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RELAXATION RESULTS FOR HIGHER ORDER INTEGRALS BELOW THE NATURAL GROWTH EXPONENT*

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Abstract. We consider higher-order variational integrals of the type

$$\mathcal{F}(u,\Omega)=\int_\Omega f(x,u,D^{[k]}u,D^{k+1}u)\;dx$$

and study relaxation and lower semicontinuity properties of such functionals. In particular, under a bound of the type

$$0\le f(x,u,z_1,z_2,\dots,z_{k+1})\le L(1+|z_{k+1}|^q)$$

the following relaxed energies are studied:

$$\begin{aligned}\mathcal{F}^{q,p}(u,\Omega) &= \inf_{\{u_n\}}\Big\{\liminf_n\int_\Omega f(x,u_n,D^{[k]}u_n,D^{k+1}u_n)\;dx\;:\\ &\qquad u_n\in W^{k+1,q}(\Omega;\mathbb{R}^d),\;u_n\rightharpoonup u\text{ in }W^{k+1,p}(\Omega;\mathbb{R}^d)\Big\}\\ \mathcal{F}^{q,p}_{\rm loc}(u,\Omega) &= \inf_{\{u_n\}}\Big\{\liminf_n\int_\Omega f(x,u_n,D^{[k]}u_n,D^{k+1}u_n)\;dx\;:\\ &\qquad u_n\in W^{k+1,q}_{\rm loc}(\Omega;\mathbb{R}^d),\;u_n\rightharpoonup u\text{ in }W^{k+1,p}(\Omega;\mathbb{R}^d)\Big\}\end{aligned}$$

with $\frac{q}{p}<\frac{Nk}{Nk-1}$.

1. Introduction

In this paper we consider variational integrals of the type

$$\mathcal{F}(u,\Omega)=\int_\Omega f(x,u,D^{[k]}u,D^{k+1}u)\;dx \tag{1.1}$$

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where $u \in W^{k+1,p}(\Omega;\mathbb{R}^d)$, Ω is a bounded, open subset, $f : \mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^{Nd} \times \cdots \times \mathbb{R}^{dN^{k+1}} \to \mathbb{R}^+$ is a nonnegative Carathéodory function and $D^{[k]}u$ stands for the vector of all derivatives of u of order $\leq k$. The issue of studying lower semicontinuity properties of functionals as in (1.1) turns out to be of fundamental importance in the calculus of variations; it is well known that under certain assumptions on the integrand f (in the simpler case when f does not depend on lower-order derivatives),

$$\int_\Omega f(D^{k+1}u)\ dx \leq \liminf_n \int_\Omega f(D^{k+1}u_n)\ dx \tag{1.2}$$

if $u_n \rightharpoonup u$ in $W^{k+1,p}(\Omega;\mathbb{R}^d)$ and f is a quasiconvex function (see Definition 4.1 below) such that

$$0 \leq f(z) \leq L(1+|z|^p)$$

for any $z \in \mathbb{R}^{dN^{k+1}}$ (see [23] and [14]). As is easily seen by direct methods of the calculus of variations, lower semicontinuity allows us to study existence of minimizers of the functional. When the functional is not lower semicontinuous, then, in order to study the behavior of the functional along minimizing sequences, one is led to consider the so-called relaxed functional, i.e., the lower-semicontinuous envelope of the functional with respect to a chosen topology. In this paper we study the following relaxed energies:

$$\begin{aligned} \mathcal{F}^{q,p}(u,\Omega) &= \inf_{\{u_n\}} \Big\{ \liminf_n \int_\Omega f(x,u_n,D^{[k]}u_n,D^{k+1}u_n)\ dx\ : \qquad (1.3)\\ &\qquad u_n \in W^{k+1,q}(\Omega;\mathbb{R}^d),\ u_n \rightharpoonup u \text{ in } W^{k+1,p}(\Omega;\mathbb{R}^d)\Big\},\\ \mathcal{F}^{q,p}_{\mathrm{loc}}(u,\Omega) &= \inf_{\{u_n\}} \Big\{ \liminf_n \int_\Omega f(x,u_n,D^{[k]}u_n,D^{k+1}u_n)\ dx\ : \qquad (1.4)\\ &\qquad u_n \in W^{k+1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^d),\ u_n \rightharpoonup u \text{ in } W^{k+1,p}(\Omega;\mathbb{R}^d)\Big\}, \end{aligned}$$

where $u \in W^{k+1,p}(\Omega;\mathbb{R}^d)$ and $q \geq p$, $k \geq 1$.

In a recent paper (see [12]) I. Fonseca and J. Malý considered the case $k = 0$ and proved measure-representation results for the quantities $\mathcal{F}^{q,p}_{\mathrm{loc}}(u,\cdot)$ and $\mathcal{F}^{q,p}(u,\cdot)$ under a growth assumption of the type

$$0 \leq f(z) \leq L(1+|z|^q) \tag{1.5}$$

(for the sake of simplicity we assume again that no dependence on x and u is involved in f) with $\frac{q}{p} < \frac{N}{N-1}$. In particular, under some natural assumptions, they prove that, with A denoting an open subset of Ω

$$\mathcal{F}^{q,p}(u,A) = \mu(A) \qquad \mathcal{F}^{q,p}_{\mathrm{loc}}(u,A) = \lambda(A),$$

for some Radon measures λ and μ, and they give a lower bound for the relaxed functional

$$\mathcal{F}^{q,p}(u,A) \geq \int_\Omega Qf(Du)\ dx, \tag{1.6}$$

where Qf denotes the quasiconvex envelope of f.

An important step in order to prove (1.6) is the following lower-semicontinuity result below the natural growth exponent q,

$$\int_\Omega f(Du)\ dx \leq \liminf_n \int_\Omega f(Du_n)\ dx, \tag{1.7}$$

where $u_n \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^d)$ and f is quasiconvex. An improvement of this result has been given by Kristensen (see [18]). We emphasize that the main point in this kind of result is that the convergence of the functions u_n in (1.7) is supposed to take place with respect to a topology (the weak-$W^{1,p}$) which is weaker than the natural one (the weak-$W^{1,q}$). This generalizes classical results on quasiconvex functions (see [23] and [2]) and later ones obtained by Marcellini, who was the first to prove lower semicontinuity theorems below the natural topology (see [23]).

In this paper we reproduce the results obtained by Fonseca and Malý for higher-order multiple integrals of the type in (1.1). More precisely we consider integral functionals $\mathcal{F}$ with the energy density of f satisfying the following growth assumptions:

$$0 \leq f(x,u,z_1,z_2,\ldots,z_{k+1}) \leq L(1+\sum_{1\leq h\leq k}|z_h|^{r_h}+|z_{k+1}|^q),$$

with $z_h \in \mathbb{R}^{N^h d}$, and we study the relaxed functionals defined in (1.3)-(1.4) under the main hypothesis

$$\frac{q}{p} < \frac{Nk}{Nk-1}, \tag{1.8}$$

and with the lower-order exponents r_h satisfying similar restrictions (see Section 3).

We then give a lower-semicontinuity result for quasiconvex functionals (see Definition 4.1 below) of the type

$$\int_\Omega f(D^{k+1}u)\ dx$$

with f satisfying a growth assumption as in (1.5):

$$0 \leq f(z_{k+1}) \leq L(1+|z_{k+1}|^q)\ .$$

More precisely we prove that the result in (1.2) is valid if $u_n \rightharpoonup u$ weakly in $W^{k+1,p}(\Omega;\mathbb{R}^d)$ and p and q are related through (1.8). This extends previous, analogous results in which $p = q$, due to Meyers and Fusco (see also [17] and [25]) and allows us to obtain a lower bound for the bulk part of the relaxed functional as in (1.6):

$$\mathcal{F}^{q,p}(u, A) \geq \int_\Omega Qf(D^{k+1}u)\, dx\, .$$

The central technical point in order to achieve the previous results is, following Fonseca and Malý, the construction of a linear continuous operator $T : W^{k+1,p} \to W^{k+1,p}$ that improves on the integrability of functions on certain thin layers. With this operator it is possible to construct suitable sequences of comparison functions in order to make the standard relaxation procedures work in our case. A point worth mentioning is that, the restriction imposed on p, q also depends on k though in the particular case $k = 1$ (which corresponds to a functional depending on second derivatives) we recover the precise bound of Fonseca and Malý, valid in the case $k = 0$.

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2. A lifting result

In the sequel p and q denote two positive numbers such that $1 < p \leq q \leq +\infty$. Moreover, when not differently specified, k and d will be two integers such that $1 \leq k, d$ while Ω will denote an open, bounded subset of $\mathbb{R}^N$, with $N \geq 2$. If $u \in W^{k+1,p}(\Omega;\mathbb{R}^d)$ and if $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_h)$ is a multi-index with length $|\alpha| := \sum_{j=1}^h \alpha_j \leq k+1$, we will denote by $D^\alpha u := (D^{\alpha_1}u, D^{\alpha_2}u, \dots, D^{\alpha_h}u)$ a derivative of order $|\alpha|$ of u, and if $1 \leq s \leq k+1$ then $D^s u$ will denote the vector $(D^\alpha u)_{|\alpha|=s}$ of all s-th order derivatives of u with the usual convention that $D^0 u \equiv u$. Moreover $D^{[s]}u$ will be the vector $(D^\alpha u)_{|\alpha|\leq s}$ so that, in particular, $D^{[k]}u = (D^\alpha u)_{|\alpha|\leq k}$. Finally, $(r_h)_{1\leq h\leq k+1}$ will be a $(k+1)$-ple of integers such that $r_h \geq 1$ for every $1 \leq h \leq k+1$. We will keep the notation

$$q = r_{k+1}\, .$$

We will often abbreviate by using $W^{k+1,p}(\Omega)$ or even $W^{k+1,p}$ instead of $W^{k+1,p}(\Omega;\mathbb{R}^d)$.

