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REMARKS ON STRICHARTZ ESTIMATES FOR NULL FORMS

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Abstract. We prove some improved Strichartz estimates for null forms, some of which were conjectured recently in [7]. The results follow from combining the usual Strichartz estimates with div-curl lemma techniques rather than space-time Fourier transform.

Introduction

Consider the bilinear form

$$Q_{ij}(f,g) = \partial_i f \partial_j g - \partial_i g \partial_j f, \tag{0.1}$$

where f and g are functions of $x \in \mathbb{R}^n$. Such forms appear in various instances connected with geometric PDEs, either elliptic ([10] and references therein) or hyperbolic ([17] and references therein). In the later context, they are called null forms (along with other bilinear forms which we will introduce later). These forms have cancellation properties. For the simple product Hölder yields

$$\nabla f, \nabla g \in L^2 \Longrightarrow \partial f \partial g \in L^1, \tag{0.2}$$

while for the null form one has

$$\nabla f, \nabla g \in L^2 \Longrightarrow Q_{ij}(f,g) \in \mathcal{H}^1, \tag{0.3}$$

where $\mathcal{H}^1$ denotes the Hardy space: $h \in \mathcal{H}^1$ if and only if $h \in L^1$ and $R_i h \in L^1$ where R_i denote the Riesz transforms. One may recast (0.3) in term of Riesz transforms,

Proposition 1 ([2])**.** *Let $f, g \in L^2$, and let $\mathcal{Q}_{ij}(f,g) = R_i f R_j g - R_i f R_j g$. Then*

$$\|\mathcal{Q}_{ij}(f,g)\|_{\mathcal{H}^1} \lesssim \|f\|_{L^2} \|g\|_{L^2}. \tag{0.4}$$

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In view of the proof of such an estimate, one is led naturally to the following question: what about f, g with negative regularity, namely $f, g \in \dot{H}^{-s}$, for $s > 0$? It turns out that one has

Proposition 2. *Let $f, g \in \dot{H}^{-s}$, for $0 < s < \frac{1}{2}$. Then*

$$\|\frac{1}{|\nabla|^{2s}}\mathcal{Q}_{ij}(f,g)\|_{L^1} \lesssim \|f\|_{\dot{H}^{-s}}\|g\|_{\dot{H}^{-s}}. \tag{0.5}$$

Before proceeding with the proof, we remark one has in fact an even better result, as $\mathcal{Q}_{ij}(f,g) \in \dot{B}_1^{-2s,1}$. Through what follows the reader is assumed to be familiar with Besov spaces ([29]). In order to prove (0.5), we perform a paraproduct decomposition ([1]), writing

$$R_i f R_j g = \Pi(R_i f, R_j g) + \Pi(R_j g, R_i f) + \Upsilon(R_i f, R_j g), \tag{0.6}$$

with

$$\Pi(F,G) = \sum_k S_{k-2}F\Delta_k G, \quad \Upsilon(F,G) = \sum_{|k-k'|\leq 1} \Delta_k F \Delta_k G$$

where Δ_k is a localization operator at frequency $|\xi| \approx 2^k$ from a Littlewood-Paley decomposition, and $S_k = \sum_{l<k}\Delta_l$ is a low frequencies localization.

Since $f \in \dot{H}^{-s}$ is equivalent to $2^{-ks}\|\Delta_k f\|_{L^2} = \varepsilon_k \in l^2$, we have (recall $-s < 0$), $2^{-ks}\|S_k f\|_{L^2} = \mu_k \in l^2$ (and similar estimates for g, with $\tilde{\varepsilon}_k$ and $\tilde{\mu}_k$), from which we get

$$\|S_{k-2}R_i f\Delta_k R_j g\|_{L^1} \lesssim 2^{2ks}\mu_k\tilde{\varepsilon}_k, \tag{0.7}$$

and then

$$\Pi(R_i f, R_j g) + \Pi(R_i f, R_j g) - \Pi(R_j f, R_i g) - \Pi(R_j f, R_i g) \in \dot{B}_1^{-2s,1}. \tag{0.8}$$

