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ASYMPTOTIC DESCRIPTION OF VANISHING IN A FAST-DIFFUSION EQUATION WITH ABSORPTION *

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Abstract. In this paper we study the Cauchy problem

$$u_t = \Delta u^m - u^m \quad \text{in } \mathbb{R}^N \times (0, \infty),$$

$u(x, 0) = u_0(x)$, with $\frac{N-2}{N} < m < 1$ and $N \geq 2$. If $u_0 \not\equiv 0$ is a non-negative, compactly supported function such that and $u_0^m \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then the solution u vanishes identically after a least finite time $T > 0$. We prove the asymptotic formula

$$u(x, t) \sim (1 - m)^{\frac{1}{1-m}} (T - t)^{\frac{1}{1-m}} w_*^m(|x - \bar{x}|)$$

as $t \uparrow T$, for certain uniquely determined $\bar{x} \in \mathbb{R}^N$. Here w_* is the unique positive radial solution of

$$\Delta w - w + w^p = 0 \quad \text{in } \mathbb{R}^N,$$

$$w(x) \to 0 \text{ as } |x| \to \infty,$$

where $p = 1/m$.

1. Introduction

This paper deals with analysis of the finite-time extinction phenomenon in solutions of the Cauchy problem,

$$u_t = \Delta u^m - u^m, \quad \text{in } \mathbb{R}^N \times (0, \infty) \tag{1.1}$$

$$u(x, 0) = u_0(x), \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $0 < m < 1$. $u_0 \not\equiv 0$ is a non-negative, compactly supported function which we additionally assume to be such that $u_0^m \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Asymptotic behavior of solutions for the more general problem

$$u_t = \Delta u^m - u^\alpha, \quad \text{in } \mathbb{R}^N \times (0, \infty) \tag{1.3}$$

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$$u(x,0) = u_0(x), \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

for $0 < m < 1$ and $\alpha > 0$ has been addressed in various works in recent years. In the case $\alpha > 1$, for suitable conditions on the initial data, solutions approach as $t \to +\infty$ a universal profile up to certain time scaling, see [14], and [13]. When $\alpha < 1$, extinction in finite time may take place. In fact, using the explicit space-independent solutions

$$u(x,t) = (1-\alpha)^{\frac{1}{1-\alpha}} (L-t)_+^{\frac{1}{1-\alpha}}$$

as barriers, we see that if $u_0 \in L^\infty(\mathbb{R}^N)$ then the solution of (1.2)-(1.3) must vanish identically after a first finite time $T > 0$. We define the *vanishing time* T of a solution of (1.3) as the number

$$T = \inf\{\tau > 0 \ / \ u(x,t) \equiv 0 \text{ for all } t > \tau\}.$$

At this point we specify that by a solution of the Cauchy problem (1.3)-(1.4) we mean a function $u^m \in C([0,\infty)\,; H^1(\mathbb{R}^N))$ which satisfies the equation in the weak sense, namely

$$\int u(x,t)\eta(x)dx + \int_0^t \int (\nabla u^m(x,s)\nabla\eta(x) + u^\alpha(x,s)\eta(x))dxds$$
$$= \int u_0(x)\eta(x)dx$$

for all $\eta \in H^1(\mathbb{R}^N)$. Existence, uniqueness, comparison principle and continuity of the solution are well established, see [3], [10].

The one-dimensional case $N = 1$ has been analyzed in the in the recent works [6], [7], and a classification of possible asymptotic behaviors has been found for a broad range of exponents in the one-dimensional case $N = 1$, under suitable constraints on the initial data. Roughly speaking, it is found that near the extinction time $T > 0$, a self-similar behavior of the form

$$u(x,t) \sim (T-t)^\gamma F(x(T-t)^\beta), \tag{1.5}$$

takes place. Here

$$\gamma = \frac{1}{1-\alpha}, \quad \beta = \frac{\alpha - m}{2(1-\alpha)}.$$

They also find that this profile is actually the same along any sequence $t_n \uparrow T$, for instance when $m = \alpha$, the initial condition is symmetric and has enough decay as $|x| \to \infty$. The methods there presented take strong advantage of an ODE analysis which does not extend to higher dimensions. In this paper we will study the asymptotic behavior of the solutions near extinction time in the special case $m = \alpha$ in the higher-dimensional case.

Observe that in the context of the works [6] and [7], $\alpha = m$ sets up exactly the treshold between "spreading" and "concentration" of the profile, since $\beta = 0$ in that case, and the asymptotic form takes exactly a separation of variables. In fact while diffusion and absorption contribute to vanishing, rate higher when $m < \alpha$, at the same time in this range, faster diffusion makes the profile "flatter".

