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ON THE CONVERGENCE OF EULER-STOKES SPLITTING OF THE NAVIER-STOKES EQUATIONS

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Abstract. We consider Euler–Stokes splitting approximation of the Navier–Stokes equations with no–slip boundary condition. This consists in alternate solving of the Euler equations with tangential boundary condition and Stokes equations with no-slip boundary condition on small time intervals of the same length k. In a previous paper, J.T. Beale and C. Greengard proved the convergence of this approximation scheme in L^p norm as k tends to zero, for smooth solutions of the Navier–Stokes equations. Here we show how a certain simplification in their arguments improves their main result in the following way: the convergence holds without any additional regularity assumption on the solution of the Navier–Stokes equations.

1. Introduction

Let Ω be an open and bounded subset of $\mathbb{R}^d$, where $d = 2$ or $d = 3$, with a smooth boundary, and let T be a fixed time. The Navier–Stokes equations for incompressible fluid flow consist in the following system of $d+1$ equations:

$$\frac{\partial v}{\partial t} - \Delta v + (v \cdot \nabla)v + \nabla \pi = 0 \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$\nabla \cdot v = 0 \quad \text{in } \Omega \times (0,T), \tag{1.2}$$

together with the boundary condition

$$v = 0 \quad \text{on } \partial\Omega \times [0,T]. \tag{1.3}$$

Here $v : \Omega \times [0,T] \to \mathbb{R}^d$ is the velocity vector field and $\pi : \Omega \times [0,T] \to \mathbb{R}$ is the (scalar) pressure. The viscosity coefficient is supposed to be 1.

In [1], J.T. Beale and C. Greengard tackled the approximation of the Navier–Stokes equations by splitting them into two pieces: the Euler equations and the Stokes equations, on small time intervals (see also [2]–[4]).

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The Euler equations of incompressible inviscid fluid flow are written as

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi = 0 \quad \text{in } \Omega, \tag{1.4}$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega, \tag{1.5}$$

and are taken together with the tangential boundary condition

$$u \cdot N = 0 \quad \text{on } \partial\Omega, \tag{1.6}$$

where N is the exterior normal vector field.

The Stokes equations consist in the system of $d+1$ linear equations

$$\frac{\partial u}{\partial t} - \Delta u + \nabla \pi = 0 \quad \text{in } \Omega, \tag{1.7}$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega, \tag{1.8}$$

together with the no-slip boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.9}$$

The approximation scheme proposed in [1] can be described as follows. Let $v_0 : \Omega \to \mathbb{R}^d$ be an initial divergence–free velocity field. Divide the time interval $[0, T]$ into m subintervals of the same length $k = T/m$. Starting with the initial velocity v_0, we alternately solve the Euler equations with tangential boundary condition and the Stokes equations with the no-slip boundary condition m times on the interval $[0, k]$. So, to compute the approximate solution of the Navier–Stokes equations at the time $t_n = nk$, for integral n with $0 \leq n \leq m$, we alternate the solution of equations (1.4)–(1.6) on $[0, k]$ and the solution of equations (1.7)–(1.9) on $[0, k]$, $2n-1$ times, at each alternation the initial value is taken to be the final value of the preceding solution. Let us denote the approximate solution by u.

The objective pursued by Beale and Greengard in [1] is to prove the convergence of the splitting approximation described before. Their main result is the following.

If the initial divergence–free velocity field v_0 satisfying the no–slip boundary condition belongs to the Sobolev space $(H^{2,p}(\Omega))^d$ with $d < p < \infty$ and, in addition, v_0 is extendible to a solution $v \in C^\alpha([0,T]; (H^{2,p}(\Omega))^d)$ of the Navier–Stokes equations, with $0 < \alpha < 1$, then, for sufficiently small k, the approximate solution u remains bounded in $(H^{2,p}(\Omega))^d$ and satisfies

$$\max_{0 \leq n \leq m} |u(t_n) - v(t_n)|_{(L^p(\Omega))^d} \leq Ck, \tag{1.10}$$

where the constant C does not depend on k.

The proof mainly contains two parts. First one proves the boundedness of u in $(H^{2,p}(\Omega))^d$. This requires a careful study of a linear version of the above approximation scheme where in the Euler step the Euler equations are replaced by the following linear variant:

$$\begin{aligned} \frac{\partial u}{\partial t} + (v \cdot \nabla)u + \nabla \pi = 0, \\ \nabla \cdot u = 0, \end{aligned}$$

where v is the given solution of (1.1)–(1.3). The boundedness of u in $(H^{2,p}(\Omega))^d$ being obtained, the estimate (1.10) follows after rewriting the Navier–Stokes equations and the splitting approximation as integral equations by means of the evolution generated by the sum of the Stokes operator and a linear advection operator. *The needed estimates for the evolution system require the above additional regularity assumption on the solution v of the Navier–Stokes equations* (that is, $v \in C^\alpha([0,T];(H^{2,p}(\Omega))^d)$ with $0 < \alpha < 1$).