Notice both the special structure of $\mathcal{Q}_{ij}$ and the value of s are irrelevant as long as $s < 0$. In fact each of these four terms belong individually to the Besov space $\dot{B}_1^{-2s,1}$. We are left with the third term, which deals with interactions at the same frequency: for a simple product of two distributions, defining Υ is in general impossible without regularity and/or decay of the two distributions. However, there are numerous instances where one can lower these requirements, and $\mathcal{Q}_{ij}$ is one of them. More generic classes of symbols can be treated in a similar way, see [5, 3, 23, 4, 9] and references therein. To take advantage of the structure, we write

$$R_i\Delta_k f R_j \Delta_k g - R_i\Delta_k g R_j\Delta_k f = \partial_i(\Lambda^{-1}\Delta_k f R_j\Delta_k g) - \partial_j(\Lambda^{-1}\Delta_k f R_i \Delta_k g), \tag{0.9}$$

where $\Lambda^{-1} = \sqrt{-\Delta}^{-1}$. We then proceed to evaluate

$$\|\Lambda^{-1}\Delta_k f R_j \Delta_k g\|_{L^1} \lesssim \|\Lambda^{-1}\Delta_k f\|_{L^2}\|\Delta_k g\|_{L^2} \lesssim 2^{-k(1-2s)}\varepsilon_k\tilde{\varepsilon}_k.$$

Assuming $s < \frac{1}{2}$ one may sum over high frequencies to get

$$\Upsilon_j = \sum_k \Lambda^{-1}\Delta_k f R_j \Delta_k g \in \dot{B}_1^{1-2s,1}, \tag{0.10}$$

and finally

$$\Upsilon(f,g) = \partial_i \Upsilon_j - \partial_j \Upsilon_i \in \dot{B}_1^{-2s,1}. \tag{0.11}$$

If $s = \frac{1}{2}$ one may slightly modify the argument and write a pointwise estimate,

$$|\Upsilon_j(x)| \leq \Big(\sum_k (2^{-\frac{k}{2}}|\Delta_k g|)^2\Big)^{\frac{1}{2}}\Big(\sum_k (2^{\frac{k}{2}}|\Delta_k \Lambda^{-1} f|)^2\Big)^{\frac{1}{2}}. \tag{0.12}$$

The two square functions on the RHS belong to L^2, thus one gets $\Upsilon_j \in L^1$, and $\Upsilon(f,g) = \nabla \cdot H$, where $H \in L^1$.

1. Application to the wave equation

Let u be a solution of the wave equation $\Box u = 0$, with initial data $u_0 = f$ and $u_1 = 0$ (this in order to simplify notations). Then we have dispersive estimates known as Strichartz estimates ([25, 8, 22, 11]), which we state localized in frequency:

$$\|\Delta_j u\|_{L_t^p(L_x^q)} \lesssim 2^{js}\|\Delta_j u_0\|_{L^2}, \tag{1.1}$$

with

$$s - \frac{n}{2} = \frac{1}{p} - \frac{n}{q},\ \frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}. \tag{1.2}$$

The condition on s can be conveniently rephrased as $s = \frac{n+1}{(n-1)p}$. The pairs (p,q) are called admissible pairs and $2 \leq p \leq \infty$, with the additional restriction $p > 2$ in 3D and $p > 4$ in 2D.

If we consider $u_0 \in \dot{H}^\alpha$, then we may recombine the frequency localized estimates to get

$$u \in \mathcal{L}_t^p(\dot{B}_q^{\alpha-s,2}) \hookrightarrow L_t^p(\dot{H}_q^{\alpha-s}), \tag{1.3}$$

where the $\mathcal{L}_t^p$ notation stands for the time norm taken before summing over dyadic blocs, namely

$$\|u\|^2_{\mathcal{L}_t^p(\dot{B}_q^{\alpha-s,2})} = \sum_j \Big(2^{j(\alpha-s)}\|\Delta_j u\|_{L_t^p(L_x^q)}\Big)^2. \tag{1.4}$$

Suppose as an example that $p = q = 2\frac{n+1}{n-1}$, $s = \alpha = \frac{1}{2}$, then for two solutions u, v of the wave equation with data u_0, v_0, we will have by Hölder

$$\|\partial u \partial v\|_{L^{\frac{n+1}{n-1}}_{t,x}} \lesssim \|u_0\|_{\dot{H}^{\frac{3}{2}}} \|v_0\|_{\dot{H}^{\frac{3}{2}}}. \tag{1.5}$$

