilon}{2}}(2^jx-k))dw\psi_{\frac{\epsilon}{2}}(2^jy-k)|\\&\le C\sum_{j\le\Theta_{j_0}}2^{jn}t^{-n}\int_{|x-w|\le t/2}(\frac{|x-w|}{t})^{-n}2^j|x-w|dw \qquad (4.7)\\&\le C\sum_{j\le j_0+M+2+\log\sqrt{n}}2^{j(n+1)}\int_{|x-w|\le t/2}|x-w|^{-n+1}dw\le Ct^{-n}.\end{aligned}$$

When $|x-y|>t$: For any $j\le\Theta_{j_0}$, the variables x,w,y in the expression in $\sqcap_{1,t}^{\alpha\beta}(x,y)$ satisfy

$$|x-w|\le t/2\le 2^{-j_0+1}\le C2^{-j} \qquad \text{and} \quad |w-y|\le 2^{-j+M}\sqrt{n}.$$

This implies $|x-y| \le C2^{-j}$. Noting that $\int_{\mathbf{R^n}} G^{\alpha\beta}_{1,\lambda}(x)x^\gamma dx = 0$ for any $|\gamma| \le 1$, we have

$$|\sqcap^{\alpha\beta}_{1,t}(x,y)| \le \sum_{j\le -\log|x-y|+C} 2^{jn}t^{-n}|\sum_{k,\epsilon}\int_{\mathbf{R^n}} G^{\alpha\beta}_{1,\lambda}(\frac{x-w}{t})[\psi_{\frac{\epsilon}{2}}(2^j w-k)$$
$$-\sum_{p:|p|\le 1}\psi^{(p)}_{\frac{\epsilon}{2}}(2^j x-k)[2^j(x-w)]^p]dw\psi_{\frac{\epsilon}{2}}(2^j y-k)| \tag{4.8}$$
$$\le \sum_{j\le -\log|x-y|+C} 2^{j(n+2)}t^{-n}\int_{|x-w|\le t/2}(\frac{|x-w|}{t})^{-n}|x-w|^2 dw \le Ct^2|x-y|^{-n-2},$$

which proves $|\sqcap^{\alpha\beta}_{1,t}(x,y)| \le Ct^{-n}(1+\frac{|x-y|}{t})^{-n-2}$ by the estimates (4.7) and (4.8).

Next we estimate $\sqcap^{\alpha\beta}_{2,t}(x,y)$. We first consider the case $|x-y|\le t$. For the functions $\psi_{\frac{\epsilon}{2}}(2^j w-k) = \prod_{i=1}^n \psi_{\frac{\epsilon_i}{2}}(2^j w_i - k_i)$ with $\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n) \in E$, without loss of generality we assume that $\epsilon_1 = 1$. Define

$$\tilde{\psi}_{\frac{\epsilon}{2}}(2^j w-k) = \int_0^{w_1}\psi_{\frac{\epsilon_1}{2}}(2^j z-k_1)dz\prod_{i=2}^n \psi_{\frac{\epsilon_i}{2}}(2^j w_i-k_i).$$

Using the facts that

$$\mathrm{supp}\tilde{\psi}_{\frac{\epsilon}{2}}(2^j w-k) \subset [0,2^M)^n, \qquad \int_{\mathbf{R^n}}\tilde{\psi}_{\frac{\epsilon}{2}}(2^j w-k)dw = 0$$

and

$$|\sum_{\epsilon\in E}\sum_{k\in\mathbf{Z^n}} 2^j\tilde{\psi}_{\frac{\epsilon}{2}}(2^j w-k)\psi_{\frac{\epsilon}{2}}(2^j y-k)| \le C,$$

we have

$$|\sqcap^{\alpha\beta}_{2,\mathrm{t}}(\mathrm{x},\mathrm{y})| \le |\sum_{\lambda:\lambda(j)\le\Theta_{j_0}} t^{-n}\int_{\mathbf{R^n}} G^{\alpha\beta}_{2,\lambda}(\frac{x-w}{t})\psi_\lambda(w)dw\psi_\lambda(y)|$$
$$+|\sum_{\lambda:\lambda(j)>\Theta_{j_0}} t^{-n}\int_{\mathbf{R^n}} G^{\alpha\beta}_{2,\lambda}(\frac{x-w}{t})\psi_\lambda(w)dw\psi_\lambda(y)| \tag{4.9}$$
$$\le C\sum_{\lambda:\lambda(j)\le j_0+M+2+\log\sqrt{n}} 2^{jn}t^{-n}\int_{|x-w|>t/2}(\frac{|x-w|}{t})^{-n-1}dw$$