In this paper, we shall present a simplified version of the second part of the proof involving only the Stokes semigroup. In other words, we rewrite the Navier–Stokes equations and the splitting approximation as integral equations by means of the semigroup generated by the Stokes operator. This allows us to obtain the convergence result of [1] *without any additional regularity assumption on v.*

It is interesting to point out that our considerations does not exceed the horizon of [1]; all the ideas and ingredients used here are extracted from [1].

As the present paper is directly connected with [1], we recommend to read it together with [1]. For the same reason, here we preserve most of the notations of [1].

2. The functional framework and the main result

As in the introduction, Ω is a bounded open set of $\mathbb{R}^d$, where $d = 2$ or $d = 3$, whose boundary is of class C^∞. To define the function spaces we shall use in our statements and arguments, we need the Sobolev spaces $H^{m,p}(\Omega)$ (of functions in $L^p(\Omega)$ whose derivatives of order less than or equal to m also belong to $L^p(\Omega)$) for $m = 1$ or $m = 2$, and $p \geq 1$. The further considerations involving estimates of fractional powers of the Stokes operator require the complex interpolation spaces $H^{s,p}(\Omega) = [L^p(\Omega), H^{m,p}(\Omega)]_{\theta = s/m}$ for $0 < s < 2$. We denote by $\overset{\circ}{H}{}^{s,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm of $H^{s,p}(\Omega)$. In what follows, we shall use the product spaces

$(L^p(\Omega))^d$, $(C_0^\infty(\Omega))^d$, $(H^{s,p}(\Omega))^d$ and $(\overset{\circ}{H}{}^{s,p}(\Omega))^d$ for $0 < s \le 2$. The norms of $(L^p(\Omega))^d$ and $(H^{s,p}(\Omega))^d$, denoted by $|\cdot|_{L^p}$ and $|\cdot|_{H^{s,p}}$, respectively, are defined as product norms.

The Navier–Stokes equations can be viewed as an evolution equation in the space H_p of weakly divergence–free vector fields in $(L^p(\Omega))^d$ which are tangential to the boundary $\partial\Omega$ in a weak sense, that is,

$$H_p = \{u \in (L^p(\Omega))^d : \operatorname{div} u = 0 \quad \text{in } \Omega \quad \text{and } u \cdot N = 0 \quad \text{on } \partial\Omega\}.$$

In fact, H_p is the closure in $(L^p(\Omega))^d$ of the divergence–free vector fields in $(C_0^\infty(\Omega))^d$. For each $u \in (L^p(\Omega))^d$ there exist unique $u_1 \in H_p$ and $u_2 = \nabla g$ for some $g \in H^{1,p}(\Omega)$ such that $u = u_1 + u_2$ (see [1] or [7]). Define the projection $P_p : (L^p(\Omega))^d \to H_p$ by $P_p u = u_1$.

The Stokes operator $A_p : D(A_p) \to H_p$ is defined as

$$A_p = -P_p\Delta$$

with domain $D(A_p) = (H^{2,p}(\Omega))^d \cap (\overset{\circ}{H}{}^{1,p}(\Omega))^d \cap H_p$. The operator A_p generates an analytic semigroup $\{e^{-tA_p} : t \ge 0\}$ on H_p called the Stokes semigroup. (For more details concerning operators P_p and A_p, we refer to [1] or [7].) When the exponent p is fixed, for the sake of simplicity, we may write P and A instead of P_p and A_p.

Now the Navier–Stokes equations (i.e., equations (1.1)–(1.3)) can be expressed as an evolution equation in H_p in the following way:

$$v' + A_p v + P_p(v \cdot \nabla)v = 0. \tag{2.1}$$

For sufficiently smooth divergence–free initial value v_0, equation (2.1) has a unique local strong solution. The following existence result can be found in W. von Wahl's book [7].

Theorem 2.1. *Let $d < p < \infty$. If $v_0 \in D(A_p)$, then there exist $T \in (0, +\infty]$ and a unique $v \in C^1([0,T]; H_p) \cap C([0,T]; (H^{2,p}(\Omega))^d)$ with $v(t) \in D(A_p)$ for all $t \in [0,T]$, which satisfies* (2.1) *and* $v(0) = v_0$.

The Euler equations (i.e., equations (1.4)–(1.6)) can be written as an evolution equation like that:

$$u' + P_p(u \cdot \nabla)u = 0. \tag{2.2}$$

A theorem established by R. Temam in [6] (see also [1]) says that equation (2.2) has a local strong solution in H_p with $d < p < \infty$ provided that the initial value u_0 is sufficiently smooth:

Theorem 2.2. *Let $d < p < \infty$. If $u_0 \in (H^{2,p}(\Omega))^d \cap H_p$ with $|u_0|_{H^{2,p}} \leq M$, then there exists a constant C_0, depending only on p and M, such that equation* (2.2) *has a unique solution $u \in L^\infty(0, T_0; (H^{2,p}(\Omega))^d \cap H_p)$, where $T_0 = 1/(C_0|u_0|_{H^{2,p}})$, which satisfies $u(0) = u_0$.*

Denote by $E(t)$ the nonlinear Euler solution operator. The local solution u given by Theorem 2.2 can be expressed as $u(t) = E(t)u_0$.