In the study of semilinear wave equations with derivative nonlinearities, being able to lower the requirement on the regularity of the data is often essential. Thus, one may wonder if (1.5) is the best possible estimate when considered as a genuine bilinear estimate. In [21] an improvement over (1.5) was proved, using a refinement of the Strichartz inequalities. In the particular example above, the estimate can be improved to (we left the ∂ operator in order to get a sense of perspective with later estimates)

$$\|\Lambda^{-2\sigma}(\partial u \partial v)\|_{L^{\frac{n+1}{n-1}}_{t,x}} \lesssim \|u_0\|_{\dot{H}^{\frac{3}{2}-\sigma}} \|v_0\|_{\dot{H}^{\frac{3}{2}-\sigma}}, \tag{1.6}$$

with $0 < \sigma < \frac{2}{n+1}$. We intend to follow a different path and replace the product $\partial u \partial v$ by the null form $Q_{ij}(u, v)$ or a variant of it. In this context, the situation is completely understood when considering an $L^2_{t,x}$ norm of such a null form, and we refer to [7] for an extensive review of such results. Remark that, when $n = 3$, $(4, 4)$ is a Strichartz pair and such an $L^2_{t,x}$ norm comes somewhat naturally when studying bilinear quantities. Such estimates are proved using space-time Fourier transform, while we will only rely on the Strichartz estimates and the space cancellation explained in the introduction. By doing this we cannot recover all $L^2_{t,x}$ bilinear estimates, but on the other hand we easily get estimates for all Strichartz pairs. Later we will see how to combine the effect of the null form and the improved Strichartz estimates mentioned earlier to improve further the result. The estimates we will prove cover some of the conjectures on bilinear estimates formulated in [7]. Let us state our first result:

Theorem 1. *Let (p_1, q_1) and (p_2, q_2) be admissible pairs, s_1 and s_2 defined from (1.2), (P, Q) defined by*

$$\frac{2}{P} = \frac{1}{p_1} + \frac{1}{p_2}, \ \frac{2}{Q} = \frac{1}{q_1} + \frac{1}{q_2}, \tag{1.7}$$

and α_i, $i = 1, 2$ such that

$$\alpha_i - 1 - s_i < 0, \ S = \alpha_1 + \alpha_2 - 1 - \frac{2(n+1)}{(n-1)P} \geq 0. \tag{1.8}$$

Then, if $u \in \mathcal{L}_t^{p_1}(\dot{B}_{q_1}^{\alpha_1 - s_1, 2}) = \mathcal{E}_1$ *and* $v \in \mathcal{L}_t^{p_2}(\dot{B}_{q_2}^{\alpha_2 - s_2, 2}) = \mathcal{E}_2$*, we have, if* $S > 0$

$$Q_{ij}(u,v) \in \mathcal{L}_t^{\frac{P}{2}}(\dot{B}_{\frac{Q}{2}}^{S-1,1}). \tag{1.9}$$

Moreover for all $S \geq 0$*,*

$$\|\Lambda^{S-1} Q_{ij}(uv)\|_{L_t^{\frac{P}{2}}(L_x^{\frac{Q}{2}})} \lesssim \|u\|_{\mathcal{E}_1} \|u\|_{\mathcal{E}_2}. \tag{1.10}$$

We make a few remarks to start with: informally the null form structure allows to deal with $Q_{ij}(u,v)$ as if it were $\partial(u\partial v)$. Hence we gained one derivative which can be distributed on u and v. We intentionally stated the Theorem for generic functions in $\mathcal{E}_1$ and $\mathcal{E}_2$. We will make no use of the wave equation in the proof, in fact the time norms do not play any role and are just carried over all estimates. Later we will provide an improvement specific to solutions of the homogeneous wave equation by combining Theorem 1 with the aforementioned result from [21]. There is a great amount of flexibility concerning the possible choice of parameters. Other choices are of course possible, for example one could require $u, v \in \mathcal{E}_1 \cap \mathcal{E}_2$ and only $\inf(\alpha_i - 1 - s_i) < 0$. The important point is that restrictions are dictated by our need to define all terms which will appear in the paraproduct later. In particular, we stated an estimate which can be put in perspective with what one would get by Hölder. However one may also use Sobolev embeddings to relax (P, Q) to higher values than Strichartz pairs, and also allow for equality to hold in (1.21) (which means relaxing the Besov space to a Sobolev space).