$$+ C \sum_{j>\Theta_{j_0}} \sum_{\epsilon,k} 2^{jn} t^{-n} | \int_{\mathbf{R^n}} D_{w_1}[G^{\alpha\beta}_{2,\lambda}(\frac{x-w}{t})] \tilde{\psi}_{\frac{\epsilon}{2}}(2^j w - k) dw \psi_{\frac{\epsilon}{2}}(2^j y - k)|$$

$$\leq C \sum_{j \leq j_0 + M + 2 + \log\sqrt{n}} 2^{jn} + C \sum_{j > j_0 + M + 2 + \log\sqrt{n}} 2^{-j} t^{-n-1} \leq C t^{-n}.$$

The case $|x-y| > t$: For any $j > \Theta_{j_0}$ the variables w, y in the term $\sqcap^{\alpha\beta}_{2,\mathrm{t}}(\mathrm{x},\mathrm{y})$ must satisfy

$$|y - w| < 2^{M-j}\sqrt{n} < 2^{-j_0-1} < t/2 < |x-y|/2,$$

which implies $|x-w| > |x-y|/2$. Then

$$\sum_{j>\Theta_{j_0}} \sum_{\epsilon,k} 2^{jn} t^{-n} | \int_{\mathbf{R^n}} G^{\alpha\beta}_{2,\lambda}(\frac{x-w}{t}) \psi_{\frac{\epsilon}{2}}(2^j w - k) dw \psi_{\frac{\epsilon}{2}}(2^j y - k)| \tag{4.10}$$

$$\leq \sum_{j>\Theta_{j_0}} \sum_{\epsilon,k} 2^{jn} t^{-n} | \int_{\mathbf{R^n}} D_{w_1}[G^{\alpha\beta}_{2,\lambda}(\frac{x-w}{t})] \tilde{\psi}_{\frac{\epsilon}{2}}(2^j w - k) dw \psi_{\frac{\epsilon}{2}}(2^j y - k)|$$

$$\leq \sum_{j > j_0 + M + 2 + \log\sqrt{n}} 2^{-j} t^{-n} |\frac{x-y}{t}|^{-n-2} t^{-1} \leq C t^2 (x-y)^{-n-2}.$$

On the other hand, for any fixed $x, y \in \mathbf{R^n}$, set $\varphi_1(w-x) + \varphi_2(w-x) = 1$, where $\varphi_1(w-x) = 1$ if $|w-x| \geq |x-y|/3$ and $\varphi_1(w-x) = 0$ if $|w-x| \leq |x-y|/2$. Since $\int_{\mathbf{R^n}} G^{\alpha\beta}_{1,\lambda}(x) x^\gamma dx = 0$ for any $|\gamma| \leq 1$, one writes

$$\begin{aligned}
&| \sum_{\lambda:\lambda(j)\leq\Theta_{j_0}} t^{-n} \int_{\mathbf{R^n}} G^{\alpha\beta}_{2,\lambda}(\frac{x-w}{t}) \psi_\lambda(w) dw \psi_\lambda(y)| \\
&= \sum_{j\leq\Theta_{j_0}} 2^{jn} | \sum_{k,\epsilon} t^{-n} \int_{\mathbf{R^n}} \varphi_1(w-x) G^{\alpha\beta}_{2,\lambda}(\frac{x-w}{t}) [\psi_{\frac{\epsilon}{2}}(2^j w - k) \\
&- \sum_{p:|p|\leq 1} \psi^{(p)}_{\frac{\epsilon}{2}}(2^j x - k)[2^j(x-w)]^p] dw \psi_{\frac{\epsilon}{2}}(2^j y - k)| \\
&+ \sum_{j\leq\Theta_{j_0}} 2^{jn} | \sum_{k,\epsilon} t^{-n} \int_{\mathbf{R^n}} \varphi_2(w-x) G^{\alpha\beta}_{2,\lambda}(\frac{x-w}{t}) [\psi_{\frac{\epsilon}{2}}(2^j w - k) \\
&- \sum_{p:|p|\leq 1} \psi^{(p)}_{\frac{\epsilon}{2}}(2^j x - k)[2^j(x-w)]^p] dw \psi_{\frac{\epsilon}{2}}(2^j y - k)| = \mathrm{I} + \mathrm{II}.
\end{aligned}$$