Before stating our main results, we recall the well-known fact that the elliptic problem

$$\Delta w - w + w^p = 0 \quad \text{in } \mathbb{R}^N \tag{1.6}$$

$$w(x) \to 0 \quad \text{as } |x| \to +\infty, \tag{1.7}$$

has a unique radially symmetric positive solution $w(x) = w_*(|x|)$, provided that $1 < p < \frac{N+2}{N-2}$, where the latter number is regarded as $+\infty$ if $N = 1, 2$. If m is such that $p \equiv \frac{1}{m}$ falls into this range, then the functions

$$u(x,t) = (1-m)^{\frac{1}{1-m}}(T-t)^{\frac{1}{1-m}} w_*(|x-\bar{x}|)^{\frac{1}{m}}$$

solve equation (1.1) for any $T > 0$ and $\bar{x} \in \mathbb{R}^N$. Our main result states that for each given compactly supported initial data, the solution of (1.1)-(1.2) selects one and only one of these solutions as limiting profile. We have not established this in the whole range for m but only if $\frac{N-2}{N} < m < 1$

Theorem 1.1. *Assume that $N \geq 2$ and $\frac{N-2}{N} < m < 1$. Let $w_*(|x|)$ be the unique positive radial solution of* (1.6)-(1.7) *for $p = \frac{1}{m}$. Assume also that u_0 is non-negative, continuous and compactly supported. Let $T > 0$ be the vanishing time for the solution $u(x,t)$ of problem* (1.1)-(1.2). *Then there exists a (unique) $\bar{x} \in \mathbb{R}^N$ such that*

$$(1-m)^{\frac{1}{1-m}} \lim_{t \to T^-} (T-t)^{-\frac{1}{1-m}} u(x,t) = w_*(|x-\bar{x}|)$$

uniformly in $x \in \mathbb{R}^N$.

We observe that problem (1.6)-(1.7) has no solution for $m = \frac{1}{p} < \frac{N-2}{N+2}$, so finding the actual behavior at vanishing below this treshold arises as an open question. It is worthwhile mentioning that $\alpha = m$ plays a role in the vanishing behavior which is similar to that of the exponent $m = \frac{N-2}{N+2}$ when one analyzes the vanishing behavior of solutions of the fast diffusion equation without absorption term

$$u_t = \Delta u^m.$$

Solutions of this problem for fast decaying initial data vanish in finite time whenever $0 < m < \frac{N-2}{N}$. In the radially symmetric case it was established in [11, 8] that the vanishing takes the form $u(x,t) = (T-t)^{\beta} F((T-t)^{\alpha} x)$ where

α changes sign precisely when $m = \frac{N-2}{N+2}$. In [4] we have recently established that in such a case, without any symmetry assumed, the asymptotic behavior of the solution is given by

$$u(x,t) \sim (T-t)^{\frac{N+2}{4}} \left\{ \frac{k_N \lambda}{\lambda^2 + |x-\bar{x}|^2} \right\}^{\frac{N+2}{2}}$$

as $t \uparrow T$, for certain uniquely determined $\bar{x} \in \mathbb{R}^N$, $\lambda > 0$.

2. Recasting the problem

Let $u(x,t)$ be the solution of problem (1.1)-(1.2), and let us make the following change of variables.

$$v(x,s) = (1-m)^{-\frac{m}{1-m}} (T-t)^{-\frac{m}{1-m}} u^m(x,t)|_{t=T(1-e^{-(1-m)s})}. \tag{2.1}$$

Then letting $p = \frac{1}{m}$, $v(x,s)$ satisfies the following equation

$$\frac{\partial}{\partial s} v^p = \Delta v - v + v^p, \quad \text{in } \mathbb{R}^N \times (0,\infty) \tag{2.2}$$

$$v(x,0) = (1-m)^{-\frac{m}{1-m}} T^{-\frac{m}{1-m}} u_0^m(x). \quad \text{in } \mathbb{R}^N. \tag{2.3}$$

Positive steady states of (2.2)-(2.3) which decay at infinity are precisely the solutions of

$$\Delta w - w + w^p = 0 \quad \text{in } \mathbb{R}^N \tag{2.4}$$

$$w(x) \to 0 \quad \text{as } |x| \to +\infty, \tag{2.5}$$

As we mentioned in the introduction, this problem has positive solutions if and only if $1 < p < \frac{N+2}{N-2}$. The result in [9] tells us that any positive solution of this problem is radially symmetric around some point. In [12] it was established that the radially symmetric solution w_* is unique. Thus, all positive solutions of (2.4)-(2.5) have the form $w(x) = w_*(|x-\bar{x}|)$ for some $\bar{x} \in \mathbb{R}^N$.