In the rest of the section we denote by $\eta \in C_0^\infty(\Omega)$ a nonnegative function and we fix $A \geq 1$ and $[t_1, t_2] \subset (0, ||\eta||_\infty)$, such that $0 < |D\eta| \leq A$, for

$t_1 \le \eta \le t_2$. Given $t_1 < a < b < t_2$, we define $Z_a^b = \{a < \eta < b\}$, and denote by $\Gamma_{t_0} = \{\eta = t_0\}$, the level set of η. Moreover τ and C will freely denote two positive constants which may vary in any two occurences, while only the important dependences will be highlighted.

Proceeding as in [12] we will need an operator on $W^{k+1,p}(\Omega)$ which improves on the integrability properties in Z_a^b, preserving the function values elsewhere. Then we recall the following two lemmas used in [12] that we will need later. The first one is a simple consequence of the Sobolev imbedding theorem on manifolds.

Lemma 2.1. *Let us fix $t_0 \in (t_1, t_2)$. Given a smooth function v we have for every $k \in N$ and for every multi-index α with $h = |\alpha| \le k$, $1 \le r < +\infty$*

$$\Big(\int_{\{\eta=t_0\}} |D^\alpha v|^r dH^{N-1}\Big)^{\frac{1}{r}} \le C\Big(\int_{\{\eta=t_0\}} \big(\sum_{h\le|l|\le k+1} |D^l v|^\beta\big)\, dH^{N-1}\Big)^{\frac{1}{\beta}}, \quad (2.1)$$

where $1 \le \beta$ and either $\beta \ge \frac{N-1}{k+1-h}$ or else $r \le \frac{\beta(N-1)}{N-1-(k+1-h)\beta}$ and $C = C(N, \beta, r, \eta, t_1, t_2)$.

Next lemma is from [12], Lemma 2.1.

Lemma 2.2. *Consider $s \in (t_1, t_2)$ and $\bar\rho > 0$ such that $[s-\bar\rho, s+\bar\rho] \subset (t_1, t_2)$. Let f be a nonnegative measurable function on Ω. Then*

$$\int_{\{\eta=s\}} \Big(\int_{B(z,\frac{\bar\rho}{A})} f(y)\, dy\Big)\, dH_z^{N-1} \le C\bar\rho^{N-1} \int_{Z_{s-\bar\rho}^{s+\bar\rho}} f(y)\, dy, \quad (2.2)$$

where $C = C(N, \eta, t_1, t_2)$.

Now we are ready to prove the main lemma of this section.

Main Lemma. *Let $t_1 < a < b < t_2$ (with $t_2 - t_1 < \frac{1}{8}$) and $k \ge 1$, $h = 0, 1, \dots, k+1$. There exists a linear operator $T : W^{k+1,p}(\Omega) \to W^{k+1,p}(\Omega)$ such that $Tu = u$ on $\Omega - Z_a^b$ and*

$$\sum_{h=0}^{k+1} \sum_{|\alpha|=h} ||D^\alpha Tu||_{L^{r_h}(Z_a^b)} + ||D^\alpha Tu||_{W^{k+1,q}(Z_a^b)} \quad (2.3)$$

$$\le C(b-a)^\tau\Big(\sup_{t\in(a,b)} [(t-a)^{-\frac{1}{p}}||u||_{W^{k+1,p}(Z_a^t)}] + \sup_{t\in(a,b)} [(b-t)^{-\frac{1}{p}}||u||_{W^{k+1,p}(Z_t^b)}]\Big)$$

where $r_h > p$ are such that $p > \frac{r_h(Nk-1)}{Nk+r_h k(k+1-h)}$, $C = C(N, p, r_h, \eta, t_1, t_2, k)$ and $\tau = \tau(N, p, r_h) > 0$.

Proof. Let us define, for $u \in W^{k+1,p}(\Omega;\mathbb{R}^d)$,

$$Tu(x) = \fint_{B(0,1)} u(x+\theta(x)y)\ dy,$$

where

$$\theta(x) = \begin{cases} 0 & \text{if } \eta(x) \le a \\ \frac{(\eta(x)-a)^k(b-\eta(x))^k}{A(b-a)^k} & \text{if } a < \eta(x) < b \\ 0 & \text{if } \eta(x) \ge b\ . \end{cases}$$

Let us note that there exists a constant $C(N,k)$ such that $|D^\alpha\theta(x)| \le C$, for every multi-index $\alpha = (\alpha_1,\dots,\alpha_h)$ with $|\alpha| \le k+1$ and for $x \in Z_a^b$. Moreover if $0 < \rho < (b-a)/2$ and if $z \in \{\eta = a+\rho\}$, then $\rho^k/2^kA \le \theta(z) \le \rho^k/A$ and $B(z,\theta(z)) \subset Z_{a+\rho-\rho^k}^{a+\rho+\rho^k}$, and so we have, with u being now a smooth function,

$$\begin{aligned} |Tu(z)|^q &\le C\rho^{-kNq}\Big(\int_{B(z,\rho^k/A)} |u(y)|dy\Big)^q \\ &\le C\rho^{-kNq+knq(1-1/p)}\Big(\int_{B(z,\rho^k/A)} |u(y)|^p\ dy\Big)^{\frac{q}{p}} \\ &\le C\rho^{\frac{-kNq}{p}}\Big(\int_{Z_{a+\rho-\rho^k}^{a+\rho+\rho^k}} |u(y)|^p dy\Big)^{\frac{q}{p}-1}\Big(\int_{B(z,\rho^k/A)} |u(y)|^p dy\Big). \end{aligned}$$

Using Lemma 2.2 with $s = a+\rho$ and $\bar\rho = \rho^k$, we obtain

$$\begin{aligned} &\int_{\{\eta=a+\rho\}} |Tu(z)|^q dH^{N-1} \qquad (2.4) \\ &\le C\rho^{\frac{-kNq}{p}}\Big(\int_{Z_{a+\rho-\rho^k}^{a+\rho+\rho^k}} |u(y)|^p dy\Big)^{\frac{q}{p}-1}\int_{\{\eta=a+\rho\}}\Big(\int_{B(z,\rho^k/A)}|u(y)|^p dy\Big)\ dH^{N-1} \\ &\le C\rho^{\frac{-kNq}{p}+k(N-1)}\Big(\int_{Z_{a+\rho-\rho^k}^{a+\rho+\rho^k}} |u(y)|^p dy\Big)^{\frac{q}{p}}. \end{aligned}$$

Now we let $c = (a+b)/2$ and perform the following choice of $\epsilon \ge 0$:

$$0 < \epsilon < \frac{b-a}{2} - (\frac{b-a}{2})^k \text{ if } k > 1 \text{ and } \epsilon := 0 \text{ if } k = 1\ .$$

Using the Coarea formula, we get

$$\int_{Z_{a+\epsilon}^c} |Tu(x)|^q dx \le C\int_\epsilon^{\frac{b-a}{2}}\Big(\int_{\{\eta=a+\rho\}} |Tu(z)|^q dH^{N-1}\Big)\ d\rho \qquad (2.5)$$

$$\leq C \int_{\epsilon}^{\frac{b-a}{2}} \rho^{-\frac{kNq}{p}+k(N-1)} \Big(\int_{Z_{a+\rho-\rho^k}^{a+\rho+\rho^k}} |u(y)|^p dy \Big)^{\frac{q}{p}} \, d\rho \, .$$

The inequality stated above has been derived for smooth functions; a simple approximation argument with Fatou's lemma shows that actually it works for any $u \in L^p$. Now, we define

$$\varphi(t) = \int_{Z_a^{a+t}} |u(y)|^p dy$$

and

$$M = \sup_{t \in (0, b-a)} t^{-1} \int_{Z_a^{a+t}} |u(y)|^p dy \, .$$

We observe that $\rho < (b-a)/2 < 1/16$ implies $\rho + \rho^k < 2\rho$, and so we have

$$Z_{a+\rho-\rho^k}^{a+\rho+\rho^k} \subset Z_a^{a+2\rho},$$

and by definition of M we get

$$\int_{Z_{a+\rho-\rho^k}^{a+\rho+\rho^k}} |u(y)|^p dy \leq \int_{Z_a^{a+2\rho}} |u(y)|^p dy \leq CM\rho \, . \tag{2.6}$$

From (2.5) and (2.6) we have that

$$\int_{Z_{a+\epsilon}^c} |Tu(x)|^q dx \leq$$
$$CM^{\frac{q}{p}-1} \int_{\epsilon}^{\frac{b-a}{2}} \left(\rho^{-\frac{kNq}{p}+k(N-1)+\frac{q}{p}-1} \int_{Z_{a+\rho-\rho^k}^{a+\rho+\rho^k}} |u(y)|^p dy \right) d\rho \, . \tag{2.7}$$

Now we distinguish between two cases.