We now turn to the proof. It proceeds exactly as for Proposition 2. Namely, the condition $\alpha_1 - 1 - s_1 < 0$ provides us with

$$2^{k(\alpha_1 - s_1 - 1)} \|\partial S_k u\|_{L_t^{p_1}(L_x^{q_1})} \lesssim \mu_k \in l^2, \tag{1.11}$$

and similarly for v. Thus we get

$$2^{k(2S - s_1 - s_2 - 2)} \|S_{k-2} \partial u \Delta_k \partial v\|_{L_t^P(L_x^Q)} \lesssim \mu_k \varepsilon_k, \tag{1.12}$$

which gives

$$\Pi(\partial u, \partial v) \in \mathcal{L}_t^{\frac{P}{2}}(\dot{B}_{\frac{Q}{2}}^{\alpha_1 + \alpha_2 - 2 - s_1 - s_2, 1}). \tag{1.13}$$

We are left with the same frequencies interaction $Q_{i,j}\Upsilon(u,v)$ (abusing slightly the notation), and thus are led to study again

$$\Upsilon_j(u,v) = \sum_k R_j \Delta_k u \partial_i \Delta_k v. \tag{1.14}$$

This sum will be well-defined provided $\alpha_1+\alpha_2-1-s_1-s_2\geq 0$, and then

$$\Upsilon_j(u,v)\in\mathcal{L}_t^{\frac{P}{2}}(\dot B^{\alpha_1+\alpha_2-1-s_1-s_2,1}_{\frac{Q}{2}}),\tag{1.15}$$

if $S>0$. for the limiting case $S=0$, we use the pointwise estimate,

$$|\Upsilon_j(u,v)|^2(x)\leq\sum_k|R_j\Delta_k u|^2(x)\sum_k|\partial_i\Delta_k v|^2(x),\tag{1.16}$$

which together with the characterization of Lebesgue spaces via square functions yields

$$\Upsilon_j(u,v)\in L_t^P(L_x^Q),\tag{1.17}$$

Using the null form structure as

$$Q_{i,j}\Upsilon(u,v)\approx\sum_k Q_{ij}(\Delta_k u,\Delta_k v)=\partial_j(\Upsilon_j(u,v))-\partial_i(\Upsilon_i(u,v)),\tag{1.18}$$

we get the desired result by deriving (1.15) or (1.17). This ends the proof.

We now turn to a slightly different null form, namely

$$\tilde Q_{i,j}(u,v)=R_iu\partial_jv-\partial_ivR_ju.\tag{1.19}$$

Notice how this new null form is no longer symmetrical in u and v. Therefore the requirement on u and v will be slightly different from Theorem 1. Informally, $\tilde Q_{i,j}(u,v)$ behaves at first glance like $u\partial v$, but again the null form allows to us to see it as $\partial(uv)$. We now state our result:

Theorem 2. *Let (p_1,q_1) and (p_2,q_2) be admissible pairs, s_1 and s_2 defined from (1.2), (P,Q) defined by*

$$\frac{2}{P}=\frac{1}{p_1}+\frac{1}{p_2},\ \frac{2}{Q}=\frac{1}{q_1}+\frac{1}{q_2},\tag{1.20}$$

and α_i, $i=1,2$ such that

$$\alpha_1-s_1<0,\ \alpha_2-1-s_2<0,\ S=\alpha_1+\alpha_2-\frac{2(n+1)}{(n-1)P}\geq 0.\tag{1.21}$$