Therefore, in terms of v defined by (2.1), Theorem 1.1 can be re-stated in the following form.

Theorem 2.1. *Assume that* $1 < p < \frac{N}{N-2}$ *Let* v *be the solution of* (2.2)-(2.3). *Then there is a* (*unique*) *positive solution* w *of* (2.4)-(2.5) *such that*

$$\lim_{s \to +\infty} v(x,s) = w(x)$$

uniformly in $x \in \mathbb{R}^N$.

We will devote the rest of the paper to the proof of this result. An important feature of equation (2.2)-(2.3) is the presence of a Lyapunov functional for it. For $z \in H^1(\mathbb{R}^N)$ we define

$$J(z) = \frac{1}{2}\int |\nabla z|^2 dx + \frac{1}{2}\int z^2 dx - \frac{1}{p+1}\int z^{p+1} dx.$$

Then J is decreasing along the solution $v(x,s)$, namely the function $s \mapsto J(v(\cdot,s))$ is decreasing. In fact, assuming enough regularity, we find that

$$\frac{d}{ds} J(v(\cdot,s)) = -p\int w^{p-1} w_s^2 dx \leq 0.$$

The desired result then follows in a standard manner from suitable approximations. A standard fact is that the presence of a Lyapunov functional guarantees that limit points of $v(\cdot,s)$ in H^1 as $s \to +\infty$ are necessarily steady states of (2.2). We shall show that such limit points indeed exist and are non-trivial. In order to prove that the limiting point is unique, the following fact is of crucial importance. The solutions of the linearized equation

$$\mathcal{L}h \equiv \Delta h - h + p w_*(|x|)^{p-1} h = 0 \tag{2.6}$$

$$h(x) \to 0 \quad \text{as } |x| \to +\infty, \tag{2.7}$$

are all constituted by linear combinations of the functions $\frac{\partial w_*}{\partial x_i}$, $i = 1, \ldots, N$. In fact, no radial solutions exist, as established in [12], and the vector space of these solutions is N-dimensional, as found in [15].

3. Uniform estimates

The main purpose of this section is to prove that the solution v to (2.2)-(2.3) is uniformly bounded.

Proposition 3.1. *There is a constant $A > 0$ such that $v(x,s) \leq A$ for all $x \in \mathbb{R}^N$ and $s > 0$.*

This result will be a consequence of several lemmas which we state and prove next. We will use the notation $B_\rho = \{x \in \mathbb{R}^N \ / \ |x| \leq \rho\}$.

Lemma 3.1. *Assume that the support of $w(x,0)$ is contained in the ball B_{R_0}. Then for any $R > 0$ the following estimate holds:*

$$\sup_{|x|=R+2R_0} v(x,s) \leq \inf_{|x|\leq R} v(x,s) \ \textit{for all } s \geq 0.$$

Proof. We follow an argument found in [2] in a related setting. Let $a = (a', a_N)$, $b = (b', b_N)$, be points such that $a \in \partial B_{R+2R_0}$ and $b \in B_R$. Here $a', b' \in \mathbb{R}^{N-1}$. We want to show that $v(a,s) \le v(b,s)$ for all $s > 0$. Since the equation satisfied by v is invariant under rotations, we may assume without loss of generality that $a' = b'$, and that $b_N > 0$. Let $\lambda = \frac{a_N + b_N}{2}$ and consider the half space

$$H_\lambda = \{x \in \mathbb{R}^N \ / \ x_N > \lambda\}.$$

Clearly H_λ does not intersect B_{R_0}. Let us define on $H_\lambda \times [0.\infty)$ the function $v^\lambda(x', x_N, s) \equiv v(x', 2\lambda - x_N, s)$. Then, both v and v^λ satisfy the same equation (2.2), and v vanishes identically on H_λ at time $s = 0$. On the other hand, their values coincide on $\partial H_\lambda \times [0, \infty)$. By comparison, we deduce then that $v(x,s) \ge v^\lambda(x,s)$ for all $s > 0$ and $x \in H_\lambda$. In particular, we get $v(b,s) \ge v^\lambda(b,s) = v(a,s)$, as desired. □

Lemma 3.2. *Given any number $A > 1$, there exists $R > 0$ such that $v(x,s) \le A$ for all $|x| \ge R$ and all $s > 0$.*