The case $k = 1$. In this case we are dealing with $W^{2,p}$. Then, from (2.7) and by definition of M, once again it follows that

$$\int_{Z_a^c} |Tu(x)|^q dx \leq CM^{\frac{q}{p}} \int_0^{\frac{b-a}{2}} \rho^{-\frac{Nq}{p}+N-1+\frac{q}{p}} d\rho \leq CM^{\frac{q}{p}} (b-a)^{\tau}$$

with $\tau = -\frac{Nq}{p} + N + \frac{q}{p} > 0$ by the fact that $\frac{q}{p} < \frac{N}{N-1}$.

The case $k > 1$. We let $d = \frac{kNq}{p} - kN - \frac{q}{p} + k + 1$, and we obtain from (2.7)

$$\int_{Z_{a+\epsilon}^c} |Tu(x)|^q dx \leq CM^{\frac{q}{p}-1} \int_{\epsilon}^{\frac{b-a}{2}} \rho^{-d} [\varphi(\rho+\rho^k) - \varphi(\rho-\rho^k)] \, d\rho \tag{2.8}$$

$$= CM^{\frac{q}{p}-1}\int_{\epsilon}^{\frac{b-a}{2}}\Big(\rho^{-d}\int_{\rho-\rho^k}^{\rho+\rho^k}\varphi'(t)\ dt\Big)\ d\rho\ .$$

Now we want to apply Fubini's theorem to the last term in (2.8); to this aim we observe that $\rho+\rho^k=\tau \Longrightarrow \rho=\tau-\rho^k>\tau-\tau^k \Longrightarrow \rho>\tau-\tau^k$, while since $\rho<1/2$, $\rho-\rho^k=\tau \Longrightarrow \rho<\tau+2^k\tau^k$, and so we have from (2.8)

$$\int_{Z^c_{a+\epsilon}}|Tu(x)|^q dx \le CM^{\frac{q}{p}-1}\int_{\epsilon}^{(\frac{b-a}{2})(1-(\frac{b-a}{2})^{k-1})}\varphi'(t)\ dt\int_{t-2^kt^k}^{t+2^kt^k}\rho^{-d}\ d\rho$$

$$+\, CM^{\frac{q}{p}-1}\int_{(\frac{b-a}{2})(1-(\frac{b-a}{2})^{k-1})}^{(\frac{b-a}{2})(1+(\frac{b-a}{2})^{k-1})}\varphi'(t)\ dt\int_{t-2^kt^k}^{\frac{b-a}{2}}\rho^{-d}d\rho = I_1+I_2. \tag{2.9}$$

In order to estimate the last terms I_1 and I_2 we begin by observing that since $\frac{q}{p}\ge 1$ and $k>1$ we have that $d>1$ so that

$$\int_{t-2^kt^k}^{t+2^kt^k}\rho^{-d}d\rho = [\frac{1}{1-d}\rho^{1-d}]_{t-2^kt^k}^{t+2^kt^k} = \frac{1}{1-d}[(t+2^kt^k)^{1-d}-(t-2^kt^k)^{1-d}]$$

$$= \frac{1}{d-1}\frac{(t+2^kt^k)^{d-1}-(t2^k-t^k)^{d-1}}{(t+2^kt^k)^{d-1}(t-2^kt^k)^{d-1}} \le c(d,k)\frac{t^{d+k-2}}{t^{2d-2}} = c(d,k)t^{k-d}\ .$$

We observe now that since φ is a continuous function and $D\varphi$ is an L^1 function in $[0,\frac{b-a}{2}]$ a simple approximation argument based on mollification justifies the following integration-by-parts formula, which we apply with the condition $d<k+1$:

$$\int_{\epsilon}^{\frac{b-a}{2}}\varphi'(t)t^{k-d}dt = \varphi(t)t^{k-d}|_{\epsilon}^{\frac{b-a}{2}} + (k-d)\int_{\epsilon}^{\frac{b-a}{2}}\varphi(t)t^{k-d-1}dt$$

$$\le \varphi(t)t^{k-d}|_{\epsilon}^{\frac{b-a}{2}} + c(k,d)M\int_{\epsilon}^{\frac{b-a}{2}}t^{k-d}dt$$

where, in order to perform the last estimate, we again applied the inequality $\varphi(t)\le Mt$; finally, again by this inequality, we arrive at

$$I_1 \le CM^{\frac{q}{p}-1}\int_{\epsilon}^{\frac{b-a}{2}}\varphi'(t)t^{k-d}dt \le CM^{\frac{q}{p}}(b-a)^{k+1-d}+o(\epsilon)\ .$$

Now, in order to estimate I_2 we observe that in the first integral in I_2, $t>(\frac{b-a}{2})-(\frac{b-a}{2})^k$ and then $t-2^kt^k \ge (\frac{b-a}{2})-2^{k+1}(\frac{b-a}{2})^k>0$ (because $(b-a)<\frac{1}{8}$), and thus we get with a computation similar to the ones for I_1

$$\int_{t-2^kt^k}^{\frac{b-a}{2}}\rho^{-d}d\rho \le \int_{(\frac{b-a}{2})-2^{k+1}(\frac{b-a}{2})^k}^{(\frac{b-a}{2})+2^{k+1}(\frac{b-a}{2})^k}\rho^{-d}d\rho < c(d,k)(b-a)^{k-d}$$

and again

$$I_2 \le CM^{\frac{q}{p}-1}\int_{\epsilon}^{b-a}(b-a)^{k-d}\varphi'(t)\ dt \le CM^{\frac{q}{p}}(b-a)^{k+1-d}+o(\epsilon)\ .$$

Merging the estimates for I_1 and I_2 with (2.9) and letting $\epsilon \to 0$, we get, with $\tau = k+1-d$,

$$\int_{Z_a^c}|Tu|^q dx \le CM^{\frac{q}{p}}(b-a)^{\tau}\ . \tag{2.10}$$

Observe that $\tau > 0$ since $d < k+1$; that is,

$$\frac{q}{p} < \frac{Nk}{Nk-1}\ . \tag{2.11}$$

Using similar arguments it is also possible to get

$$\int_{Z_c^b}|Tu|^q dx \le CM^{\frac{q}{p}}(b-a)^{\tau}. \tag{2.12}$$

Keeping in mind the last formulas we finally obtain

$$\begin{aligned} ||Tu||_{L^q(Z_a^b)} \quad \le \quad & C(b-a)^{\tau}\Big(\sup_{t\in(a,b)}\big[(t-a)^{-\frac{1}{p}}||u||_{L^p(Z_a^t)}\big] \\ & + \sup_{t\in(a,b)}\big[(b-t)^{-\frac{1}{p}}||u||_{L^p(Z_t^b))}\big]\Big)\ . \end{aligned} \tag{2.13}$$

Now we compute the derivatives of Tu and we observe that for every multi-index α with $|\alpha| \le k+1$ and for $x \in Z_a^b$

$$\begin{aligned} & D^{\alpha}Tu(x) = \fint_{B(0,1)} D^{\alpha}u(x+\theta(x)y)\ dy \\ & + \fint_{B(0,1)} \sum_{|\beta|\le|\alpha|}\Big(D^{\beta}u(x+\theta(x)y)\sum_{1\le|l|<|\alpha|}\lambda_l\ y^{\sigma_l}(D^l\theta)^{s_l}\Big)\ dy \\ & + \fint_{B(0,1)} \sum_{h=1}^{N}\frac{\partial u}{\partial x_h}(x+\theta(x)y)y_h D^{\alpha}\theta(x)\ dy \end{aligned} \tag{2.14}$$

for suitable coefficients λ_l, s_l and multi-indices σ_l; thus,

$$|D^{\alpha}Tu| \le CT\Big(\sum_{|\beta|\le|\alpha|}|D^{\beta}u|\Big)\ . \tag{2.15}$$

It follows that estimate (2.13) also holds for derivatives, and we have

$$||Tu||_{W^{k+1,q}(Z_a^b)} \quad \le \quad C(b-a)^{\tau}\Big(\sup_{t\in(a,b)}\big[(t-a)^{-\frac{1}{p}}||u||_{W^{k+1,p}(Z_a^t)}\big] \tag{2.16}$$

$$+ \sup_{t\in(a,b)} \left[(b-t)^{-\frac{1}{p}}||u||_{W^{k+1,p}(Z_t^b)}\right]\Big) \, .$$

Now we prove the continuity of T in $W^{k+1,p}(\Omega)$, and so we observe that for any smooth function u, as for (2.5), with q replaced by p,

$$\int_{Z_a^c} \sum_{|\alpha|\le k+1} |D^\alpha Tu|^p \le C \int_0^{\frac{b-a}{2}} \Big(\int_{\{\eta=a+\rho\}} |T\Big(\sum_{|\alpha|\le k+1} |D^\alpha u|\Big)|^p dH^{N-1}\Big) d\rho$$

$$\le C \int_0^{\frac{b-a}{2}} \Big(\int_{Z_{a+\rho-\rho^k}^{a+\rho+\rho^k}} \rho^{-k}\Big(\sum_{|\alpha|\le k+1} |D^\alpha u|^p dy\Big)\Big) \, d\rho \, .$$

