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VITALI CONVERGENCE THEOREM AND PALAIS–SMALE CONDITION

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Abstract. In this article, we present several new results for Palais–Smale sequences. Consequently, we unify the Vitali convergence theorem and many main concepts in the variational methods by Lions, Lien–Tzeng–Wang, del Pino–Felmer and Chabrowski.

1. Introduction

Let $N \geq 2$ and $2 < p < 2^*$, where $2^* = \frac{2N}{N-2}$ for $N \geq 3$, $2^* = \infty$ for $N = 2$. Consider the semilinear elliptic equation

$$\begin{cases} -\Delta u + u = |u|^{p-2}u \text{ in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^N$ and $H_0^1(\Omega)$ is the Sobolev space in Ω.

To prove the existence of solutions of semilinear elliptic equations in unbounded domains, the standard variational techniques such as minimization and critical-point theory do not apply directly because of a lack of compactness. Remarkable progress in the study of this kind of problem has been made by P. L. Lions [11]. He developed the concentration–compactness principle for solving a large class of minimization problems with constraints in unbounded domains. Later developments in this direction have been made by Lien–Tzeng–Wang [10], Chabrowski [5], and del Pino and Felmer [8], [9].

In this article, in Section 2 we study the Vitali convergence theorem for $L^1(\Omega)$: if $\{u_n\}_{n=1}^\infty \subset L^1(\Omega)$, $u \in L^1(\Omega)$, and $u_n \to u$ almost everywhere in Ω satisfy the uniformly integrable (denoted by (UI)) and the uniformly continuous (denoted by (UC)) conditions, then $u_n \to u$ in $L^1(\Omega)$. If we replace $L^1(\Omega)$ by $H_0^1(\Omega)$, and use the interpolation theorem, the Vitali convergence theorem for $H_0^1(\Omega)$ may be simply stated as the following: if $\{u_n\}_{n=1}^\infty \subset H_0^1(\Omega)$, $u \in H_0^1(\Omega)$, and $u_n \to u$ almost everywhere in Ω satisfy the (UI) condition,

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then $u_n \to u$ in $H_0^1(\Omega)$. In recent years, one of the main variational techniques is to analyze the Palais–Smale (denoted by (PS)) sequences for an energy functional J obtained from an equation, and try to see how the (PS)-condition becomes possible. In Section 3, we present several new results for (PS)-sequences. We prove that the (PS)-condition, the Vitali convergence theorem for $H_0^1(\Omega)$, and the methods by Lions, Lien–Tzeng–Wang, Chabrowski, and del Pino–Felmer are equivalent.

2. Compactness theorems

The Lebesgue dominated convergence theorem is a well-known compactness theorem.

Theorem 1. (Lebesgue dominated convergence theorem) *Suppose Ω is a domain in $\mathbb{R}^N$, and $\{u_n\}_{n=1}^{\infty}$ and u are measurable functions in Ω such that $u_n \to u$ almost everywhere in Ω. If there is $\varphi \in L^1(\Omega)$ such that for each $n \in \mathbb{N}$*

$$|u_n| \leq \varphi \text{ a.e. in } \Omega,$$

then $u_n \to u$ in $L^1(\Omega)$.

The converse of the Lebesgue dominated convergence theorem fails.

Example 2. For $n = 1, 2, \ldots$, let $u_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$u_n(x) = \begin{cases} 0 & \text{for } x \leq n; \\ 2 & \text{for } x = n + 1/2n; \\ 0 & \text{for } x \geq n + 1/n; \\ \text{linear} & \text{otherwise.} \end{cases}$$

Then we have

$$\int_{\mathbb{R}} u_n(x)dx = \frac{1}{n} < \infty \text{ for each } n \in \mathbb{N}.$$

Then $u_n \to 0$ almost everywhere in $\mathbb{R}$ and strongly in $L^1(\mathbb{R})$. Suppose that there exists $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying

$$|u_n| \leq \varphi \text{ a.e. in } \mathbb{R} \text{ for each } n \in \mathbb{N}.$$

Then $\infty = \sum \frac{1}{n} = \int_{\mathbb{R}} \sum u_n \leq \int_{\mathbb{R}} \varphi$. Consequently, $\varphi \notin L^1(\mathbb{R})$.

However, the Vitali convergence theorem is a necessary and sufficient result for L^1 convergence.

Theorem 3. (Vitali convergence theorem for $L^1(\Omega)$) *Suppose* Ω *is a domain in* $\mathbb{R}^N$, $\{u_n\}_{n=1}^{\infty} \subset L^1(\Omega)$, $u \in L^1(\Omega)$ *and* $u_n \to u$ *almost everywhere in* Ω. *Then* $\|u_n - u\|_{L^1} \to 0$ *if and only if the following two conditions hold:*
(UI) (Uniformly integrable) For each $\varepsilon > 0$, *there exists a measurable set* $E \subset \Omega$ *such that* $|E| < \infty$ *and*

$$\int_{E^c} |u_n| \, d\mu < \varepsilon \text{ for each } n \in \mathbb{N}.$$

(UC) (Uniformly continuous) For each $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $|E| < \delta$ *implies*

$$\int_{E} |u_n| \, d\mu < \varepsilon \text{ for each } n \in \mathbb{N}.$$

Proposition 4. *In the (UI) condition of the Vitali convergence Theorem 3, the condition* $|E| < \infty$ *can be replaced by "E is bounded."*

Proof. Let $E_n = E \cap B(0;n)$ for $n = 1, 2, \ldots$; then $E_1 \subset E_2 \subset \cdots \nearrow E$. Thus $|E_1| \leq |E_2| \leq \cdots \nearrow |E|$. For $\delta > 0$ as in the (UC) condition of Theorem 3, there is E_N such that $|E \backslash E_N| < \delta$. Now $\int_{E_N^c} |u_n| \, dx = \int_{E^c} |u_n| \, dx + \int_{E \backslash E_N} |u_n| \, dx < 2\varepsilon$ for each $n \in \mathbb{N}$. □

If we replace $L^1(\Omega)$ in the Vitali convergence theorem by $H_0^1(\Omega)$, we can drop the (UC) condition.