Proof. Let λ_R and ϕ_R be respectively the first eigenvalue and first eigenfunction of the Laplacian with zero boundary data in B_R, where ϕ_R is normalized so that $\int_{B_R} \phi_R(x)dx = 1$. Fix a number $A > 1$, and a sufficiently large $\bar R > R_0$ such that $A > [\lambda_{\bar R} + 1)]^{\frac{1}{p-1}}$. We claim that the result of the lemma holds with the choice $R = 2R_0 + \bar R$. Let us assume the opposite, so that there are numbers $s_0, R > 0$ such that

$$\sup_{|x| = R + 2R_0 + \bar R} w(x, s_0) > A.$$

The previous lemma implies that

$$w(x, s_0) > A \ \forall x \in B_{\bar R + R}.$$

Let us consider the function $\psi(s) = \int_{B_{\bar R}} w^p(x,s)\phi_{\bar R}(x)dx$. Then we have $\psi(s_0) > A^p$. By direct integration, we find that

$$\psi'(s) \ge \psi(s) - (\lambda_{\bar R} + 1)\psi^{\frac{1}{p}}(s) \text{ for all } \ s \ge 0.$$

This implies in particular that the right hand side of the above inequality remains positive for all $s \ge s_0$. Then, it is easily checked that for some $C > 0$, $\psi(s) > Ce^s \ \forall s \ge s_0$. By the definitions of v in (2.1) and $\psi(s)$, we have that

$$(Te^{-(1-m)s})^{\frac{-1}{1-m}} \int_{B_{\bar R}} u(x, T(1 - e^{-(1-m)s}))\phi_{\bar R}(x)dx > Ce^s,$$

so that letting $s \to \infty$ we conclude

$$\int_{B_{\bar{R}}} u(x,T)\phi_{\bar{R}}(x)dx > 0,$$

which contradicts the definition of T, the vanishing time of u. This finishes the proof. □

Corollary 3.1. *For all $R > 0$ the quantity $\int_{B_R} w(x,s)^p dx$ is uniformly bounded on $s \in [0,\infty)$.*

This result follows directly from the proof of Lemma 3.2, remembering that if $\int_{B_R} w^p dx$ is not upper bounded, so is not $\int_{B_{2R}} w^p \phi_{2R} dx$ (because ϕ_{2R} is bounded from below in B_R).

Lemma 3.3. *There exist constants $M > 0$ and $R > 0$ such that*

$$v(x,s) \le Me^{-(|x|-R)} \text{ for all } |x| \ge R \text{ and all } s \ge 0.$$

Proof. Let $g(y)$ be the positive solution of

$$g''(y) + g^p(y) - g(y) = 0 \text{ in } \mathbb{R}$$

which maximizes at $y = 0$ and tends to zero as $|y| \to \infty$. Then g is decreasing on $[0,\infty)$, $g(0) > 1$ and g decays like e^{-y} at infinity. Since the support of $v(x,0)$ is contained in B_R, Lemma 3.2, implies the existence of $\bar{R} > R$ such that $v(x,s) \le g(0)$ for all $|x| \ge \bar{R}$ and all $s > 0$. Let $h(x) = g(|x| - \bar{R})$. Then we see that

$$\Delta h + h^p - h \le 0.$$

By the definition of h, we also have that $w(x,s) \le h(x)$ for all $|x| = \bar{R}$ and all $s > 0$ and $0 = w(x,0) \le h(x)$ for every $|x| \ge \bar{R}$. Thus, by comparison, $v(x,s) \le h(x)$ for all $|x| \ge \bar{R}$ and all $s > 0$, and the desired result follows. □

Corollary 3.2. *There is a constant C such that $\int v^p(x,s)dx \le C$ for all $s > 0$.*

This result follows immediately from Lemma 3.3 and Corollary 3.1.

Proof of Proposition 3.1. We will argue by contradiction. If the Proposition was false, there would be a sequence $s_n \to +\infty$ such that

$$A_n =: ||v(\cdot, s_n)||_{L^\infty} \to \infty \text{ as } n \to \infty.$$

For each n, let us choose a point x_n such that

$$\frac{A_n}{2} \le v(x_n, s_n) \le A_n.$$

Defining

$$w_n(y,\tau) =: \varepsilon_n^{2/(p-1)} v(x_n + \varepsilon_n y, s_n + \tau) \tag{3.1}$$

with $\varepsilon_n^{2/(p-1)} = A_n^{-1}$. It easy to verify that w_n satisfies

$$\frac{\partial w_n^p}{\partial \tau} = \Delta w_n + w_n^p - \varepsilon_n^2 w_n \text{ in } \mathbb{R}^N \times (-s_n, 0].$$