Once again, we first treat the case $k = 1$, that is,

$$\int_{Z_a^c} \sum_{|\alpha|\le 2} |D^\alpha Tu|^p dy \le C \int_0^{\frac{b-a}{2}} \Big(\int_{Z_a^{a+2\rho}} \rho^{-1} \sum_{|\alpha|\le 2} |D^\alpha u|^p dH^N\Big) \, d\rho$$

$$\le C \int_0^{\frac{b-a}{2}} (\int_a^{a+2\rho} (\int_{\{\eta=t\}} \rho^{-1} \sum_{|\alpha|\le 2} |D^\alpha u|^p dH^{N-1}) \, dt) \, d\rho$$

$$\le C \int_a^b \, dt \int_{\{\eta=t\}} (\int_{\frac{t-a}{2}}^{\min\{t-a, \frac{b-a}{2}\}} \rho^{-1} (\sum_{|\alpha|\le 2} |D^\alpha u|^p) \, d\rho) \, dH^{N-1}$$

$$\le C \int_{Z_a^b} \sum_{|\alpha|\le 2} |D^\alpha u|^p dy \, .$$

In the case $k > 1$ we let

$$\varphi(t) = \int_{Z_a^{a+t}} \sum_{|\alpha|\le k+1} |D^\alpha u|^p dy,$$

and as for (2.9) and (2.10) (and keeping in mind that $k = d$ when $q = p$) we obtain

$$\int_{Z_a^c} \sum_{|\alpha|\le k+1} |D^\alpha Tu|^p \, dy \le C \int_0^{\frac{b-a}{2}} \rho^{-k} (\varphi(\rho+\rho^k) - \varphi(\rho-\rho^k)) \, d\rho$$

$$\le C \int_0^{\frac{b-a}{2}} \rho^{-k} d\rho \int_{\rho-\rho^k}^{\rho+\rho^k} \varphi'(t) \, dt \le C \int_{Z_a^b} \sum_{|\alpha|\le k+1} |D^\alpha u|^p \, dy.$$

A similar estimate holds for

$$\int_{Z_c^b} \sum_{|\alpha|\le k+1} |D^\alpha Tu|^p \; dy \; ,$$

so that, keeping in mind that $Tu(x) = u(x)$ if $x \in \Omega - Z_a^b$,

$$\int_{Z_a^b} \sum_{|\alpha|\le k+1} |D^\alpha Tu|^p dy \le C \int_{Z_a^b} \sum_{|\alpha|\le k+1} |D^\alpha u|^p dy \tag{2.17}$$

$$\int_{\Omega - Z_a^b} \sum_{|\alpha|\le k+1} |D^\alpha Tu|^p dy \le C \int_{\Omega - Z_a^b} \sum_{|\alpha|\le k+1} |D^\alpha u|^p dy \; .$$

We actually proved the latter estimate for smooth functions only, but once again a simple approximation argument gives the same estimate for any $u \in W^{k+1,p}(\Omega)$.

In order to show that T is a continuous linear operator on $W^{k+1,p}(\Omega)$ we have only to check that $Tu \in W^{k+1,p}(\Omega; \mathbb{R}^d)$. In order to do this, by (2.17), the continuity of the higher-order trace operators and the density of smooth functions in $W^{k+1,p}$ it is enough to prove that

$$D^\alpha Tu_{|\partial Z_a^b} = D^\alpha u_{|\partial Z_a^b}$$

for any smooth function u and $|\alpha| \le k$. To see this we recall (2.14) and note that if $x \to x_0 \in \partial Z_a^b$ then the first integral tends to $D^\alpha u(x_0)$, while the second and the third disappear because $D^{|\alpha|}\theta = 0$ on ∂Z_a^b if $|\alpha| \le k$ and $\int_{B(0,1)} y_h dy = 0$ for any $1 \le h \le N$ respectively. This fact and (2.17) prove that T is a linear continuous operator from $W^{k+1,p}$ to $W^{k+1,p}$.

We only have to prove the L^{r_h} estimates for the lower-order derivatives. Let us fix $h = 0, 1, \ldots, k$ and $\beta \ge 1$ such that

$$\frac{1}{p} \cdot \frac{(Nk-1)}{k(N-1)} - \frac{1}{r_h(N-1)} < \frac{1}{\beta} \le \min\{\frac{1}{p} \; ; \; \frac{(k+1-h)}{N-1} + \frac{1}{r_h}\}, \tag{2.18}$$

where r_h satisfies

$$p > \frac{r_h(Nk-1)}{kN + r_h k(k+1-h)}.$$

Given a smooth function u, for every multiindex α with $|\alpha| = h$, by (2.1) and (2.4) we get, with q replaced by β,

$$\Big(\int_{\{\eta=a+\rho\}} |D^\alpha Tu|^{r_h}\Big)^{\frac{\beta}{r_h}} \le C \int_{\{\eta=a+\rho\}} \sum_{h\le|l|\le k+1} |D^l Tu|^\beta \; dH^{N-1}$$

$$\leq C\rho^{-\frac{Nk\beta}{p}+k(N-1)}\Big(\int_{Z^{a+\rho+\rho^k}_{a+\rho-\rho^k}}\sum_{h\leq|l|\leq k+1}|D^l u|^p\Big)^{\frac{\beta}{p}} .$$

Now we raise the previous inequality to the power $\frac{r_h}{\beta}$ and integrate on $(\epsilon, \frac{b-a}{2})$, thus obtaining

$$\int_{Z^c_{a+\epsilon}}|D^\alpha Tu|^{r_h} \leq C\int_\epsilon^{\frac{b-a}{2}}\rho^{-\frac{Nkr_h}{p}+\frac{r_h k(N-1)}{\beta}}\Big(\int_{Z^{a+\rho+\rho^k}_{a+\rho-\rho^k}}\sum_{h\leq|l|\leq k+1}|D^l u|^p\Big)^{\frac{r}{p}}d\rho .$$

Arguing as before and letting

$$M_h = \sup_{t\in(0,b-a)} t^{-1}\int_{Z^{a+t}_a}\big(\sum_{h\leq|l|\leq k+1}|D^l u|^p\big)\, dy$$

we have, if $k=1$, that

$$\int_{Z^c_a}|D^\alpha Tu|^{r_h} \leq M^{\frac{r_h}{p}}\int_0^{\frac{b-a}{2}}\rho^{-\frac{Nr_h}{p}+\frac{r_h(N-1)}{\beta}+\frac{r_h}{p}}d\rho \leq cM^{\frac{r_h}{p}}(b-a)^{\tau_h},$$

with $\tau_h = 1 - r_h(N-1)(\frac{1}{p}-\frac{1}{\beta}) \geq 0$, whereas if $k \geq 2$ we argue as for (2.9) and we obtain, with $d_h = \frac{Nkr_h}{p} - \frac{r_h k(N-1)}{\beta} - \frac{r_h}{p} + 1$,

$$\begin{aligned}\int_{Z^c_{a+\epsilon}}|D^\alpha Tu|^{r_h} &\leq CM_h^{\frac{r_h}{p}-1}\int_\epsilon^{\frac{b-a}{2}}\rho^{-d_h}\Big(\int_{Z^{a+\rho+\rho^k}_{a+\rho-\rho^k}}\sum_{h\leq|l|\leq k+1}|D^l u|^p\Big)\, d\rho\\ &\leq cM^{\frac{r_h}{p}}(b-a)^{\tau_h} + o(\epsilon)\end{aligned}$$

where we notice that $\tau_h = k+1-d_h$, by (2.18). Now, exactly as for (2.16), the proof can be concluded. □

We observe that the right-hand side of (2.3) may not be finite, and then we use the following lemma to choose a region where it is suitably bounded.

Lemma 2.3. *Let ψ be a continuous nondecreasing function on an interval $[a,b]$, $a<b$. There exist $a' \in [a, a+\frac{1}{3}(b-a)]$ and $b' \in [b-\frac{1}{3}(b-a), b]$, such that $a \leq a' < b' \leq b$, and*

$$\frac{\psi(t)-\psi(a')}{t-a'} \leq 3\frac{\psi(b)-\psi(a)}{b-a}, \qquad \frac{\psi(b')-\psi(t)}{b'-t} \leq 3\frac{\psi(b)-\psi(a)}{b-a},$$

for all $t \in (a', b')$.

