Differential and Integral Equations Volume 15, Number 8, August 2002, Pages 973–1008

UNIQUENESS FOR MULTI-DIMENSIONAL STEFAN PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS DESCRIBED BY MAXIMAL MONOTONE OPERATORS

Toyohiko Aiki
Department of Mathematics, Faculty of Education, Gifu University
Yanagido 1-1, Gifu 501-1193, Japan

(Submitted by: Reza Aftabizadeh)

Dedicated to Professor Karl-Heinz Hoffmann on his 60th birthday

Abstract. In this paper we discuss weak solutions of the multi-dimensional two-phase Stefan problem with the boundary condition including the subdifferential operator of the convex function on $\mathbb{R}$ so that the boundary condition is nonlinear and contains a multi-valued operator, in general. Kenmochi and Pawlow already established the uniqueness and existence of a solution to our problem in the sense of the vanishing viscosity solution. The purpose of this paper is to prove the uniqueness theorem for a solution defined in the usual variational sense. Our proof is due to the standard method in which the dual problem of the original problem plays a very important role.

1. Introduction

In this paper we consider the following two-phase Stefan problem: Let $T > 0$ and Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a boundary $\Gamma := \partial\Omega$ consisting of two disjoint, smooth and compact surfaces Γ_0 and Γ_1 with $meas\,\Gamma_0 > 0$ and $meas\,\Gamma_1 > 0$. The problem $\mathrm{SP} := \mathrm{SP}(\beta; f; \gamma_0, \gamma_1, g_0, g_1; u_0)$ is to find a function u on $Q(T) := (0,T) \times \Omega$ satisfying

$$u_t - \Delta\beta(u) = f \quad \text{in } Q(T), \tag{1.1}$$

$$-\frac{\partial\beta(u)}{\partial\nu} \in \partial\gamma_i(\beta(u) - g_i) \quad \text{on } \Sigma_i(T) := (0,T) \times \Gamma_i,\ i = 0,1, \tag{1.2}$$

$$u(0,x) = u_0(x) \quad \text{for } x \in \Omega, \tag{1.3}$$

Accepted for publication: October 2001.
AMS Subject Classifications: 35K15, 80A22, 35B50.

where β is a given non-decreasing function on $\mathbb{R}$; f is a given heat source on $Q(T)$; for $i = 0, 1$ γ_i is a proper l.s.c. and convex function on $\mathbb{R}$ and $\partial\gamma_i$ denotes its subdifferential; for $i = 0, 1$ g_i is a given function on $\Sigma_i(T)$; u_0 is a given initial function; $(\partial/\partial\nu)$ denotes the outward normal derivative on Γ.

The above equation (1.1) gives the so-called enthalpy formulation which is well-known as the weak formulation for the Stefan problem, when

$$\beta(r) = \begin{cases} c_1(r-1) & \text{for } r \geq 1, \\ 0 & \text{for } 0 < r < 1, \\ c_2 r & \text{for } r \geq 0 \end{cases}$$

for some positive constants c_1, c_2. In such a case u and $\beta(u)$ represent the enthalpy density and the temperature field, respectively. The mathematical literature dealing with (1.1) so large that we show only the book [17, Chapter 4] by Visintin as a general reference source for the background information.

Our interest in this paper is how to handle a weak solution of the initial boundary value problem with nonlinear boundary conditions since the regularity of the solution is not enough to apply the L^2-theory, in general. Here, we give a typical example of the nonlinear boundary conditions in our problem, which is called Signorini type as follows:

$$\begin{cases} \beta(u) \geq g_0 & \text{on } \Sigma_0(T), \\ \dfrac{\partial\beta}{\partial\nu} = 0 & \text{on } \{\beta(u) > g_0\}, \\ \dfrac{\partial\beta}{\partial\nu} > 0 & \text{on } \{\beta(u) = g_0\}, \end{cases} \qquad \begin{cases} \beta(u) \leq g_1 & \text{on } \Sigma_1(T), \\ \dfrac{\partial\beta}{\partial\nu} = 0 & \text{on } \{\beta(u) < g_1\}, \\ \dfrac{\partial\beta}{\partial\nu} < 0 & \text{on } \{\beta(u) = g_1\}, \end{cases}$$

these conditions are represented in the form (1.2) for γ_0 (resp. γ_1) given by

$$\gamma_0(r)(\text{ resp. } \gamma_1(r)) = \begin{cases} 0 & \text{if } r \geq 0(\text{ resp. } r \leq 0), \\ \infty & \text{otherwise.} \end{cases}$$

The boundary condition of Signorini type is known as one of the mathematical models describing a thermostat device.

In [11] Kenmochi and Pawlow studied the above problem SP and showed the existence and the uniqueness of a solution by using the vanishing viscosity method. We note that the vanishing viscosity solution is always the weak solution in the usual variational sense. The purpose of this paper is to give a proof of the uniqueness for a weak solution, that is, we establish the uniqueness theorem for the more general class of solutions. In our proof we shall use the dual equation for the original equation (1.1). Such an idea was found in the book [12, Chapter 3] and used in order to prove the uniqueness for weak solutions of parabolic equations without L^2-theory. Also, Niezgódka and Pawlow in [16] investigated the multi-dimensional Stefan problem with the following nonlinear boundary condition:

$$-\frac{\partial\beta(u)}{\partial\nu} = g(t, x, \beta(u)) \quad \text{on } \Sigma(T) := (0, T) \times \partial\Omega,$$

where $g(\cdot, \cdot, \xi)$ is measurable on $\Sigma(T)$ and $g(t, x, \xi)$ is locally Lipschitz continuous in ξ uniformly with respect to $(t, x) \in \Sigma(T)$. They improved the dual equation method and obtained the well-posedness of their problem.

As mentioned before the existence of a solution of SP was already proved by Kenmochi and Pawlow in [11]. However, there are some gaps in their argument so that we give a complete proof of the existence, again. In order to fill the gaps we assume the stronger condition for data than their one. In the proof we approximate β by $\beta_\mu(r) = \beta(r) + \mu r$ for $r \in \mathbb{R}$, $\mu > 0$, and solve the problem with β_μ by applying the abstract theory for the doubly nonlinear evolution equation on a Hilbert space(cf. Kenmochi [10]). The similar problems have been treated in Lions [13], Grange and Mignot [8], Barbu [4], and DiBenedetto and Showalter [7].

Here, we list some interesting results concerned with multi-dimensional Stefan problems with boundary conditions including heat flux. In [14] Magenes, Verdi and Visintin considered the Stefan problems with the time-independent boundary condition given by

$$-\frac{\partial\beta(u)}{\partial\nu} \in \gamma(\beta(u)) \quad \text{on } \Sigma(T),$$

where γ is a multivalued operator on $\mathbb{R}$. Their approach was based on the semigroup theory for L^1-space. In case with the usual Neumann boundary condition Haraux and Kenmochi [9] reformulated the Stefan problem as a nonlinear evolution equation in the dual space of some subspace of $H^1(\Omega)$ then the uniqueness was easily obtained. The author showed the uniqueness for the problem with the following nonlinear dynamic boundary condition,

$$-\frac{\partial\beta(u)}{\partial\nu} - \frac{\partial\beta(u)}{\partial t} = g(t, x, \beta(u)) \quad \text{on } \Sigma(T),$$

by applying their techniques to the problem in [1].

Finally, we give a brief sketch of the proof of the uniqueness, in which the dual equation method employed by Niezgódka and Pawlow is used. Let u_1 and u_2 be solutions of SP and $u = u_1 - u_2$; $v = \beta(u_1) - \beta(u_2)$; $a(t, x) = \frac{v(t,x)}{u(t,x)}$ if $u(t, x) \neq 0$, $= 0$ otherwise; $\xi_j = -\frac{\partial\beta(u_j)}{\partial\nu} \in \partial\gamma_i(\beta(u_j) - g_i)$ on $\Sigma_i(T)$, $i = 0, 1$, $j = 1, 2$; $\xi = \xi_1 - \xi_2$; $E_e = \{(t, x) \in \Sigma(T); v(t, x) = 0\}$, $E_n = \{(t, x) \in \Sigma(T); v(t, x) \neq 0\}$; $G(t, x) = \frac{\xi(t,x)}{v(t,x)}$ on E_n, $= 0$ on E_e.