We may assume that $w_n \leq 1$ globally in the above region, without loss of generality. The fact that $w_n(x,0) \geq \frac{1}{2}$ for all n and the compactness result found in Lemma 5.2 in the appendix yields that

$$\int_{-1}^{0} ds \int_{B_1} w_n(y,s)^p dy \geq c > 0,$$

On the other hand, the definition of w_n gives that

$$\int w_n^p(y,s)dy = \varepsilon^\theta \int v^p(x, s_n + \tau)dx$$

with $\theta = \frac{2p}{p-1} - 2N$. Since $1 < p < \frac{N}{N-2}$, θ is positive. Then letting $n \to \infty$ and recalling that $\int v^p(x,s)dx$ is uniformly bounded in s by Corollary 3.2, we conclude that in particular

$$\lim_{n\to\infty} \int_{B_1} w_n^p(y,s)dy = 0$$

for all $s > 0$. We have reached a contradiction that yields the desired result.

4. Convergence

A preliminary result we need for the proof of convergence of v to a steady state of (2.2) is the following.

Lemma 4.1. *$J(v(\cdot,s)) \geq 0$ for all $s > 0$.*

Proof. Assume that for a certain $s_0 > 0$, $J(v(\cdot,s_0)) < 0$. Let us consider the quantity $F(s) = \int v^{p+1}(y,s)dy$. Differentiating and using the equation for v we obtain the identity

$$\frac{p}{p+1}\frac{dF}{ds}(s) = -\int |\nabla v|^2 dx - \int v^2 dx + \int v^{p+1} dx = -2J(v(\cdot,s)) + \frac{p-1}{p+1}F(s).$$

Since $J(v(\cdot,s))$ is non-increasing, it follows that

$$\frac{dF}{ds}(s) > \frac{p-1}{p}F(s)$$

for all $s > s_0$. Integrating the above inequality, we obtain that $F(s) > F(s_0)e^{\frac{p-1}{p}(s-s_0)}$. On the other hand, Proposition 5.1 tells us that v is uniformly bounded for $s > s_0$ and hence so is $F(s)$ because of the uniform exponential decay predicted by Lemma 3.3. This is a contradiction which concludes the proof. $\square$

Proposition 4.1. *Every sequence $s_n \uparrow \infty$ has a subsequence, which we call again s_n, such that for some w, positive solution of* (2.4)-(2.5), *we have that $v(\cdot, s_n) \to w$ in L^2 and uniform senses. as $n \to \infty$.*

Proof. By Lemma 3.3 and Proposition 3.1, we know that $\int w^{p+1}dx$ is bounded. We also notice that $J(v(\cdot, s)) \le J(v(\cdot, 0))$. Then $\|v(\cdot, s)\|_{H^1}$ is uniformly bounded. Thus, a given sequence $s_n \to \infty$ has a subsequence, which we call the same way, such that $v(\cdot, s_n)$, converges weakly in H^1 to a function w. This implies that $v(\cdot, s_n) \to w$ in the L^2-sense. Lemma 3.3 and the compactness Lemma 5.1 in the appendix, we conclude that the convergence is also in uniform sense. Next we will prove that w is a non trivial solution of the stationary problem. Let us recall that

$$\frac{d}{ds}J(v(\cdot, s)) = -\frac{4p}{(p+1)^2}\int |(w^{\frac{p+1}{2}}(x, s))_s|^2 dx.$$

Integrating from s_n to $s_n + \tau$ and using Cauchy-Schwarz's inequality we obtain

$$\int |w^{\frac{p+1}{2}}(x, s_n+\tau) - w^{\frac{p+1}{2}}(x, s_n)|^2 dx \le \frac{(p+1)^2\tau}{4p}[J(w)(s_n) - J(w)(s_n+\tau)].$$

By Lemma 4.1, $J(v(\cdot, s))$ is bounded below and decreasing, hence it has a finite limit as $s \to \infty$. Then, for any $\tau > 0$, $v(\cdot, s_n + \tau) \to w$ in L^{p+1}. From Lemma 3.3, it also converges in L^2 and L^∞. Since $v(x, s)$ solves the equation (2.2), we get that for any $\varphi \in H^1(\mathbb{R}^N)$

$$\int (w^p(x, s_n+1) - w^p(x, s_n))\varphi(x)dx = \int_{s_n}^{s_n+1}\int -\nabla v(x, s_n+\tau)\nabla\varphi(x)$$
$$+ (\frac{1}{1-m}w^p(x, s_n+\tau) - v(x, s_n+\tau))\varphi(x)dxd\tau.$$