Theorem 5. (Vitali convergence theorem for $H_0^1(\Omega)$) (1) *Let* Ω *be a domain in* $\mathbb{R}^N$ *of finite measure. Then the embedding of* $H_0^1(\Omega)$ *into* $L^p(\Omega)$ *is compact.*

(2) *Let* Ω *be a domain in* $\mathbb{R}^N$. *Suppose that* $\{u_n\}_{n=1}^{\infty} \subset H_0^1(\Omega)$ *is a bounded sequence such that* $u_n \to u$ *almost everywhere in* Ω *for some* $u \in H_0^1(\Omega)$. *If for each* $\varepsilon > 0$, *there exists a measurable set* E *such that* $|E| < \infty$ *and* $\int_{E^c} |u_n|^p dx < \varepsilon$ *for each* $n \in \mathbb{N}$, *then* $\|u_n - u\|_{L^p} \to 0$.

Proof. (1) This holds by Willem [16]. (2) By the Fatou lemma, $\int_{E^c} |u|^p \, dx \leq \varepsilon$. Since $|E| < \infty$, by part (1), we have

$$\int_{E} |u_n - u|^p \, dx = \mathrm{o}(1).$$

Therefore,

$$\int_{\Omega} |u_n - u|^p \, dx = \int_{E} |u_n - u|^p \, dx + \int_{E^c} |u_n - u|^p \, dx = \mathrm{o}(1). \qquad \square$$

Let $H_r^1(\mathbb{R}^N) = \left\{u \in H^1(\mathbb{R}^N) : u \text{ is radially symmetric}\right\}$. We state the following famous result of Strauss [13].

Lemma 6. *For $N \geq 2$, every $u \in H_r^1(\mathbb{R}^N)$ is equal to a continuous function U in $\mathbb{R}^N \backslash \{0\}$ such that for $x \neq 0$*

$$|U(x)| \leq \left(\frac{2}{\omega_N}\right)^{1/2} |x|^{\frac{1-N}{2}} \left(\int_{|y|\geq|x|} |u(y)|^2 dy\right)^{1/4} \left(\int_{|y|\geq|x|} |\nabla u(y)|^2 dy\right)^{1/4}.$$

Let Ω be an annulus, say $\Omega = \left\{x \in \mathbb{R}^N \ : 1 < |x|\right\}$ with $N \geq 3$. Let $H_r^1(\Omega) = \left\{u \in H_0^1(\Omega) : u \text{ is radially symmetric}\right\}$. Moreover, if we replace $L^1(\Omega)$ in the Vitali convergence theorem by $H_r^1(\Omega)$, we can drop both the (UI) and the (UC) conditions. Then

Theorem 7. (Vitali convergence theorem for $H_r^1(\Omega)$) *The embedding of $H_r^1(\Omega)$ into $L^p(\Omega)$ is compact.*

Proof. This holds by Berestycki–Lions [3, p. 341]. □

3. Palais–Smale values

Corresponding to equation (1.1), let the energy functionals a, b, and J in $H_0^1(\Omega)$ be given by

$$a(u) = \int_\Omega \left(|\nabla u|^2 + u^2\right), \ b(u) = \int_\Omega |u|^p, \quad J(u) = \frac{1}{2}a(u) - \frac{1}{p}b(u).$$

By Rabinowitz [12, Proposition B. 10], a, b, and J are of class $C^{1,1}$. Clearly J satisfies the mountain-pass hypothesis: that is, there are r, $\delta > 0$ and $e \in H_0^1(\Omega)$ such that $e \notin \overline{B(0;r)}$, $J(e) = 0$ and $J(u) \geq \delta > 0$ for each $u \in \partial B(0;r)$.

We define the (PS)-sequences for J.

Definition 8. (1) For $\beta \in \mathbb{R}$, a sequence $\{u_n\} \subset H_0^1(\Omega)$ is a $(\mathrm{PS})_\beta$-sequence for J if $J(u_n) \to \beta$ and $J'(u_n) \to 0$ strongly in $H^{-1}(\Omega)$ as $n \to \infty$.

(2) $\beta \in \mathbb{R}$ is a (PS)-value for J if there is a $(\mathrm{PS})_\beta$-sequence for J.

(3) J satisfies the $(\mathrm{PS})_\beta$-condition if every $(\mathrm{PS})_\beta$-sequence for J contains a convergent subsequence.

(4) J satisfies the (PS)-condition if J satisfies the $(\mathrm{PS})_\beta$-condition for every $\beta \in \mathbb{R}$.

A $(\mathrm{PS})_\beta$-sequence for J is bounded.

Lemma 9. *Let $\{u_n\} \subset H_0^1(\Omega)$ be a (PS)$_\beta$-sequence for J; then there is a positive, bounded sequence $\{c_n(\beta)\}$ such that $||u_n||_{H^1} \leq c_n(\beta) \leq c$ for each n and $c_n(\beta) = \mathrm{o}(1)$ as $n \to \infty$ and $\beta \to 0$. Furthermore, $a(u_n) = b(u_n) + \mathrm{o}(1) = \frac{2p}{p-2}\beta + \mathrm{o}(1)$ and $\beta \geq 0$.*

Proof. Since $\{u_n\} \subset H_0^1(\Omega)$ is a (PS)$_\beta$-sequence for J, we have $\eta_n = \mathrm{o}(1)$ and $\varepsilon_n = \mathrm{o}(1)$ such that

$$|\beta| + \eta_n + \frac{\varepsilon_n ||u_n||_{H^1}}{p} \geq J(u_n) - \frac{1}{p}\left\langle J'(u_n), u_n \right\rangle = (\frac{1}{2} - \frac{1}{p})\|u_n\|_{H^1}^2.$$

Take $c_n(\beta) = \frac{1}{p-2}(\varepsilon_n + \sqrt{\varepsilon_n^2 + 2p(p-2)(|\beta| + \eta_n)})$. Then $||u_n||_{H^1} \leq c_n(\beta) \leq c$ for each n and $c_n(\beta) = \mathrm{o}(1)$ as $n \to \infty$ and $\beta \to 0$. Consequently, $\{u_n\}$ is bounded. Thus $\mathrm{o}(1) = \langle J'(u_n), u_n \rangle = a(u_n) - b(u_n)$, or

$$\beta + \mathrm{o}(1) = J(u_n) = \frac{1}{2}a(u_n) - \frac{1}{p}b(u_n) = \frac{p-2}{2p}a(u_n) + \mathrm{o}(1).$$

Therefore, $a(u_n) = b(u_n) + \mathrm{o}(1) = \frac{2p}{p-2}\beta + \mathrm{o}(1)$, and consequently, $\beta \geq 0$. □

Consider the following four important positive values.