Finally, the Stokes equations (i.e., equations (1.7)-(1.9)) can be written as

$$u' + A_p u = 0. \tag{2.3}$$

If $u_0 \in H_p$, we can express the solution of (2.3) which satisfies the initial condition $u(0) = u_0$ by means of the Stokes semigroup: $u(t) = e^{-tA_p}u_0$.

Now we are prepared to define the splitting scheme for the Navier–Stokes equations; this will be done recursively. Let $k = T/m$, where T is given by Theorem 2.1 and m is a positive integer, and let $t_n = nk$ for integer n with $0 \leq n \leq m$. We set

$$u(t) = e^{-\tau A_p}E(\tau)u(t_n) \text{ for } t = t_n + \tau,\ \tau \in (0, k],\ 0 \leq n \leq m-1, \tag{2.4}$$

$$u(0) = v_0. \tag{2.5}$$

Comparing with the main result of [1], the following theorem does not require any additional regularity assumption on the solution v of equation (2.1) having the initial value v_0 in $D(A_p)$.

Theorem 2.3. *Let p be given such that $d < p < \infty$. If $v_0 \in D(A_p)$, then, for sufficiently small k, u (given by* (2.4), (2.5)) *is well defined and bounded in $(H^{2,p}(\Omega))^d$, and satisfies*

$$\sup_{t \in [0,T]} |u(t) - v(t)|_{L^p} \leq Ck,$$

where v is the solution of equation (2.1) *satisfying $v(0) = v_0$ (given by Theorem* 2.1) *and C is a positive constant.*

The question of the well–definedness of u comes from the fact that the solutions of the Euler equations involved in the definition of u are only local (see Theorem 2.2).

3. Proof of Theorem 2.3

Here we shall maintain the main lines of the proof given in [1]. But, as we have already mentioned in the introduction, certain arguments of that proof will be simplified (which will have as an effect the improvement of the main result from [1]). For these reasons, we shall merely describe the proof, developing only those parts which are really distinct.

As an intermediate stage in the proof given in [1], one proves the convergence of an analogous splitting scheme for the linear Navier–Stokes equations

$$u' + A_p u + P_p(v \cdot \nabla)u = 0,$$

where v is the solution of equation (2.1) satisfying $v(0) = v_0$. This *linear* splitting scheme is defined as follows:

$$\tilde{u}(t) = e^{-\tau A} E_v(t_n+\tau, t_n)\tilde{u}(t_n) \quad \text{for } t{=}t_n{+}\tau,\ \tau{\in}(0,k],\ 0{\le}n{\le}m{-}1, \tag{3.1}$$

$$\tilde{u}(0) = v_0. \tag{3.2}$$

Here, for $u \in H_p$ and $t \geq s \geq 0$, $E_v(t,s)u$ denotes the solution of the advection equation

$$\frac{\partial}{\partial t} E_v(t,s)u + P(v(t)\cdot\nabla)E_v(t,s)u = 0,\ t > s,$$

which satisfies the initial condition $E_v(s,s)u = u$. We also define

$$\tilde{u}_E(t) = E_v(t,t_n)\tilde{u}(t_n) \text{ for } \ t = t_n + \tau, \tau \in (0,k], 0 \leq n \leq m-1.$$

Obviously, we have $\tilde{u}(t) = e^{-\tau A}\tilde{u}_E(t)$.

The following assertion expressing the boundedness of $\tilde{u}$ in $(H^{2,p}(\Omega))^d$ is just Lemma 3.2 from [1].

Lemma 3.1 *There exists a constant C independent of $k = T/m$ such that*

$$|\tilde{u}(t)|_{H^{2,p}} \leq C \quad \textit{for all } t \in [0,T].$$

The only information on v used in the proof of Lemma 3.1 is the fact that $v \in L^\infty(0,T;(H^{1,p}(\Omega))^d)$. Actually, in Lemma 3.2 from [1], the boundedness of $\tilde{u}(t)$ in $H^{2,p}$ norm is stated only for $t = t_n$, $0 \leq n \leq m$. However, the proof is the same for any $t \in [0,T]$; besides, in the sequel, we need the boundedness of $\tilde{u}(t)$ for all $t \in [0,T]$.

Now, applying Lemma 3.1, it is easy to derive the boundedness of $\tilde{u}_E$ in $H^{2,p}$ norm. Indeed, using the definition of $\tilde{u}_E$, the boundedness of $E_v(t,t_n)$ in $H^{2,p}$ norm (see (2.23) in [1]) and, of course, Lemma 3.1, we obtain

$$|\tilde{u}_E(t)|_{H^{2,p}} \leq C \quad \text{for all } t \in [0,T].$$

The convergence of the linear splitting scheme in $(L^2(\Omega))^d$ is stated in Proposition 3.4 from [1] under the hypothesis that $v \in C^\alpha([0,T];(H^{2,p}(\Omega))^d)$. Here no additional regularity assumption on v is supposed. Set $\tilde{w} = \tilde{u} - v$. We have:

Lemma 3.2. *There exists a positive constant C such that for all $k > 0$ and all $t \in [0, T]$,*

$$|\tilde{w}(t)|_{L^p} \leq Ck.$$

Proof. Let us differentiate (3.1) with respect to t: At time $t = t_n + \tau$ with $0 < \tau < k$, we have

$$\tilde{u}'(t) + A\tilde{u}(t) = -e^{-\tau A} P(v(t) \cdot \nabla)\tilde{u}_E(t).$$

From this equation we subtract (2.1), that is,

$$v'(t) + Av(t) = -P(v(t) \cdot \nabla)v(t).$$

We get

$$\tilde{w}'(t)+A\tilde{w}(t)=-Z(\tau)P(v(t)\cdot\nabla)\tilde{u}_E(t)+P(v(t)\cdot\nabla)Z(\tau)\tilde{u}_E(t)-P(v(t)\cdot\nabla)\tilde{w}(t),$$

where the operator $Z(\tau)$ is defined by $Z(\tau) = e^{-\tau A} - I$. Define

$$\begin{aligned} f_1(t) &= -Z(\tau)P(v(t) \cdot \nabla)\tilde{u}_E(t), \\ f_2(t) &= P(v(t) \cdot \nabla)Z(\tau)\tilde{u}_E(t), \\ f_3(t) &= -P(v(t) \cdot \nabla)\tilde{w}(t), \end{aligned}$$

and set $f = f_1 + f_2 + f_3$. So, we can write

$$\tilde{w}' + A\tilde{w} = f. \tag{3.3}$$

On the other hand, $f \in L^1(0, T; (L^p(\Omega))^d)$. Even more, we have $f \in L^\infty(0, T; (L^p(\Omega))^d)$. Indeed, using inequality (2.2) from [1], we get

$$\begin{aligned} |f_1(t)|_{L^p} &\leq C_1|(v(t) \cdot \nabla)\tilde{u}_E(t)|_{L^p} \leq C_2|v(t)|_{H^{1,p}}|\tilde{u}_E(t)|_{H^{1,p}} \\ &\leq C_3|\tilde{u}_E(t)|_{H^{1,p}} \leq C_4. \end{aligned}$$

(For the sake of simplicity, in this paper, we shall frequently denote different constants by the same symbol. However, in the same chain of inequalities, we always use different symbols for different constants.) In the same way (as before), we have

$$\begin{aligned} |f_2(t)|_{L^p} &\leq C_1|v(t)|_{H^{1,p}}(|u(t)|_{H^{1,p}} + |\tilde{u}(t)|_{H^{1,p}}) \leq C_2, \\ |f_3(t)|_{L^p} &\leq C_1|v(t)|_{H^{1,p}}(|\tilde{u}(t)|_{H^{1,p}} + |v(t)|_{H^{1,p}}) \leq C_2. \end{aligned}$$

Thus, we can express (3.3) in integral form with the aid of the Stokes semigroup (see, for instance, Corollary 2.2, Ch.4 from [5]):

$$\tilde{w}(t) = \int_0^t e^{-(t-s)A} f(s) ds. \tag{3.4}$$

Now we shall estimate the right-hand side of (3.4) in L^p norm. (For the sake of completeness, we shall repeat some considerations from [1].)

Let $g \in H_q$, where q denotes the exponent conjugate to $p : 1/p+1/q = 1$. In what follows, the brackets denote the pairing between elements of $(L^p(\Omega))^d$ and those of $(L^q(\Omega))^d$. Let $s = t_j + \sigma$ with $0 < \sigma \leq k$. We have

$$\begin{aligned} < e^{-(t-s)A} f_1(s), g > &= - < P(v(s)\cdot\nabla)\tilde{u}_E(s), Z(\sigma)e^{-(t-s)A} g > \\ &= < P(v(s)\cdot\nabla)\tilde{u}_E(s), \int_0^\sigma Ae^{-\sigma' A}e^{-(t-s)A} g\, d\sigma' > \\ &= < P(v(s)\cdot\nabla)\tilde{u}_E(s), A^\varepsilon \int_0^\sigma e^{-\sigma' A}A^{1-\varepsilon}e^{-(t-s)A} g\, d\sigma' >, \end{aligned} \tag{3.5}$$

where ε is chosen to satisfy $0 < \varepsilon < 1/2p$. (Here and further on, for $0 < \alpha < 1$, A^α denotes the fractional power of A.) Using inequality (2.2) from [1] and the boundedness of $\tilde{u}_E$ in $H^{1+2\varepsilon,p}$ norm, we see that

$$|P(v(s)\cdot\nabla)\tilde{u}_E(s)|_{H^{2\varepsilon,p}} \leq C_1|v(s)|_{H^{1,p}}|\tilde{u}_E(s)|_{H^{1+2\varepsilon,p}} \leq C_2,$$

therefore, $P(v(s)\cdot\nabla)\tilde{u}_E(s) \in (H^{2\varepsilon,p}(\Omega))^d$. But one knows that for $0 < \varepsilon < 1/2p$, $(H^{2\varepsilon,p}(\Omega))^d \cap H_p = D(A^\varepsilon)$ (see (2.11) from [1]). Consequently, we may move A^ε from the second factor to the first. Thus we have obtained

$$\begin{aligned} &| < e^{-(t-s)A} f_1(s), g > | \\ &\leq |A^\varepsilon P(v(s)\cdot\nabla)\tilde{u}_E(s)|_{L^p} \Big| \int_0^\sigma e^{-\sigma' A}A^{1-\varepsilon}e^{-(t-s)A} g\, d\sigma' \Big|_{L^p}. \end{aligned}$$