Then we have, for any $\eta \in \mathcal{L}(Q(T))$,

$$-\int_{Q(T)} u(\eta_t + a\Delta\eta)dxdt + \int_{E_n} v(\frac{\partial\eta}{\partial\nu} + G\eta)d\Gamma dt + \int_{E_e} \xi\eta d\Gamma dt = 0,$$

where $\mathcal{L}(Q(T)) := \{\eta \in W^{1,2}(0,T;H); \eta \in L^2(0,T;H^2(\Omega)), \eta(T) = 0\}$ and $d\Gamma$ stands for the usual surface element on Γ.

Here, we consider the following dual problem for SP:

$$\begin{aligned}
&\eta_t + a\Delta\eta = \zeta \quad \text{in } Q(T),\\
&\frac{\partial\eta}{\partial\nu} + G\eta = 0 \quad \text{on } E_n,\\
&\eta = 0 \quad \text{on } E_e,\\
&\eta(T,x) = 0 \quad \text{for } x \in \Omega,
\end{aligned}$$

where $\zeta \in C_0^\infty(\overline{Q(T)})$.

If the above problem has a solution $\eta \in \mathcal{L}(Q(T))$, then immediately we obtain $\int_{Q(T)} u\zeta dxdt = 0$ for any $\zeta \in C_0^\infty(\overline{Q(T)})$, that is, $u = 0$ a.e. on $Q(T)$. However, the function a may vanish and we do not know the regularity of the function G and the boundaries of the sets E_n and E_e. So, it is not easy to or impossible to solve the problem. Accordingly, instead of solving we approximate the dual problem in the following way;

$$\begin{aligned}
&\eta_t + a_k\Delta\eta = \zeta \quad \text{in } Q(T_1),\\
&\frac{\partial\eta}{\partial\nu} + G_m\eta = 0 \quad \text{on } \Sigma_\varepsilon^0(T_1),\\
&\eta = 0 \quad \text{on } \Sigma_\varepsilon^1(T_1),\\
&\eta(T_1,x) = 0 \quad \text{for } x \in \Omega,
\end{aligned}$$

where $0 < T_1 \le T$; $\{a_k\} \subset C^\infty(\overline{Q(T)})$, a_k is strictly positive on $\overline{Q(T)}$ for each k and $a_k \to a$ in $L^2(Q(T))$ as $k \to \infty$; $\{G_m\} \subset C^\infty(\overline{\Sigma(T)})$ and $G_m \to G$ in the appropriate sense; $\Sigma_\varepsilon^0(T_1) \supset E_n$ and $\Sigma_\varepsilon^1(T_1) \subset E_e$ with smooth boundaries, and the measures of the differences of $\Sigma_\varepsilon^0(T_1)$ and E_n, $\Sigma_\varepsilon^1(T_1)$ and E_e are sufficient small. We shall obtain solutions of approximate problems and uniform estimates of solutions in Section 3. By virtue of those uniform estimates we get the uniqueness of solutions of SP(see Section 5).

In this paper we refer the book [5] for the theory on maximal monotone operators and subdifferentials of convex functions in Hilbert space.

Acknowledgement. We thank the referee for bringing references and helpful comments.

2. Main results

We begin with the assumptions on data under which SP is discussed.

(A1) $\beta : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous and non-decreasing function with Lipschitz constant C_β, and $\beta(0) = 0$.

(A1-1) The function β satisfies

$$|\beta(r)| \geq M_\beta |r| - m_\beta \quad \text{for any } r \in \mathbb{R},$$

where M_β and m_β are positive constants.

(A1-2) The function β vanishes on the interval $[0, 1]$ and β^{-1} is Lipschitz continuous on $(-\infty, 0]$ and $[1, \infty)$, that is,

$$M'_\beta |r_1 - r_2| \leq |\beta(r_1) - \beta(r_2)| \text{ for } r_i \in (-\infty, 0] \text{ or } r_i \in [1, \infty), i = 1, 2,$$

where M'_β is a positive constant.

(A2) For $i = 0, 1$ γ_i is a proper l.s.c. and convex function on $\mathbb{R}$.

(A3) $g_i \in L^2(\Sigma_i(T))$ for $i = 0, 1$.

(A3-1) For $i = 0, 1$ g_i is the trace of a function g on Σ_i such that $g \in W^{1,2}(0, T; H^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega)) \cap L^\infty(\Sigma_i(T))$.

(A4) $f \in L^\infty(Q(T))$, that is, there exist two constants b_0 and b_1 such that

$$b_0 \leq f \leq b_1 \quad \text{a.e. on } Q(T).$$

(A5) $u_0 \in L^\infty(\Omega)$ and $v_0 := \beta(u_0) \in H^1(\Omega)$.

(A6) The initial and boundary data satisfy the following compatibility condition:

$$\int_{\Gamma_i} \gamma_i(v_0(x) - g_i(0, x)) d\Gamma < \infty \quad \text{for } i = 0, 1.$$

Furthermore, we need the following conditions in the argument for the existence.

(A7) $meas\{x \in \Omega; 0 \leq u_0(x) \leq 1\} = 0$; $D(\gamma_0) \subset [0, \infty)$ and $D(\gamma_1) \subset (-\infty, 0]$; there exist positive constants m_* and M_* such that

$$0 < m_* \leq g_0 \leq M_* \text{ a.e. on } \Sigma_0(T) \text{ and}$$
$$- M_* \leq g_1 \leq -m_* < 0 \text{ a.e. on } \Sigma_1(T),$$

and the initial function v_0 satisfies the following condition (*):

(*) There exist a positive constant M'_* and functions $V^{(i)} \in H^2(\Omega) \cap C(\overline{\Omega})$, $i = 0, 1$ such that $V^{(0)} \leq v_0 \leq V^{(1)}$ a.e. on Ω, and $V^{(0)}$ and $V^{(1)}$ are solutions of the elliptic problems given by:

$$(\mathrm{EP})^{(0)}: \begin{cases} -\Delta V^{(0)} = b_0 & \text{in } \Omega, \\ V^{(0)} = m_* & \text{on } \Gamma_0, \\ -\dfrac{\partial V^{(0)}}{\partial \nu} \in \partial\gamma_1(V^{(0)} + M'_*) & \text{on } \Gamma_1, \end{cases}$$

$$(\mathrm{EP})^{(1)}: \begin{cases} -\Delta V^{(1)} = b_1 & \text{in } \Omega, \\ V^{(1)} = -m_* & \text{on } \Gamma_1, \\ -\dfrac{\partial V^{(1)}}{\partial \nu} \in \partial\gamma_0(V^{(1)} - M'_*) & \text{on } \Gamma_0. \end{cases}$$

Remark 2.1. By Brézis [6, Theorems 1.10 and 1.11] for $i = 0, 1$ the above problem $(\mathrm{EP})^{(i)}$ has a unique solution $V^{(i)} \in H^2(\Omega) \cap C(\overline{\Omega})$.

Here, for simplicity we shall use the following notations:

$$H := L^2(\Omega), X = H^1(\Omega);$$

$$(u, v) := \int_\Omega uv dx \quad \text{for } u, v \in H;$$

$$A(u, v) := \int_\Omega \nabla u \cdot \nabla v dx \quad \text{for } u, v \in X;$$

X^* is the dual space of X and $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{R}$ stands for the duality pairing between X^* and X;

$$(u, v)_i := \int_{\Gamma_i} uv d\Gamma \quad \text{for } i = 0, 1.$$

Now, we give a precise definition of a solution of SP.