(1) Consider the constrained value $\alpha_\theta = (\frac{1}{2} - \frac{1}{p})\theta^{\frac{2p}{2-p}}$, where

$$\theta = \sup\left\{\|u\|_{L^p(\Omega)} : u \in H_0^1(\Omega),\ a(u) = 1\right\}.$$

Clearly, α_θ is a positive value.

(2) Consider the Nehari value $\alpha_{\mathbf{M}} = \inf_{u \in \mathbf{M}(\Omega)} J(u)$, where

$$\mathbf{M}(\Omega) = \left\{u \in H_0^1(\Omega) \setminus \{0\} : a(u) = b(u)\right\}.$$

As a consequence of the following lemma, $\alpha_{\mathbf{M}}$ is a positive value.

Lemma 10. *Let $\mathbf{S}(\Omega) = \{u \in H_0^1(\Omega) : ||u||_{H^1} = 1\}$ be the unit sphere. Then there is a bijective $C^{1,1}$ map m from $\mathbf{S}(\Omega)$ to $\mathbf{M}(\Omega)$. Moreover, $\mathbf{M}(\Omega)$ is path-connected and there exists a constant $c > 0$ such that for $u \in \mathbf{M}(\Omega)$, $||u||_{H^1} \geq c$ and $J(u) \geq c$.*

Proof. See Chen–Wang [7, Lemma 2.2]. □

(3) Consider the minimax value $\alpha_\Gamma = \inf_{v \in \Gamma} \max_{t \in [0,1]} J(v(t))$, where

$$\Gamma = \{v \in C([0,1], H_0^1(\Omega)) : v(0) = 0, v(1) = e\} \text{ and } J(e) = 0.$$

Since J satisfies the mountain-pass hypothesis, α_Γ is a positive value.

(4) Consider the minimal value $\alpha_P = \inf_{\beta \in P(\Omega)} \beta$, where $P(\Omega)$ is the set of all positive (PS)-values for J in Ω. As a consequence of the following lemma, α_P is a positive value.

Lemma 11. *There is a $\beta_0 > 0$ such that $\beta \geq \beta_0$ for every positive (PS)-value β.*

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be a $(\mathrm{PS})_\beta$-sequence for J for $\beta > 0$. By Lemma 9, $a(u_n) \leq c_n(\beta)^2$. By the Sobolev embedding theorem, there is a constant $d > 0$ such that

$$b(u_n) \leq d a(u_n)^{p/2}.$$

By the above two inequalities, we have

$$\mathrm{o}(1) = a(u_n) - b(u_n) \geq a(u_n) \left[1 - dc_n(\beta)^{p-2}\right].$$

Take $\beta_0 > 0$ and $n_0 > 0$ such that if $\beta < \beta_0$ and $n \geq n_0$, then $1 - dc_n(\beta)^{p-2} > \frac{1}{2}$. Consequently, $a(u_n) = b(u_n) + \mathrm{o}(1) = \mathrm{o}(1)$. Thus $\beta = 0$, a contradiction.

We have the following useful lemma.

Lemma 12. *Let $\{u_n\} \subset H_0^1(\Omega)$ be a $(PS)_\beta$-sequence for J with $\beta > 0$. Then there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $\{s_n u_n\} \subset \mathbf{M}(\Omega)$ and $J(s_n u_n) = \beta + \mathrm{o}(1)$.*

Proof. See Wang [15, Lemma 8]. □

We now study several important (PS)-values.

Lemma 13. *α_θ, $\alpha_{\mathbf{M}}$, α_Γ and α_P are positive (PS)-values for J. Moreover, every minimizing sequence for $\alpha_{\mathbf{M}}$ is a $(PS)_{\alpha_{\mathbf{M}}}$-sequence for J.*

Proof. (1) By Lien–Tzeng–Wang [10, Theorem 2.1], α_θ is a positive (PS)-value for J.

(2) By Stuart [14, Lemma 3.4], $\alpha_{\mathbf{M}}$ is a positive (PS)-value for J. By Chen–Wang [7, Lemma 2.1], every minimizing sequence for $\alpha_{\mathbf{M}}$ is a $(\mathrm{PS})_{\alpha_{\mathbf{M}}}$-sequence for J.

(3) By Brezis–Nirenberg [4], α_Γ is a positive (PS)-value for J.

(4) For each $n \in \mathbb{N}$, take $u_n \in H_0^1(\Omega)$ and $\beta_n \in P(\Omega)$ such that

$$|\beta_n - \alpha_P| < \frac{1}{2n}, \quad |J(u_n) - \beta_n| < \frac{1}{2n}, \quad ||J'(u_n)|| < \frac{1}{2n}.$$

Then $J(u_n) = \alpha_P + \mathrm{o}(1)$ and $J'(u_n) = \mathrm{o}(1)$ strongly in $H^{-1}(\Omega)$. Thus $\alpha_P \in P(\Omega)$. □

In the following, we present a comparison lemma.

Lemma 14. *Let $\{u_n\} \subset H_0^1(\Omega)$ be a $(PS)_\beta$-sequence for J with $\beta > 0$. Then $\beta \geq \alpha_\theta$, $\beta \geq \alpha_{\mathbf{M}}$, $\beta \geq \alpha_\Gamma$ and $\beta \geq \alpha_P$.*

Proof. By Wang [15, Lemma 9], $\beta \geq \alpha_\theta$, $\beta \geq \alpha_{\mathbf{M}}$ and $\beta \geq \alpha_\Gamma$. Clearly, $\beta \geq \alpha_P$. □

By Lemmas 13 and 14, we have the following interesting result.

Theorem 15. *Four important (PS)-values are equal:* $\alpha_\theta = \alpha_{\mathbf{M}} = \alpha_\Gamma = \alpha_P$.