We now estimate term I. For any $j \in \mathbf{Z}$ the variables x, w, y in its expression satisfy $|x-y| \leq |x-w| + |w-y| \leq |x-y|/2 + 2^{M-j}\sqrt{n}$ when $j \leq \Theta_{j_0}$, which implies $j \le -\log|x-y| + C$. Then,

$$\begin{aligned} \mathrm{I} &\le C \sum_{j\le -\log|x-y|+C} 2^{j(n+2)} t^{-n} \int_{t/2<|x-w|} \left|\frac{x-w}{t}\right|^{-n-3} |x-w|^2 dw \\ &\le Ct^2 |x-y|^{-n-2}. \end{aligned} \tag{4.11}$$

Since $\varphi_2(x-w) = 0$ if $|x-w| \le |x-y|/3$, we have

$$\begin{aligned} \mathrm{II} \le & \sum_{j\le\Theta_{j_0}} 2^{jn} t^{-n} \int_{|x-w|>|x-y|/3} \left|\frac{x-w}{t}\right|^{-2n-2} dw \qquad (4.12) \\ & + \sum_{j\le\Theta_{j_0}} 2^{jn} t^{-n} \sum_{p:|p|\le 1} 2^{j|p|} \int_{|x-w|>|x-y|/3} \left|\frac{x-w}{t}\right|^{-(2n+2+p)} |x-w|^p dw \\ & \le Ct^2 |x-y|^{-n-2}, \end{aligned}$$

which proves $|\sqcap_{2,t}^{\alpha\beta}(x,y)| \le Ct^{-n}(1+\frac{|x-y|}{t})^{-n-2}$ by combining the estimates (4.9)-(4.12). This gives the proof of (4.2).

Similarly to the above estimates (4.7)-(4.12), we have the estimates (4.3) and those of the kernel $\overline{\sqcap}_t(x,y)$. We omit the details here.

The proof of Lemma 4.1 is complete.

Lemma 4.2. *Let Q_t defined as in* (3.8), $U_t = Q_t(1-t^2\triangle)$ *and* $V_t = (1-t^2\triangle)Q_t$.

(*i*) *The distribution kernel $Q_t(x,y)$ of the operator Q_t satisfies*

$$|Q_t(x,y)| + t|\nabla_x Q_t(x,y)| \le \begin{cases} Ct^{-n}(\frac{|x-y|}{t})^{1-n}, & \text{if } |x-y| \le t; \\ Ct^{-n}(1+\frac{|x-y|}{t})^{-n-2}, & \text{if } |x-y| > t. \end{cases} \tag{4.13}$$

(*ii*) *The distribution kernel $U_t(x,y)$ of the operator U_t admits a decomposition $U_t(x,y)(x,y) = \rho_t(x,y) + \sqcap_t(x,y)$, where the distribution kernel $\rho_t(x,y)$ is supported in $\{(x,y) \in \mathbf{R^n}\times\mathbf{R^n} : |x-y| \le t\}$, and $\sqcap_t(x,y)$ satisfies*

$$|\sqcap_t(x,y)| + t|\bigtriangledown_x \sqcap_t(x,y)| + t|\bigtriangledown_y \sqcap_t(x,y)| \le Ct^{-n}(1+\frac{|x-y|}{t})^{-n-2},$$

The above assertions hold for the distribution kernel $V_t(x,y)$ of the operator V_t.

Proof. The proof of this lemma is similar to those in Lemma 4.1. We omit its details here.