Definition 2.1. A function $u : [0, T] \to H$ is called a weak solution (in the variational sense) of SP, if it satisfies the following conditions:

(S1) $u \in W^{1,2}(0, T; X^*) \cap L^\infty(Q(T))$, $\beta(u) \in L^2(0, T; X)$;

(S2) for $i = 0, 1$ there exists a function $\xi_i \in L^2(\Sigma_i(T))$ such that

$$\xi_i \in \partial\gamma_i(\beta(u) - g_i) \quad \text{a.e. on } \Sigma_i, \quad \text{and}$$

$$\langle u'(t), \eta \rangle + A(\beta(u(t)), \eta) + \sum_{i=0}^{1} (\xi_i(t), \eta)_i = 0 \text{ for any } \eta \in X \text{ and a.e. } t \in [0, T];$$

(S3) (1.3) holds.

Remark 2.2. Let u be a function on $Q(T)$. Then u is a solution of SP if and only if the following conditions:

(S1)$'$ $u \in C_w([0, T]; H) \cap L^\infty(Q(T))$, $\beta(u) \in L^2(0, T; X)$;

(S2)′ for $i = 0,1$ there exists a function $\xi_i \in L^2(\Sigma_i(T))$ such that $\xi_i \in \partial\gamma_i(\beta(u) - g_i)$ a.e. on $\Sigma_i(T)$ and

$$-\int_0^T (u(t), \eta_t(t))dt + \int_0^T A(\beta(u)(t), \eta(t)))dt + \sum_{i=0}^{1} \int_0^T (\xi_i(t), \eta(t))_i dt$$
$$= (u_0, \eta(0)) \quad \text{for any } \eta \in \mathcal{L}(Q(T)),$$

where $\mathcal{L}(Q(T)) := \{\eta \in W^{1,2}(0,T;H); \eta \in L^2(0,T;H^2(\Omega)), \eta(T) = 0\}$.

Here, we give the following existence theorem. Before the statement of the theorem we put the set of data, (H1) := {(A1), (A1-1), (A1-2), (A2), (A3), (A3-1), (A4), (A5), (A6), (A7)}.

Theorem 2.1. *Under the assumption* (H1)*, SP has a solution u such that*

$$\beta(u) \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X),$$

$$t \to \int_{\Gamma_i} \gamma_i(\beta(u)(t) - g_i(t))d\Gamma \in L^\infty(0,T) \quad \textit{for } i = 0,1.$$

In the proof of Theorem 2.1, we consider the following approximate problems $(\mathrm{SP})_{\lambda,\mu}$ for $\lambda > 0$ and $\mu > 0$:

$$u_t - \Delta\beta_\mu(u) = f \quad \text{in } Q(T), \tag{2.1}$$
$$-\frac{\partial\beta_\mu(u)}{\partial\nu} = \partial\gamma_{i\lambda}(\beta_\mu(u) - g_i) \quad \text{on } \Sigma_i(T), i = 0,1, \tag{2.2}$$
$$u(0,x) = u_{0,\mu}(x) \quad \text{for } x \in \Omega, \tag{2.3}$$

where $\beta_\mu(r) := \beta(r) + \mu r$ for $r \in \mathbb{R}$; for $i = 0,1$ and $\lambda > 0$ $\gamma_{i\lambda}$ is the Yosida approximation for γ_i; $u_{0,\mu}(x) = \beta_\mu^{-1}(v_0(x))$ for $x \in \Omega$.

These problems $(\mathrm{SP})_{\lambda,\mu}$ are solved by applying the abstract theory of nonlinear evolution equations in Section 6 (cf. [10]). After we get some uniform estimates for solutions of $(\mathrm{SP})_{\lambda,\mu}$, letting $\mu \downarrow 0$ and $\lambda \downarrow 0$ yields the existence of solutions of SP. In the proof it is essential that $\{\partial\gamma_{i\lambda}(\beta_\mu(u_{\lambda,\mu}) - g_i)\}$ is a bounded set in $L^2(\Sigma_i(T))$, $i = 0,1$, where $u_{\lambda,\mu}$ is a solution of $(\mathrm{SP})_{\lambda,\mu}$. In order to obtain uniform estimates we divide the proof into the following several steps.

Step I. By routine works some uniform estimates are obtained (Section 6).

Step II. Choosing sufficiently small positive numbers λ_0 and μ_0 for $0 < \lambda < \lambda_0$ and $0 < \mu < \mu_0$ there exist functions $U^{(i)}_{\lambda,\mu} \in C(\overline{\Omega})$, $i = 0, 1$ such that

$$U^{(0)}_{\lambda,\mu} \le u_0 \le U^{(1)}_{\lambda,\mu} \quad \text{a.e. on } \Omega, \tag{2.4}$$

$$(\mathrm{EP})^{(0)}_{\lambda,\mu}: \begin{cases} -\Delta\beta_\mu(U^{(0)}_{\lambda,\mu}) = b_0 & \text{in } \Omega, \\ -\dfrac{\partial\beta_\mu(U^{(0)}_{\lambda,\mu})}{\partial\nu} = \partial q_\lambda(\beta_\mu(U^{(0)}_{\lambda,\mu}) - \dfrac{m_*}{2}) & \text{a.e. on } \Gamma_0, \\ -\dfrac{\partial\beta_\mu(U^{(0)}_{\lambda,\mu})}{\partial\nu} = \partial\gamma_{1\lambda}(\beta_\mu(U^{(0)}_{\lambda,\mu}) + M'_* + \dfrac{m_*}{2}) & \text{a.e. on } \Gamma_1, \end{cases} \tag{2.5}$$

$$(\mathrm{EP})^{(1)}_{\lambda,\mu}: \begin{cases} -\Delta\beta_\mu(U^{(1)}_{\lambda,\mu}) = b_1 & \text{in } \Omega, \\ -\dfrac{\partial\beta_\mu(U^{(1)}_{\lambda,\mu})}{\partial\nu} = \partial q_\lambda(\beta_\mu(U^{(1)}_{\lambda,\mu}) + \dfrac{m_*}{2}) & \text{a.e. on } \Gamma_1, \\ -\dfrac{\partial\beta_\mu(U^{(1)}_{\lambda,\mu})}{\partial\nu} = \partial\gamma_{0\lambda}(\beta_\mu(U^{(1)}_{\lambda,\mu}) - M'_* - \dfrac{m_*}{2}) & \text{a.e. on } \Gamma_0, \end{cases} \tag{2.6}$$

where q_λ is Yosida-approximation of q and $q(r) = 0$ if $r = 0$, $= \infty$ otherwise. Immediately, we have $q_\lambda = \frac{1}{2\lambda}|r|^2$ for $r \in \mathbb{R}$. Here, we note that for $i = 0, 1$ the problem $(\mathrm{EP})^{(i)}_{\lambda,\mu}$ admits a unique solution since $meas\,\Gamma_0 > 0$, $meas\,\Gamma_1 > 0$ and ∂q_λ is strictly monotone on $\mathbb{R}$ (Section 7).

Step III. In Section 7 by using the comparison principle and Step II it implies that $u_{\lambda,\mu} \ge m'_* + 1$ a.e. on $\Omega_{0,\delta}$ and $u_{\lambda,\mu} \le -m'_*$ a.e. on $\Omega_{1,\delta}$ for $0 < \lambda < \lambda_1$ and $0 < \mu < \mu_1$, where $0 < m'_* \le m_*$, $0 < \lambda_1 \le \lambda_0$, $0 < \mu_1 \le \mu_0$, δ is some positive number and and $\Omega_{i,\delta} := \{x \in \Omega; dist(x, \Gamma_i) < \delta\}$, $i = 0, 1$.