Definition 16. By Theorem 15, the positive (PS)-values α_θ, α_Γ, $\alpha_{\mathbf{M}}$ and α_P for J are the same. Any one of them is called the index of J in Ω and denoted by $\alpha(\Omega)$ (simply by α). By the definition of $\alpha_{\mathbf{M}}$, if u is a nonzero solution of equation (1.1), then $J(u) \geq \alpha$. Following Berestycki–Lions [3], we say that a solution u of equation (1.1) is a ground state solution if $J(u) = \alpha$ and is a higher energy solution if $J(u) > \alpha$.

The next result shows that minimizers are solutions.

Lemma 17. *Let* $u \in \mathbf{M}(\Omega)$ *such that* $J(u) = \min_{v \in \mathbf{M}(\Omega)} J(v)$. *Then* u *is a solution of equation* (1.1).

Proof. Set $g(v) = a(v) - b(v)$ for $v \in H_0^1(\Omega)$. The minimum of J is achieved at u constrained in $\mathbf{M}(\Omega)$. Note that $\langle g'(u), u\rangle = (2-p)a(u) \neq 0$. By the Lagrange multiplier theorem, there is λ such that $J'(u) = \lambda g'(u)$. We have

$$0 = \langle J'(u), u\rangle = \lambda \langle g'(u), u\rangle.$$

Hence $\lambda = 0$ or $J'(u) = 0$. We conclude that u is a solution of equation (1.1). □

In Lemma 18, below, we show that a ground-state solution is of constant sign.

Lemma 18. *Let* $u \in H_0^1(\Omega)$ *be a solution of equation* (1.1) *which changes sign and* α *the index of* J *in* Ω. *Then* $J(u) > 2\alpha$.

Proof. Let $u^- = \max\{-u, 0\}$. Then u^- is nonzero. Multiply equation (1.1) by u^- and integrate to get

$$\int_\Omega \nabla u \nabla u^- + \int_\Omega u u^- = \int_\Omega |u|^{p-2} u u^-.$$

Consequently,

$$\int_\Omega \left|\nabla u^-\right|^2 + \int_\Omega \left|u^-\right|^2 = \int_\Omega |u^-|^p.$$

Thus, $u^- \in \mathbf{M}(\Omega)$, and hence $J(u^-) \geq \alpha$. Suppose that $J(u^-) = \alpha$. By Lemma 17, u^- is a nonzero solution of equation (1.1). By the maximum principle, $u = u^-$, which contradicts the sign assumption on u. Thus $J(u^-) > \alpha$.

Similarly $J(u^+) > \alpha$, where $u^+ = \max\{u, 0\}$. Thus

$$J(u) = J(u^+) + J(u^-) > 2\alpha. \qquad \square$$

Now we have the following interesting theorem.

Theorem 19. (1) *Let $\{u_n\} \subset H_0^1(\Omega)$ be a (PS)$_\alpha$-sequence for J satisfying $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Then u is a solution of equation* (1.1)*. Furthermore, if u is nonzero, then u is a positive ground state solution of equation* (1.1) *and $u_n \to u$ strongly in $H_0^1(\Omega)$.*

(2) *J satisfies the (PS)$_\alpha$-condition in Ω if and only if every (PS)$_\alpha$-sequence $\{u_n\}$ in $H_0^1(\Omega)$ admits a subsequence $\{u_n\}$ and $u \neq 0$ in $H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$.*

Proof. (1) By Zeidler [17, II/A, p. 303], $u_n \rightharpoonup u$ and $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega)$. Moreover, there is a subsequence $\{u_n\}$ such that $u_n \to u$ almost everywhere in Ω and strongly in $L^{p-1}_{loc}(\Omega)$. For $\phi \in C_c^\infty(\Omega)$, we obtain

$$\int_\Omega \nabla u_n \cdot \nabla\phi \to \int_\Omega \nabla u \cdot \nabla \phi, \quad \int_\Omega u_n\phi \to \int_\Omega u\phi,$$
$$\int_\Omega |u_n|^{p-2}u_n\phi \to \int_\Omega |u|^{p-2}u\phi.$$

Thus $\langle J'(u_n), \phi\rangle \to \langle J'(u), \phi\rangle$ for each $\phi \in C_c^\infty(\Omega)$. Since $\langle J'(u_n), \phi\rangle = \mathrm{o}(1)$ for each $\phi \in C_c^\infty(\Omega)$, we have $J'(u) = 0$ in $H^{-1}(\Omega)$. Therefore u is a solution of equation (1.1). Let u be nonzero; then $u \in M(\Omega)$ and $J(u) \geq \alpha$. By Lemma 9,

$$a(u_n) = \frac{2p}{p-2}\alpha + \mathrm{o}(1).$$

Since a is weakly lower semicontinuous, we have

$$\alpha \leq J(u) = (\frac{1}{2} - \frac{1}{p})a(u) \leq (\frac{1}{2} - \frac{1}{p})\liminf_{n\to\infty} a(u_n) = \alpha,$$

or $J(u) = \alpha$. Since u is a ground-state solution of equation (1.1), by Lemma 18, u is of constant sign. Note that if u is a solution of equation (1.1), then $-u$ is a solution of equation (1.1). Thus by the maximum principle we may assume that u is positive. Let $p_n = u_n - u$ for each n. By Bahri–Lions [1], $\{p_n\}$ is a (PS)-sequence for J :

$$J(p_n) = J(u_n) - J(u) + \mathrm{o}(1) = \mathrm{o}(1), \qquad J'(p_n) = \mathrm{o}(1).$$

Moreover, we have

$$a(p_n) = \frac{2p}{p-2}J(p_n) + \mathrm{o}(1) = \mathrm{o}(1).$$

Thus,

$$u_n \to u \quad \text{strongly in } H_0^1(\Omega).$$

(2) This holds by part (1). □

Let $\Omega^1 \subsetneqq \Omega^2$ and $\alpha_i = \alpha(\Omega^i)$ for $i = 1, 2$; then clearly $\alpha_2 \le \alpha_1$. If $\alpha_2 = \alpha_1$, then we have the following useful results.