Thus letting $n \to \infty$ we obtain

$$\int -\nabla w \nabla\varphi(x) + (w^p(x) - w(x))\varphi(x)dx = 0,$$

hence w is a non-negative solution of the stationary problem (2.4) in H^1. It remains to show that $w \not\equiv 0$. In fact we claim that there is a $k_1 > 0$ such that

$$\sup_{x\in\mathbb{R}^N} v(x,s) \geq k_1. \tag{4.1}$$

Let us assume, by contradiction, that this is not true, namely that for each $\varepsilon > 0$, there is a $s_\varepsilon > 0$ such that $v(x,s_\varepsilon) < \varepsilon$ for all $x \in \mathbb{R}^N$. Given ε, let us consider the function

$$U(s) = K(1 + s_\varepsilon - s)_+^{\frac{m}{1-m}}.$$

It is easily seen that if k is chosen less than certain constant which depends only on m then U satisfies

$$(U^p)_s + U - U^p < 0, \quad \forall s > s_\varepsilon.$$

Hence, this function constitutes a supersolution of equation (2.2) if $s > s_\varepsilon$. Finally, if the number ε is chosen sufficiently small (depending only on K), we will also have that $U(s_\varepsilon) > \varepsilon$. Since $v(\cdot, s_\varepsilon) < \varepsilon$, comparison for $s > s_\varepsilon$ then yields that $v(x,s) < U(s)$. But since U vanishes identically for $s > s_\varepsilon + 1$, so does $v(x,s)$. But this violates the definition of v for this would imply that the original u vanished identically before time $t = T$. Hence (4.1) holds, which implies that $w \neq 0$, then Proposition 4.1 is proved. □

To conclude the proof of Theorem 2.1, it remains to prove that there is one and only one limit point as those predicted by the above result. A main ingredient to establish this is the following estimate, which is essentially known in the one-dimensional case from the work [16], and for a related semilinear parabolic problem from [5].

Lemma 4.2. *There is a constant $\gamma > 0$, depending only of an upper bound of w, such that*

$$\int_s^\infty \| \left(w^{\frac{p+1}{2}} \right)_s \|_{L^2}^2 (\tau) d\tau \leq c e^{-\gamma s}.$$

Proof. We begin with establishing the following fact: Given $\varepsilon > 0$ there is a time $s_0 > 0$ such that for all $s > s_0$ exists a solution of stationary problem $w^s(x) = w_*(|x - y(s)|)$ which satisfies

$$\|v(\cdot,s) - w^s\|_{L^2} \leq \varepsilon, \quad \int (v(x,s) - w^s(x)) w^s_{x_i}(x) dx = 0. \tag{4.2}$$

In fact, let us consider the functional

$$I(y,s) = \int (v(x,s) - w_*(|x-y|))^2 dx.$$

Proposition 4.1 implies that I has a minimum value in y, which becomes arbitrarily small for all s sufficiently large. Let us assume that this minimum value is reached at some $y = y(s)$. Differentiating I with respect to y and evaluating at this point, relation (4.2) follows.

Let us choose w^s as in the above claim, and consider the linearized operator around w^s, $\mathcal{L}_s$ as in (2.6), then

$$\mathcal{L}_s(v(\cdot,s) - w^s) = -(v(\cdot,s)^p)_s + h(v,w^s)|v - w^s|^{1+\mu},$$

Here h is bounded for bounded arguments and $0 < \mu < p-1$. Since $v(\cdot,s)$ and $w^s(\cdot)$ are uniformly bounded for all s, then Proposition 4.1 yields

$$\|h(w,w^s)|w - w^s|^{1+\mu}\|_{L^2} \leq \varepsilon^\mu c\|w - w^s\|_{L^2}.$$

As we mentioned earlier, by an result in [15], the solutions of the linearized equation are linear combinations of the functions $\frac{\partial w^s}{\partial x_i}$. Then by the above claim, $(w - w^s)$ is orthogonal to the kernel of $\mathcal{L}_s$. Hence we conclude

$$\|v(\cdot,s) - w^s\|_{H^1} \leq \|(v^p)_s\|_{L^2}. \tag{4.3}$$

On the other hand, since w^s is a solution of stationary problem, a direct computation yields,

$$|J(v(\cdot,s)) - J(w^s)| = \int \frac{1}{2}|\nabla v - \nabla w^s|^2 + \int F(v) - F(w^s)$$
$$-f(w^s)(w - w^s)dx \leq \|v(\cdot,s) - w^s\|_{H^1},$$