Step IV. Since β is bi-Lipschitz continuous on $(-\infty, 0]$ and $[1, \infty)$, we can easily get uniform estimates for $|u_{\lambda,\mu t}|_{L^2(Q_{i,\delta})}$, $i = 0, 1$, where $Q_{i,\delta} = (0, T) \times \Omega_{i,\delta}$ (Section 8).

Step V. The theory for elliptic problems implies that the set $\{\beta_\mu(u_{\lambda,\mu})\}$ is bounded in $L^2(0, T; H^2(\Omega_{i,\delta}))$, that is, $\{\partial\gamma_{i\lambda}(\beta_\mu(u_{\lambda,\mu}) - g_i)\}$ is bounded in $L^2(\Sigma_i(T))$, $i = 0, 1$ (Section 8).

Next, we discuss the uniqueness of solutions of SP. To do so we can weaken the assumptions of Theorem 2.1 as follows.

Theorem 2.2. *If the condition* (H2) := {(A1), (A2), (A3), (A4), (A5), (A6)} *holds, then SP admits at most one solution.*

We shall use the following notations in the sequel and prove Theorem 2.2 in the following way. Let u_1 and u_2 be solutions of SP, and ξ_{1i}, $i = 0, 1$ and

ξ_{2i}, $i = 0, 1$ be functions in Definition 2.1 (S2) corresponding to u_1 and u_2, respectively. Observe first that from (S2)$'$, we have

$$\begin{aligned}
&-\int_0^T (u_1(t) - u_2(t), \eta_t(t))dt + \int_0^T (\beta(u_1(t)) - \beta(u_2(t)), \Delta\eta(t))dt \\
&+ \sum_{i=0}^{1} \int_0^T (\beta(u_1(t)) - \beta(u_2(t)), \frac{\partial \eta(t)}{\partial \nu})_i dt + \sum_{i=0}^{1} \int_0^T (\xi_{1i}(t) - \xi_{2i}(t), \eta(t))_i dt \\
&= 0 \quad \text{for any } \eta \in \mathcal{L}(Q(T)). \qquad (2.7)
\end{aligned}$$

Here, we put $u := u_1 - u_2$, $v := \beta(u_1) - \beta(u_2)$ on $Q(T)$, $\xi(t, x) := \xi_{1i}(t, x) - \xi_{2i}(t, x)$ if $(t, x) \in \Sigma_i(T)$, $i = 0, 1$, and

$$a(t, x) := \begin{cases} \dfrac{v(t, x)}{u(t, x)} & \text{if } u(t, x) \neq 0, \\ 0 & \text{otherwise;} \end{cases}$$

$E_e(t) := \{x \in \Gamma; v(t, x) = 0\}$ and $E_n(t) := \{x \in \Gamma; v(t, x) \neq 0\}$ for each $t \in [0, T]$,

$$E_e := \bigcup_{t\in(0,T)} E_e(t), \quad E_n := \bigcup_{t\in(0,T)} E_n(t); \quad G(t, x) := \begin{cases} \dfrac{\xi(t, x)}{v(t, x)} & \text{on } E_n, \\ 0 & \text{on } E_e. \end{cases}$$

By Definition 2.1 and assumptions (A1) and (A2), we see that $v = au$ on $Q(T)$, $u \in L^\infty(Q(T))$ and

$$0 \leq a \leq C_\beta \text{ in } Q(T). \qquad (2.8)$$

Also, we have

$$G \geq 0 \text{ a.e. on } \Sigma(T), Gv = \xi \text{ a.e. on } E_n \text{ and } Gv \in L^2(\Sigma(T)). \qquad (2.9)$$

Using these above notations we can rewrite (2.7) in the form

$$\begin{aligned}
&-\int_{Q(T)} u(\eta_t + a\Delta\eta)dxdt + \int_{E_n} v\frac{\partial \eta}{\partial \nu} d\Gamma dt + \int_{E_n} Gv\eta d\Gamma dt + \int_{E_e} \xi\eta d\Gamma dt \\
&= 0 \quad \text{for any } \eta \in \mathcal{L}(Q(T)). \qquad (2.10)
\end{aligned}$$

By virtue of (2.8) we can choose the sequence $\{a_k\} \subset C^\infty(\overline{Q(T)})$ satisfying

$$|a_k - a|_{L^2(Q)} \leq C_0 k^{-1} \text{ and } a_k \geq k^{-1} \quad \text{for each } k,$$

where C_0 is some positive constant.

Now, we consider the behavior of v on the boundary $\Sigma(T)$. It is clear that there are the following three possibilities for v on $\Sigma(T)$:

(C1) There exists some positive number T_0 such that $v(t,x) = 0$ a.e. on $\Sigma(T_0)$;

(C2) there exists some positive number T_0 such that $v(t,x) \neq 0$ a.e. on $\Sigma(T_0)$;

(C3) otherwise, that is, for some positive number T_0 $meas\, E_e(t) > 0$ and $meas\, E_n(t) > 0$ for a.e. $t \in [0, T_0]$.

First, we shall show that the case (C3) never occurs in Section 4 and the both cases (C1) and (C2) imply $u = 0$ a.e. on $Q(T_0)$ in Section 5. This means that Theorem 2.2 is true. In order to prove the above facts we prepare some lemmas on uniform estimates for solutions of the approximation problems for the dual problems of SP introduced in Section 3.

3. APPROXIMATION OF THE DUAL PROBLEMS

In this section we investigate the three approximate problems of the dual problems for SP. First, we denote by $\mathrm{DP}_3(\Gamma^0, \Gamma^1, T; b; \zeta; g)$ the following initial boundary value problem:

$$
\begin{aligned}
& z_t + b\Delta z = \zeta \quad \text{in } Q(T),\\
& \frac{\partial z}{\partial \nu} + gz = 0 \quad \text{on } \Sigma^0(T) := (0,T) \times \Gamma^0,\\
& z = 0 \quad \text{on } \Sigma^1(T) := (0,T) \times \Gamma^1,\\
& z(T,x) = 0 \quad \text{for } x \in \Omega,
\end{aligned}
$$

where Γ^0 and Γ^1 are subsets of Γ, measurable and disjoint; $b \in C^\infty(\overline{Q(T)})$ and b is strictly positive on $\overline{Q(T)}$; ζ and g are given functions on $Q(T)$ and $\Sigma^0(T)$, respectively.

Next, the second problem $\mathrm{DP}_2(T; b; \zeta; g)$ is as follows:

$$
\begin{aligned}
& z_t + b\Delta z = \zeta \quad \text{in } Q(T),\\
& \frac{\partial z}{\partial \nu} + gz = 0 \quad \text{on } \Sigma(T) := (0,T) \times \Gamma,\\
& z(T,x) = 0 \quad \text{for } x \in \Omega.
\end{aligned}
$$

Furthermore, we consider the following problem $\mathrm{DP}_1(T; b; \zeta)$:

$$
\begin{aligned}
& z_t + b\Delta z = \zeta \quad \text{in } Q(T),\\
& z = 0 \quad \text{on } \Sigma(T),\\
& z(T,x) = 0 \quad \text{for } x \in \Omega.
\end{aligned}
$$

Remark 3.1. Putting $w(t,x) = z(T-t,x)$, z is a solution of DP_3 if and only if w satisfies the following conditions:

$$w_t - \hat{b}\Delta w = \hat{\zeta} \quad \text{in } Q(T), \tag{3.1}$$

$$\frac{\partial w}{\partial \nu} + \hat{g}w = 0 \quad \text{on } \Sigma^0(T), \tag{3.2}$$

$$w = 0 \quad \text{on } \Sigma^1(T), \tag{3.3}$$

$$w(0,x) = 0 \quad \text{for } x \in \Omega, \tag{3.4}$$

where $\hat{b}(t,x) = b(T-t,x)$, $\hat{\zeta}(t,x) = -\zeta(T-t,x)$ and $\hat{g}(t,x) = g(T-t,x)$.