Lemma 20. *Let $\Omega^1 \subsetneqq \Omega^2$ and $J : H_0^1(\Omega^2) \to \mathbb{R}$ be the energy functional. Suppose that $\alpha_2 = \alpha_1$. Then*

(1) *J does not satisfy the (PS)$_{\alpha_1}$-condition in Ω^1;*

(2) *J does not satisfy the (PS)$_{\alpha_2}$-condition in Ω^2.*

Proof. (1) Suppose that J satisfies the (PS)$_{\alpha_1}$-condition in Ω^1. Let $\{u_n\} \subset H_0^1(\Omega^1)$ satisfy $J(u_n) \to \alpha_1$ and $J'(u_n) \to 0$ strongly in $H^{-1}(\Omega^1)$. Then there is a subsequence $\{u_n\}$ and $u \neq 0$ in $H_0^1(\Omega^1)$ such that $u_n \to u$ strongly in $H_0^1(\Omega^1)$, $J(u) = \alpha_1$. Thus $u \in \mathbf{M}(\Omega^1)$. Consequently, $u \in \mathbf{M}(\Omega^2)$ and $J(u) = \alpha_2 = \min_{v \in \mathbf{M}(\Omega^2)} J(v)$. By Lemma 17 and Theorem 19, u is a positive ground-state solution of equation (1.1) in Ω^2. This contradicts the fact that $u \in H_0^1(\Omega^1)$.

(2) Let $\{u_n\} \subset H_0^1(\Omega^1)$ satisfy $J(u_n) \to \alpha_1$ and $J'(u_n) \to 0$ strongly in $H^{-1}(\Omega^1)$. By Lemma 12, there exists $\{s_n\} \subset \mathbb{R}^+$ such that $s_n = 1 + \mathrm{o}(1)$, $w_n = s_n u_n \in \mathbf{M}(\Omega^1)$ and $J(w_n) \to \alpha_1$ and $J'(w_n) \to 0$ strongly in $H^{-1}(\Omega^1)$. Since $\mathbf{M}\left(\Omega^1\right) \subset \mathbf{M}\left(\Omega^2\right)$, $\{w_n\} \subset \mathbf{M}\left(\Omega^2\right)$ and $J(w_n) \to \alpha_2$. By Lemma 13, we have

$$\begin{aligned} &J(w_n) = \alpha_2 + \mathrm{o}(1), \\ &J'(w_n) = \mathrm{o}(1) \quad \text{strongly in } H^{-1}(\Omega^2). \end{aligned}$$

Suppose that J satisfies the (PS)$_{\alpha_2}$-condition in Ω^2. Then there is a subsequence $\{w_n\}$ and a $w \in H_0^1(\Omega^2)$ satisfying $w_n \to w$ strongly in $H_0^1(\Omega^2)$ and $J(w) = \alpha_2$. Hence $w \neq 0$. By Theorem 19, w is a positive ground-state solution of equation (1.1) in Ω^2. Since $\{w_n\} \subset \mathbf{M}\left(\Omega^1\right)$ and $w_n \to w$ strongly in $H_0^1(\Omega^2)$, we have $w = 0$ in $(\Omega^1)^c$; this contradicts the fact that w is a positive solution of equation (1.1) in Ω^2. Thus J does not satisfy the (PS)$_{\alpha_2}$-condition in Ω^2. □

Example 21. Let $\mathbf{A}$, $\mathbf{A}_0$, $\mathbb{R}^N$, and $\mathbb{R}^N_+$ be the infinite strip, the upper half strip, the entire space, and the upper half space, respectively, defined by

$$\begin{aligned} \mathbf{A} &= \left\{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| < 1\right\}, \\ \mathbf{A}_0 &= \left\{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| < 1,\, x_N > 0\right\}, \\ \mathbb{R}^N_+ &= \left\{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0\right\}. \end{aligned}$$

It is easy to see that $\alpha(\mathbf{A}) = \alpha(\mathbf{A}_0)$ and $\alpha(\mathbb{R}^N) = \alpha(\mathbb{R}^N_+)$. Hence by Lemma 20, J does not satisfy the $(\mathrm{PS})_{\alpha(\mathbf{A})}$-condition in either $\mathbf{A}$ or $\mathbf{A}_0$ and J does not satisfy the $(\mathrm{PS})_{\alpha(\mathbb{R}^N)}$-condition in either $\mathbb{R}^N$ or $\mathbb{R}^N_+$.

Let $\xi \in C^\infty([0,\infty))$ such that $0 \le \xi \le 1$, and

$$\xi(t) = \begin{cases} 0 & \text{for } t \in [0,1], \\ 1 & \text{for } t \in [2,\infty). \end{cases} \tag{3.1}$$

Let $\xi_n(z) = \xi(\frac{2|z|}{n})$ for each n and $v_n = \xi_n u_n$. Then we have the following lemma.

Lemma 22. *Let $\{u_n\}$ be (PS)$_\beta$-sequence for J such that*

$$\int_{\Omega_n} |u_n|^p = \mathrm{o}(1),$$

where $\Omega_n = \Omega \cap B^N(0;n)$ for each n. If $r \in \mathbb{N}$, then

(1) $\displaystyle\int_\Omega \xi_n^r |u_n|^p = \int_\Omega |u_n|^p + \mathrm{o}(1) = \frac{2p}{p-2}\beta + \mathrm{o}(1)$;

(2) $\displaystyle\int_\Omega \xi_n^r (|\nabla u_n|^2 + u_n^2) = \int_\Omega \xi_n^r |u_n|^p + \mathrm{o}(1) = \frac{2p}{p-2}\beta + \mathrm{o}(1)$;

(3) $\displaystyle\int_\Omega (\xi_n - 1) u_n \varphi = \mathrm{o}(1) \|\varphi\|_{H^1}$ *for every* $\varphi \in H_0^1(\Omega)$;

(4) $\displaystyle\left|\int_\Omega (\xi_n^r - 1) |u_n|^{p-2} u_n \varphi\right| = \mathrm{o}(1) \|\varphi\|_{H^1}$ *for every* $\varphi \in H_0^1(\Omega)$;

(5) $\displaystyle\left|\int_\Omega (\xi_n^r - 1) \nabla u_n \nabla \varphi\right| = \mathrm{o}(1) \|\varphi\|_{H^1}$ *for every* $\varphi \in H_0^1(\Omega)$.