Lemma 4.3. *The operators ρ_t and $\sqcap_t$ in* (4.1) *are continuous on L^p when $1<p<\infty$. Moreover, we have*

$$\|\rho_t(f)\|_p + \|\sqcap_t(f)\|_p \le C\|A\|_*\|f\|_p, \tag{4.14}$$

where C is a constant independent of t.

Proof. We first recall that the distributional kernel $\rho_t(x,y)$ is defined by

$$\rho_t(x,y) = \sum_{\alpha,\beta=1}^{n} \sum_{\lambda:\lambda(j)>j_0+M+2} \big[a_{\alpha\beta}(x) - m_\lambda(a_{\alpha\beta})\big]\tilde{\theta}_{t,\lambda}^{\alpha\beta}(x)\psi_\lambda(y),$$

where

$$\tilde{\theta}_{t,\lambda}^{\alpha\beta}(x) = t^{-n}\int_{\mathbf{R^n}} G_{1,\lambda}^{\alpha\beta}(\frac{x-w}{t})\psi_\lambda(w)dw$$

for any $\lambda \in \mathbf{I}$ and $\alpha,\beta \in \{1,2,\cdots,n\}$.

As in the proof of Corollary 1 in [3], the functions $(\tilde{\theta}_{t,\lambda}^{\alpha\beta})_{\lambda\in\mathbf{I}}$ form a family of vaguelettes in Definition 2.2. It follows that ρ_t is continuous on L^p when $1 < p < \infty$. Thanks to (3.9), we also have the estimate (4.14) for the operator $\sqcap_t$.

5. Kernel estimates

In this section we give estimates on the kernels of operators similar to T_m in (3.11). Let T be a bounded operator from the class $\mathcal{S}(\mathbf{R^n})$ of Schwartz functions to its dual $\mathcal{S}'(\mathbf{R^n})$. As in [7], a continuous function K defined on $\Omega = \mathbf{R^n}\times\mathbf{R^n}\backslash\Theta$, where $\Theta = \{(x,y); x=y\}$, is called a standard kernel if there exist constants $\delta\in(0,1]$ and $C_K > 0$ such that

(i) For all $(x,y)\in\Omega$, $|K(x,y)| \le C_K|x-y|^{-n}$, and

(ii) For all x,x',y such that $|x-x'| < \frac{1}{2}|x-y|$,

$$|K(x',y) - K(x,y)| + |K(y,x') - K(y,x)| \le C_K\frac{|x-x'|^\delta}{|x-y|^{n+\delta}},$$

(iii) For all functions f and $g\in\mathcal{C}_0^\infty(\mathbf{R^n})$ with disjoint supports,

$$\langle Tf,g\rangle = \int_{\mathbf{R^n}}(Tf)(x)g(x)dx = \int_{\mathbf{R^n}}\int_{\mathbf{R^n}} K(x,y)f(y)g(x)dxdy.$$

We now state the following proposition, which will be used in the proof of Proposition 3.3.

Proposition 5.1. *For $0 < t < \infty$. Let $Q_t(x,y)$ be the distribution kernel of the operator Q_t which satisfies the estimate* (4.13) *and $\overline{[R_t,A]}$ be defined as in* (3.8). *Denote $\mu(t)$ a measurable function such that $\|\mu\|_\infty \le 1$. Then for any $m\in\mathbf{N}$ the kernel $K(x,y)$ of the operator $T_m := \int_0^\infty Q_t\overline{[R_t,A]}^m\mu(t)\frac{dt}{t}$ satisfies*

$$\int_{|x-y|>2|x'-x|}|K(x,y)-K(x',y)|dy \le C^{m+1}||A||_*^{\frac{m}{n+2}}. \tag{5.1}$$

Let W_t be the operator of convolution with the function $w_t(x) = t^{-n}w(x/t)$, where w satisfies

$$|w(u)| + |\nabla w(u)| \leq C(1+|u|)^{-n-2}, \tag{5.2}$$

and let $B(x)$ be a bounded measurable function such that $||B||_\infty \leq 1$. Then, the kernel of the operator $\int_0^\infty Q_t\overline{[R_t, A]}^m BW_t\mu(t)\frac{dt}{t}$ satisfies the standard kernel estimates (i) and (ii) with a constant $C_K = C^{m+1}||A||_*^{\frac{m}{n+2}}$.