Of course, this remark holds for DP_1 and DP_2.

We give a lemma which guarantees the existence and uniqueness of solutions of DP_1, DP_2 and DP_3.

Lemma 3.1. (cf. [2, Appendix]) *Assume that $b \in C^\infty(\overline{Q(T)})$ and $b \geq \mu > 0$ on $Q(T)$ for some positive constant μ, and $\zeta \in L^2(Q(T))$. Then the following existence and uniqueness results hold.*

(i) If Γ^0 and Γ^1 are smooth, and meas $\Gamma^0 > 0$ and meas $\Gamma^1 > 0$, $g \in C_0^\infty(\overline{\Sigma^0(T)})$ and g is non-negative on $\Sigma^0(T)$, then there exists one and only one solution z of DP_3 such that $z \in \mathcal{L}(Q(T))$.

(ii) If $g \in C^\infty(\overline{\Sigma(T)})$ and $g \geq \delta$ on $\Sigma(T)$ for some positive number δ, then DP_2 has a unique solution $z \in \mathcal{L}(Q(T))$.

(iii) DP_1 admits a unique solution $z \in \mathcal{L}(Q(T))$.

Lemma 3.2. *Let $\zeta \in L^\infty(Q(T))$.*

(1) *Under the same assumptions as in Lemma* 3.1(*i*) *let z be a solution of $DP_1(\Gamma^0, \Gamma^1, T; b; \zeta; g)$. Then there exists a positive number K_1 depending only on T and $|\zeta|_{L^\infty(Q(T))}$ such that*

$$|z|_{L^\infty(Q(T))} \leq K_1. \tag{3.5}$$

(2) *Under the same assumptions as in Lemma* 3.1(*ii*) (*resp.* (*iii*)) *let z be a solution of $DP_2(T; b; \zeta; g)$ (resp. $DP_1(T; b; \zeta)$). Then there exists a positive number K_1 depending only on T and $|\zeta|_{L^\infty(Q(T))}$ satisfying* (3.5).

Proof. First, we prove the assertion (1) in this lemma. We put $w(t,x) = z(T-t,x)$ on $Q(T)$. By Remark 3.1, w satisfies (3.1) $\sim$ (3.4).

Let M be any positive number and $y(t,x) = M(t+1)$ for $(t,x) \in Q(T)$. We see that

$$\int_\Omega w_t(t)[w(t) - y(t)]^+ dx = \int_\Omega (\hat{b}(t)\Delta w(t) + \hat{\zeta}(t))[w(t) - y(t)]^+ dx$$

$$= -\int_\Omega \nabla w(t)\cdot\nabla(\hat{b}(t)[w(t)-y(t)]^+)dx + \int_\Gamma \frac{\partial w(t)}{\partial\nu}\hat{b}(t)[w(t)-y(t)]^+ d\Gamma$$
$$+ \int_\Omega \hat{\zeta}(t)[w(t)-y(t)]^+ dx =: I_1(t)+I_2(t)+I_3(t) \quad \text{for a.e. } t\in[0,T].$$

Immediately, we obtain that for a.e. $t \in [0,T]$

$$\begin{aligned}
I_1(t) &= -\int_\Omega (\nabla w(t)\cdot\nabla\hat{b}(t))[w(t)-y(t)]^+ dx \\
&- \int_\Omega \hat{b}(t)(\nabla w(t)\cdot\nabla[w(t)-y(t)]^+)dx \\
&= -\int_\Omega (\nabla[w(t)-y(t)]^+\cdot\nabla\hat{b}(t))[w(t)-y(t)]^+ dx \\
&- \int_\Omega \hat{b}(t)|\nabla[w(t)-y(t)]^+|^2 dx \\
&\le \frac{\mu}{2}\int_\Omega |\nabla[w(t)-y(t)]^+|^2 dx + \frac{1}{2\mu}\int_\Omega |\nabla\hat{b}(t)|^2|[w(t)-y(t)]^+|^2 dx \\
&- \mu\int_\Omega (\nabla[w(t)-y(t)]^+)^2 dx \\
&= \frac{1}{2\mu}\int_\Omega |\nabla\hat{b}(t)|^2|[w(t)-y(t)]^+|^2 dx - \frac{\mu}{2}\int_\Omega |\nabla[w(t)-y(t)]^+|^2 dx; \\
I_2(t) &= \int_{\Gamma^0}\frac{\partial w(t)}{\partial\nu}\hat{b}(t)[w(t)-y(t)]^+ d\Gamma + \int_{\Gamma^1}\frac{\partial w(t)}{\partial\nu}\hat{b}(t)[w(t)-y(t)]^+ d\Gamma \\
&= -\int_{\Gamma^0}\hat{b}(t)\hat{g}(t)w(t)[w(t)-y(t)]^+ d\Gamma + \int_{\Gamma^1}\frac{\partial w(t)}{\partial\nu}\hat{b}(t)[-y(t)]^+ d\Gamma \\
&= -\int_{\Gamma^0}\hat{b}(t)\hat{g}(t)(w(t)-y(t))[w(t)-y(t)]^+ d\Gamma \\
&- \int_{\Gamma^0}\hat{b}(t)\hat{g}(t)y(t)[w(t)-y(t)]^+ d\Gamma \le 0; \\
I_3(t) &= \int_\Omega (\hat{\zeta}(t)-M)[w(t)-y(t)]^+ dx + M\int_\Omega [w(t)-y(t)]^+ dx
\end{aligned}$$

for a.e. $t \in [0,T]$. On the other hand, we have, for a.e. $t \in [0,T]$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega ([w(t)-y(t)]^+)^2 dx = \int_\Omega (w_t(t)-M)[w(t)-y(t)]^+ dx.$$

Therefore, from the above inequalities it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega([w(t)-y(t)]^+)^2dx+\frac{\mu}{2}\int_\Omega|\nabla[w(t)-y(t)]^+|^2dx\\
\le\ &\frac{1}{2\mu}\int_\Omega|\nabla\hat b(t)|^2|[w(t)-y(t)]^+|^2dx+\int_\Omega(\hat\zeta(t)-M)[w(t)-y(t)]^+dx\\
\le\ &\frac{1}{2\mu}|\nabla\hat b|^2_{L^\infty(Q(T))}\int_\Omega|[w(t)-y(t)]^+|^2dx\\
&+\int_\Omega(\hat\zeta(t)-M)[w(t)-y(t)]^+dx\quad\text{ for a.e. } t\in[0,T].
\end{aligned}$$

Here, we take a number M satisfying $|\zeta|_{L^\infty(Q(T))}\le M$ so that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega([w(t)-y(t)]^+)^2dx\le\frac{1}{2\mu}|\nabla\hat b|^2_{L^\infty(Q(T))}\int_\Omega|[w(t)-y(t)]^+|^2dx$$

for a.e. $t\in[0,T]$. By applying Gronwall's inequality we infer that

$$\int_\Omega|[w(t)-y(t)]^+|^2dx\le 0\quad\text{ for any } t\in[0,T]$$

since $w(0)=0$. Hence, we have $w\le y\le M(T+1)$ on $Q(T)$. Similarly, we can obtain that $w\ge -M(T+1)$ on $Q(T)$. We can prove (2) in a similar way to that of the above one. □

Lemma 3.3. *Assume that all the assumptions in Lemma 3.2 hold and $\zeta\in C_0^\infty(Q(T))$. Let z be a solution of $DP_1(\Gamma^0,\Gamma^1;T;b;\zeta;g)$. Then there exists a positive constant $K_2=K_2(K_1,|\Delta\zeta|_{L^1(Q(T))},|\hat g|_{L^\infty(\Sigma^0(T))},|\hat g_t|_{L^\infty(\Sigma^0(T))})$ such that*

$$|\Delta z|_{L^2(Q(T))}\le\frac{K_2}{\sqrt{\mu}}.\tag{3.6}$$

Also, for DP_1 and DP_2 the similar estimate holds.