Proof. See [12], Section 2. □

Lemma 2.4. *Let $V \subset\subset \Omega$ and $W \subset \Omega$ be open sets with $\Omega = V \cup W$, $v \in W^{k+1,q}(V)$, $w \in W^{k+1,q}(W)$ and $\frac{q}{p} < \frac{Nk}{Nk-1}$. For every $m \in \mathbb{N}$ there exist a function $z \in W^{k+1,q}_{\mathrm{loc}}(\Omega)$ and open sets $V' \subset V$ and $W' \subset W$, such that $V' \cup W' = \Omega$, $z = v$ in $\Omega - W'$, $z = w$ in $\Omega - V'$,*

$$\mathcal{L}^N(V' \cap W') \le \frac{C}{m} \tag{2.19}$$

$$\begin{aligned} &\sum_{h=0}^{k} \sum_{|\alpha|=h} ||D^\alpha z||_{L^{r_h}(V'\cap W')} + ||z||_{W^{k+1,q}(V'\cup W')} \\ &\le Cm^{-\tau}(||v||_{W^{k+1,p}(V\cap W)} + ||w||_{W^{k+1,p}(V\cap W)} + m||w-v||_{W^{k,p}(V\cap W)}), \end{aligned} \tag{2.20}$$

where r_h are as in the Main Lemma, $C = C(p, q, r_h, N, k, V, W)$ and $\tau = \tau(p, q, r_h, N, k) > 0$.

Proof. Let $\eta \in C_c^\infty(\Omega)$ be such that

$$\eta = 0 \quad \text{on } \Omega - V \quad \text{and } \eta = 1 \quad \text{on } \Omega - W. \tag{2.21}$$

Now we apply Sard's lemma to the critical points of η and we find $0 < a < b < 1$ such that $[a, b] \subset (0, 1) - \eta(\{D\eta = 0\})$. Given $m \in \mathbb{N}$ we define

$$f = \sum_{|\alpha|\le k+1} |D^\alpha v|^p + \sum_{|\alpha|\le k+1} |D^\alpha v|^p + m^p \sum_{|\alpha|\le k} |D^\alpha (w - v)|^p \,.$$

Given $m \in \mathbb{N}$ we also define for $s \in \{1, 2, \ldots, m\}$ the numbers

$$a_h = a + \frac{(s-1)(b-a)}{m}, \qquad b_h = a + \frac{s(b-a)}{m},$$

and observe that since $\{a < \eta < b\} \subset V \cap W$, it is possible to find $h \in \{1, \ldots, m\}$ in such a way that

$$\int_{\{a_h<\eta<b_h\}} f \; dx \le \frac{1}{m} \int_{V\cap W} f \; dx. \tag{2.22}$$

Then we apply Lemma 2.3 to the function

$$\psi(t) = \int_{\{\eta<t\}} f \; dx,$$

and we find $[a', b'] \subset [a_h, b_h]$ such that $b' - a' \ge \frac{1}{3}(b_h - a_h)$ and

$$\int_{\{a'<\eta<t\}} f \; dx \le 3\frac{t-a'}{b'-a'} \int_{\{a'<\eta<b'\}} f \; dx \tag{2.23}$$

$$\int_{\{t<\eta<b'\}} f\ dx \leq 3\frac{b'-t}{b'-a}\int_{\{a'<\eta<b'\}} f\ dx \tag{2.24}$$

for every $t \in (a', b')$. Finally we define

$$V' = \Omega \cap \{\eta > a'\}, \qquad W' = \Omega \cap \{\eta < b'\}$$

$$u = \begin{cases} v & \text{if } \eta(x) \geq b' \\ \frac{(\eta-a')v+(b'-\eta)w}{b'-a'} & \text{if } a' < \eta(x) < b' \\ w & \text{if } \eta(x) \leq a' \ . \end{cases}$$

Clearly $V' \subset V$, $W' \subset W$ and $V' \cup W' = \Omega$. Also (2.19) is verified because $|D\eta| > c > 0$ on $\{a < \eta < b\}$ and $b' - a' \leq \frac{b-a}{m}$. Moreover by direct computation we get

$$\sum_{|\alpha|\leq k+1} |D^\alpha u|^p \leq Cf \qquad \text{on } \{a' < \eta < b'\}.$$

Finally we can apply (2.22)–(2.24) and by the Main Lemma we get $z \in W^{k+1,p}(\Omega)$ such that

$$z = u = v \quad \text{on} \quad \Omega - W'$$
$$z = u = w \quad \text{on} \quad \Omega - V',$$

and (2.20) is satisfied. □

3. Measure representation of relaxed energies

Here we consider integral functionals of the Calculus of Variations of the type

$$\mathcal{F}(u,\Omega) = \int_\Omega f(x, u, D^{[k]}u, D^{k+1}u)\ dx$$

where the energy density f satisfies the following growth assumptions:

$$\begin{aligned} 0 \leq f(x, u, z_1, z_2, \ldots, z_{k+1}) &\leq L(1 + \sum_{1\leq h\leq k} |z_h|^{r_h} + |z_{k+1}|^q) \qquad (3.1) \\ &= L(1 + \sum_{1\leq h\leq k+1} |z_h|^{r_h}) \end{aligned}$$

for every $z_h \in \mathbb{R}^{N^h d}$ and with the exponents r_h satisfying the bounds

$$p > \frac{r_h(Nk-1)}{Nk + r_h k(k+1-h)} \tag{3.2}$$

so that, in particular,

$$\frac{q}{p} < \frac{Nk}{Nk-1}.$$

Our goal is to give a suitable measure representation of the relaxed energies $\mathcal{F}^{q,p}(u,\cdot)$ and $\mathcal{F}^{q,p}_{\rm loc}(u,\cdot)$ as defined in (1.3) and (1.4); to this purpose we give the following:

Definition 3.1. Let $u \in W^{k+1,p}(\Omega)$. We say that a Radon measure μ represents $\mathcal{F}^{q,p}(u,\cdot)$ (respectively $\mathcal{F}^{q,p}_{\rm loc}(u,\cdot)$) if and only if

$$\mu(U) = \mathcal{F}^{q,p}(u,U) \quad (\text{respectively } \mathcal{F}^{q,p}_{\rm loc}(u,U)),$$

for all open subsets $U \subset \Omega$. A Radon measure μ weakly represents $\mathcal{F}^{q,p}(u,\cdot)$ (respectively $\mathcal{F}^{q,p}_{\rm loc}(u,\cdot)$) if and only if

$$\mu(U) \le \mathcal{F}^{q,p}(u,U) \le \mu(\overline{U}) \quad (\text{respectively } \mu(U) \le \mathcal{F}^{q,p}_{\rm loc}(u,U) \le \mu(\overline{U}))$$

for all open subsets $U \subset \Omega$.

We state the following representation results:

Theorem 3.1. *Let $u \in W^{k+1,p}(\Omega;\mathbb{R}^d)$ and assume (3.1)–(3.2) hold. If $\mathcal{F}^{q,p}_{\rm loc}(u,\Omega) < +\infty$, then there exists a nonnegative finite Radon measure μ defined on Ω representing $\mathcal{F}^{q,p}_{\rm loc}(u,\Omega)$.*

Theorem 3.2. *Let $u \in W^{k+1,p}(\Omega;\mathbb{R}^d)$ and assume (3.1)–(3.2) hold. If $\mathcal{F}^{q,p}(u,\Omega) < +\infty$, then there exists a nonnegative finite Radon measure μ defined on $\bar{\Omega}$ weakly representing $\mathcal{F}^{q,p}(u,\Omega)$.*

The key to the proof of both theorems is the subadditivity of $\mathcal{F}^{q,p}(u,\cdot)$ and $\mathcal{F}^{q,p}_{\rm loc}(u,\cdot)$, which we establish in the following lemma.

Lemma 3.1. *Under the hypotheses (3.1)–(3.2), let $V, W \subset \Omega$ be open subsets of Ω such that $V \subset\subset \Omega$ and $V \cup W = \Omega$, and let $u \in W^{k+1,p}(\Omega;\mathbb{R}^d)$. Then*

$$\begin{aligned}\mathcal{F}^{q,p}(u,\Omega) &\le \mathcal{F}^{q,p}(u,V) + \mathcal{F}^{q,p}(u,W)\\ \mathcal{F}^{q,p}_{\rm loc}(u,\Omega) &\le \mathcal{F}^{q,p}_{\rm loc}(u,V) + \mathcal{F}^{q,p}_{\rm loc}(u,W)\,.\end{aligned}$$

Proof. We confine ourselves to proving the first assertion, the proof of the second being similar. We fix $\varepsilon > 0$ and let $V' \subset V$, $W' \subset W$ be open subsets such that $\Omega = V' \cup W'$ and $\overline{V' \cap W'} \subset V \cap W$. By the Rellich embedding theorem it is possible to find $v_n \in W^{k+1,q}(V;\mathbb{R}^d)$ and $w_n \in W^{k+1,q}(\Omega;\mathbb{R}^d)$ such that

$$v_n \rightharpoonup u \text{ weakly in } W^{k+1,p}(V;\mathbb{R}^d)$$

$$||v_n - u||_{W^{k,p}(V' \cap W')} \le \frac{1}{n}$$

$$\int_V f(x, v_n, D^{[k]}v_n, D^{k+1}v_n)\, dx \le \mathcal{F}^{q,p}(u,V) + \varepsilon$$

$$w_n \rightharpoonup u \text{ weakly in } W^{k+1,p}(W;\mathbb{R}^d)$$
$$||w_n - u||_{W^{k,p}(v'\cap W')} \leq \frac{1}{n}$$
$$\int_W f(x, w_n, D^{[k]}w_n, D^{k+1}w_n)\; dx \leq \mathcal{F}^{q,p}(u,W) + \varepsilon.$$