Analogous statements hold for $-\infty < t < 0$.

To prove Proposition 5.1, we first introduce a lemma. Define the kernels $\tilde{p}_t(x,y)$ and $p_t(x,y)$ by

$$\tilde{p}_t(x,y) = t^{-n}(1+\frac{|x-y|}{t})^{-n-2}, \quad p_t(x,y) = t^{-n}(1+\frac{|x-y|}{t})^{-n-1}.$$

Lemma 5.2. *Let $0 < t < \infty$ and $1 \leq k \leq m$ be fixed. Let S_t and V_t be two operators given by its kernels $S_t(x,y)$ and $V_t(x,y)$ satisfying*

$$\begin{aligned} |S_t(x,y)| &\leq Ct^{-n}(\frac{|x-y|}{t})^{1-n}, \quad \text{when } |x-y| \leq t; \\ |S_t(x,y)| &\leq C\tilde{p}_t(x,y), \quad \text{when } |x-y| \geq t; \\ |V_t(x,y)| &\leq C\tilde{p}_t(x,y), \quad \text{for all } (x,y) \text{ in } \mathbf{R^{2n}}. \end{aligned}$$

Then, the kernel $K_t(x,y)$ of the operator $S_t\overline{[R_t, A]}^k V_t$ satisfies

$$|K_t(x,y)| \leq C^{k+1}||A||_*^{\frac{k}{n+2}} p_t(x,y). \tag{5.3}$$

Let W_t be the convolution operator and $B(x)$ be a bounded function as in Proposition 5.1, and Q_t be defined as in Lemma 4.2, then the kernel $K_t(x,y)$ of the operator $Q_t\overline{[R_t, A]}^k BW_t$ satisfies

$$|K_t(x,y)| \leq C^{k+1}||A||_*^{\frac{k}{n+2}} p_t(x,y); \tag{5.4}$$

$$|K_t(x',y) - K_t(x,y)| \leq C^{k+1}||A||_*^{\frac{k}{n+2}}(\frac{|x'-x|}{t})^\delta p_t(x,y), \quad \text{when } |x-x'| \leq t; \tag{5.5}$$

$$|K_t(x,y') - K_t(x,y)| \leq C^{k+1}||A||_*^{\frac{k}{n+2}}\frac{|y'-y|}{t} p_t(x,y), \quad \text{when } |y-y'| \leq t, \tag{5.6}$$

where $\delta > 0$ is a constant independent of t.

Proof. We first prove (5.3). Choosing an index r such that $1 < r < \frac{1}{n-1}$, and r' satisfying $\frac{1}{r} + \frac{1}{r'} = 1$. Since the operator $\overline{[R_t, A]}$ is a bounded operator (see (3.9)), we obtain

$$\begin{aligned}|K_t(x,y)| &= |S_t\overline{[R_t,A]}^k V_t(x,y)| \le \int_{\mathbf{R^n}} |S_t(x,w)\overline{[R_t,A]}^k V_t(w,y)|dw \qquad (5.7)\\ &\le (\int_{\mathbf{R^n}} |S_t(x,w)|^r dw)^{1/r} (\int_{\mathbf{R^n}} |\overline{[R_t,A]}^k V_t(w,y)|^{r'} dw)^{1/r'}\\ &\le Ct^{-n+\frac{n}{r}} ||\overline{[R_t,A]}^k V_t(w,y)||_{L^{r'}(w)} \le C^{k+1}||A||_*^k t^{-n+\frac{n}{r}} t^{-n+\frac{n}{r'}}\\ &\le C^{k+1}||A||_*^k t^{-n}.\end{aligned}$$

Similarly to the proof of Lemma 8 in [7], using the decomposition on the operator $\overline{[R_t,A]}$ in Lemma 4.1 and the following integral estimate

$$\int_{\mathbf{R^n}} \tilde{p}_t(x-u)\tilde{p}_t(u-y)du \le C\tilde{p}_t(x,y),$$

we obtain

$$|K_t(x,y)| \le C^{k+1}\tilde{p}_t(x,y). \qquad (5.8)$$

By combining (5.7) and (5.8) we have

$$|K_t(x,y)| = |K_t(x,y)|^{\frac{1}{n+2}} \cdot |K_t(x,y)|^{\frac{n+1}{n+2}} \le C^{k+1}||A||_*^{\frac{k}{n+2}} p_t(x,y),$$

which proves (5.3).