Proof. We use the same transformation and notations as in the proof of Lemma 3.2.

$$\begin{aligned}
\int_{Q(T)}\hat b(t)(\Delta w(t))^2dxdt&=\int_{Q(T)}(w_t(t)-\hat\zeta(t))\Delta w(t)dx\,dt\\
&=\int_{Q(T)}w_t(t)\Delta w(t)dx\,dt-\int_{Q(T)}\hat\zeta(t)\Delta w(t)dx\,dt=:J_1-J_2.
\end{aligned}$$

Put $w(t)=0$ and $\hat g(t)=\hat g(0)$ for $t<0$. Since $w_t\in L^2(Q(T))$ we have

$$J_1^h:=\int_{Q(T)}\frac{w(t)-w(t-h)}{h}\Delta w(t)dx\,dt\to J_1\quad\text{ as } h\downarrow 0.$$

Here, it is easy to see that

$$\begin{aligned}
J_1^h &= -\int_{Q(T)} \frac{\nabla w(t) - \nabla w(t-h)}{h} \nabla w(t) dxdt \\
&+ \int_{\Sigma(T)} \frac{w(t) - w(t-h)}{h} \frac{\partial w(t)}{\partial \nu} d\Gamma dt \\
&\leq \frac{1}{2h} \int_{Q(T)} |\nabla w(t-h)|^2 dxdt - \frac{1}{2h} \int_{Q(T)} |\nabla w(t)|^2 dxdt \\
&- \int_{\Sigma^0(T)} \frac{w(t) - w(t-h)}{h} \hat{g}(t) w(t) d\Gamma dt \\
&\leq -\frac{1}{2h} \int_{\Sigma^0(T)} \hat{g}(t) |w(t)|^2 d\Gamma dt + \frac{1}{2h} \int_{\Sigma^0(T)} \hat{g}(t-h) |w(t-h)|^2 d\Gamma dt \\
&+ \frac{1}{2h} \int_{\Sigma^0(T)} (\hat{g}(t-h) - \hat{g}(t)) |w(t-h)|^2 d\Gamma dt \\
&\leq \tfrac{1}{2} |w|^2_{L^\infty(Q(T))} |\hat{g}|_{L^\infty(\Sigma^0(T))} meas\, \Gamma + \tfrac{1}{2} |w|^2_{L^\infty(Q(T))} |\hat{g}_t|_{L^\infty(\Sigma^0(T))} meas\, \Sigma(T)
\end{aligned}$$

for $h > 0$. Letting $h \downarrow 0$ in the above inequality we have

$$J_1 \leq \frac{K_1^2}{2} (|g_t|_{L^\infty(\Sigma^0(T))} meas\, \Sigma^0(T) + |\hat{g}|_{L^\infty(\Sigma^0(T))} meas\, \Gamma),$$

where K_1 is a positive constant given by Lemma 3.2. Also, we observe that

$$J_2 = \int_{Q(T)} w \Delta \hat{\zeta} dxdt \leq K_1 |\Delta \zeta|_{L^1(Q(T))}.$$

Therefore, we conclude that

$$\begin{aligned}
&\mu \int_{Q(T)} |\Delta w|^2 dxdt \\
&\leq \frac{K_1^2}{2} (|g_t|_{L^\infty(\Sigma(T))} meas \Sigma(T) + |g|_{L^\infty(\Sigma(T))} meas \Gamma) + K_1 |\Delta \zeta|_{L^1(Q(T))} =: K_3,
\end{aligned}$$

that is, $|\Delta w|_{L^2(Q(T))} \leq \sqrt{K_3}/\sqrt{\mu}$. Easily, we can get the similar estimates for the solutions of DP_1 and DP_2. □

4. CASE (C3)

Throughout this section we shall use the notations defined in previous sections. The purpose of this section is to prove the following proposition.

Proposition 4.1. *If all the assumptions as in Theorem* 2.2 *hold, then the case* $(C3)$ *never occurs, that is,* $v = 0$ *a.e. on* $\Sigma(T_0)$. *Moreover, we have* $u = 0$ *a.e. on* $Q(T_0)$.

For the proof of Proposition 4.1 we prepare Lemma 4.1. It is important to treat the smoothness of the boundary in the proof of Proposition 4.1. According to the book by Necăs [15] the boundary Γ is smooth so that there exist the sets $\{\Gamma^{(i)}\}_{i=1}^{r}$ of subsets of Γ, $\{C_i\}_{i=1}^{r}$ of (open) cubes in $\mathbb{R}^{N-1}$ and $\{\Phi_i\}_{i=1}^{r}$ of smooth functions on $\overline{C}_i$, such that $\Gamma = \cup_{i=1}^{r}\Gamma^{(i)}$, $\Phi_i : C_i \to \Gamma^{(i)}$ is homeomorphism and $\Gamma = \cup_{i=1}^{r}\Phi_i(C_i)$. Obviously, the set $U \subset \Gamma$ is open if and only if

$$U = \bigcup_{1\le i\le r, \lambda\in\Lambda_i} \Phi_i(B_\lambda^{(i)}),$$

where $B_\lambda^{(i)}$ is an open ball in $\mathbb{R}^{N-1}$ and Λ_i is the set of indices λ for each i.

Lemma 4.1. *If the condition* $(C3)$ *holds, then for any positive number* ε *there exist positive constant* T_1 *and a subset* $\Gamma^\varepsilon \subset \Gamma$ *such that* $\Sigma^\varepsilon := \Gamma^\varepsilon \times (0, T_1) \supset E_n$, $\Gamma^\varepsilon = \cup_{1\le j\le j_0} B_j$ *where* B_j $(j = 1, 2, \cdots, j_0)$ *is the image of the open ball in* $\mathbb{R}^{N-1}$ *by some* Φ_i, *and*

$$meas(\Sigma^\varepsilon - (E_n \cap \{0 < t < T_1\})) < \varepsilon.$$

Proof. In this proof we denote by $\overline{K}$ the closure of the subset $K \subset \Sigma(T)$. Let ε be any positive number. Then the measurability of E_n implies that there exists an open set U_0 with $E_n \subset U_0 \subset \Sigma(T)$ such that $meas(U_0 - E_n) < \varepsilon$, moreover, there exists an open set U_1 with $\overline{U}_0 \subset U_1 \subset \Sigma(T)$ such that $meas(U_1 - \overline{U}_0) < \varepsilon$. Now, by virtue of the compactness of $\overline{U}_0$ we can choose the sets $\{B_j\}_{j=1}^{j_1}$ of the images of some open ball in $\mathbb{R}^{N-1}$ by some Φ_i and $\{(s_j, t_j)\}_{j=1}^{j_1}$ of the open subintervals of $(0, T)$ where j_1 is a finite number satisfying

$$\bar{U}_0 \subset \bigcup_{j=1}^{j_1} B_j \times (s_j, t_j) \subset U_1.$$

Clearly, we have

$$meas((\bigcup_{j=1}^{j_1} B_j \times (s_j, t_j)) - E_n) < 2\varepsilon.$$

Next, noting that by (C3) there exists at least one s_j such that $s_j = 0$, for such j we write B_j and $(0, t_j)$, $j = 1, 2, \cdots, j_0$, again. We put $T_1 =$

$\min\{t_1, t_2, \cdots, t_{j_0}\}$, $\Gamma^\varepsilon = \cup_{j=1}^{j_0} B_j$ and $\Sigma^\varepsilon = \Gamma \times (0, T_1)$. Therefore, Γ^ε and T_1 are the required ones in this lemma. □

Lemma 4.2. *Under the same assumptions and notations as in Proposition 4.1, we can choose a sequence* $\{G_m\} \subset C^\infty(\overline{\Sigma(T_1)})$ *satisfying that* $G_m \geq 0$ *on* $\Sigma(T_1)$ *and* $G_m \in C_0^\infty(\Sigma_\varepsilon^0(T_1))$ *for each* m*, and* $G_m v \to Gv$ *in* $L^2(\Sigma(T_1))$ *as* $m \to \infty$.

