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POSITIVE SOLUTIONS WITH PRESCRIBED PATTERNS IN SOME SIMPLE SEMILINEAR EQUATIONS

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1. Introduction

Recently, for various elliptic problems of the type

$$-\epsilon\Delta u = f(u), \quad x \in \Omega;\ Bu = 0,\ x \in \partial\Omega$$

where $Bu = u$ (Dirihclet boundary conditions) or $Bu = \partial u/\partial \nu$ (Neumann boundary conditions), it has been proved that positive solutions with a single sharp peak or multiple sharp peaks exist when $\epsilon > 0$ is sufficiently small. See, for example, [5, 12, 14] and the references therein. Since the space variable x does not appear in the nonlinearity, the topology and geometry of the domain Ω plays a central role in these problems.

For problems of the type

$$-\Delta u = f_\epsilon(x,u),\ u|_{\partial\Omega} = 0,$$

where

$$f_\epsilon(x,u) = Q(x)u^{\frac{N+2}{N-2}} + \epsilon u \quad (\text{ see, e.g., [3]}),$$

or

$$f_\epsilon(x,u) = Q(x)u^p - \epsilon^{-1}u \quad (\text{ see, e.g., [4]}),$$

or

$$f_\epsilon(x,u) = \epsilon^{-1}[u^p + Q(x)u] \quad (\text{ see, e.g., [10]}),$$

with $1 < p < (N+2)/(N-2)$, $\Omega \subseteq R^N$ $(N \geq 3)$, single and multiple peaked positive solutions have also been established for small positive ϵ. Here the peaks are determined largely by the function $Q(x)$.

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The peaks of these solutions to both classes of problems have the common special feature that they concentrate at isolated points on the underlying domain and hence the measure of the set of peaks goes to zero as $\epsilon \to 0$. Thus the solutions exhibit the so called spike-layers. Moreover, these solutions are usually unstable as stationary solutions of the corresponding parabolic problem, due mainly to the superlinear nature of the nonlinearities involved.

In this paper, we look at sublinear nonlinearities $f_\epsilon(x,u)$ and demonstrate that positive solutions with a different kind, but quite arbitrary, patterns can be found in problems of the type

$$-\Delta u = f_\epsilon(x,u), \quad x \in \Omega;\; Bu = 0,\, x \in \partial\Omega, \tag{1.1}$$

where the boundary operator B can be Dirichlet type, Neumann type or Robin type. Moreover, the nonlinearities can be chosen from well-known models and the patterned solutions are globally asymptotically stable.

To be more precise, we show that given a bounded domain Ω in R^N ($N \geq 2$) with smooth boundary $\partial\Omega$, and an arbitrary set of finitely many disjoint closed subdomains $D_1, ..., D_m$ of Ω, we can find a simple continuous function $f(x,u,\epsilon) = f_\epsilon(x,u)$ (sublinear in nature) such that the boundary value problem (1.1) has a unique positive solution u_ϵ, and for ϵ small, u_ϵ has peaks concentrating exactly on $D_1 \cup ... \cup D_m$ (see Section 2 for a precise definition). Moreover, any positive solution (regardless of the initial value) of the parabolic problem

$$u_t - \Delta u = f_\epsilon(x,u), \quad x \in \Omega,\; t > 0;\; Bu = 0,\; x \in \partial\Omega, t > 0$$

satisfies $\lim_{t\to\infty} u(x,t) = u_\epsilon(x)$, uniformly for $x \in \overline{\Omega}$.

Note that by properly choosing $D_1, ..., D_m$, the set of peaks of the solution u_ϵ can resemble a rather arbitrary pattern given on the set Ω. As the set $\cup_{j=1}^m D_j$ has positive measure, we see that the measure of the set of peaks of u_ϵ goes to a positive number as $\epsilon \to 0$. Moreover, we will show that when u_ϵ is rescaled properly, the rescaled u_ϵ has a limit $\tilde{u}$ as $\epsilon \to 0$, and $\tilde{u}$ is positive in the interior of each D_j but is identically zero on the rest of Ω, and in contrast to the spike-layered solutions, $\tilde{u}$ is continuous on Ω and hence the rescaled u_ϵ exhibits no layers (see Remark 2.8). Thus these solutions are very different in nature from those peaked solutions found in superlinear problems mentioned above.

The above stated result will be established in Section 2 (see Theorem 2.2 and Remark 2.8) as a consequence of more general results on problems which include as a special case the logistic model

$$-\Delta u = \lambda u - [b(x) + \epsilon]u^2, \quad Bu|_{\partial\Omega} = 0.$$

Indeed, the function f in (1.1) can be chosen as

$$f_\epsilon(x,u) = \lambda u - [b(x) + \epsilon]u^2$$

with λ a suitable constant and $b(x) = d(x, D_1 \cup ... \cup D_m)$, the distance between the point x and the set $D_1 \cup ... \cup D_m$. The profile of the solution u_ϵ over the peaks D_j will also be determined in this section (Theorem 2.7).

2. Realization of prescribed patterns in logistic type models

Throughout this section, we assume that Ω is a bounded domain in R^N $(N \geq 2)$ with C^2-boundary $\partial\Omega$, and $\mathcal{D} = \{D_1, ..., D_m\}$ a finite set of disjoint closed subdomains of Ω, that is, the interior of D_j is connected, $D_j \subset \Omega$, $D_i \cap D_j = \emptyset$ when $i \neq j$. We assume further that each D_j has C^2 boundary ∂D_j.

Definition. Let $\mathcal{D}$ be as above. A one parameter family of functions $\phi_\epsilon \in C(\overline{\Omega})$, $\epsilon \in (0, \epsilon_0)$, is said to have pattern $\mathcal{D}$ as $\epsilon \to 0$, if $\phi_\epsilon > 0$ on Ω and for any compact subset Ω_0 of $\overline{\Omega} \setminus D$ with $D = (D_1 \cup ... \cup D_m)$, it holds

$$\lim_{\epsilon \to 0} \frac{\min_{x \in D} \phi_\epsilon(x)}{\max_{x \in \Omega_0} \phi_\epsilon(x)} = \infty.$$

Let

$$Lu = \sum_{i,j=1}^{N} (a_{ij} u_{x_i})_{x_j} + cu$$

with $a_{ij} = a_{ji} \in C^2(\overline{\Omega})$, $c \in C(\overline{\Omega})$, and

$$\sigma_1 |\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij} \xi_i \xi_j \leq \sigma_2 |\xi|^2, \ \forall \xi \in R^N,$$

where σ_1, σ_2 are positive constants.

If Ω' stands for Ω or any subdomain of Ω, we use $\lambda_1^D(\Omega')$ to denote the first eigenvalue of the Dirichlet problem

$$-Lu = \lambda u, \ u|_{\partial\Omega'} = 0$$

and use $\lambda_1^B(\Omega')$ to denote the first eigenvalue of the above problem but with the Dirichlet boundary condition replaced by $Bu|_{\partial\Omega'} = 0$, where B can be of Dirichlet type, Neumann type or Robin type. By changing the subscripts when needed, we may assume that

$$\lambda_1^D(D_1) \leq \lambda_1^D(D_2) \leq ... \leq \lambda_1^D(D_m).$$

Clearly $\lambda_1^B(\Omega) \leq \lambda_1^D(\Omega) < \lambda_1^D(D_1)$.