where $f(v) = v - v^p$ and $F(v) = \int_0^v f(z)dz$. This implies that

$$0 \leq J(w(s)) - J(w^s) \leq \|w - w^s\|_{H^1}. \tag{4.4}$$

From (4.3), (4.4) and Proposition 3.1

$$\frac{d}{ds}(J(w(s)) - J(w^s)) = -\frac{4p}{(p+1)^2}\|(w^{\frac{p+1}{2}})_s\|^2_{L^2} \leq -\frac{1}{pA^{p-1}}\|(w^p)_s\|^2_{L^2}$$
$$\leq -\frac{1}{pA^{p-1}}\|w - w^s\|^2_{H^1} \leq \frac{1}{pA^{p-1}}(J(w(s)) - J(w^s)).$$

where A is any upper bound of w. Noticing that $J(w^s) = J_0$ is constant for every solution of stationary problem and that $\lim_{s\to\infty} J(w(s)) = J_0$ we conclude that

$$\int_s^\infty \|(w^{\frac{p+1}{2}})_s\|^2_{L^2}(\tau)d\tau = J(w(s)) - J(w^s) \leq ce^{-\gamma s}$$

with $\gamma = \frac{1}{pA^{p-1}}$ and $c = J(w(0)) - J_0$. This concludes the proof. □

Now we are in a position to prove our main result.

Proof of Theorem 2.1. By Proposition 4.1 we already know that each sequence $s_n \to \infty$ has a subsequence which converges uniformly and in L^2 to a positive solution of (2.4)-(2.5). So it is enough to prove that v has only one such limit point. This is equivalent to showing that $v^{\frac{p+1}{2}}(\cdot, s)$ has at most one limit point in L^2. Assume first that $0 \le s_2 - s_1 \le 1$. By Lemma 4.2 we obtain then that

$$\|v^{\frac{p+1}{2}}(\cdot, s_1) - w^{\frac{p+1}{2}}(\cdot, s_2)\|_{L^2} \le \int_{s_1}^{s_2} \|(w^{\frac{p+1}{2}})_s(\tau)\|_{L^2} d\tau$$

$$\le \sqrt{s_2 - s_1} \left(\int_{s_1}^{s_2} \|(w^{\frac{p+1}{2}})_s(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{2}} \le ce^{-\gamma \frac{s_1}{2}}.$$

Now, if for some positive integer l, $l \le s_2 - s_1 < l$, we see that

$$\|w^{\frac{p+1}{2}}(s_1) - w^{\frac{p+1}{2}}(s_2)\|_{L^2}^2 \le \sum_{j=1}^{l} \|(w^{\frac{p+1}{2}})_s(s_1 + j) - (w^{\frac{p+1}{2}})_s(s_1 + j - 1)\|_{L^2}^2$$

$$+\|(w^{\frac{p+1}{2}})_s(s_2) - (w^{\frac{p+1}{2}})_s(s_1 + l)\|_{L^2}^2.$$

Then

$$\|w^{\frac{p+1}{2}}(s_1) - w^{\frac{p+1}{2}}(s_2)\|_{L^2}^2 \le c \sum_{j=l}^{\infty} e^{-\gamma j}. \tag{4.5}$$

From here it follows that $w^{\frac{p+1}{2}}(\cdot, s)$ has just one limit point in L^2, and the proof is thus concluded.

5. Appendix

Let us consider the original problem

$$u_t = \Delta u^m - u^m.$$

We begin with the following observation: Solutions of this problem (with bounded initial data) satisfy the Aronson-Benilan inequality

$$u_t \le \frac{1}{1-m} \frac{u}{t},$$

which translates into the fact that for $t_2 > t_1 > 0$ the inequality

$$u(x, t_2) \le \left(\frac{t_2}{t_1} \right)^{\frac{1}{1-m}} u(x, t_1)$$

for all x. The proof of this fact can be made similarly to that in [1]. At the formal level, let us observe that $u_\lambda(t, x) = \lambda^{-\frac{1}{1-m}} u(\lambda t, x)$ satisfies the same equation. Hence setting

$$h = \frac{\partial v_\lambda}{\partial \lambda}|_{\lambda=1} = tu_t - \frac{1}{1-m} u,$$

we find that h satisfies

$$h_t = m\Delta u^{m-1}h - mu^{m-1}h$$

and the result follows by comparison since $h \leq 0$ at $t = 0$. Of course this argument only works if enough regularity is assumed. This can be achieved using approximations, after adding a small number to the initial data.