Proof. (1) This is clear.
(2) Let $w_n = \xi_n^r u_n$. Since $\{w_n\}$ is bounded in $H_0^1(\Omega)$, we have

$$\begin{aligned} \mathrm{o}(1) &= \langle J'(u_n), w_n \rangle \\ &= \int_\Omega (\xi_n^r |\nabla u_n|^2 + r\xi_n^{r-1} u_n \nabla \xi_n \cdot \nabla u_n + \xi_n^r u_n^2) - \int_\Omega \xi_n^r |u_n|^p . \end{aligned}$$

Note that $|\nabla \xi_n(z)| \le \frac{c}{n}$ for each n and $\{u_n\}$ is bounded in $H_0^1(\Omega)$, so

$$\int_\Omega \xi_n^{r-1} u_n \nabla \xi_n \cdot \nabla u_n = \mathrm{o}(1).$$

We conclude that

$$\int_\Omega \xi_n^r (|\nabla u_n|^2 + u_n^2) = \int_\Omega \xi_n^r |u_n|^p = \frac{2p}{p-2}\beta + \mathrm{o}(1).$$

(3) By the Hölder and Sobolev inequalities, we have

$$\left|\int_{\Omega}(\xi_n^r-1)\,u_n\varphi\right|\leq\left(\int_{\Omega_n}|u_n|^2\right)^{\frac{1}{2}}\left(\int_{\Omega}|\varphi|^2\right)^{\frac{1}{2}}\leq \mathrm{o}(1)\,\|\varphi\|_{H^1}.$$

(4) This holds similarly by the Hölder and Sobolev inequalities.
(5) By the hypothesis and part (1), we have

$$\begin{aligned}\mathrm{o}(1)&=\langle J'(u_n),w_n\rangle=\langle J'(u_n),w_n\rangle-\langle J'(u_n),u_n\rangle+\langle J'(u_n),u_n\rangle\\&=\int_{\Omega}(\xi_n^r-1)\,|\nabla u_n|^2+\int_{\Omega}(\xi_n^r-1)\,u_n^2-\int_{\Omega}(\xi_n^r-1)\,|u_n|^p+\mathrm{o}(1)\\&=\int_{\Omega}(\xi_n^r-1)\,|\nabla u_n|^2+\mathrm{o}(1).\end{aligned}$$

Thus

$$\left|\int_{\Omega}(\xi_n^r-1)\,|\nabla u_n|^2\right|=\int_{\Omega}(1-\xi_n^r)\,|\nabla u_n|^2=\mathrm{o}(1).$$

Therefore, by the Hölder inequality, we have

$$\begin{aligned}\left|\int_{\Omega}(\xi_n^r-1)\,\nabla u_n\nabla\varphi\right|&\leq\left(\int_{\Omega}(\xi_n^r-1)^2\,|\nabla u_n|^2\right)^{\frac{1}{2}}\|\varphi\|_{H^1}\\&\leq\left(\int_{\Omega}(1-\xi_n^r)\,|\nabla u_n|^2\right)^{\frac{1}{2}}\|\varphi\|_{H^1}\leq\mathrm{o}(1)\,\|\varphi\|_{H^1}.\end{aligned}$$ □

Let Ω be an unbounded domain in $\mathbb{R}^N$ and Ω^i a proper subdomain of Ω for $i=1,2,\ldots,k$ such that $\Omega^i\cap\Omega^j$ is bounded if $i\neq j$ and $\Omega=\Omega^1\cup\cdots\cup\Omega^k$. Let $\alpha=\alpha(\Omega)$, $\alpha_i=\alpha(\Omega^i)$, $\mathbf{M}(\Omega)=\left\{u\in H_0^1(\Omega)\setminus\{0\}:a(u)=b(u)\right\}$ and $\mathbf{M}(\Omega^i)=\left\{u\in H_0^1(\Omega^i)\setminus\{0\}:a(u)=b(u)\right\}$ for $i=1,2,\ldots,k$. Since $H_0^1(\Omega^i)\subset H_0^1(\Omega)$ and $\mathbf{M}(\Omega^i)\subset\mathbf{M}(\Omega)$, for $i=1,2,\ldots,k$, we have $\alpha\leq\min\{\alpha_1,\alpha_2,\ldots,\alpha_k\}$. If one of Ω^i is bounded, say Ω^1 is bounded, then J satisfies the $(\mathrm{PS})_{\alpha_1}$-condition. By Lemma 20 (1), $\alpha<\alpha_1$. Hence we can always assume that each Ω^i is unbounded. Let

$$\begin{aligned}&\widetilde{\Omega}_n=\Omega\setminus\overline{B^N(0;n)}\subsetneqq\Omega;\quad\widetilde{\mathbf{M}}_n=\left\{u\in H_0^1(\widetilde{\Omega}_n)\setminus\{0\}:a\,(u)=b\,(u)\right\};\\&\widetilde{\alpha}_n=\alpha(\widetilde{\Omega}_n)=\inf_{u\in\widetilde{\mathbf{M}}_n}J\,(u)\,.\end{aligned}$$

If $\{u_n\}\subset H_0^1\,(\Omega)$ is a $(\mathrm{PS})_\alpha$-sequence for J, then $\{u_n\}$ is bounded. There are a subsequence $\{u_n\}$ and $u\in H_0^1\,(\Omega)$ such that $u_n\rightharpoonup u$ weakly in $H_0^1\,(\Omega)$, almost everywhere in Ω and strongly in $L^p_{loc}(\Omega)$. Define

$$\alpha_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\Omega\cap\{|x|>R\}} |u_n|^p.$$

For the quantity α_∞, which measures a loss of mass at infinity of a weakly convergent sequence, see Chabrowski [5], Ben–Naoum–Troestler–Willem [2], and Willem [16]. A lot of information on α_∞ and its significance for weak convergence methods can be found in the book of Chabrowski [6].

We have the following main theorem.