We consider now the more general problem

$$-Lu = \lambda u - [b(x) + \epsilon]g(u), x \in \Omega;\ Bu = 0, x \in \partial\Omega, \tag{2.1}$$

where $g(u) = u^p$ $(p > 1)$, $b(x)$ is a continuous function satisfying $b(x) > 0$ on $\overline{\Omega} \setminus D$ and $b(x) \equiv 0$ on $D = \cup_{j=1}^m D_j$; and the boundary operator B is either Dirichlet type: $Bu = u$, or Neumann/Robin type:

$$Bu = \partial u/\partial \mu + \beta(x)u$$

where $\beta \geq 0$, $\beta \in C^1(\partial\Omega)$ and $\mu = A\nu$, $A = [a_{ij}]_{N\times N}$, ν is the unit outward normal of $\partial\Omega$.

The following result is folklore.

Theorem 2.1. *For each $\epsilon > 0$, the following holds.*

(i) (2.1) *has no positive solution when $\lambda \leq \lambda_1^B(\Omega)$, and it has a unique positive solution u_ϵ when $\lambda > \lambda_1^B(\Omega)$.*

(ii) *Let $u(x,t)$ be an arbitrary positive solution of*

$$u_t - Lu = \lambda u - [b(x) + \epsilon]g(u),\ x \in \Omega,\ t > 0;\ Bu = 0,\ x \in \partial\Omega, t > 0. \tag{2.2}$$

Then u is defined for all $t > 0$ and it satisfies

(a) $\lim_{t\to\infty} u(x,t) = 0$ *uniformly for $x \in \overline{\Omega}$ if $\lambda \leq \lambda_1^B(\Omega)$,*

(b) $\lim_{t\to\infty} u(x,t) = u_\epsilon(x)$ *uniformly for $x \in \overline{\Omega}$ if $\lambda > \lambda_1^B(\Omega)$.*

Proof. We present here a simple proof of this result for completeness.

(i) We show first that (2.1) has no positive solution when $\lambda \leq \lambda_1^B(\Omega)$. Indeed, if u is a positive solution of (2.1), and ϕ_1 is a positive eigenfunction corresponding to $\lambda_1^B(\Omega)$, then we can multiply (2.1) by ϕ_1 and integrate over Ω to obtain

$$\lambda_1^B(\Omega)\int_\Omega u\phi_1 dx = \lambda\int_\Omega u\phi_1 dx - \int_\Omega [b(x) + \epsilon]g(u)\phi_1 dx < \lambda\int_\Omega u\phi_1 dx.$$

It follows that $\lambda > \lambda_1^B(\Omega)$.

We show next that when $\lambda > \lambda_1^B(\Omega)$, (2.1) has at least one positive solution. This can be achieved by an upper and lower solution argument. In this case, it is easily seen that for all small and positive constant δ, $\delta\phi_1$ is a lower solution of (2.1) while any positive constant M satisfying $M > M_0$, where $g(M_0)/M_0 > (\lambda + \|c\|_\infty)\epsilon^{-1}$, is an upper solution of (2.1). Hence (2.1) has at least one positive solution.

We show finally that (2.1) has at most one positive solution. Arguing indirectly, we assume that (2.1) has two different positive solutions u_1 and u_2. Choose $M > \max\{M_0, \|u_1\|_\infty, \|u_2\|_\infty\}$, and consider the iteration

$$-Lv_n + Cv_n = Cv_{n-1} + f(x, v_{n-1}),\ Bv_n|_{\partial\Omega} = 0,\ n = 1, 2, ...$$

where $v_0 = M$ and $C > 0$ is chosen such that $-L+C$ is positive definite and $Cu+f(x,u) = Cu+\lambda u-[b(x)+\epsilon]g(u)$ is increasing in u for $u \in [0, M]$. It is easily checked that $v_n \geq u_i$ for $i = 1,2$ and as $n \to \infty$, v_n decreases to some function v which is a positive solution of (2.1) satisfying $v \geq u_i$, $i = 1,2$. Since u_1 and u_2 are different, we may assume that $v \not\equiv u_1$. Now from

$$\int_\Omega f(x,u_1)v = \int_\Omega (-Lu_1)v = \int_\Omega u_1(-Lv) = \int_\Omega f(x,v)u_1$$

we deduce

$$0 = \int_\Omega [b(x)+\epsilon][g(u_1)v - g(v)u_1] = \int_\Omega [b(x)+\epsilon](\frac{g(u_1)}{u_1} - \frac{g(v)}{v})u_1 v < 0.$$

This contradiction proves the uniqueness of positive solutions of (2.1).

(ii) Let $u(x,t)$ be any positive solution of (2.2). Fix a $t_0 > 0$, and assume that $\lambda > \lambda_1^B(\Omega)$, then we can find $\delta > 0$ small and $M > 0$ large such that $\delta\phi_1(x) \leq u(x,t_0) \leq M$ for $x \in \Omega$. We may assume that δ and M have been chosen such that $\delta\phi_1$ and M are lower and upper solutions of (2.1), respectively. Let $v(x,t)$ and $V(x,t)$ denote the unique positive solutions of (2.2) satisfying $v(x,t_0) = \delta\phi_1(x)$ and $V(x,t_0) = M$, respectively. Then $v(x,t)$ is increasing in t, $V(x,t)$ is decreasing in t and $v(x,t) \leq V(x,t)$. It follows that they are defined for all $t > t_0$ and $v(x) = \lim_{t\to\infty} v(x,t)$ (uniformly in x), $V(x) = \lim_{t\to\infty} V(x,t)$ (uniformly in x) are positive solutions of (2.1) (see Sattinger [13] and Matano [11, Prop. 2.4]). As u_ϵ is the only such solution of (2.1), we must have $v = V = u_\epsilon$. By the parabolic maximum principle, $v(x,t) \leq u(x,t) \leq V(x,t)$ for $t > t_0$. Hence $u(x,t)$ is defined for all $t > t_0$ and $\lim_{t\to\infty} u(x,t) = u_\epsilon(x)$ (uniformly in x).

When $\lambda \leq \lambda_1^B(\Omega)$, $\delta\phi_1$ is no longer a lower solution of (2.1), but large constants M are still upper solutions of (2.1). Therefore, we can still construct the solution $V(x,t)$ which decreases to a nonnegative solution $V(x)$ of (2.1) as $t \to \infty$. Since 0 is the only nonnegative solution in this case, we deduce $\lim_{t\to\infty} V(x,t) = 0$ uniformly in x. But we have $0 \leq u(x,t) \leq V(x,t)$ for $t > t_0$. Hence we must have $\lim_{t\to\infty} u(x,t) = 0$ uniformly in x. This completes the proof of Theorem 2.1. □

Our main result is the following theorem.