Let us consider now the function $v(x, s)$ defined from u by (2.1). Then the Aronson-Benilan inequality, translates into the fact that for $0 < s_1 < s_2$ we get

$$\frac{e^{-s_2}v(x, s_2)}{e^{-s_1}v(x, s_1)} \leq \Big(\frac{1 - e^{-(1-m)s_2}}{1 - e^{-(1-m)s_1}}\Big)^{\frac{1}{1-m}}.$$

Hence for $s_1 >> 1$,

$$v(x, s_2) \leq e^{s_2 - s_1}(1 + Ce^{-(1-m)s_1})v(x, s_1). \tag{5.1}$$

Lemma 5.1. *Assume that the sequence $s_n \to \infty$ is such that $v(x, s_n + \tau) \to w(x)$ in L^2-sense. Then the convergence is uniform.*

Proof. Let us set $v_n(x, \tau) = v(x, s_n + \tau)$. Consider a fixed $R > 0$ and the cylinders $Q_2 = B_2 \times [0, 2]$, $Q_1 = B_1 \times [1, 2]$. Let $G(x, y)$ be the Green's function of the ball B_2. Then we have the following identity:

$$\begin{aligned}&\frac{\partial}{\partial s}\int_{B_2} G(x, y)v_n^p(y, s)dy \\ &= \int_{\partial B_2} \frac{\partial G}{\partial \nu}(x, y)v_n(y, s)d\sigma(y) - v_n(x, s) + \int_{B_2} G(x, y)(v_n^p - v_n)(y, s)dy.\end{aligned}$$

Let us restrict ourselves to $x \in B_1$. After integration in time in (a, b), we get

$$\begin{aligned}&\int_{B_2} G(x, y)v_n^p(y, b)dy - \int_{B_2} G(x, y)v_n^p(y, a)dy \\ &+ \int_a^b\int_{\partial B_2} \frac{\partial G}{\partial \nu}(x, y)v_n(y, s)d\sigma(y) - \int_a^b\int_{B_2} G(x, y)(v_n^p - v_n)(y, s)dyds \\ &= \int_a^b v_n(x, s)ds.\end{aligned}$$

Since v_n is uniformly bounded, it follows that on B_1, the sequence of functions $\int_a^b v_n(x, s)ds$ is uniformly equicontinuous, from the regularity properties of G. Hence it follows that

$$\int_a^b v_n(x, s)ds \to (b - a)w(x).$$

uniformly in B_1. Finally, relation (5.1) tells us that for some $\gamma > 0$,

$$e^{-\gamma(b-a)} v_n(x,a) \le \frac{1}{b-a} \int_a^b v_n(x,s)ds \le e^{\gamma(b-a)} v_n(x,b).$$

Hence

$$\limsup_{n\to\infty} e^{-\gamma(b-a)} v_n(x,a) \le w(x),$$

and similarly

$$\liminf_{n\to\infty} e^{\gamma(b-a)} v_n(x,b) \ge w(x).$$

From these relations, which hold for arbitrary $a < b$ we conclude that $v_n(x,a) \to w(x)$ uniformly on B_1. □

Let us consider now the situation described in the proof of Theorem 1.1, with the function $w_n(y,\tau)$ defined in (3.1). Let us recall that w_n satisfies

$$\frac{\partial w_n^p}{\partial \tau} = \Delta w_n + w_n^p - \varepsilon_n^2 w_n \text{ in } \mathbb{R}^N \times (-s_n, 0],$$

that w_n is uniformly bounded in this range, and that $w_n(0,0) \ge 1/2$. Arguing similarly as in the above proof we see then that for each $\tau \le 0$ the quantity $\int_{-1}^0 w_n(y,s)ds$ converges uniformly on B_1 to some continuous function $w(y)$. Inequality (5.1) gives us on the other hand,

$$w_n(y,0) \le C w_n(y,s), \quad \forall s \in [-1,0],$$

hence in particular we have $w(0) > 0$, so that then $w \not\equiv 0$. In particular we find that

$$\liminf_{n\to\infty} \int_{-1}^0 ds \int_{B_1} w_n(y,s)^p dy \ge \lim_{n\to\infty} \int_{B_1} dy (\int_{-1}^0 w_n(y,s)ds)^p$$
$$= \int_{B_1} w(y)^p dy > 0.$$

Summarizing, we have proven

Lemma 5.2. *w_n satisfies that for some $c > 0$*

$$\int_{-1}^0 ds \int_{B_1} w_n(y,s)^p dy \ge c > 0,$$

for all n.

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