Now in order to connect v_n to w_n we make use of Lemma 3.1. Accordingly, we find $V_n \subset V'$, $W_n \subset W'$ and functions $z_n \in W^{k+1,q}(\Omega;\mathbb{R}^d)$ such that $V_n \cup W_n = \Omega$, $z_n = v_n$ on $\Omega - W_n$, $z_n = w_n$ on $\Omega - V_n$ and

$$\int_{V_n\cap W_n} f(x, z_n, D^{[k]}z_n, D^{k+1}z_n)\; dx$$
$$\leq c\int_{V_n\cap W_n} (1 + \sum_{0\leq h\leq k} |D^h z_n|^{r_h} + |D^{k+1}z_n|^q)\; dx \leq \frac{c}{n^{p\tau}}\ ,$$
$$\mathcal{L}^N(V_n \cap W_n) \leq \frac{c}{n}\ ,$$

so that

$$\int_\Omega f(x, z_n, D^{[k]}z_n, D^{k+1}z_n)\; dx \leq \int_V f(x, v_n, D^{[k]}v_n, D^{k+1}v_n)\; dx$$
$$+ \int_W f(x, w_n, D^{[k]}w_n, D^{k+1}w_n)\; dx + \frac{c}{n^{p\tau}},$$

and hence

$$\liminf_n \int_\Omega f(x, u, D^{[k]}z_n, D^{k+1}z_n)\; dx \leq \mathcal{F}^{q,p}(u,V) + \mathcal{F}^{q,p}(u,W) + 2\varepsilon.$$

Finally, using the boundedness of z_n in $W^{k+1,p}$ and the fact that $\mathcal{L}^N(V_n \cap W_n) \to 0$, it is easy to see that z_n converges weakly in $W^{k+1,p}$ to u, and the thesis follows. □

Proof of Theorem 3.2. Step 1. We first prove the theorem under the additional coercivity assumption

$$f(x, z_0, z_1, \dots, z_{k+1}) \geq c(1 + \sum_{0\leq h\leq k+1} |z_h|^p) \tag{3.3}$$

for some $c > 0$. This assumption will be removed in step 2. By (3.3) it is possible to find a minimizing sequence $\{u_n\} \subset W^{k+1,q}(\Omega;\mathbb{R}^d)$ such that $u_n \rightharpoonup u$ weakly in $W^{k+1,p}(\Omega;\mathbb{R}^d)$ and

$$\lim_n \int_\Omega f(x, u_n, D^{[k]}u_n, D^{k+1}u_n)\; dx = \mathcal{F}^{q,p}(u,\Omega).$$

Up to a not-relabeled subsequence we may suppose that there exists a non-negative Radon measure defined on $\overline{\Omega}$ such that

$$f(x, u, D^k u_n, D^{k+1} u_n)\mathcal{L}^N \lfloor \Omega \rightharpoonup \mu$$

weakly in the sense of measures; it also follows that

$$\mu(\overline{\Omega}) = \mathcal{F}^{q,p}(u, \Omega), \tag{3.4}$$

and if V is an open subset of Ω then

$$\mathcal{F}^{q,p}(u, V) \leq \liminf_n \int_V f(x, u_n, D^{[k]} u_n, D^{k+1} u_n)\, dx \leq \mu(\overline{V})\,. \tag{3.5}$$

In order to prove that also

$$\mu(V) \leq \mathcal{F}^{q,p}(u, V) \tag{3.6}$$

we fix $\varepsilon > 0$ and select an open subset $Z \subset\subset V$ such that

$$\mu(V) - \mu(Z) < \varepsilon;$$

then, using Lemma 3.1 along with (3.4) and (3.5) we have

$$\begin{aligned} \mu(V) &\leq \mu(Z) + \varepsilon = \mu(\overline{\Omega}) - \mu(\overline{\Omega} - Z) + \varepsilon \\ &\leq \mathcal{F}^{q,p}(u, \overline{\Omega}) - \mathcal{F}^{q,p}(u, \Omega - \bar{Z}) + \varepsilon \leq \mathcal{F}^{q,p}(u, V) + \varepsilon. \end{aligned}$$

This is valid for each $\epsilon > 0$, so that (3.6) follows.

Step 2. In order to remove hypothesis (3.3) we argue as follows. We define, for each open subset $V \subseteq \Omega$ and $\varepsilon > 0$,

$$\mathcal{F}_\varepsilon(u, V) = \int_V f(x, u, D^{[k]} u, D^{k+1} u) + \varepsilon(1 + \sum_{0 \leq h \leq k} |D^h u|^p + |D^{k+1} u|^p)\, dx$$

and denote by $\mathcal{F}_\varepsilon^{q,p}(u, \cdot)$ the relaxation of $\mathcal{F}_\varepsilon$ in the sense of (1.3)–(1.4); according to Step 1, let μ_ε be the Radon measure weakly representing $\mathcal{F}_\varepsilon^{q,p}$; we have that

$$\mu_\varepsilon(\overline{\Omega}) = \mathcal{F}_\varepsilon^{q,p}(\Omega) \leq \mathcal{F}^{q,p}(u, \Omega) + \varepsilon \sup_n ||u_n||_{W^{k+1,p}} + \varepsilon \leq C.$$

So we may select a sequence ε_h such that $\mu_{\varepsilon_h} \rightharpoonup \mu$ weakly in the sense of measures, and so

$$\mathcal{F}^{q,p}(u, U) \leq \liminf_\varepsilon \mu_\epsilon(\overline{U}) = \mu(\overline{U}).$$

On the other hand, for each $\delta > 0$, there exists a sequence v_n such that $v_n \rightharpoonup u$ weakly in $W^{k+1,p}(U)$ and

$$\int_U f(x, v_n, D^{[k]} v_n, D^{k+1} v_n)\, dx \leq \mathcal{F}^{q,p}(u, U) + \delta,$$

so that if we take h large enough

$$\mathcal{F}_{\varepsilon_h}(v_n, U) \le \mathcal{F}^{q,p}(u, U) + 2\delta$$

whence

$$\mu_{\varepsilon_h}(U) \le \mathcal{F}^{q,p}(u, U) + 2\delta,$$

and letting first $h \to \infty$ and then $\delta \to 0$ we have

$$\mu(U) \le \mathcal{F}^{q,p}(u, U),$$

and the proof is concluded. □

Remark. Exactly as in [12], by a simple use of Lemma 3.1 it can be proved that if μ is a measure that weakly represents $\mathcal{F}^{q,p}(u,\cdot)$ and u is an open subset of Ω then $\mu(U) = \mathcal{F}^{q,p}(u, U)$ whenever the following inner approximation property holds:

$$\inf_K \{\mathcal{F}^{q,p}(u, U \setminus K) : K \subset U \ \text{ is compact}\} = 0 \,. \tag{3.7}$$

Proof of Theorem 3.1. By looking at the proof of Theorem 3.2 there is no loss of generality in supposing that (3.3) holds. This assumption can be easily removed as done previously. By Theorem 3.2 we define on $\overline{\Omega}$ the Radon measure μ, weakly representing $\mathcal{F}^{q,p}(u,\cdot)$. Now we need only to prove that actually

$$\mu(U) \ge \mathcal{F}^{q,p}(u, U) \tag{3.8}$$

for each open subset $U \subset \Omega$.

Let us consider an increasing sequence of smooth, open subsets $\{U_h\}_h$ such that $\overline{U_h} \subset\subset U_{h+1}$ for all h and $U = \bigcup_{h=1}^{\infty} U_h$. We take, for each $h \ge 3$, a sequence $\{u_{h,n}\}_n \subset W^{k+1,q}_{\mathrm{loc}}(U_h \setminus \overline{U_{h-2}})$ weakly convergent to u in $W^{k+1,p}(U_h \setminus \overline{U_{h-2}})$ and such that

$$\int_{U_h \setminus \overline{U_{h-2}}} f(x, u_{h,n}, D^{[k]}u_{h,n}, D^{k+1}u_{h,n})\, dx \le \mathcal{F}^{q,p}_{\mathrm{loc}}(u, U_h \setminus \overline{U_{h-2}}) + \frac{1}{2^h} \,. \tag{3.9}$$

We denote by $\{\alpha_h\}_h$ a sequence of positive integers that we are going to determine later and up to (not-relabeled) subsequences we may assume that $u_{h,n} \to u$ almost everywhere in $U_h \setminus \overline{U_{h-2}}$ and

$$||u_{h,n} - u||_{W^{k,p}(U_h \setminus \overline{U_{h-2}})} \le \frac{1}{2^{h+n}\alpha_h} \,.$$