By (2.12) from [1], we have

$$|A^\varepsilon P(v(s)\cdot\nabla)\tilde{u}_E(s)|_{L^p} \leq |P(v(s)\cdot\nabla)\tilde{u}_E(s)|_{H^{2\varepsilon,p}} \leq C.$$

Then, using the analyticity of A (in fact, inequality (2.16) from [1]), we obtain

$$\begin{aligned} \Big| \int_0^\sigma e^{-\sigma' A}A^{1-\varepsilon}e^{-(t-s)A} g\, d\sigma' \Big|_{L^q} &\leq C_1 k |A^{1-\varepsilon}e^{-(t-s)A} g|_{L^q} \\ &\leq C_2 k (t-s)^{-(1-\varepsilon)}|g|_{L^q}. \end{aligned}$$

(Here, inequality (2.16) from [1], derived from the analyticity of A, replaces the more elaborated estimate (2.30) from [1] for the evolution generated by $A+P(v\cdot\nabla)$, used there in the same place of the proof. It is just this estimate which requires the assumption that $v \in C^\alpha([0,T];(H^{2,p}(\Omega))^d)$ in [1].) The last three inequalities give

$$| < e^{-(t-s)A} f_1(s), g > | \leq Ck(t-s)^{-(1-\varepsilon)}|g|_{L^q} \quad \text{for all } g \in (L^q(\Omega))^d.$$

Hence

$$|e^{-(t-s)A} f_1(s)|_{L^p} \le Ck(t-s)^{-(1-\varepsilon)}. \tag{3.6}$$

Take once again $g \in H_p$ and $s = t_j + \sigma$ with $0 < \sigma \le k$. We have

$$\begin{aligned} < e^{-(t-s)A} f_2(s), g > &= < (v(s)\cdot\nabla) Z(\sigma)\tilde{u}_E(s), Pe^{-(t-s)A} g > \\ &= - < Z(\sigma)\tilde{u}_E(s), (v(s)\cdot\nabla) Pe^{-(t-s)A} g > . \end{aligned}$$

To estimate the first factor, we rewrite it as in [1]:

$$Z(\sigma)\tilde{u}_E(s) = Z(\sigma)(\tilde{u}_E(s) - \tilde{u}(t_j)) + Z(\sigma)\tilde{u}(t_j).$$

Next, using the definition of $\tilde{u}_E$, inequality (2.2) from [1] and the boundedness of $\tilde{u}_E$ in $H^{1,p}$ norm, we see that

$$\begin{aligned} |\tilde{u}_E(s) - \tilde{u}(t_j)|_{L^p} &= \Big| \int_{t_j}^{s} P(v(s')\cdot\nabla)\tilde{u}_E(s') ds' \Big|_{L^p} \\ &\le C_1 \int_{t_j}^{s} |v(s')|_{H^{1,p}} |\tilde{u}_E(s')|_{H^{1,p}} ds' \le C_2 k \sup_{t_j \le s' \le s} |\tilde{u}_E(s')|_{H^{1,p}} \le C_3 k. \end{aligned}$$

This together with a use of (2.12) from [1] leads to the inequality

$$\begin{aligned} |Z(\sigma)\tilde{u}_E(s)|_{L^p} &\le C_1 k + \Big| \int_0^{\sigma} e^{-\sigma' A} A\tilde{u}(t_j) d\sigma' \Big|_{L^p} \\ &\le C_1 k + C_2 k |A\tilde{u}(t_j)|_{L^p} \le C_3 k(1 + |\tilde{u}(t_j)|_{H^{2,p}}) \le C_4 k. \end{aligned}$$

For the second factor, notice that $H^{1,p}(\Omega) \subset L^\infty(\Omega)$ because $p > d$. The corresponding Sobolev inequality, (2.12) from [1] and the analyticity of the Stokes semigroup (that is, (2.16) from [1]) give

$$\begin{aligned} &|(v(s)\cdot\nabla) Pe^{-(t-s)A} g|_{L^q} \le C_1 |v(s)|_{L^\infty} |\nabla Pe^{-(t-s)A} g|_{L^q} \qquad (3.7) \\ &\le C_2 |v(s)|_{H^{1,p}} |Pe^{-(t-s)A} g|_{H^{1,q}} \le C_3 |A^{1/2} e^{-(t-s)A} g|_{L^q} \le C_4 (t-s)^{-1/2} |g|_{L^q}. \end{aligned}$$

Thus,

$$| < e^{-(t-s)A} f_2(s), g > | \le Ck(t-s)^{-1/2} |g|_{L^q} \quad \text{for } g \in (L^q(\Omega))^d,$$

whence

$$|e^{-(t-s)A} f_2(s)|_{L^q} \le Ck(t-s)^{-1/2}. \tag{3.8}$$

Finally, for $g \in H_q$ and $s = t_j + \sigma$ with $0 < \sigma \le k$, we have as before:

$$\begin{aligned} < e^{-(t-s)A} f_3(s), g > &= - < (v(s)\cdot\nabla)\tilde{w}(s), Pe^{-(t-s)A} g > \\ &= < \tilde{w}(s), (v(s)\cdot\nabla) Pe^{-(t-s)A} g > . \end{aligned}$$