Theorem 2.2. *Let $\lambda > \lambda_1^B(\Omega)$ and u_ϵ be the unique positive solution of* (2.1). *Then the following holds.*

(i) *If $\lambda_1^B(\Omega) < \lambda < \lambda_1^D(D_1)$, then as $\epsilon \to 0$, u_ϵ converges uniformly to the unique positive solution of*

$$-Lu = \lambda u - b(x)g(u), \ Bu|_{\partial\Omega} = 0. \tag{2.3}$$

(ii) *If* $\lambda_1^D(D_k) \le \lambda < \lambda_1^D(D_{k+1})$ *for some* $1 \le k \le m-1$, *then*

(a) $\lim_{\epsilon\to 0} u_\epsilon(x) = \infty$ *uniformly on* $\cup_{j=1}^k D_j$,

(b) $\lim_{\epsilon\to 0} u_\epsilon(x) = U(x) < \infty$ *uniformly on any compact subset of* $\overline{\Omega}\setminus(\cup_{j=1}^k D_j)$, *where* $U(x)$ *is the minimal positive solution of the boundary blow-up problem*

$$-Lu = \lambda u - b(x)g(u), x \in \Omega\setminus(\cup_{j=1}^k D_j);\ Bu|_{\partial\Omega} = 0, u|_{\partial D_j} = \infty,\ j = 1, ..., k. \tag{2.4}$$

(iii) *If* $\lambda \ge \lambda_1^D(D_m)$, *then*

(a) $\lim_{\epsilon\to 0} u_\epsilon(x) = \infty$ *uniformly on* $D = \cup_{j=1}^m D_j$,

(b) $\lim_{\epsilon\to 0} u_\epsilon(x) = U(x) < \infty$ *uniformly on any compact subset of* $\overline{\Omega}\setminus D$, *where* $U(x)$ *is the minimal positive solution of the boundary blow-up problem*

$$-Lu = \lambda u - b(x)g(u),\ x \in \Omega\setminus D;\ Bu|_{\partial\Omega} = 0, u|_{\partial D} = \infty. \tag{2.5}$$

We should remark that unless each D_j is simply connected, $\Omega\setminus\cup_{j=1}^k D_j$ ($1 \le k \le m$) may have more than one components. By a positive solution of (2.4) or (2.5), we mean a solution which is positive on every components of the underlying region.

Let us note that when $\lambda \ge \lambda_1^D(D_m)$, by part (iii) of Theorem 2.2, the unique positive solution u_ϵ of (2.1) has pattern $\mathcal{D}$ as $\epsilon \to 0$.

The rest of this section is mainly devoted to the proof of Theorem 2.2. At the end of this section, we will also determine the blow-up profiles of u_ϵ on D_j where $\lambda_1^D(D_j) < \lambda$.

2.1. **A comparison result.** The following useful comparison result will be repeatedly used later.

Lemma 2.3. *If* $u_1, u_2 \in C^1(\overline{\Omega}\setminus\cup_{j=1}^k D_j)$ *are positive on* $\Omega\setminus\cup_{j=1}^k D_j$ *and satisfy in the weak sense*

$$Lu_1 + \lambda u_1 - b(x)g(u_1) \le 0 \le Lu_2 + \lambda u_2 - b(x)g(u_2)\ \text{in}\ \Omega\setminus\cup_{j=1}^k D_j, \tag{2.6}$$

$$Bu_1 \ge 0 \ge Bu_2\ \text{on}\ \partial\Omega, \quad \lim_{d(x,\partial D_j)\to 0}(u_1 - u_2) \ge 0,\ j = 1, ..., k$$

and $\lambda < \lambda_1^D(D_{k+1})$ *when* $1 \le k < m$. *Then* $u_1 \ge u_2$ *on* $\Omega\setminus\cup_{j=1}^k D_j$.

Proof. Let w_1, w_2 be C^2 nonnegative functions on $\overline{\Omega}\setminus\cup_{j=1}^k D_j$ vanishing near $\cup_{j=1}^k \partial D_j$. Using (2.6), applying integration by parts and subtracting, we obtain

$$\begin{aligned} & -\int_{\tilde{\Omega}} \sum_{i,j} a_{ij}[(u_2)_{x_i}(w_2)_{x_j} - (u_1)_{x_i}(w_1)_{x_j}] - \beta \int_{\partial\Omega} (u_2 w_2 - u_1 w_1) \\ & \geq \int_{\tilde{\Omega}} b(x)[g(u_2)w_2 - g(u_1)w_1] + \int_{\tilde{\Omega}} (\lambda + c)(u_1 w_1 - u_2 w_2), \end{aligned} \tag{2.7}$$

where $\tilde{\Omega} = \Omega \setminus \cup_{j=1}^k D_j$, and when the boundary operator B is of Dirichlet type, we understand that $\beta = 0$ in (2.7).

For $\epsilon > 0$, denote $\epsilon_1 = \epsilon$ and $\epsilon_2 = \epsilon/2$ and let

$$v_i = [(u_2 + \epsilon_2)^2 - (u_1 + \epsilon_1)^2]^+ / (u_i + \epsilon_i), \ i = 1, 2.$$

Since v_i can be approximated arbitrarily closely in the $W^{1,2} \cap L^\infty$ norm on $\Omega \setminus \cup_{j=1}^k D_j$ by C^2 functions vanishing near $\partial(\cup_{j=1}^k D_j)$, we see that (2.7) holds when w_i is replaced by v_i, $i = 1, 2$.

Denote

$$\Omega_+(\epsilon) = \{x \in \tilde{\Omega} : u_2(x) + \epsilon_2 > u_1(x) + \epsilon_1\}.$$

We see immediately that the integrands of $\int_{\tilde{\Omega}}$ in (2.7) (with $w_i = v_i$) vanish outside $\Omega_+(\epsilon)$. Moreover, using the simple observation that

$$\sum a_{ij}(u_{x_i} v_{x_j} + v_{x_i} u_{x_j}) = 2 \sum a_{ij} u_{x_i} v_{x_j},$$

one easily checks that the first integral on the left hand side of (2.7) equals

$$\begin{aligned} & -\int_{\Omega_+(\epsilon)} \sum a_{ij}\Big((u_2)_{x_i} - \frac{u_2+\epsilon_2}{u_1+\epsilon_1}(u_1)_{x_i}\Big)\Big((u_2)_{x_j} - \frac{u_2+\epsilon_2}{u_1+\epsilon_1}(u_1)_{x_j}\Big) \\ & \quad + \sum a_{ij}\Big((u_1)_{x_i} - \frac{u_1+\epsilon_1}{u_2+\epsilon_2}(u_2)_{x_i}\Big)\Big((u_1)_{x_j} - \frac{u_1+\epsilon_1}{u_2+\epsilon_2}(u_2)_{x_j}\Big), \end{aligned}$$

which is nonpositive. On the other hand, as $\epsilon \to 0$, the first term on the right hand side of (2.7) converges to

$$\int_{\Omega_+(0)} b(x)\Big[\frac{g(u_2)}{u_2} - \frac{g(u_1)}{u_1}\Big](u_2^2 - u_1^2),$$

while the other two terms in (2.7) converge to 0. Therefore, we would have a contradiction unless $\Omega_+(0) \cap (\Omega \setminus \cup_{j=1}^m D_j)$ has measure zero, which implies that $u_1 \geq u_2$ on $\Omega \setminus \cup_{j=1}^m D_j$.