Now we use Lemma 2.3 to connect $u_{h,n}$ to $u_{h+1,n}$ across $U_h \setminus \overline{U_{h-1}}$; so there exist open sets $V^+_{h,n}$, $V^-_{h+1,n}$ such that $V^+_{h,n} \subset U_h \setminus \overline{U_{h-2}}$ and $V^-_{h+1,n} \subset U_{h+1} \setminus \overline{U_{h-1}}$ with $U_{h+1} \setminus \overline{U_{h-2}} = V^+_{h,n} \cup V^-_{h+1,n}$,

$$L^n(V^+_{h,n} \cap V^-_{h+1,n}) \le \frac{C_h}{2^{h+n}\alpha_h}$$

and there exist functions $z_{h,n} \in W^{k+1,p}(U_{h+1} \setminus \overline{U_{h-1}})$ such that $z_{h,n} = u_{h,n}$ in $(U_h \setminus \overline{U_{h-1}}) \setminus V^-_{h+1,n}$ and $z_{h,n} = u_{h+1,n}$ in $(U_{h+1} \setminus \overline{U_{h-1}}) \setminus V^+_{h,n}$ and

$$\int_{V^+_{h,n} \cap V^-_{h+1,n}} f(x, u_{h,n}, D^{[k]}u_{h,n}, D^{k+1}u_{h,n})\, dx$$
$$\le C \int_{V^+_{h,n} \cap V^-_{h+1,n}} (1 + \sum_{h=0}^{k+1} |D^h z_{h,n}|)\, dx \le \frac{C_h}{2^{p\tau(h+n)}\alpha_h^{\tau p}}\ .$$

Now the numbers α_h are specified depending on the constants C_h appearing in the previous formula, in such a way that $\alpha^{-\tau q} C_h \le 1$. Let $z_n \in W^{k+1,q}_{\mathrm{loc}}(\Omega \setminus \overline{U_1})$ be given by $z_n = z_{h,n}$ on $V^+_{h,n} \cap V^-_{h+1,n}$ and $z_n = u_{h+1,n}$ on $(U_{h+1} \setminus U_{h-1}) \setminus (V^+_{h,n} \cup V^-_{h+2,n})$.

With $s \ge 2$, we have

$$\int_{\Omega \setminus \overline{U_s}} f(x, z_n, D^{[k]}z_n, D^{k+1}z_n)\, dx$$
$$\le \sum_{h=s+1}^{\infty} \int_{U_h \setminus \overline{U_{h-1}}} f(x, z_n, D^{[k]}z_n, D^{k+1}z_n)\, dx$$
$$\le \sum_{h=s+1}^{\infty} \int_{U_{h+1} \setminus \overline{U_{h-1}}} f(x, u_{h,n}, D^{[k]}u_{h,n}, D^{k+1}u_{h,n})\, dx$$
$$+ \sum_{h=s+1}^{\infty} \int_{U_h \setminus \overline{U_{h-1}}} f(x, u_{h,n}, D^{[k]}u_{h,n}, D^{k+1}u_{h,n})\, dx$$
$$+ \sum_{h=s+1}^{\infty} \int_{V^+_{h,n} \cap V^-_{h+1,n}} f(x, z_{h,n}, D^{[k]}z_{h,n}, D^{k+1}z_{h,n})\, dx$$
$$\le \sum_{h=s+1}^{\infty} \left(2\mathcal{F}^{q,p}(u, U_h \setminus \overline{U_{h-1}}) + \frac{1}{2^{h-1}}\right) + \sum_{h=s+1}^{\infty} \frac{1}{2^{q\tau(n+h)}}$$
$$\le \sum_{h=s+1}^{\infty} 2\mu(U_{h+2} \setminus U_{h-1}) + \frac{1}{2^{s+1}} + \frac{C}{2^{q\tau(n+s)}}$$

$$\leq 6\mu(U \setminus U_{s-1}) + \frac{C}{2^{q\tau(n+s)}}.$$

By (3.3) we may suppose that the sequence $\{z_n\}$ is bounded in $W^{k+1,p}(U\setminus\overline{U_s})$ and moreover $z_n \rightharpoonup u$ weakly in $W^{k+1,p}(U \setminus \overline{U_s})$, so that

$$\mathcal{F}^{q,p}_{\mathrm{loc}}(u, U \setminus \overline{U_k}) \leq 6\mu(U \setminus \overline{U_{s-1}}) + \frac{C}{2^{s+1}}.$$

Hence, (3.7) is satisfied, and by the previous remark we may finally conclude that (3.8) holds. □

4. A lower-semicontinuity result

In this section we concentrate our attention on functionals depending on highest-order derivatives

$$\mathcal{F}(u, \Omega) = \int_\Omega f(D^{k+1}u)\; dx\;. \tag{4.1}$$

We recall that for such functionals an appropriate notion of quasiconvexity, studied by Meyers (see [23] and also [14]), is available.

Definition 4.1. Let $k \geq 0$. A function $f : \mathbb{R}^{dN^{k+1}} \to \mathbb{R}$ is quasiconvex if and only if

$$\int_\Omega f(\xi + D^{k+1}z) - f(\xi)\; dx \geq 0 \tag{4.2}$$

for each $\xi \in \mathbb{R}^{dN^{k+1}}$ and $z \in C^{k+1}_0(\Omega; \mathbb{R}^d)$, where Ω is an open subset of $\mathbb{R}^{dN^{k+1}}$. A functional $\mathcal{F}$ as in (4.1) is said to be quasiconvex if and only if the function f is quasiconvex.

Remark. In the literature, the definition of such a function is supplied with the domain (of f) being not $\mathbb{R}^{dN^{k+1}}$, but an appropriate domain of tensors, keeping in mind the symmetries of the matrices of the type $D^s u$ (see [23], [14] and [17]). Here, to avoid the overburdening of notation, we preferred to consider the integrand f to be defined on the whole of $\mathbb{R}^{dN^{k+1}}$, making an obvious abuse of notation.

The previous definition may need some comment. Indeed, one can check that it is well posed in the sense that if condition (4.2) is satisfied for an open subset Ω, then it still holds for any other open subset $\overline{\Omega} \subset \mathbb{R}^N$ (see [23] and [14]). Moreover, since each $\xi \in \mathbb{R}^{dN^{k+1}}$ can be seen as the $(k+1)$-order derivative of a polynomial w of order $k+1$, Definition 4.1 can be immediately reformulated by saying that each $(k+1)$-order polynomial is a minimizer of a quasiconvex functional in the Dirichlet class $w + C^{k+1}_0(\Omega; \mathbb{R}^d)$. Finally, we observe that, in the case $k = 0$, Definition 4.1 recaptures the classical notion of quasiconvexity in the sense of Morrey. We devote this section to the proof of a lower-semicontinuity result that extends in different ways previous results in [2, 12, 14, 17, 23, 25].

Theorem 4.1. *Suppose $q \geq 1$ and $\frac{q}{p} < \frac{Nk}{Nk-1}$. Let $f : \mathbb{R}^{dN^{k+1}} \to \mathbb{R}$ be a quasiconvex function such that*

$$0 \leq f(\xi) \leq L(1 + |\xi|^q) \tag{4.3}$$

for each $\xi \in \mathbb{R}^{dN^{k+1}}$. Let $u \in W^{k+1,p}(\Omega; \mathbb{R}^d)$, $u_n \in W^{k+1,q}(\Omega; \mathbb{R}^d)$ and $u_n \rightharpoonup u$ in $W^{k+1,p}(\Omega; \mathbb{R}^d)$; then

$$\int_\Omega f(D^{k+1}u)\; dx \leq \liminf_n \int_\Omega f(D^{k+1}u_n)\; dx. \tag{4.4}$$

Before proving Theorem 4.1, we state some preliminaries about the validity of the Taylor formula in Sobolev spaces. For detailed proofs we refer to the book by W.P. Ziemer [26]. If $u \in W^{k+1,p}(\Omega)$ then it is possible to define its Taylor polynomial centered at point x_0,

$$P_{k+1}u(x, x_0) = \sum_{0 \leq |\alpha| \leq k+1} \frac{1}{\alpha!} D^\alpha u(x_0)(x - x_0)^\alpha \;;$$

then the following result holds (see [26], Theorem 3.4.2, p. 129, for the proof).