Theorem 23. *The following properties are equivalent:*

(1) *J satisfies the $(PS)_\alpha$-condition.*

(2) *For every $(PS)_\alpha$-sequence $\{u_n\} \subset H^1_0(\Omega)$ for J, there are a subsequence $\{u_n\}$ and $u \neq 0$ in $H^1_0(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H^1_0(\Omega)$.*

(3) *For every $(PS)_\alpha$-sequence $\{u_n\} \subset H^1_0(\Omega)$ for J, there are $c > 0$, a subsequence $\{v_m\}$ of $\{u_n\}$, and a positive integer $K > 0$ such that for each $k \geq K$ there is a positive integer $N(k)$ such that for $m \geq N(k)$ we have*

$$\int_{\Omega_k} |v_m|^p \geq c.$$

(4) *For every $(PS)_\alpha$-sequence $\{u_n\} \subset H^1_0(\Omega)$ for J, there is a subsequence $\{u_n\}$ such that for $\varepsilon > 0$ there is a measurable set E such that $|E| < \infty$ and $\int_{E^c} |u_n|^p dx < \varepsilon$ for each $n \in \mathbb{N}$.*

(5) *$\alpha < \tilde{\alpha}_n$ for each $n \in \mathbb{N}$.*

(6) *$\alpha < \min\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$.*

(7) *$\alpha_\infty < \frac{2p}{p-2}\alpha$.*

Proof. (1) $\Longrightarrow$ (2) Suppose that J satisfies the $(\mathrm{PS})_\alpha$-condition and that $\{u_n\} \subset H^1_0(\Omega)$ is a $(\mathrm{PS})_\alpha$-sequence for J; then there are a subsequence $\{u_n\}$ and u in $H^1_0(\Omega)$ such that $u_n \to u$ strongly in $H^1_0(\Omega)$. Since $J(u) = \alpha > 0$, we have $u \neq 0$.

(2) $\Longrightarrow$ (3) Suppose that $\{u_n\} \subset H^1_0(\Omega)$ is a $(\mathrm{PS})_\alpha$-sequence for J which has a subsequence $\{u_n\}$ and $u \neq 0$ in $H^1_0(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H^1_0(\Omega)$. Let $\int_\Omega |u|^p = d$ for some $d > 0$. First take $K > 0$ satisfying $\int_{\Omega_k} |u|^p \geq \frac{d}{2}$ for $k \geq K$. We then may obtain a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ satisfying $\int_{\Omega_K} |u_{n_i}|^p \to \int_{\Omega_K} |u|^p$. Hence there is $N(K) > 0$ such that $\int_{\Omega_K} |u_{n_i}|^p \geq \frac{d}{4}$ for $i \geq N(K)$. Second we obtain a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ such that $\int_{\Omega_{K+1}} |u_{n_{i_j}}|^p \to \int_{\Omega_{K+1}} |u|^p$. Therefore there is $N(K+1) > 0$ such that

$\int_{\Omega_{K+1}}|u_{n_{i_j}}|^p \geq \frac{d}{4}$ for $i_{N(K+1)} \geq N(K)$ and $j \geq N(K+1)$. Continuing this process, let $c = \frac{d}{4}$ and $v_K = u_{n_{N(K)}}$, $v_{K+1} = u_{n_{i_{N(K+1)}}}, \ldots$; then (3) follows.

(3) $\Longrightarrow$ (4) Given a $(\mathrm{PS})_\alpha$-sequence $\{u_n\} \subset H_0^1(\Omega)$ for J such that there are $c > 0$, a subsequence $\{v_m\}$ of $\{u_n\}$ and u in $H_0^1(\Omega)$ satisfying $v_m \rightharpoonup u$ weakly, a positive integer $K > 0$ such that for each $k \geq K$ there is a positive integer $N(k)$ satisfying for $m \geq N(k)$, we have

$$\int_{\Omega_k} |v_m|^p \geq c.$$

Since

$$\int_{\Omega_K} |u|^p = \lim_{m\to\infty} \int_{\Omega_K} |v_m|^p \geq c,$$

we have $u \neq 0$. By Theorem 19, $v_m \to u$ in L^p. By the Vitali convergence Theorem 3, we have that for $\varepsilon > 0$, there is a set E such that $|E| < \infty$ and $\int_{E^c} |v_m|^p dx < \varepsilon$ for each $m \in \mathbb{N}$.

(4) $\Longrightarrow$ (5) For every $(\mathrm{PS})_\alpha$-sequence $\{u_n\} \subset H_0^1(\Omega)$ for J, there is a subsequence $\{u_n\}$ such that for $\varepsilon > 0$, there is a set E such that $|E| < \infty$ and $\int_{E^c} |u_n|^p dx < \varepsilon$ for each $n \in \mathbb{N}$. Then $\{u_n\}$ is bounded and there is a subsequence $\{u_n\}$ such that $u_n \to u$ almost everywhere in Ω. By the Vitali convergence Theorem 5, $u_n \to u$ in L^p.
Note that

$$\alpha + \mathrm{o}(1) = J(u_n) = \big(\frac{1}{2} - \frac{1}{p}\big) b(u_n) = \big(\frac{1}{2} - \frac{1}{p}\big) b(u) + \mathrm{o}(1).$$

Thus $u \neq 0$. Therefore J satisfies the $(\mathrm{PS})_\alpha$-condition in Ω. Since $\widetilde{\Omega}_{n_0} \subsetneqq \Omega$, if $\widetilde{\alpha}_{n_0} = \alpha$ for some $n_0 \in \mathbb{N}$, by Lemma 20 (2), J does not satisfy the $(\mathrm{PS})_\alpha$-condition in Ω, a contradiction. We have $\alpha < \tilde{\alpha}_n$ for each n.

(5) $\Longrightarrow$ (6) On the contrary, suppose that $\alpha = \min\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, say $\alpha = \alpha_1$. Since $\Omega^1 \subsetneqq \Omega$, by Lemma 20 (2), J does not satisfy the $(\mathrm{PS})_\alpha$-condition in Ω. By Theorem 19, there is a $(\mathrm{PS})_\alpha$-sequence $\{u_n\}$ such that $u_n \rightharpoonup 0$ weakly. Clearly, there is a subsequence $\{u_n\}$ and a sequence $\{\Omega_n\}$ such that

$$\int_{\Omega_n} |u_n|^p = \mathrm{o}(1).$$

Let ξ_n be as in equality (3.1) and $v_n = \xi_n u_n$; by Lemma 22, we have

$$J(v_n) = \alpha + \mathrm{o}(1), \quad J'(v_n) = \mathrm{o}(1) \text{ strongly in } H^{-1}(\Omega).$$

Then by Lemma 12, there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $w_n = s_n u_n$, $\{w_n\} \in \widetilde{\mathbf{M}}_n$ and $J(w_n) = J(v_n) + \mathrm{o}(1) = \alpha + \mathrm{o}(1)$. Note that $\widetilde{\alpha}_n \leq J(w_n)$

for each $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} \widetilde{\alpha}_n \leq \alpha$. Since $\Omega \supset \widetilde{\Omega}_n \supset \widetilde{\Omega}_{n+1}$, we have $\alpha \leq \widetilde{\alpha}_n \leq \widetilde{\alpha}_{n+1}$ for each $n \in \mathbb{N}$. We then conclude that $\alpha = \widetilde{\alpha}_n$ for each $n \in \mathbb{N}$, a contradiction.