We still need to show that $u_1 \geq u_2$ on $\cup_{j=k+1}^m D_j$ when $1 \leq k < m$. On D_j with $k + 1 \leq j \leq m$,

$$Lu_1 + \lambda u_1 \leq 0 \leq Lu_2 + \lambda u_2$$

and by what has just been proved above, $u_1 \geq u_2$ on ∂D_j. Since $\lambda < \lambda_1^D(D_j)$ by assumption, it follows from the maximum principle that $u_1 \geq u_2$ on D_j. This completes the proof. $\square$

Remark 2.4. Let us note that Lemma 2.3 applied to the case $m = 0$, i.e., $b(x) > 0$ on Ω, implies that (2.1) with $\epsilon > 0$ has at most one positive solution. This provides an alternative proof of the uniqueness result in Theorem 2.1.

2.2. **Boundary blow-up solutions.**

Lemma 2.5. *For any $\lambda \in (-\infty, \infty)$, any positive constant ξ, and any subdomain Ω' of Ω with smooth boundary, the problem*

$$-Lu = \lambda u - \xi g(u),\ x \in \Omega',\ u|_{\partial\Omega'} = \infty$$

has a positive solution in $C^2_{loc}(\Omega')$.

Here, and in what follows, by $u|_{\partial\Omega'} = \infty$, we mean $u(x_n) \to \infty$ along any sequence $x_n \in \Omega'$ satisfying $d(x_n, \partial\Omega') \to 0$.

Proof. For each positive integer k, we can find a constant $\alpha_k > k$ such that $g(\alpha_k)/\alpha_k = (\alpha_k)^{p-1} > (\lambda + \|c\|_\infty)/\xi$. It is easily checked that $u = 0$ and $v = \alpha_k$ are lower and upper solutions to the problem

$$-Lu = \lambda u - \xi g(u),\ u|_{\partial\Omega'} = k.$$

Hence this problem has a positive solution u_k. Applying Lemma 2.3 we see that such solution u_k is unique and increases with k. If for each $x_0 \in \Omega'$, we can find a small ball $B_R(x_0)$ with center x_0 and radius R such that $B_R(x_0)$ is contained in Ω' and $u_k|_{B_R(x_0)}$ has an upper bound independent of k, then it follows from standard consideration that $u(x) = \lim_{k\to\infty} u_k(x)$ is a positive solution as required. Thus, it suffices to find such an upper bound.

Let x_0 be an arbitrary point in Ω' and let $B_R(x_0)$ be contained in Ω'. Consider now $w(x) = (R - |x - x_0|)^\alpha$, where $\alpha = 2/(1-p) < 0$. For simplicity of notations, we assume that $x_0 = 0$ and hence $w(x) = (R - |x|)^\alpha$. For $0 < r < |x| < R$, a simple calculation gives

$$\begin{aligned} L_0 w &= \sum a_{ij} w_{x_i x_j} + \sum (a_{ij})_{x_i} w_{x_j} \\ &= \left[\alpha(\alpha-1)(R-|x|)^{\alpha-2} + \alpha(R-|x|)^{\alpha-1}\right]|x|^{-3} \sum a_{ij} x_i x_j \\ &\quad - \alpha(R-|x|)^{\alpha-1}|x|^{-1}\Big(\sum_i a_{ii}\Big) - \alpha(R-|x|)^{\alpha-1} \sum (a_{ij})_{x_i} x_j |x|^{-1} \\ &\leq \xi_1(|x|)(R-|x|)^{\alpha-2} + \xi_2(|x|)(R-|x|)^{\alpha-1} \end{aligned}$$

for some positive functions $\xi_1(s)$ and $\xi_2(s)$ which are continuous on the closed interval $[r, R]$.

Since $w(x) \to \infty$ as $|x| \to R$, for any positive constant η such that $\xi\eta^{p-1} > 2\xi_1(R)$, we can find $r < R$ but close to R such that

$$\begin{aligned} &L(\eta w) + \lambda(\eta w) - \xi g(\eta w) \\ &\leq \xi_1(|x|)\eta w^p + \xi_2(|x|)\eta w^{(p+1)/2} + (\lambda + c)\eta w - \xi\eta^p w^p \\ &\leq 0, \quad r \leq |x| < R. \end{aligned}$$

We now fix such an r and extend $w(x)$ smoothly into $|x| < r$ but keep w positive in the ball $B_R(x_0)$. We can easily check that for all large positive constant η,

$$L(\eta w) + \lambda(\eta w) - \xi g(\eta w) \leq 0, \ \forall x \in B_R(x_0).$$

Thus ηw is an upper solution of $-Lu = \lambda u - \xi g(u)$ on $B_R(x_0)$. Clearly $\lim_{|x|\to R}(u_k - \eta w) = -\infty$, $\forall k \geq 1$. By Lemma 2.3, we deduce $u_k \leq \eta w$ for all $k \geq 1$ and $x \in B_R(x_0)$. Thus $\{u_k\}$ has an upper bound independent of k in $B_{R/2}(x_0)$. This completes the proof. □

Lemma 2.6. *(i) For any $\lambda \in (-\infty, \infty)$, the boundary blow-up problem* (2.5) *has a minimal positive solution $\underline{U}$ and a maximal positive solution $\overline{U}$ in the sense that any positive solution u of* (2.5) *satisfies $\underline{U} \leq u \leq \overline{U}$;*

(ii) For any $\lambda \in (-\infty, \lambda_1^D(D_{k+1}))$, the boundary blow-up problem (2.4) *has a minimal positive solution $\underline{U}$ and a maximal positive solution $\overline{U}$, while for $\lambda \geq \lambda_1^D(D_{k+1})$,* (2.4) *has no positive solution.*

Proof. (i) Since Ω and each D_j has C^2 boundary, $\Omega \setminus \cup_{j=1}^m D_j$ has finitely many components. Let $\Omega_1, ..., \Omega_l$ denote these components with $\partial\Omega \subset \partial\Omega_1$.

On each Ω_i, $i = 1, ..., l$, we can basically follow the method in Section 2 of [7] to obtain a minimal positive solution and a maximal positive solution for the boundary blow-up problem inherited from (2.5). More precisely, for each $n \geq 1$, by Lemma 2.3 in [7], one easily sees that the following l problems

$$\begin{aligned} &-Lu = \lambda u - b(x)g(u), \ x \in \Omega_1; \ Bu|_{\partial\Omega} = 0, \ u|_{\partial\Omega_1\setminus\partial\Omega} = n, \\ &-Lu = \lambda u - b(x)g(u), \ x \in \Omega_i; \ u|_{\partial\Omega_i} = n, \ i = 2, ..., l, \end{aligned}$$

have unique positive solutions $u_n^{(i)}$, $i = 1, ..., l$, respectively.