Proposition 4.1. *Let $u \in W^{k+1,p}(\Omega; \mathbb{R}^d)$; then for almost every $x_0 \in \Omega$*

$$\lim_{\rho \to 0} \rho^{-(k+1)} \left(\fint_{B_\rho(x_0)} |u(x) - P_{k+1}u(x; x_0)|\, dx \right) = 0.$$

Proof of Theorem 4.1. Step 1. Here we make a preliminary reduction; that is, we suppose that Ω is the unit ball $\Omega = B(0,1) \equiv B$ and u is a polynomial of order $k+1$, $u = \sum_{|\alpha|=k+1} c_\alpha x^\alpha$, with $c_\alpha \in \mathbb{R}$. By Rellich's theorem there is no loss of generality in assuming that

$$||u_n - u||_{W^{k,p}(B)} \leq \frac{1}{n}.$$

Let $R < 1$, $\rho = \frac{R+1}{2}$. Applying Lemma 3.1 to $v = u_n$, $w = u$ and $V = \rho B$, $W = B - RB$, we obtain a sequence of functions $z_n \in W^{k+1,q}(B; \mathbb{R}^d)$ and open subset $V_n \subset\subset V$, $W_n \subset W$ such that $V_n \cup W_n = B$, $z_n = u_n$ on $B - W_n$, $z_n = u$ on $B - V_n$ and moreover

$$\mathcal{L}^N(V_n \cap W_n) \quad \leq \quad \frac{c(R)}{n}$$

$$\int_{V_n \cap W_n} |D^{k+1} z_n|^q dx \quad \leq \quad \frac{c(R)}{n^{q\tau}},$$

for some $\tau > 0$. Now we observe that $z_n - u \in W_0^{k+1,q}(B;\mathbb{R}^d)$. By the growth conditions in (4.3) we observe that the quasiconvexity inequality remains valid if we test it with functions from $W^{k+1,q}(B;\mathbb{R}^d)$; then we have

$$\int_B f(D^{k+1}u)\ dx \leq \int_B f(D^{k+1}z_n)\ dx.$$

It then follows that

$$\begin{aligned}
&\int_B f(D^{k+1}u) - f(D^{k+1}u_n)\ dx \leq \int_B f(D^{k+1}z_n) - f(D^{k+1}u_n)\ dx \\
&\int_{B-V_n} f(D^{k+1}u) - f(D^{k+1}u_n)\ dx + \int_{V_n\cap W_n} f(D^{k+1}z_n) - f(D^{k+1}u_n)\ dx \\
&\leq \int_{B-V_n} f(D^{k+1}u)\ dx + \int_{V_n\cap W_n} f(D^{k+1}z_n)\ dx \\
&\leq c[\mathcal{L}^N(B-RB) + \frac{c(R)}{n^{q\tau}}] \leq c(1-R) + \frac{c(R)}{n^{q\tau}};
\end{aligned}$$

then it suffices to let first $n \to +\infty$ and then $R \to 1$ to obtain the assertion in this case.

Step 2. Here we treat the general case $u \in W^{k+1,p}(\Omega;\mathbb{R}^d)$, $u_n \in W^{k+1,q}(\Omega;\mathbb{R}^d)$, $u_n \rightharpoonup u$ weakly in $W^{k+1,p}(\Omega;\mathbb{R}^d)$. We assume, without loss of generality, that

$$\lim_n \int_\Omega f(D^{k+1}u_n)\ dx$$

exists and is finite. Moreover we define two nonnegative Radon measures μ and ν in such a way that, up to subsequences,

$$f(D^{k+1}u_n)\mathcal{L}^N \rightharpoonup \mu \qquad |D^{k+1}u_n|^p\mathcal{L}^N \rightharpoonup \nu$$

weakly in the sense of measures. Finally, in a standard way, we introduce

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0}\frac{\mu(B(x_0,\varepsilon))}{\mathcal{L}^N(B(x_0,\varepsilon))} \qquad \frac{d\nu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0}\frac{\nu(B(x_0,\varepsilon))}{\mathcal{L}^N(B(x_0,\varepsilon))}. \tag{4.5}$$

We are going to show that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq f(D^{k+1}u(x_0)) \tag{4.6}$$

for almost every $x_0 \in \Omega$.

To this aim we recall that for $(\mathcal{L}^N)$-almost-every $x_0 \in \Omega$ the quantities in (4.5) exist and are finite, and moreover, by Proposition 4.1,

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{k+1}} \fint_{B(x_0,\varepsilon)} |u(x) - P_{k+1}u(x, x_0)| \, dx = 0; \tag{4.7}$$

so, let $x_0 \in \Omega$ such that both (4.5) and (4.7) are satisfied. We select $\varepsilon_h \to 0$ such that $\mu(\partial B_{\varepsilon_h}(x_0)) = \nu(\partial B_{\varepsilon_h}(x_0)) = 0$. Now we rescale the functions u_n in $B(0,1)$ and define

$$u_{n,h}(y) = \frac{u_n(x_0 + \varepsilon_h y) - \sum_{0\le|\alpha|\le k} \frac{1}{\alpha!} D^\alpha u(x_0)(\varepsilon_h y)^\alpha}{\varepsilon_h^{k+1}}.$$

with $y \in B(0,1)$. With such a notation we have

$$\begin{aligned} \lim_{h\to+\infty} \frac{\mu(B(x_0,\varepsilon_h))}{\mathcal{L}^N(B(x_0,\varepsilon_h))} &= \lim_{h\to+\infty} \lim_{n\to+\infty} \fint_{B(x_0,\varepsilon_h)} f(D^{k+1}u_n(x)) \, dx \\ &= \lim_{h\to+\infty} \lim_{n\to+\infty} \fint_{B_1(0)} f(D^{k+1}u_{n,h}(y)) \, dy. \end{aligned}$$

Now we observe that $u_{n,h} \in W^{k+1,q}(B_1(0); \mathbb{R}^d)$ and that

$$\lim_{h\to+\infty} \lim_{n\to+\infty} ||u_{n,h} - u_0||_{L^1(B(0,1))} = 0$$

$$\limsup_{h\to+\infty} \limsup_{n\to+\infty} ||D^{k+1}u_{n,h}||_{L^p(B(0,1))} \le \frac{d\nu}{d\mathcal{L}^N}(x_0) < +\infty$$

where

$$u_0(y) := \sum_{|\alpha|=k+1} \frac{1}{\alpha!} D^{k+1}u(x_0) y^\alpha \,.$$

By a simple diagonalization argument we may extract a subsequence $v_h = u_{n_h,h}$ such that $v_h \rightharpoonup u_0$ weakly in $W^{1,p}(B(0,1); \mathbb{R}^d)$ and

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{h\to+\infty} \fint_{B(0,1)} f(D^{k+1}v_h(y)) \, dy \,.$$

By the result proved in Step 1 we deduce that

$$\begin{aligned} \frac{d\mu}{d\mathcal{L}^N}(x_0) &= \lim_{h\to+\infty} \fint_{B_1(0)} f(D^{k+1}v_h(y)) \, dy \\ &\ge \fint_{B_1(0)} f(D^{k+1}u_0(y)) \, dy \ge f(D^{k+1}u(x_0)), \end{aligned}$$

and (4.6) is finally proved.

We are now ready to conclude the proof. Indeed, let $\varphi \in C_c^\infty(\Omega)$ such that $0 \leq \varphi \leq 1$; then we have

$$\lim_{n\to+\infty} \int_\Omega f(D^{k+1}u_n)\, dx \geq \lim_{n\to+\infty} \int_\Omega \varphi f(D^{k+1}u_n)\, dx$$
$$= \int_\Omega \varphi\, d\mu \geq \int_\Omega \varphi \frac{d\mu}{dL^n}\, dx \geq \int_\Omega \varphi f(D^{k+1}u(x))\, dx,$$

and letting $\varphi \uparrow 1$ we have (4.4). □

Theorem 4.1 has some corollaries. Let f be a Carathéodory integrand of the type considered in (4.1) but not necessarily quasi convex; we denote by Qf the quasi convex envelope of f, that is, the greatest quasi convex function (in the sense of Definition 4.1) that lies below f, i.e.,

$$Qf := \sup\{g : g \leq f;\ g \text{ is quasiconvex}\}\ .$$

We observe that, by the very definition of Qf, it follows that $Qf = f$ if and only if f is a quasiconvex function. Moreover, it is an easy consequence of the definition of quasiconvex envelope that if f satisfies the bound in (4.3) then also Qf satisfies the same bound:

$$0 \leq Qf(\xi) \leq L(1 + |\xi|^q)\ . \tag{4.8}$$

Proposition 4.2. *Let f be a Carathéodory function such that* (4.3) *is satisfied. Let $u \in W^{k+1,p}(\Omega)$ with p, q related as in Theorem* 4.1; *then*

$$\mathcal{F}^{q,p}(u,\Omega) \geq \int_\Omega Qf(D^{k+1}u)\, dx.$$

Proof. By (4.8), Theorem 4.1, if $\{u_n\}$ is a sequence from $W^{k+1,q}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{k+1,p}(\Omega)$, then

$$\liminf_n \int_\Omega f(D^{k+1}u_n)\, dx \geq \liminf_n \int_\Omega Qf(D^{k+1}u_n)\, dx \geq \int_\Omega Qf(D^{k+1}u)\, dx.$$

This is valid for any sequence $\{u_n\}$, and the assertion follows by the definition of $\mathcal{F}^{q,p}(u,\cdot)$. □

Remark. In the paper [6] the authors give a sharp version of the result of Proposition 4.1 in the case $k = 0$; that is, they deal with $W^{1,p}(\Omega)$ and with first-order integrals,

$$\int_\Omega f(Du)\, dx,$$

as described in the Introduction.

More precisely, if $u \in W^{1,p}(\Omega)$, $\mathcal{F}^{q,p}(u,\Omega) < +\infty$ and μ is the Radon measure that weakly represents $\mathcal{F}^{q,p}(u,\cdot)$ then a precise characterization of the bulk part of the relaxed energy can be given; indeed,

$$\frac{d\mu}{dL^n}(x) = Qf(Du(x)) \tag{4.9}$$

for almost every $x \in \Omega$. This result is achieved by applying techniques from [12] with a new relaxation method introduced in [7]. Likely, a result of the type in (4.9) also holds in our case, following the methods from [6] and combining them with the ones used here, and in particular with the main lemma of Section 2.

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