Proof. By the definition of G we have $G = 0$ on E_e and $G \geq 0$ on E_n. Since G is measurable, there exists a sequence $\{G^{(k)}\}$ of step functions on $\Sigma(T_1)$ such that $0 \leq G^{(k)} \leq G$ on $\Sigma(T_1)$ and $G^{(k)}(t,x) \to G(t,x)$ as $k \to \infty$ for a.e. $(t,x) \in \Sigma(T_1)$. Lebesgue dominate convergence theorem and (2.9) imply that $G^{(k)} v \to Gv$ in $L^2(\Sigma(T_1))$ as $k \to \infty$. Furthermore, for each k we can take a smooth function $G_1^{(k)}$ such that $|G_1^{(k)} - G^{(k)}|_{L^2(\Sigma(T_1))} < \frac{1}{k}$ and $supp G_1^{(k)} \subset \Sigma_\varepsilon^0(T_1)$ since $E_n \cap \{0 < t < T_1\} \subset \overline{U}_0 \cap \{0 < t < T_1\} \subset \Sigma_\varepsilon^0(T_1)$. Thus, $\{G_1^{(k)}\}$ satisfies all the required conditions. □

Proof of Proposition of 4.1. Let T_1 be a positive constant, $\Gamma_\varepsilon^0 = \Gamma^\varepsilon$ and $\Gamma_\varepsilon^1 = \Gamma - \Gamma^\varepsilon$ defined by Lemma 4.1, and $\zeta \in C_0^\infty(\overline{Q(T_1)})$. Clearly, Γ_ε^0 and Γ_ε^1 are smooth. Here, we put $\Sigma_\varepsilon^0(T_1) := \Gamma_\varepsilon^0 \times (0, T_1)$ and $\Sigma_\varepsilon^1(T_1) := \Gamma_\varepsilon^1 \times (0, T_1)$. The definition of $\Sigma_\varepsilon^1(T_1)$ guarantees that $meas((E_e \cap \{0 < t < T\}) - \Sigma_\varepsilon^1(T_1)) < \varepsilon$. Also, we can take a sequence $\{G_m\} \subset C^\infty(\overline{\Sigma(T_1)})$ satisfying that $G_m \geq 0$ on $\Sigma(T_1)$ and $G_m \in C_0^\infty(\overline{\Sigma_\varepsilon^0(T_1)})$ for each m, and $G_m v \to Gv$ in $L^2(\Sigma(T_1))$ as $m \to \infty$.

For $\varepsilon > 0$ and each k and m let $z_{k,\varepsilon,m}$ be a solution of $\mathrm{DP}_3(\Gamma_\varepsilon^0, \Gamma_\varepsilon^1, T_1; a_k; \zeta; G_m)$. By Lemmas 3.1, 3.2 and 3.3 we infer that $z_{k,\varepsilon,m} \in \mathcal{L}(Q(T_1))$ and there exist positive constants M_1 independent of ε, k, m and $M_2(\varepsilon, m)$ such that

$$\left.\begin{array}{l} |z_{k,\varepsilon,m}|_{L^\infty(Q(T_1))} \leq M_1 \\ |\Delta z_{k,\varepsilon,m}|_{L^2(Q(T_1))} \leq M_2(\varepsilon, m)\sqrt{k} \end{array}\right\} \text{ for any } \varepsilon \in (0,1] \text{ and each } k, m. \tag{4.1}$$

For simplicity, we put $z = z_{k,\varepsilon,m}$, $E_n' = E_n \cap \{0 < t < T_1\}$ and $E_e' = E_e \cap \{0 < t < T_1\}$. Then substituting z in (2.10), we observe that

$$-\int_{Q(T_1)} u(z_t + a\Delta z)dxdt + \int_{E_n'} v\frac{\partial z}{\partial \nu} d\Gamma dt + \int_{E_n'} Gvz d\Gamma dt + \int_{E_e'} \xi z d\Gamma dt = 0.$$

Accordingly, we obtain

$$|\int_{Q(T_1)} u\zeta dxdt| = |\int_{Q(T_1)} u(z_t + a_k\Delta z)dxdt - \int_{Q(T_1)} u(z_t + a\Delta z)dxdt$$

$$+\int_{E_n'} v\frac{\partial z}{\partial\nu}d\Gamma dt+\int_{E_e'}\xi zd\Gamma dt+\int_{E_n'}Gvzd\Gamma dt-\int_{\Sigma_\varepsilon^0(T_1)}v(\frac{\partial z}{\partial\nu}+G_m)d\Gamma dt|$$
$$\le\int_{Q(T_1)}|a-a_k||u||\Delta z|dxdt+|\int_{\Sigma_\varepsilon^0(T_1)-E_n'}v(\frac{\partial z}{\partial\nu}+G_mz)d\Gamma dt|$$
$$+\int_{E_n'}|Gv-G_mv||z|d\Gamma dt+\int_{E_e'-\Sigma_\varepsilon^1(T_1)}|\xi z|d\Gamma dt.$$

It from (4.1) follows that

$$|\int_{Q(T_1)}u\zeta dxdt|\le|u|_{L^\infty(Q(T_1))}|a-a_k|_{L^2(Q(T_1))}|\Delta z|_{L^2(Q(T_1))}$$
$$+|Gv-G_mv|_{L^2(\Sigma(T_1))}|z|_{L^2(\Sigma(T_1))}+|\xi|_{L^2(\Sigma)}|z|_{L^\infty(\Sigma)}meas(E_e'-\Sigma_\varepsilon^1(T_1))^{1/2}$$
$$\le|u|_{L^\infty(Q(T_1))}C_0M_2(\varepsilon,m)k^{-1/2}$$
$$+M_1meas\Sigma(T_1)|Gv-G_mv|_{L^2(\Sigma(T_1))}+M_1|\xi|_{L^2(\Sigma(T_1))}\varepsilon^{1/2}.$$

This implies that $\int_{Q(T_1)}u\zeta dxdt=0$ for any $\eta\in C_0^\infty(\overline{Q(T_1)})$, that is, $u=0$ a.e. on $Q(T_1)$. This is a contradiction. Hence, the case (C3) never occurs.

5. Proof of the uniqueness

In this section we give a complete proof of Theorem 2.2. First, we show that the case (C2) never occurs.

Proposition 5.1. *If the condition* $(C2)$ *holds, then* $u=0$ *a.e. on* $Q(T_0)$. *This means that the case* $(C2)$ *never occurs.*

Proof. We assume that (C2) holds. Then we see that $meas(E_e\cap\{0<t<T_0\})=0$ and $Gv=\xi$, $G\ge 0$ a.e. on $\Sigma(T_0)$. Hence, similarly to Lemma 4.2 we can choose a sequence $\{G_m\}$ of smooth functions such that $G_m\ge\frac{1}{m}$ on $\Sigma(T_0)$ and $G_mv\to Gv$ in $L^2(\Sigma(T_0))$ as $m\to\infty$. Let $\zeta\in C_0^\infty(\overline{Q(T_0)})$ and $z_{k,m}$ be a solution of $\mathrm{DP}_2(T_0;a_k;\zeta;G_m)$ for each k and m. Lemmas 3.1 $\sim$ 3.3 imply that $z_{k,m}\in\mathcal{L}(Q(T_0))$ and there are positive constants L_1 and $L_2(m)$ such that

$$|z_{k,m}|_{L^\infty(Q(T_0))}\le L_1\text{ and }|\Delta z_{k,m}|_{L^2(Q(T_0))}\le L_2(m)\sqrt{k}\quad\text{for each }k\text{ and }m.$$