By Lemma 2.3 above, $n \to u_n^{(i)}(x)$ is increasing. Hence,

$$\underline{U}_i(x) = \lim_{n\to\infty} u_n^{(i)}(x)$$

is either finite or ∞. For each $x_0 \in \Omega_i$, we can find a small ball $B(x_0)$ centered at x_0 such that $B(x_0) \subset\subset \Omega_i$. Let $\delta = \min_{x\in\overline{B(x_0)}} b(x)$. Then

$\delta > 0$. By Lemma 2.5, the problem

$$-Lu = \lambda u - \delta g(u),\ x \in B(x_0),\ u|_{\partial B(x_0)} = \infty$$

has a positive solution U_0. Applying Lemma 2.3 we see that $u_n^{(i)} \le U_0$ on $B(x_0)$. It follows that $\underline{U}_i(x_0) \le U_0(x_0) < \infty$. Since $x_0 \in \Omega_i$ is arbitrary, we see that $\underline{U}_i(x) < \infty$ on Ω_i. It follows now from standard interior regularity result for elliptic equations that $\underline{U}_i$ is a positive solution of the boundary blow-up problem inherited from (2.5). Moreover, by its construction and Lemma 2.3, it is the minimal positive solution. The existence of the maximal positive solution $\overline{U}_i$ on Ω_i is constructed in the same way as in the proof of Theorem 2.4 in [7], which is the limit of the minimal positive solutions on a sequence of increasing domains approaching Ω_i.

Now we define $\underline{U}(x) = \underline{U}_i(x)$ for $x \in \Omega_i$, $i = 1, ..., l$, and define $\overline{U}(x) = \overline{U}_i(x)$ for $x \in \Omega_i$, $i = 1, ..., l$. It is easily seen that $\underline{U}$ and $\overline{U}$ are the minimal and maximal positive solutions of (2.5).

(ii) We first show that (2.4) has no positive solution when $\lambda \ge \lambda_1^D(D_{k+1})$. If u_0 is a positive solution of (2.4), then

$$-Lu_0 = \lambda u_0, x \in D_{k+1},\ u_0|_{\partial D_{k+1}} > 0.$$

Multiplying the above identity by the positive eigenfunction ϕ corresponding to $\lambda_1^D(D_{k+1})$ and integrate over D_{k+1}, we easily obtain

$$\lambda_1^D(D_{k+1}) \int_{D_{k+1}} u_0\phi > \lambda \int_{D_{k+1}} u_0\phi.$$

Thus, $\lambda < \lambda_1^D(D_{k+1})$.

To show the existence of a minimal positive solution for $\lambda < \lambda_1^D(D_{k+1})$, we first follow the argument used in part (i) above. Let $\omega_1, ..., \omega_r$ be the components of $\Omega \setminus \cup_{j=1}^k D_j$ with $\partial\Omega \subset \partial\omega_1$, and for each $n \ge 1$, consider

$$\begin{aligned} &-Lu = \lambda u - b(x)g(u),\ x \in \omega_1;\ Bu|_{\partial\Omega} = 0,\ u|_{\partial\omega_1\setminus\partial\Omega} = n,\\ &-Lu = \lambda u - b(x)g(u),\ x \in \omega_i;\ u|_{\partial\omega_i} = n,\ i = 2, ..., r. \end{aligned}$$

By Lemma 3.3 of [6], these problems have unique positive solutions $u_n^{(i)}$, $i = 1, ..., r$, respectively. The same argument used in (i) above can be employed to show that $u_n^{(i)}(x)$ is bounded from above on any compact subset of $\overline{\Omega} \setminus \cup_{j=1}^m D_j$. If we can also obtain an upper bound for $u_n^{(i)}(x)$ on a small neighbourhood of $\cup_{j=k+1}^m D_j$, then we can argue as in (i) to see that $\underline{U}_i(x) = \lim_{n\to\infty} u_n^{(i)}(x)$ is a minimal positive solution on ω_i, $i = 1, ..., r$. The existence of a maximal positive solution can be constructed as before.

Let N_j^δ denote the closed δ-neighbourhood of D_j, $j = k+1, ..., m$, and assume that for all $0 < \delta < \delta_0$, $D_1, ..., D_k, N_{k+1}^\delta, ..., N_m^\delta$ are disjoint and connected, and $\lambda < \lambda_1^D(N_j^\delta)$, $j = k+1, ..., m$. For $j = k+1, ..., m$, let ϕ_j^δ be determined by

$$-L\phi_j^\delta = \lambda_1^D(N_j^\delta)\phi_j^\delta, \phi_j^\delta > 0,\ x \in N_j^\delta;\ \phi_j^\delta|_{\partial N_j^\delta} = 0; \|\phi_j^\delta\|_\infty = 1.$$

By what we have already proved, we can find positive constants M_j such that $u_n^{(j)} \le M_j$ for all $n \ge 1$ and $x \in \partial N_j^{\delta/2}$, $j = k+1, ..., m$. We can now find a large positive constant M such that $M\phi_j^\delta(x) > M_j$ for $x \in \partial N_j^{\delta/2}$ and $j = k+1, ..., m$. Thus we obtain

$$-Lu_n^{(j)} \le \lambda u_n^{(j)},\ -L(M\phi_j^\delta) \ge \lambda(M\phi_j^\delta),\ x \in N_j^{\delta/2},$$

and $M\phi_j^\delta \ge u_n^{(j)}$ on $\partial N_j^{\delta/2}$. As $\lambda_1^D(N_j^{\delta/2}) > \lambda_1^D(N_j^\delta) > \lambda$, we deduce from the maximum principle that $u_n^{(j)}(x) \le M\phi_j^\delta(x)$ for all $x \in N_j^{\delta/2}$. Thus $u_n^{(j)}(x)$ is upper bounded on a small neighbourhood of D_j, $j = k+1, ..., m$. This finishes the proof. □

2.3. **Proof of Theorem 2.2.**

Proof of conclusion (i). In this case, it is known ([8, Theorems 3.5, 3.7]) that (2.3) has a unique positive solution $u_0(x)$. By a simple upper and lower solution argument, one easily sees that u_ϵ increases as ϵ decreases and $u_\epsilon \le u_0$. It follows easily that $\lim_{\epsilon\to0} u_\epsilon$ is a positive solution of (2.4). As u_0 is the only such solution, we necessarily have $\lim_{\epsilon\to0} u_\epsilon = u_0$. By standard L^p-theory and the Sobolev imbedding theorem, we see that $u_\epsilon \to u_0$ in the C^1 norm on $\overline{\Omega}$.

Proof of conclusion (iii). As before, we use $\Omega_1, ..., \Omega_l$ to denote the components of $\Omega\setminus\cup_{j=1}^m D_j$. We denote $D = \cup_{j=1}^m D_j$ and $\Omega_+ = \Omega\setminus D$. By Lemma 2.6, (2.5) has a minimal positive solution $\underline{U}$. Applying Lemma 2.3 to each Ω_i, $i = 1, ..., l$, we conclude that $u_\epsilon(x) \le \underline{U}(x)$ for all $x \in \Omega \setminus D$. We will show at the end that $u_\epsilon \to \underline{U}$ locally uniformly on $\overline{\Omega} \setminus D$, but we will first prove the following claim:

$$m_\epsilon = \min_{x\in D} u_\epsilon \to \infty \quad \text{as} \quad \epsilon \to 0.$$

We break the proof of this claim into several steps.