The proofs of (5.4)-(5.6) are similar as above, see Lemma 10 in [7]. Hence, the proof of Lemma 5.2 is complete.

Proof of Proposition 5.1. The second statement of Proposition 5.1 follows easily from (5.4)-(5.6). For the first statement, it can be verified that Proposition 2 in [7] remains valid in our situation using the estimates in Lemma 4.1 and the argument of the Lemma 5.2. We omit its details here.

6. Proof of Proposition 3.3

We first prove (i). For any $\lambda \in \mathbf{I}$, Let L_λ and $l_\lambda(\xi)$ defined as in Lemma 3.2. Set

$$F(l_\lambda)(\xi) := \int_\Gamma (\eta I - l_\lambda(\xi))^{-1} F(\eta) d\eta. \qquad (6.1)$$

Note that the distribution kernel $K_0(x,y)$ of T_0 is

$$K_0(x,y) = -\frac{1}{2}\sum_\lambda \int_\Gamma (\eta I - L_\lambda)^{-1} F(\eta) d\eta (\psi_\lambda)(x)\psi_\lambda(y) \overset{\text{def}}{=} \sum_\lambda \theta_\lambda(x)\psi_\lambda(y).$$

It was proved in [10] that for every $\lambda \in \mathbf{I}$, the operator

$$F(L_\lambda) := \int_\Gamma (\eta I - L_\lambda)^{-1} F(\eta) d\eta \tag{6.2}$$

has a bounded H_∞-calculus associated with the elliptic operator L_λ. Similar arguments as in the proof of [2] Lemma 5.4 show that

$$|\xi|^{|\alpha|} |\partial^\alpha F(l_\lambda)(\xi)| \leq C_\alpha ||F||_{H_\infty}, \quad \xi \in \mathbf{R^n}, \quad \alpha \in \mathbf{N^n}. \tag{6.3}$$

Moreover, it is easy to verify that $F(L_\lambda)$ is a Fourier multiplier operator with symbol $F(l_\lambda)(\xi)$. As in Lemma 3.1 we see that the functions $(\theta_\lambda(x))_{\lambda \in \mathbf{I}}$ form a family of vaguelettes, and for $1 < p < \infty$, the operator T_0 is a bounded operator on L^p. This proves (i).

We now prove (ii). When $m \geq 1$, we study the operator T_m by means of Calderón-Zygmund techniques. We refer reader to [7, 10, 18] for theoretical background on singular integral operators and T(1)-theorem.

Let $Q_t, \overline{[R_t, A]}$ defined as in (3.8), and $P_t = (1 - t^2 \triangle)^{-1}$. For any $\lambda \in \mathbf{I}$, we define

$$\begin{cases} R_{\alpha\beta}(\psi_\lambda)(x) := D_{\alpha\beta}\Big[\sum\limits_{p,q=1}^n m_\lambda(a_{pq}) D_{pq}\Big]^{-1}(\psi_\lambda)(x), \\ \overline{[R, A]}(\psi_\lambda)(x) := \sum\limits_{\alpha,\beta=1}^n \big[a_{\alpha\beta}(x) - m_\lambda(a_{\alpha\beta})\big] R_{\alpha\beta}(\psi_\lambda)(x), \\ U_t^{\alpha\beta}(\psi_\lambda)(x) := (P_t - [1 + \sigma(t) t^2 \sum\limits_{u,v=1}^n m_\lambda(a_{uv}) D_{uv}\big]^{-1}) R_{\alpha\beta}(\psi_\lambda)(x). \end{cases} \tag{6.4}$$

We have the following lemma.