(6) $\Longrightarrow$ (7) Let $\{u_n\}$ be a $(\mathrm{PS})_\alpha$-sequence in Ω. Then there exists a subsequence $\{u_n\}$ and a u in $H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. By Lemma 9, $a(u_n) = b(u_n) + \mathrm{o}(1) = \frac{2p}{p-2}\alpha + \mathrm{o}(1)$. For each $R > 0$

$$\limsup_{n\to\infty} \int_\Omega |u_n|^p = \int_{\Omega\cap\{|x|<R\}} |u|^p + \limsup_{n\to\infty} \int_{\Omega\cap\{|x|>R\}} |u_n|^p .$$

Letting $R \to \infty$, we have

$$(\frac{2p}{p-2})\alpha = \int_\Omega |u|^p + \alpha_\infty.$$

On the contrary, suppose that $\alpha_\infty = \frac{2p}{p-2}\alpha$; we have $\int_\Omega |u|^p = 0$, or $u = 0$. Thus $u_n \rightharpoonup 0$ weakly. Clearly, there are a subsequence $\{u_n\}$ and a sequence $\{\Omega_n\}$ such that

$$\int_{\Omega_n} |u_n|^p = \mathrm{o}(1).$$

Note that $J(u_n) = \alpha + \mathrm{o}(1)$. Let $\xi_n \in C^\infty([0,\infty))$ be as in equality (3.1) and w_n be as in Lemma 22. By Lemma 22, we have

$$\int_\Omega \xi_n^2(|\nabla u_n|^2 + u_n^2) = \frac{2p}{p-2}\alpha + \mathrm{o}(1). \tag{3.2}$$

Let $v_n = \xi_n u_n$. By (3.2),

$$\begin{aligned} J(v_n) &= \frac{1}{2}\int_\Omega (|\nabla \xi_n|^2 u_n^2 + \xi_n^2(|\nabla u_n|^2 + u_n^2) + 2\xi_n u_n \nabla\xi_n \cdot \nabla u_n) - \frac{1}{p}\int_\Omega \xi_n^p |u_n|^p \\ &= \frac{1}{2}\frac{2p}{p-2}\alpha - \frac{1}{p}\frac{2p}{p-2}\alpha + \mathrm{o}(1) = \alpha + \mathrm{o}(1). \end{aligned}$$

By Lemma 22, we have

$$\begin{aligned} &\langle J'(v_n), \varphi\rangle = \langle J'(v_n), \varphi\rangle - \langle J'(u_n), \varphi\rangle + \langle J'(u_n), \varphi\rangle \\ &= \int_\Omega (\xi_n \nabla u_n \nabla\varphi + u_n \nabla\xi_n \nabla\varphi + \xi_n u_n \varphi - \xi_n^r |u_n|^{p-2} u_n \varphi) \\ &- \langle J'(u_n), \varphi\rangle + \langle J'(u_n), \varphi\rangle \\ &= \int_\Omega \left[(\xi_n - 1)\nabla u_n \nabla\varphi + (\xi_n - 1)u_n\varphi - (\xi_n^r - 1)|u_n|^{p-2}u_n\varphi\right] + \langle J'(u_n), \varphi\rangle \\ &\leq \mathrm{o}(1)\, \|\varphi\|_{H^1} . \end{aligned}$$

Thus, $J'(v_n) = \mathrm{o}(1)$ strongly in $H^{-1}(\Omega)$. Since $\Omega^i \cap \Omega^j$ is bounded for $i \neq j$, there is $N > 0$, $v_n = 0$ in $B(0; N)$, where $B(0; N) \supset \Omega^i \cap \Omega^j$ for $i \neq j$. Set $v_n = v_n^1 + v_n^2 + \cdots + v_n^k$, where $v_n^i \in H_0^1(\Omega^i)$, and for $i = 1, 2, \ldots, k$,

$$v_n^i(x) = \begin{cases} v_n(x) & \text{for } x \in \Omega^i, \\ 0 & \text{otherwise.} \end{cases}$$

As in the proof of $J'(v_n) = \mathrm{o}(1)$ strongly in $H^{-1}(\Omega)$, we obtain

$$J'(v_n^i) = \mathrm{o}(1) \text{ strongly in } H^{-1}(\Omega) \text{ for } i = 1, 2, \ldots, k.$$

Assume

$$J(v_n^i) = c_i + \mathrm{o}(1) \text{ for } i = 1, 2, \cdots, k,$$

where $c_1 + c_2 + \cdots + c_k = \alpha$. Since c_i are (PS)-values, by Lemma 9, they are nonnegative. There is at least one of c_i is positive, say $c_1 > 0$. By Lemma 14, $c_1 \geq \alpha_1$; thus $\alpha \geq c_1 \geq \alpha_1$. This proves $\alpha \geq \min\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. We conclude that $\alpha = \min\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$.

(7) $\Longrightarrow$ (1) Suppose that $\alpha_\infty < \frac{2p}{p-2}\alpha$. Let $\{u_n\}$ be a $(\mathrm{PS})_\alpha$-sequence in Ω. Then there exists a subsequence $\{u_n\}$ and a u in $H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. As in (6) $\Longrightarrow$ (7), we have

$$\left(\frac{2p}{p-2}\right)\alpha = \int_\Omega |u|^p + \alpha_\infty.$$

Thus $\int_\Omega |u|^p = (\frac{2p}{p-2})\alpha - \alpha_\infty > 0$. Thus $u \neq 0$. We then apply Theorem 19 to conclude the proof. □

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