Using (3.7) again, we get

$$| < e^{-(t-s)A} f_3(s), g > | \le C(t-s)^{-1/2} |\tilde{w}(s)|_{L^p} |g|_{L^q} \quad \text{for all } g \in (L^q(\Omega))^d$$

and so,

$$|e^{-(t-s)A} f_3(s)|_{L^p} \leq C(t-s)^{-1/2} |\tilde{w}(s)|_{L^p}. \tag{3.9}$$

Now, in (3.4) we take the L^p norm of $\tilde{w}(t)$ and then we use the inequalities (3.6), (3.8) and (3.9). So we are led to the following integral inequality:

$$\begin{aligned} |\tilde{w}(s)|_{L^p} \ &\leq C_1 k \int_0^t (t-s)^{-(1-\varepsilon)} ds + C_2 k \int_0^t (t-s)^{-1/2} ds \\ &+ C_3 \int_0^t (t-s)^{-1/2} |\tilde{w}(s)|_{L^p} ds. \end{aligned}$$

Applying the Gronwall inequality, we conclude that $|\tilde{w}(t)|_{L^p} \leq Ck$, and the proof of Lemma 3.2 is finished. □

The following lemma (which collects all the information of Theorem 4.3 from [1]) asserts that the nonlinear splitting approximation scheme u is bounded in $(H^{2,p}(\Omega))^d$; this is the pivot of the proof.

Lemma 3.3. *For k sufficiently small, $u(t)$ is well defined for all $t \in [0, T]$ and there exist constants C and θ, with $0 < \theta < 1$, independent of $k = T/m$, such that for all $t \in [0, T]$,*

$$|u(t)|_{H^{2,p}} \leq C, \ |u(t) - \tilde{u}(t)|_{H^{2,p}} \leq Ck^\theta, \ |u(t) - \tilde{u}(t)|_{L^2} \leq Ck.$$

Let us notice that, in Theorem 4.3 from [1], the assertions of Lemma 3.3 are formulated only for $t = t_n$, $0 \leq n \leq m$. Here we shall use the boundedness of $u(t)$ in $H^{2,p}$ norm for all $t \in [0, T]$. However, the proof of Lemma 3.3 repeats that of Theorem 4.3 from [1] in all details. We point out that the proof uses all the facts concerning the linear splitting scheme stated before (that is, Lemmas 3.1 and 3.2).

The convergence of the linear splitting scheme in L^p norm (stated in Lemma 3.2) and the last inequality of Lemma 3.3 imply the convergence of the nonlinear splitting scheme in L^2 norm (because Ω is bounded). Now, let us study the convergence of the nonlinear splitting scheme in L^p norm.

As in the case of the convergence of the linear splitting scheme, here we have to estimate (in various norms) the result of the Euler step in the nonlinear splitting scheme, i.e.,

$$u_E(t) = E(\tau) u(t_n) \ \text{ for } \ t = t_n + \tau, \tau \in (0, k], 0 \leq n \leq m-1.$$

Clearly, we can write

$$u(t) = e^{-\tau A} u_E(t).$$

Applying Theorem 2.2, we immediately obtain the boundedness of u_E in $H^{2,p}$ norm from that of $u(t_n)$ for $0 \le n \le m$ (stated by Lemma 3.3):

$$|u_E(t)|_{H^{2,p}} \le C \quad \text{for } t \in [0,T].$$

Set $w = u - v$. As in Lemma 3.2, the use of the Stokes semigroup instead of the more complicated evolution generated by $A + P(v \cdot \nabla)$ allows us to dispense with any additional regularity assumption on v.

Lemma 3.4. *There exists a positive constant C such that for all $k > 0$ and all $t \in [0,T]$,*

$$|w(t)|_{L^p} \le Ck.$$

Proof. Differentiating (2.4), it is easy to see that, at time $t = t_n + \tau$ with $0 < \tau < k$, u satisfies

$$u'(t) + Au(t) = -e^{-\tau A} P(u_E(t) \cdot \nabla) u_E(t).$$

Subtracting the equation

$$v'(t) + Av(t) = -P(v(t) \cdot \nabla) v(t)$$

from the preceding one, we can write

$$\begin{aligned} w'(t) + Aw(t) = &-Z(\tau) P(u_E(t) \cdot \nabla) u_E(t) + P(Z(\tau) u_E(t) \cdot \nabla) u_E(t) \\ &+ P(u(t) \cdot \nabla) Z(\tau) u_E(t) - P(w(t) \cdot \nabla) u(t) - P(v(t) \cdot \nabla) w(t), \end{aligned}$$