Lemma 6.1. *(i) The operators* $\overline{[R, A]}$ *and* $R_{\alpha\beta}, \alpha, \beta \in \{1, 2, \cdots, n\}$, *are bounded on* L^2 *such that*

$$||\overline{[R, A]}||_{\mathcal{L}(L^2)} \leq C ||A||_*, \quad \text{and} \quad \sum_{\alpha,\beta=1}^n ||R_{\alpha\beta}||_{\mathcal{L}(L^2)} \leq C.$$

(ii) The kernel $U_t^{\alpha\beta}(x, y)$ *of the operator* $U_t^{\alpha\beta}$ *satisfies*

$$|U_t^{\alpha\beta}(x, y)| \leq \begin{cases} C t^{-n} (\frac{|x-y|}{t})^{1-n}, & \text{if} \quad |x - y| \leq t; \\ C t^{-n} (1 + \frac{|x-y|}{t})^{-n-2}, & \text{if} \quad |x - y| > t. \end{cases} \tag{6.5}$$

The same estimate (6.5) *holds for the kernel* $P_t(x, y)$ *of the operator* P_t.

Proof. For the proof of part (i), see Theorem 4′ in [3]. For (ii), the proof is similar to that of (4.2) and (4.3) of Lemma 4.1. We leave details to reader.

We now continue the proof of (ii). Since

$$\frac{-\sigma(t)t^2\xi_\alpha\xi_\beta}{1-\sigma(t)t^2\sum_{u,v=1}^{n} m_\lambda(a_{uv})\xi_u\xi_v}$$
$$=\frac{\xi_\alpha\xi_\beta}{\sum_{p,q=1}^{n} m_\lambda(a_{pq})\xi_p\xi_q}-\frac{\xi_\alpha\xi_\beta}{[1+t^2|\xi|^2][\sum_{p,q=1}^{n} m_\lambda(a_{pq})\xi_p\xi_q]}-\frac{\xi_\alpha\xi_\beta}{\sum_{p,q=1}^{n} m_\lambda(a_{pq})\xi_p\xi_q}$$
$$\times\Big(\frac{1}{1-\sigma(t)t^2\sum_{u,v=1}^{n} m_\lambda(a_{uv})\xi_u\xi_v}-\frac{1}{1+t^2|\xi|^2}\Big).$$

we have that the operator $\overline{[R_t,A]}$ can be written as

$$\overline{[R_t,A]}=-\overline{[R,A]}-Q_t+\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(x)P_tR_{\alpha\beta}-\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(x)U_t^{\alpha\beta}. \tag{6.6}$$

We will now prove Proposition 3.3 by induction on m. The proof follows the argument of [7], pages 387-389. We will first show that

$$||T_m||_{\mathcal{L}(L^2)}\le C^{m+1}||A||_*^{\frac{m}{n+2}}||F||_{H_\infty}. \tag{6.7}$$

Let $m\ge 1$ and suppose that (6.7) has been established for all operators of order $m-1$. Then, by (3.11) and (6.6) we have

$$T_m=-T_{m-1}\overline{[R,A]}-T_{1,m}+\sum_{\alpha,\beta=1}^{n} T_{2,m}^{\alpha\beta}R_{\alpha\beta}-\sum_{\alpha,\beta=1}^{n} T_{3,m}^{\alpha\beta},$$

where

$$\begin{aligned} T_{1,m}: &= \int_{-\infty}^{\infty} Q_t\overline{[R_t,A]}^{m-1}Q_tF(\tau(t))\frac{dt}{t},\\ T_{2,m}^{\alpha\beta}: &= \int_{-\infty}^{\infty} Q_t\overline{[R_t,A]}^{m-1}a_{\alpha\beta}(x)\cdot P_tF(\tau(t))\frac{dt}{t},\\ T_{3,m}^{\alpha\beta}: &= \int_{-\infty}^{\infty} Q_t\overline{[R_t,A]}^{m-1}a_{\alpha\beta}(x)\cdot U_t^{\alpha\beta}F(\tau(t))\frac{dt}{t}. \end{aligned}$$

It remains to show that $T_{1,m}$, $T_{2,m}^{\alpha\beta}$ and $T_{3,m}^{\alpha\beta}$ are L^2-bounded thanks to the induction hypothesis and (i) in Lemma 6.1.