Step 1. $\lim_{\epsilon\to0} u_\epsilon(x) = \infty$ *uniformly on any compact subset of the interior of* D.

If $\lambda > \lambda_1^D(D_m)$, then we consider the problems

$$-Lu = \lambda u - \epsilon g(u),\ x \in D_j;\ u|_{\partial D_j} = 0,\ j = 1, ..., m.$$

It follows from standard argument that for each j, there is a unique positive solution v_ϵ^j, and as $\epsilon \to 0$, $v_\epsilon^j \to \infty$ uniformly on any compact set $D_j' \subset\subset D_j$. Since $u_\epsilon|_{\partial D_j} > 0$, we apply Lemma 2.3 and conclude that $u_\epsilon \geq v_\epsilon^j$ on D_j and hence $u_\epsilon \to \infty$ uniformly on any subset $D' \subset\subset D = \cup_{j=1}^m D_j$.

If $\lambda = \lambda_1^D(D_m)$, then the proof is more delicate. For $\delta \geq 0$, let $N_\delta(D_j) = \{x \in \Omega : d(x, D_j) < \delta\}$, and we understand that $N_0(D_j) = D_j$. Then $\lambda > \lambda_1^D(N_\delta(D_j))$ for $j = 1, ..., m$ and all $\delta > 0$. Let ϕ_j^δ ($\delta \geq 0$) be the unique solution to

$$-L\phi = \lambda_1^D(N_\delta(D_j))\phi,\ \phi > 0,\ x \in N_\delta(D_j);\ \phi|_{\partial N_\delta(D_j)} = 0;\ \|\phi\|_\infty = 1.$$

By the continuous dependence of the first eigenfunction on the underlying domain, we know that as $\delta \to 0^+$, $\phi_j^\delta \to \phi_j^0$ locally uniformly in D_j, and $\phi_j^\delta \to 0$ uniformly on ∂D_j. Thus we can find positive constant M_δ such that $M_\delta \to \infty$ as $\delta \to 0$ while $M_\delta\phi_j^\delta \leq \eta_j$ on ∂D_j, where $\eta_j = \min_{x\in\partial D_j} u_{\epsilon_0}(x) > 0$, and $\epsilon_0 > 0$ is an arbitrarily fixed constant. By Lemma 2.3, we know $u_\epsilon \geq u_{\epsilon_0}$ on Ω whenever $\epsilon \in (0, \epsilon_0]$. Hence

$$u_\epsilon \geq M_\delta\phi_j^\delta,\ \forall x \in \partial D_j, \forall\epsilon \in (0, \epsilon_0].$$

On the other hand, for any fixed small $\delta > 0$, and all small ϵ,

$$-L(M_\delta\phi_j^\delta) = \lambda_1^D(N_\delta(D_j))(M_\delta\phi_j^\delta) \leq \lambda(M_\delta\phi_j^\delta) - \epsilon g(M_\delta\phi_j^\delta), \forall x \in D_j.$$

Therefore, we can apply Lemma 2.3 to conclude that

$$\underline{\lim}_{\epsilon\to 0} u_\epsilon \geq M_\delta\phi_j^\delta \text{ uniformly on } D_j.$$

Let $\delta \to 0$, we obtain $\phi_j^\delta \to \phi_j^0$ and $M_\delta \to \infty$. Hence $M_\delta\phi_j^\delta \to \infty$ uniformly on any compact subset $D_j' \subset\subset D_j$. It follows that $\lim_{\epsilon\to 0} u_\epsilon = \infty$ uniformly on any compact subset $D_j' \subset\subset D_j$. This is true for every $j = 1, ..., m$. Hence the proof of step 1 is complete.

Step 2. *Suppose $u_\epsilon(x_\epsilon) = m_\epsilon$, $x_\epsilon \in D$ and $\epsilon_n \to 0$. Then the assumption that $\{m_{\epsilon_n}\}$ is bounded implies that $\{\partial u_{\epsilon_n}(x_{\epsilon_n})/\partial\nu_n\}$ is bounded from above, where ν_n is a unit vector in R^N to be specified below.*

As before we will use the simplified notations $u_{\epsilon_n} = u_n$, $m_{\epsilon_n} = m_n$ and $x_{\epsilon_n} = x_n$. By step 1 and the boundedness of $\{m_n\}$, we deduce $d(x_n, \partial D) \to 0$ as $n \to \infty$. Let $D^n = \{x \in D : d(x, \partial D) \geq d(x_n, \partial D)\}$. Clearly $D^n = D$ when $x_n \in \partial D$ and D^n is close to D but is a proper subset of D when

$x_n \notin \partial D$ and n is large. In any case, ∂D^n is close to ∂D, and they have the same degree of smoothness, at least for all large n.

Applying Lemma 2.3 of [7] to our situation here, the problem

$$-Lu = \lambda u - [b(x) + \epsilon_n]g(u),\ x \in \Omega \setminus D^n,\ Bu|_{\partial\Omega} = 0,\ u|_{\partial D^n} = u_n(x_n) \quad (2.8)$$

has a unique positive solution v_n. As before, (2.8) with large n actually represents l boundary value problems with underlying domains Ω_j^n, $j = 1, ..., l$, respectively. By Lemma 2.3, it follows from $v_n \le u_n$ on ∂D^n that $v_n \le u_n$ in $\Omega \setminus D^n$. As $u_n(x_n) = v_n(x_n)$, it follows that

$$\partial u_n(x_n)/\nu_n \le \partial v_n(x_n)/\partial \nu_n,$$

where ν_n is the unit normal vector of ∂D^n at x_n pointing inward of D^n. Thus it suffices to show that $\{\partial v_n(x_n)/\partial \nu_n\}$ is bounded.

Let us now choose an open subset D' of D which is so close to D that $\lambda_1^D(\tilde{D}) > \max\{\lambda, \lambda_1^D(\Omega_1), ..., \lambda_1^D(\Omega_l)\}$ holds for every component $\tilde{D}$ of $D \setminus D'$, and that $\Omega \setminus D$ and $\Omega \setminus D'$ have the same number of components. Thus $\Omega \setminus D'$ has l components, which we denote by $\Omega_1', ..., \Omega_l'$, and they satisfy $\Omega_i \subset \Omega_i'$, $i = 1, ..., l$. It follows that

$$\max_{1 \le i \le l} \lambda_1^D(\Omega_i) \ge \max_{1 \le i \le l} \lambda_1^D(\Omega_i').$$

Choose λ' satisfying $\max\{\lambda, \lambda_1^D(\Omega_1), ..., \lambda_1^D(\Omega_l)\} < \lambda' < \min\{\lambda_1^D(\tilde{D}) : \tilde{D}$ is a component of $D \setminus D'\}$. By a simple variant of [8, Theorem 3.5],

$$-Lu = \lambda' u - b(x)g(u),\ x \in \Omega \setminus D',\ Bu|_{\partial\Omega} = 0,\ u|_{\partial D'} = 0$$

has a unique positive solution u^*, or more precisely, the l boundary value problems with underlying domains Ω_j', $j = 1, ..., l$, each has a unique positive solution u_j, and $u^*(x) = u_j(x)$ for $x \in \Omega_j$.