Then, if $u\in\mathcal{L}_t^{p_1}(\dot B^{\alpha_1-s_1,2}_{q_1})=\mathcal{E}_1$ and $v\in\mathcal{L}_t^{p_2}(\dot B^{\alpha_2-s_2,2}_{q_2})=\mathcal{E}_2$, we have

$$\tilde Q_{ij}(u,v)\in\mathcal{L}_t^{\frac{P}{2}}(\dot B^{S-1,1}_{\frac{Q}{2}}),\tag{1.22}$$

and the third index has to be replaced by a 2 when $S=0$. In particular, one has

$$\|\Lambda^{S-1}\tilde Q_{ij}(uv)\|_{L_t^{\frac{P}{2}}(L_x^{\frac{Q}{2}})}\lesssim\|u\|_{\mathcal{E}_1}\|u\|_{\mathcal{E}_2}.\tag{1.23}$$

The proof proceeds as the previous one and we will therefore omit it. The two Theorems looks deceptively similar, thus we illustrate the difference between the two on a particular case in $3D$: let $p_1 = p_2 = q_1 = q_2 = P = Q = 4$. Then we have, if we moreover assume u and v to be solutions of the wave equation,

$$\|\Lambda^{-1}Q_{ij}(u,v)\|_{L^4_{t,x}} \lesssim \|u_0\|_{\dot{H}^1}\|v_0\|_{\dot{H}^1}. \tag{1.24}$$

Such an estimate was originally proved in [12] using space-time Fourier transform. Actually, a similar estimate is also proved there for the $\tilde{Q}_{ij}$ null form, but in contrast to the previous one we cannot recover it by our method (one would need the failing end point of the Strichartz estimates). We do however get an estimate if one is willing to distribute regularity differently on u, v:

$$\|\Lambda^{-2}\tilde{Q}_{ij}(u,v)\|_{L^4_{t,x}} \lesssim \|u_0\|_{\dot{H}^{1-\eta}}\|v_0\|_{\dot{H}^{1+\eta}}, \tag{1.25}$$

where $\eta > 0$.

2. REFINEMENTS SPECIFIC TO SOLUTIONS OF THE HOMOGENEOUS WAVE EQUATION

As explained in the introduction of the previous section, one may improve on the usual Strichartz estimates, by further angular localization on the cone. Then in the bilinear context, one obtains

Theorem 3 ([21]). *Let u and v be solutions to the homogeneous wave equation, with data u_0 and v_0. Let (p,q) be a Strichartz admissible pair, s the corresponding regularity, and $\sigma < \sigma_0 = \frac{4}{(n-1)p}$, then*

$$\|\Lambda^{-\sigma}(uv)\|_{L_t^{\frac{p}{2}}(L_x^{\frac{q}{2}})} \lesssim \|u_0\|_{\dot{H}^{s-\frac{\sigma}{2}}}\|{}_0\|_{\dot{H}^{s-\frac{\sigma}{2}}}. \tag{2.1}$$

In light of this Theorem, we can substantially improve Theorems 1 and 2. Indeed, for any low frequency-high frequency interaction, namely the $\Pi(u,v)$ terms, we already noticed that one may write the estimates without any lower restriction on the regularity of the data. Thus, (2.1) helps us with the term $\Upsilon(Q_{ij}(u,v))$. This term is treated as the difference of two derivatives, Υ_i and Υ_j, which turn out to be defined as long as (say for Theorem 1) $\alpha_1 + \alpha_2 - 1 - \frac{2(n+1)}{(n-1)P} \geq 0$. Thanks to (2.1) one can improve this to

$$\alpha_1 + \alpha_2 + \sigma - 1 - \frac{2(n+1)}{(n-1)P} \geq 0. \tag{2.2}$$

Let us state the new version of Theorem 1:

Theorem 4. *Let (p_1, q_1) and (p_2, q_2) be admissible pairs, s_1 and s_2 defined from (1.2), (P, Q) defined by*

$$\frac{2}{P} = \frac{1}{p_1} + \frac{1}{p_2},\ \frac{2}{Q} = \frac{1}{q_1} + \frac{1}{q_2}, \tag{2.3}$$

and α_i, $i = 1, 2$ such that

$$\alpha_i - 1 - s_i - \sigma < 0,\ S = \alpha_1 + \alpha_2 - 1 - \frac{2}{P} > 0. \tag{2.4}$$