Let us consider the operator $T_{1,m}$. As in Lemma 4.2, we define $U_t=Q_t(1-t^2\triangle)$ and $V_t=(1-t^2\triangle)Q_t$. We shall use the decomposition $P_t=Z_t+W_t$, where W_t is a Fourier multiplier operator with symbols $C_N(1+t^2|\xi|^2)^{-N}$ for some fixed N. If N is chosen large enough, W_t satisfies the condition (5.2). On the other hand, $Z_t = P_t - W_t$ is the convolution with $z_t(x) = t^{-n}z(x/t)$, where the function $z(x)$ satisfies $\int z(x)dx = 0$. Noting that

$$Q_t = Q_t(1 - t^2\triangle)(P_t - W_t) + Q_t(1 - t^2\triangle)W_t = U_tZ_t + U_tW_t, \tag{6.8}$$

we have

$$T_{1,m} = T^1_{1,m} + T^2_{1,m}$$

where

$$T^1_{1,m} := \int_{-\infty}^{\infty} Q_t\overline{[R_t, A]}^{m-1}U_tW_tF(\tau(t))\frac{dt}{t},$$
$$T^2_{1,m} := \int_{-\infty}^{\infty} Q_t\overline{[R_t, A]}^{m-1}U_tZ_tF(\tau(t))\frac{dt}{t}.$$

We use an argument in [7]. It follows from $T(1)$-theorem and Proposition 5.1 that

$$||T^1_{1,m}||_{\mathcal{L}(L^2)} \leq C^{m+1}||A||_*^{\frac{m}{n+2}}||F||_{H_\infty}. \tag{6.9}$$

We now turn our attention to the operator $T^2_{1,m}$. We no longer need the analyticity of the function F and consider separately the case $t > 0$ and $t < 0$. Let us show the estimate for the case $t > 0$. Noting that

$$Q_t = (P_t - W_t)(1 - t^2\triangle)Q_t + W_t(1 - t^2\triangle)Q_t = Z_tV_t + W_tV_t,$$

we have

$$\int_0^{\infty} Q_t\overline{[R_t, A]}^{m-1}U_tZ_tF(\tau(t))\frac{dt}{t} = T^3_{1,m} + T^4_{1,m}$$

where

$$T^3_{1,m} := \int_0^{\infty} W_tV_t\overline{[R_t, A]}^{m-1}U_tZ_tF(\tau(t))\frac{dt}{t},$$
$$T^4_{1,m} := \int_0^{\infty} Z_tV_t\overline{[R_t, A]}^{m-1}U_tZ_tF(\tau(t))\frac{dt}{t}.$$

Observe that the kernel of the operator $(T^3_{1,m})^*$ satisfies the standard kernel (i) and (ii), and we can check that $(T^3_{1,m})^*$ satisfies the assumptions of T(1) theorem, hence it is bounded on L_2 and

$$||(T^3_{1,m})^*||_{\mathcal{L}(L^2)} \leq C^{m+1}||A||_*^{\frac{m}{n+2}}||F||_{H_\infty}. \tag{6.10}$$

For the operator $T^4_{1,m}$, since

$$\int_0^{\infty} ||Z_t(f)||_2^2\frac{dt}{t} \leq C||f||_2^2$$

and

$$\sup_{t>0} ||V_t\overline{[R_t, A]}^{m-1}U_t||_{\mathcal{L}(L^2)} \leq C^{m+1}||A||_*^{m-1},$$

a standard argument then shows that

$$||T^4_{1,m}||_{\mathcal{L}(L^2)} \leq C^{m+1}||A||_*^{\frac{m}{n+2}}||F||_{H_\infty}, \quad (\text{ when } \ ||A||_* << 1), \qquad (6.11)$$

see [7], [10]. This proves that the operator $T_{1,m}$ is bounded on $L^2(\mathbf{R^n})$ by combining the estimates (6.9),(6.10) and (6.11).

For the operators $T^{\alpha\beta}_{2,m}$ and $T^{\alpha\beta}_{3,m}$, using (ii) in Lemma 6.1 and the above argument the proofs of their L^2-boundedness are similar. we omit the details here.

Since the kernel $K(x, y)$ of T_m also satisfies the Hörmander condition (5.1), we have the estimates (ii) in Proposition 3.3. See [7] and [10] for the details. This completes the proof of Proposition 3.3.

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