We may assume that $D' \subset\subset \cup_{n \ge n_0} D^n$ for some $n_0 > 1$. Then we can find a large positive constant M_0 such that $M_0 u^*(x) \ge M \ge u_n(x_n)$ on ∂D^n. It is now easily checked that $M_0 u^*$ is an upper solution to (2.8) for every $n \ge n_0$, i.e., restricted to each component of $\Omega \setminus D^n$, it is an upper solution of the corresponding problem inherited from (2.8). By Lemma 2.3, we deduce $v_n \le M_0 u^*$ on $\Omega \setminus D^n$ for all $n \ge n_0$. This implies that the unique positive solution v_n of (2.8) has an L^∞ bound on $\Omega \setminus D^n$ that is independent of n.

Let us now observe the following further facts:

(a) $v_n|_{\partial D^n}$ is a constant which has a bound independent of n;

(b) ∂D^n is as smooth as ∂D and the degree of smoothness does not depend on n for all large n;

(c) $-Lv_n$ has an L^∞ bound over $\Omega \setminus D^n$ which is independent of n.

Using all these facts and the L^p-theory of elliptic operators up to the boundary ([9, pp. 190-193]), we conclude that for every $p > 1$, $\|v_n\|_{W^{2,p}(\Omega\setminus D^n)}$ has a bound that is independent of n. By the Sobolev imbedding theorem and the independence of the imbedding constant on domains like $\Omega\setminus D^n$ ([A]), we finally conclude that $\|v_n\|_{C^1(\overline{\Omega}\setminus D^n)}$ has a bound that is independent of n. In particular, $\{|\nabla v_n(x_n)|\}$ is bounded and hence $\{\partial v_n(x_n)/\partial\nu_n\}$ is bounded. This completes step 2.

Step 3. *$m_\epsilon \to \infty$ as $\epsilon \to \infty$.*

Otherwise we can find a sequence $\epsilon_n \to 0$ such that $m_n = m_{\epsilon_n} \le M < \infty$ for some constant M. Let x_n and ν_n be as in step 2. Then by what was proved there, $\{\partial u_n(x_n)/\partial\nu_n\}$ is bounded from above. We show that this leads to a contradiction. Let D^n be as in step 2. Then for all large n, ∂D^n is as smooth as ∂D and hence it satisfies a uniform interior ball condition: for all large n and every $x \in \partial D^n$, one can find a closed ball B_x of radius R such that $B_x \subset D^n$ and $B_x \cap \partial D^n = \{x\}$. Let $z_n = (z_1^n, ..., z_N^n)$ denote the center of B_{x_n} and define

$$\psi(x) = e^{-\sigma|x-z_n|^2} - e^{-\sigma R^2},$$

where σ is a positive constant to be specified later.

Choose $c_n \to \infty$ satisfying $\epsilon_n g(M + c_n) \le M_0 < \infty$ for some constant M_0 and all $n \ge 1$ and define $w_n = m_n + c_n\psi = u_n(x_n) + c_n\psi$. For $x \in B_{x_n} \setminus B^n$, where $B^n = \{x : |x - z_n| < R/2\}$, a simple calculation gives

$$\begin{aligned}
&Lw_n + \lambda w_n - \epsilon_n g(w_n) \ge c_n L\psi + \lambda w_n - \epsilon_n g(M + c_n)\\
&= \Big[\Sigma a_{ij}(x_i - z_i^n)(x_j - z_j^n)\Big] 4\sigma^2 c_n e^{-\sigma|x-z_n|^2}\\
&- \Big[(\Sigma a_{ii}) + \Sigma(a_{ij})_{x_j}(x_i - z_i^n)\Big] 2\sigma c_n e^{-\sigma|x-z_n|^2} + (\lambda + c)w_n - \epsilon_n g(M + c_n)\\
&\ge (\eta_1\sigma^2 - \eta_2\sigma - \eta_3)c_n e^{-\sigma R^2/4} - \eta_4,
\end{aligned}$$

where η_1-η_4 are positive constants independent of n. We now fix $\sigma > 0$ large such that

$$\eta_1\sigma^2 - \eta_2\sigma - \eta_3 > 0.$$

Then it follows from $c_n \to \infty$ that for all large n,

$$Lw_n + \lambda w_n - \epsilon_n g(w_n) \ge 0.$$

Choose a compact set $K \subset\subset D$ such that $\cup_{n=1}^\infty B^n \subset K$. By step 1, $\lim_{n\to\infty} u_n = \infty$ uniformly on K. Hence we can find a sequence $c_n' \to \infty$ such that $c_n' \le c_n$ and

$$u_n(x) \ge m_n + c_n'\psi(x) = \tilde{w}_n(x),\ \forall x \in \partial B^n \subset K.$$

Clearly u_n is an upper solution to the problem

$$-Lu = \lambda u - \epsilon_n g(u),\ x \in B_{x_n} \setminus B^n,\ u|_{\partial B_{x_n}} = m_n,\ u|_{\partial B^n} = \tilde{w}_n|_{\partial B^n}.$$

By our choice of σ and c_n', the differential inequality we have obtained for w_n above also holds for $\tilde{w}_n$. Thus, for all large n, $\tilde{w}_n$ is a lower solution to the above problem. We can now apply Lemma 2.3 to conclude that $\tilde{w}_n \le u_n$ on $B_{x_n} \setminus B^n$. It follows that

$$\partial u_n(x_n)/\partial \nu_n \ge \partial \tilde{w}_n(x_n)/\partial \nu_n = c_n' \partial \psi(x_n)/\partial \nu_n = c_n' 2\sigma R e^{-\sigma R^2} \to \infty$$

as $n \to \infty$. This is in contradiction to the conclusion of step 2. Thus step 3 is proved.

To finish the proof of conclusion (iii), it remains to show that $u_\epsilon \to \underline{U}$ locally uniformly on $\overline{\Omega} \setminus D$ as $\epsilon \to 0$. Since $u_\epsilon \le \underline{U}$ on this set, we easily see that as ϵ decreases to 0, u_ϵ increases to a positive solution u_0 of the equation $-Lu = \lambda u - b(x)g(u)$ on this set, and by standard elliptic regularity, one sees that $u_\epsilon \to u_0$ is locally uniform on $\overline{\Omega} \setminus D$. If we can show that $u_0|_{\partial D} = \infty$, then it follows from $u_0 \le \underline{U}$ and that $\underline{U}$ is the minimal positive solution of (2.5) that $u_0 = \underline{U}$, as required.