Then, for all $\sigma < \frac{4}{(n-1)P}$, if $u_0 \in \dot{H}^{\alpha_1 - \frac{\sigma}{2}}$ and $v_0 \in \dot{H}^{\alpha_2 - \frac{\sigma}{2}}$, u, v the corresponding solutions of the free wave equation, we have

$$Q_{ij}(u, v) \in \mathcal{L}_t^{\frac{P}{2}}(\dot{B}_{\frac{Q}{2}}^{S-1-\sigma,1}). \tag{2.5}$$

In particular, one has

$$\|\Lambda^{S-1-\sigma} Q_{ij}(uv)\|_{L_t^{\frac{P}{2}}(L_x^{\frac{Q}{2}})} \lesssim \|u_0\|_{\dot{H}^{\alpha_1 - \frac{\sigma}{2}}} \|u_0\|_{\dot{H}^{\alpha_2 - \frac{\sigma}{2}}}. \tag{2.6}$$

In the same vein, one may improve Theorem 2 to

Theorem 5. *Let (p_1, q_1) and (p_2, q_2) be admissible pairs, s_1 and s_2 defined from (1.2), (P, Q) defined by*

$$\frac{2}{P} = \frac{1}{p_1} + \frac{1}{p_2},\ \frac{2}{Q} = \frac{1}{q_1} + \frac{1}{q_2}, \tag{2.7}$$

and α_i, $i = 1, 2$ such that

$$\alpha_1 - s_1 - \sigma < 0,\ \alpha_2 - 1 - s_2 - \sigma < 0,\ S = \alpha_1 + \alpha_2 - \frac{2}{P} > 0. \tag{2.8}$$

Then, for all $\sigma < \frac{4}{(n-1)P}$, if $u_0 \in \dot{H}^{\alpha_1 - \frac{\sigma}{2}}$ and $v_0 \in \dot{H}^{\alpha_2 - \frac{\sigma}{2}}$, u, v the corresponding solutions of the free wave equation, we have

$$\tilde{Q}_{ij}(u, v) \in \mathcal{L}_t^{\frac{P}{2}}(\dot{B}_{\frac{Q}{2}}^{S-1-\sigma,1}). \tag{2.9}$$

In particular, one has

$$\|\Lambda^{S-1-\sigma} \tilde{Q}_{ij}(uv)\|_{L_t^{\frac{P}{2}}(L_x^{\frac{Q}{2}})} \lesssim \|u_0\|_{\dot{H}^{\alpha_1 - \frac{\sigma}{2}}} \|u_0\|_{\dot{H}^{\alpha_2 - \frac{\sigma}{2}}}. \tag{2.10}$$

As in the previous section, one may also distribute the σ effect in a different way on u_0 and u_1.

From these last two results, one gets the corresponding results for the hyperbolic Sobolev spaces $H^{s,\theta}$([14, 16, 18, 19]) replacing $\dot{H}^s$ on the right handside. As an example of a situation where the regularity of the data is small, we give

Proposition 3. *Let $n = 3$ and $u, v \in H^{\frac{1}{2}^+,\frac{1}{2}^+}$, then*

$$\|\frac{1}{\Lambda^{2^-}} Q_{ij}(uv)\|_{L_t^{1+}(L_x^{\infty-})} + \|\frac{1}{\Lambda^{1^-}} \tilde{Q}_{ij}(uv)\|_{L_t^{1+}(L_x^{\infty-})} \lesssim \|u\|_{H^{\frac{1}{2}^+,\frac{1}{2}^+}} \|v\|_{H^{\frac{1}{2}^+,\frac{1}{2}^+}}. \tag{2.11}$$

3. OTHER NULL FORMS

In dealing with the Q_{ij} null forms, we only took advantage of the null structure with respect to the space variable. However other null forms appear in the context of the wave equation. There is

$$Q_0(u, v) = \partial_t u \partial_t u - \nabla u \cdot \nabla v, \tag{3.1}$$

in connection with the wave maps equation, and

$$Q_{0,i}(u, v) = \partial_t u \partial_i v - \partial_i u \partial_t v, \tag{3.2}$$

which is connected with models related to wave maps ([20]). For these later null forms, we immediately see that Theorems 4, 5 hold. Indeed for free solutions to the wave equation, $\partial_t u$ verifies the same estimates as ∇u, and then one may perform the same trick to write