where, as before, $Z(\tau) = e^{-\tau A} - I$. Set

$$\begin{aligned} &f_1(t) = -Z(\tau) P(u_E(t) \cdot \nabla) u_E(t), \quad f_2(t) = P(Z(\tau) u_E(t) \cdot \nabla) u_E(t), \\ &f_3(t) = P(u(t) \cdot \nabla) Z(\tau) u_E(t), \qquad f_4(t) = -P(w(t) \cdot \nabla) u(t), \\ &f_5(t) = -P(v(t) \cdot \nabla) w(t) \end{aligned}$$

and $f = f_1 + f_2 + f_3 + f_4 + f_5$. Thus, w satisfies

$$w' + Aw = f. \tag{3.10}$$

As in the proof of Lemma 3.2, we can show that $f \in L^\infty(0,T;(L^p(\Omega))^d)$. Indeed, we have

$$\begin{aligned} &|f_1(t)|_{L^p} \le C_1 |u_E(t)|^2_{H^{1,p}} \le C_2, \\ &|f_2(t)|_{L^p} \le C_1 (|u(t)|_{H^{1,p}} + |u_E(t)|_{H^{1,p}}) |u_E(t)|_{H^{1,p}} \le C_2, \\ &|f_3(t)|_{L^p} \le C_1 |u(t)|_{H^{1,p}} (|u(t)|_{H^{1,p}} + |u_E(t)|_{H^{1,p}}) \le C_2, \\ &|f_4(t)|_{L^p} \le C_1 (|u(t)|_{H^{1,p}} + |v(t)|_{H^{1,p}}) |u(t)|_{H^{1,p}} \le C_2, \\ &|f_5(t)|_{L^p} \le C_1 |v(t)|_{H^{1,p}} (|u(t)|_{H^{1,p}} + |v(t)|_{H^{1,p}}) \le C_2, \end{aligned}$$

So, we may write (3.10) in integral form:

$$w(t) = \int_0^t e^{-(t-s)A} f(s) ds,$$

whence, taking the L^p norm of $w(t)$,

$$|w(t)|_{L^p} \le \int_0^t |e^{-(t-s)A} f(s)|_{L^p} ds. \tag{3.11}$$

Now, let us estimate the right–hand side of (3.11).

Let $g \in H_q$ (where q is the exponent conjugate to p) and let $s = t_j + \sigma$ with $0 < \sigma \le k$. We have

$$\begin{aligned} &< e^{-(t-s)A} f_1(s), g >= - < P(u_E(s) \cdot \nabla) u_E(s), Z(\sigma) e^{-(t-s)A} g > \\ &=< P(u_E(s) \cdot \nabla) u_E(s), A^\varepsilon \int_0^\sigma e^{-\sigma' A} A^{1-\varepsilon} e^{-(t-s)A} g \, d\sigma' >, \end{aligned}$$

where ε is taken to satisfy $0 < \varepsilon < 1/2p$ in order to move A^ε to the first factor. We may do this since, using (2.2) from [1],

$$|P(u_E(s) \cdot \nabla) u_E(s)|_{H^{2\varepsilon,p}} \le C_1 |u_E(s)|_{H^{1,p}} |u_E(s)|_{H^{1+2\varepsilon,p}} \le C_2,$$

therefore, $P(u_E(s) \cdot \nabla) u_E(s) \in (H^{2\varepsilon,p}(\Omega))^d \cap H_p = D(A^\varepsilon)$ (because $0 < \varepsilon < 1/2p$). So, we can write

$$< e^{-(t-s)A} f_1(s), g > = < A^\varepsilon P(u_E(s) \cdot \nabla) u_E(s), \int_0^\sigma e^{-\sigma' A} A^{1-\varepsilon} e^{-(t-s)A} g \, d\sigma' > .$$

But we have

$$|A^\varepsilon P(u_E(s) \cdot \nabla) u_E(s)|_{L^p} \le C_1 |P(u_E(s) \cdot \nabla) u_E(s)|_{H^{2\varepsilon,p}} \le C_2$$

(by using (2.12) from [1]) and

$$\Big| \int_0^\sigma e^{-\sigma' A} A^{1-\varepsilon} e^{-(t-s)A} g \, d\sigma' \Big|_{L^q} \le Ck(t-s)^{-(1-\varepsilon)} |g|_{L^q}$$

(as we have already shown in the proof of Lemma 3.2). Thus,

$$| < e^{-(t-s)A} f_1(s), g > | \le Ck(t-s)^{-(1-\varepsilon)} |g|_{L^q} \quad \text{for all } g \in (L^q(\Omega))^d,$$

whence

$$|e^{-(t-s)A} f_1(s)|_{L^p} \le Ck(t-s)^{-(1-\varepsilon)}. \tag{3.12}$$

Set $s = t_j + \sigma$ with $0 < \sigma \le k$. By (2.2) from [1], we obtain

$$\begin{aligned} |e^{-(t-s)A} f_2(s)|_{L^p} &\le C_1 |(Z(\sigma) u_E(s) \cdot \nabla) u_E(s)|_{L^p} \\ &\le C_2 |Z(\sigma) u_E(s)|_{L^p} |u_E(s)|_{H^{2,p}} \le C_3 |Z(\sigma) u_E(s)|_{L^p}. \end{aligned}$$