Let us now show that $u_0(x) \to \infty$ as $x \to \partial D$. Otherwise we can find $x_n \in \Omega \setminus D$, $x_n \to x_0 \in \partial D$ such that $u_0(x_n) \le M < \infty$ for some constant M and all $n \ge 1$. Since $u_\epsilon \le u_0$, we deduce $u_\epsilon(x_n) \le M$ and therefore $u_\epsilon(x_0) \le M$. But on the other hand, by step 3 proved above, $u_\epsilon(x_0) \to \infty$ as $\epsilon \to 0$. This contradiction completes the proof of conclusion (iii).

Proof of conclusion (ii). We use part (ii) of Lemma 2.6 and the rest of the proof is similar to that of conclusion (ii) above. We leave the details to the interested reader.

This finishes the proof of Theorem 2.2. □

2.4. **Blow-up profiles.**

Theorem 2.7. *Suppose that* $\lambda > \lambda_1^D(D_j)$ *and* u_ϵ *is the unique positive solution of* (2.1). *Let* θ_λ^j *denote the unique positive solution to*

$$-Lu = \lambda u - u^p,\ x \in D_j;\ u|_{\partial D_j} = 0.$$

Then

$$\epsilon^{1/(p-1)} u_\epsilon|_{D_j} \to \theta_\lambda^j \quad \text{in } L^q(D_j) \cap C^1_{loc}(D_j)\ (\forall q > 1).$$

Proof. It suffices to show that for any sequence $\epsilon_n \to 0$, there is a subsequence such that $\epsilon_n^{p-1} u_{\epsilon_n} \to \theta_\lambda^j$ in $L^q(D_j) \cap C^1_{loc}(D_j)(\forall q > 1)$.

Let $\delta_n = \epsilon_n^{1/(p-1)}$ and let θ_λ denote the unique positive solution of

$$-Lu = \lambda u - u^p,\ x \in \Omega;\ Bu|_{\partial\Omega} = 0.$$

Then it is easily checked that $w_n = \theta_\lambda/\delta_n$ and $v_n^j = \theta_\lambda^j/\delta_n$ satisfy

$$-Lw_n = \lambda w_n - \epsilon_n w_n^p,\ x \in \Omega;\ Bw_n|_{\partial\Omega} = 0, \tag{2.9}$$

and

$$-Lv_n^j = \lambda v_n^j - \epsilon_n (v_n^j)^p,\ x \in D_j;\ v_n^j|_{\partial D_j} = 0, \tag{2.10}$$

respectively.

Clearly, $u_n = u_{\epsilon_n}$ is a lower solution to (2.9) and $u_n|_{D_j}$ is an upper solution to (2.10). Applying Lemma 2.3 we conclude that

$$u_n \le w_n,\ \forall x \in \Omega;\ u_n \ge v_n^j,\ \forall x \in D_j.$$

Or, equivalently,

$$\delta_n u_n \le \theta_\lambda,\ \forall x \in \Omega;\ \delta_n u_n \ge \theta_\lambda^j,\ \forall x \in D_j. \tag{2.11}$$

Consider now the L^∞-bounded sequence $\hat{u}_n = \delta_n u_n$. It satisfies

$$-L\hat{u}_n = \lambda \hat{u}_n - [b(x) + \epsilon_n]\delta_n^{1-p}\hat{u}_n^p,\ B\hat{u}_n|_{\partial\Omega} = 0.$$

Hence $-L_0\hat{u}_n \le (\lambda + c)\hat{u}_n$, and

$$\int_\Omega \Sigma a_{ij}(\hat{u}_n)_{x_i}(\hat{u}_n)_{x_j} \le (\lambda + \|c\|_\infty)\int_\Omega \hat{u}_n^2 \le (\lambda + \|c\|_\infty)\|\theta_\lambda\|^2_{L^2(\Omega)}.$$

This implies that $\{\hat{u}_n\}$ is a bounded sequence in $W^{1,2}(\Omega)$. Therefore, subject to a subsequence, $\hat{u}_n \to \hat{u}$ weakly in $W^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$. Since $\{\hat{u}_n\}$ is L^∞-bounded, $\hat{u}_n \to \hat{u}$ in $L^q(\Omega)$ $(\forall q > 1)$. By Theorem 2.2, we know that

$$0 \le \hat{u}_n = \delta_n u_n \le \delta_n U \to 0$$

as $n \to \infty$ on $\Omega \setminus (\cup_{i=1}^m D_i)$. It follows that $\hat{u} \equiv 0$ on this set. Therefore $\hat{u}|_{D_j} \in W_0^{1,2}(D_j)$. Moreover, by (2.11), $\hat{u} \ge \theta_\lambda^j$ on D_j.

We now multiply the equation for $\hat{u}_n$ by an arbitrary $\psi \in C_0^\infty(D_j)$, integrate over D_j and pass to the limit $n \to \infty$. We obtain

$$\int_{D_j} [\Sigma a_{kl}\hat{u}_{x_k}\psi_{x_l} + c\hat{u}\psi] = \int_{D_j} [\lambda\hat{u} - \hat{u}^p]\psi.$$

That is to say that $\hat{u}$ is a weak positive solution to

$$-Lu = \lambda u - u^p,\ x \in D_j,\ u|_{\partial D_j} = 0.$$

But this problem has a unique positive solution θ_λ^j. Thus we must have $\hat{u}|_{D_j} = \theta_\lambda^j$. That is to say that $\delta_n u_n|_{D_j} \to \theta_\lambda^j$ in $L^q(D_j)$, $\forall q > 1$. Applying standard interior L^p estimates for the equation for $\hat{u}_n$ on D_j and using the Sobolev imbedding theorem, we obtain further that $\delta_n u_n|_{D_j} \to \theta_\lambda^j$ in $C^1(D_j')$ for each $D_j' \subset\subset D_j$. This completes the proof. □

Remark 2.8. Let us note that $v_\epsilon = \epsilon^{1/(p-1)} u_\epsilon$ satisfies the equation

$$-Lv_\epsilon = \lambda v_\epsilon - [\epsilon^{-1} b(x) + 1] v_\epsilon^p, \ Bv_\epsilon|_{\partial\Omega} = 0,$$

and, by Theorems 2.2 and 2.7, as $\epsilon \to 0$,

$v_\epsilon \to 0$ uniformly on any compact subset of $\overline{\Omega} \setminus (\cup_{j=1}^m D_j)$;
$v_\epsilon \to \theta_\lambda^j$ uniformly on any compact subset of interior(D_j), $j = 1, ..., m$,

provided that $\lambda > \lambda_1^D(D_m)$. Therefore, we may regard the function v_0 defined by $v_0 = \theta_\lambda^j$ for $x \in D_j, j = 1, ..., m$ and $v_0 = 0$ for $x \in \Omega \setminus (\cup_{j=1}^m D_j)$ as the limit of v_ϵ when $\epsilon \to 0$. (Indeed, it is not hard to prove directly that $v_\epsilon \to v_0$ uniformly on $\overline{\Omega}$.) We note that v_0 is continuous on Ω and hence v_ϵ does not exhibit layers.

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