$$Q_{0,i}(u, v) = \partial_t(u \partial_i v) - \partial_i(u \partial_t v), \tag{3.3}$$

and proceed as for the Q_{ij} null forms. The case of the Q_0 null form appears to be less tractable in this context, in contrast of say, the $L^2_{t,x}$ situation when one is dealing with estimates involving the symbol of the wave equation ([7] and references therein). One expects the Q_0 null form to behave better than the other forms, as Q_0 vanishes at a higher order on the null cone. However we have not really made a full use of the cone structure, but simply of symbol cancellation for antipodal directions (space directions in the previous sections, mixed space-time for the $Q_{0,i}$). Hence we do not have anything very useful to say about the Q_0 form besides a few remarks. First of all, if we restrict ourselves to opposite traveling waves, we can then say something as the symbol of Q_0 becomes a symbol with space cancellations:

$$Q_0(u^+, v^-) = -\int e^{ix\cdot(\xi+\eta)+it(|\xi|-|\eta|)}(|\xi||\eta| + \xi \cdot \eta)\hat{u}_0(\xi)\hat{v}_0(\eta)d\xi d\eta. \tag{3.4}$$

The important point here is that the symbol vanishes when $\xi = -\eta$. Though one can no longer apply the simple trick with the derivatives, this symbol nevertheless falls into the generic class of symbols for which one can prove good continuity properties ([3]). Then one gets Theorems 4, 5 for $Q_0(u^+, v^-)$. We will not provide a proof, which becomes considerably more technical. Essentially one needs to decompose the symbol into a normally convergent sum of "reduced" symbols, to which the previous trick applies. The reduction is done through Fourier series expansion together with conical partitions. We refer the interested reader to [5, 23] for an extensive presentation of such techniques. In the case of $Q_0(u^+, v^+)$, where the direction of propagation is the same, this approach fails totally. If we think about same frequency interaction in this case, this is not surprising since the symbol is not killing low frequencies but killing frequencies close to the cone. In other words, one has to make use of the derivative normal to the cone to achieve any improvement (or to say it differently in connection with recent work [30, 26], we have to consider interactions which come from nearly orthogonal ξ and η directions, for which the support will be well separated from the cone, the symbol taking care of the others).

4. A word about applications

The null forms Q_{ij} and $\tilde{Q}_{ij}$ appear in connection with the Yang-Mills equation. More precisely, it was observed in [13, 15] how one could use the Lorentz gauge to rewrite the equations in a convenient way, namely

$$\Box U = \Lambda^{-1} Q_{i,j}(U, U) + \tilde{Q}_{i,j}(U, U), \tag{4.1}$$

where U is to be understood as a vector and the right handside as any linear combination of null forms applied to the coordinates of U. This together with L^2 estimates for null forms was the cornerstone of the proof of global existence of finite energy solutions to the Yang-Mills equations, [15]. Observe how in 3D the scale invariant norm for (4.1) is $\dot{H}^{\frac{1}{2}}$. Hence H^1 is half a derivative above the scaling, and nearly missed by if one is willing to use Strichartz estimates only, as in [24]. None of the estimates we proved earlier will fix the problem, which has to deal with controlling low frequencies of u (this requires $L^\infty_x(L^2_t)$ estimates in [15], since the reversed time-space estimate in known to fail). Observe however that such a low frequency term only appears when dealing with $\tilde{Q}_{ij}(u, v)$. Hence, if one is willing to drop such terms from (4.1), our Strichartz estimates for the Q_{ij} null form will allow to prove local well-posedness in $\dot{H}^1$. However the full system (4.1) is known to be well-posed for $s > \frac{3}{4}$ using $H^{s,\theta}$ spaces ([6] for the Maxwell-Klein-Gordon system, and [31] for the model Yang-Mills equations, as well as some recent work of T. Tao, [28, 27]). For the sake of completeness, we mention that (4.1) is known to be well-posed up to scaling level in higher dimensions ([21]), while in 2D little appears to be available in the literature, though Strichartz estimates will provide well-posedness up to $s = \frac{3}{4}$ while one expects to go down to $s = \frac{1}{4}$ using hyperbolic Sobolev spaces. At the moment it doesn't seem possible to lower these requirements with the current technology.

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