In the same way as in the proof of Lemma 3.2, we write

$$Z(\sigma)u_E(s) = Z(\sigma)(u_E(s) - u(t_j)) + Z(\sigma)u(t_j).$$

By the definition of u_E and inequality (2.2) from [1], we have

$$\begin{aligned} |u_E(s) - u(t_j)|_{L^p} &= \Big| \int_{t_j}^{s} P(u_E(s')\cdot\nabla)u_E(s')ds' \Big|_{L^p} \\ &\le C_1\sigma \sup_{t_j\le s'\le s} |u_E(s')|^2_{H^{1,p}} \le C_2 k. \end{aligned}$$

Using this last estimate together with (2.12) from [1], we get

$$\begin{aligned} |Z(\sigma)u_E(s)|_{L^p} &\le C_1 k + \int_0^{\sigma} |e^{-\sigma' A}Au(t_j)|_{L^p} d\sigma' \\ &\le C_1 k + C_2 k|Au(t_j)|_{L^p} \le C_3 k(1 + |u(t_j)|_{H^{2,p}}) \le C_4 k. \end{aligned}$$

Coming back, we conclude:

$$|e^{-(t-s)A} f_2(s)|_{L^p} \le Ck. \tag{3.13}$$

Take $g \in H_q$ and $s = t_j + \sigma$ with $0 < \sigma \le k$. We have

$$\begin{aligned} < e^{-(t-s)A} f_3(s), g > &= < (u(s)\cdot\nabla)Z(\sigma)u_E(s), Pe^{-(t-s)A}g > \\ &= - < Z(\sigma)u_E(s), (u(s)\cdot\nabla)Pe^{-(t-s)A}g > . \end{aligned}$$

We have already obtained estimate of the first factor in L^p norm. For the second factor, as in the proof of Lemma 3.2, we successively use the Sobolev inequality corresponding to the inclusion $H^{1,p}(\Omega)\subset L^\infty(\Omega)$ (mind that $p>d$), (2.12) from [1] and the analyticity of the Stokes semigroup:

$$|(u(s)\cdot\nabla)Pe^{-(t-s)A}g|_{L^q} \le C_1|u(s)|_{L^\infty}|\nabla Pe^{-(t-s)A}g|_{L^q}$$

$$\le C_2|u(s)|_{H^{1,p}}|Pe^{-(t-s)A}g|_{H^{1,q}} \le C_3|A^{1/2}e^{-(t-s)A}g|_{L^q} \le C_4(t-s)^{-1/2}|g|_{L^q}.$$

Thus,

$$| < e^{-(t-s)A} f_3(s), g > | \le Ck(t-s)^{-1/2}|g|_{L^q} \quad \text{for all } g \in (L^q(\Omega))^d$$

and consequently,

$$|e^{-(t-s)A} f_3(s)|_{L^p} \le Ck(t-s)^{-1/2}. \tag{3.14}$$

Now we shall estimate the terms in the right–hand side of (3.11) containing w. Using the Sobolev inequality corresponding to the inclusion $H^{1,p}(\Omega) \subset L^\infty(\Omega)$ once again, we get

$$\begin{aligned} |e^{-(t-s)A} f_4(s)|_{L^p} &\le C_1|(w(s)\cdot\nabla)u(s)|_{L^p} \le C_2|\nabla u(s)|_{L^\infty}|w(s)|_{L^p} \\ &\le C_3|\nabla u(s)|_{H^{1,p}}|w(s)|_{L^p} \le C_4|u(s)|_{H^{2,p}}|w(s)|_{L^p} \le C_5|w(s)|_{L^p}. \end{aligned} \tag{3.15}$$

Finally, let $g \in H_q$ and set $s = t_j + \sigma$ with $0 < \sigma \le k$. As before, we have

$$\begin{aligned} < e^{-(t-s)A} f_5(s), g > &= - < (v(s) \cdot \nabla) w(s), P e^{-(t-s)A} g > \\ &= < w(s), (v(s) \cdot \nabla) P e^{-(t-s)A} g > \end{aligned}$$

and

$$|(v(s) \cdot \nabla) P e^{-(t-s)A} g|_{L^q} \le C (t-s)^{-1/2} |g|_{L^q}.$$

Hence

$$| < e^{-(t-s)A} f_5(s), g > | \le C (t-s)^{-1/2} |w(s)|_{L^p} |g|_{L^q} \quad \text{for all } g \in (L^q(\Omega))^d$$

and so,

$$|e^{-(t-s)A} f_5(s)|_{L^p} \le C (t-s)^{-1/2} |w(s)|_{L^p}. \tag{3.16}$$

Taking inequalities (3.11)–(3.16) together, we obtain

$$\begin{aligned} |w(t)|_{L^p} \le C_1 k \int_0^t (t-s)^{-(1-\varepsilon)} ds + C_2 k t + C_3 k \int_0^t (t-s)^{-1/2} ds \\ + C_4 \int_0^t |w(s)|_{L^p} ds + C_5 \int_0^t (t-s)^{-1/2} |w(s)|_{L^p} ds, \end{aligned}$$

and the conclusion of Lemma 3.4 follows by applying the Gronwall inequality.

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