Now, by substituting $z:=z_{k,m}$ as η in (2.10) we infer that

$$-\int_{Q(T_0)}u(z_t+a\Delta z)dxdt+\int_{\Sigma(T_0)}v\frac{\partial z}{\partial\nu}d\Gamma dt+\int_{\Sigma(T_0)}Gvzd\Gamma dt=0$$

for each k and m. Therefore, we have

$$
\begin{aligned}
&|\int_{Q(T_0)} u\zeta dxdt| = |\int_{Q(T_0)} u(z_t + a_k\Delta z)dxdt - \int_{Q(T_0)} u(z_t + a\Delta z)dxdt \\
&+ \int_{\Sigma(T_0)} Gvzd\Gamma dt + \int_{\Sigma(T_0)} v\frac{\partial z}{\partial \nu}d\Gamma dt - \int_{\Sigma(T_0)} v(\frac{\partial z}{\partial \nu} + G_m z)d\Gamma dt| \\
&\le \int_{Q(T_0)} |u||a_k - a||\Delta z|dxdt + \int_{\Sigma(T_0)} |G_m v - Gv||z|d\Gamma dt \\
&\le C_0|u|_{L^\infty(Q(T_0))}\frac{L_2(m)}{\sqrt{k}} + L_1\sqrt{meas\Sigma(T_0)}|G_m v - Gv|_{L^2(\Sigma(T_0))}
\end{aligned}
$$

for each k and m. Accordingly, we obtain that $\int_{Q(T_0)} u\zeta dxdt = 0$ for all $\zeta \in C_0^\infty(\overline{Q(T_0)})$ so that $u = 0$ a.e. on $Q(T_0)$. Here, we get the conclusion of this proposition. □

Proof of Theorem 2.2. By virtue of Propositions 4.1 and 5.1, we have $v = 0$ a.e. on $\Sigma(T_0)$ for some positive constant T_0. Let $\zeta \in C^\infty(\overline{Q(T_0)})$, and z_k be a solution of $DP_2(T_0; a_k; \zeta)$ for each k. Then Lemmas 3.1 ∼ 3.3 guarantee that $z_k \in \mathcal{L}(Q(T_0))$ and there exists a positive constant C_1 independent of k such that

$$|z_k|_{L^\infty(Q(T_0))} \le C_1 \text{ and } |\Delta z_k|_{L^2(Q(T_0))} \le C_1\sqrt{k} \quad \text{for each } k.$$

Then taking z_k as η in (2.10) we observe that

$$-\int_{Q(T_0)} u(z_{kt} + a\Delta z_k)dxdt = 0 \quad \text{for each } k,$$

since $meas(E_n \cap \{0 < t < T_0\}) = 0$ and $z_k = 0$ on $\Sigma(T_0)$ for all k.

Therefore, it holds that

$$
\begin{aligned}
&|\int_{Q(T_0)} u\zeta dxdt| = |\int_{Q(T_0)} u(z_{kt} + a_k\Delta z_k)dxdt - \int_{Q(T_0)} u(z_{kt} + a\Delta z_k)dxdt| \\
&\le \int_{Q(T_0)} |a - a_k||\Delta z_k||u|dxdt \le |a - a_k|_{L^2(Q(T_0))}|\Delta z_k|_{L^2(Q(T_0))}|u|_{L^\infty(Q(T_0))} \\
&\le C_0C_1|u|_{L^\infty(Q(T_0))}k^{-1/2} \quad \text{for each } k.
\end{aligned}
$$

This implies that $\int_{Q(T_0)} u\zeta dxdt = 0$, that is, $u = 0$ a.e. on $Q(T_0)$.

Next, we assume that there are numbers t_0, t_1 with $0 < t_0 < t_1 \le T$ such that $meas(\cup_{t_0 \le t \le t_1} E_n(t)) > 0$. Then from Propositions 4.1 and 5.1 and $u(t_0) = 0$ it follows that $v = 0$ a.e. on $\Gamma \times (t_0, t_1)$ so that $v = 0$ a.e. on

$\Sigma(T)$. By the above argument we conclude that $u = 0$ a.e. on $Q(T)$. Thus, we have proved Theorem 2.2.

6. Approximate problems

From now on, we shall discuss the existence of solutions of SP. The purpose of this section is to obtain the existence proposition and uniform estimates for solutions of the approximate problems $(\mathrm{SP})_{\lambda,\mu}$ for $\lambda > 0$ and $\mu > 0$.

Proposition 6.1. *If the assumption* $(H1)$ *holds, then for each* $\lambda > 0$ *and* $\mu > 0$ $(SP)_{\lambda,\mu}$ *has a unique solution. Precisely, there exists one and only one function* $u_{\lambda,\mu}$ *on* $Q(T)$ *satisfying* $u_{\lambda,\mu} \in W^{1,2}(0,T;H)$, $\beta_\mu(u_{\lambda,\mu}) \in L^\infty(0,T;X)$ *and* (2.1)–(2.3) *in the usual sense.*

By applying the theory for the subdifferential operator we prove this proposition so that we introduce a family $\{\varphi^t_\lambda\}$, $\lambda > 0$, of functions on H formulated by

$$\varphi^t_\lambda(z) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx + \sum_{i=0}^{1} \int_{\Gamma_i} \gamma_{i\lambda}(z - g(t))d\Gamma & \text{if } z \in X, \\ \infty & \text{otherwise.} \end{cases}$$

As to the family $\{\varphi^t_\lambda\}$ we have:

Lemma 6.1. (cf. [10, Chapter 3]) (1) *For each* $t \in [0,T]$ *and* $\lambda > 0$ φ^t_λ *is a proper l.s.c. and convex function on* H.

(2) *For each* $t \in [0,T]$ $\lambda > 0$ *and* $r \geq 0$ *the set* $\{z \in H; |z|_H \leq r, |\varphi^t_\lambda(z)| \leq r\}$ *is relatively compact in* H.

(3) *For any* $\lambda > 0$ *there exists a positive constant* K_λ *such that*

$$\varphi^t_\lambda(z) - \varphi^s_\lambda(z) \leq K_\lambda(|z|_H + |\varphi^t_\lambda(z)| + 1)|g(t) - g(s)|_X$$

for $0 \leq s \leq t \leq T$ *and* $z \in X$.

(4) *For each* $t \in [0,T]$ *the subdifferential of* φ^t_λ *is single-valued and characterized as follows:* $z^* = \partial\varphi^t_\lambda(z)$ *if and only if* $z^* \in H$, $z \in X$ *and*

$$(z^*, \eta) = A(z, \eta) + \sum_{i=0}^{1} (\partial\gamma_{i\lambda}(z - g(t)), \eta)_i \quad \text{for } \eta \in X.$$

Next, for $\mu > 0$ we denote by B_μ the operator from $D(B_\mu) = H$ into itself given by $B_\mu(z)(x) = \beta_\mu(z(x))$ for $x \in \Omega$ and $z \in H$. Clearly, B_μ is bi-Lipschitz continuous on H, and is the subdifferential of the convex function

j_μ given by

$$j_\mu(z) = \int_\Omega \hat{\beta}_\mu(z(x))dx \quad \text{for } z \in H,$$

where $\hat{\beta}_\mu(r) = \int_0^r \beta_\mu(s)ds$. Also, we put $\theta(z) = \int_\Omega z^+ dx$ for $z \in H$.

Lemma 6.2. (1) $\theta(z) + \theta(-z) = |z|_{L^1(\Omega)}$ *for* $z \in H$.

(2) *For each* $t \in [0, T]$, *if* $z_i \in H$ *with* $B_\mu z_i \in D(\partial\varphi_\lambda^t)$, *then*

$$(\partial\varphi_\lambda^t(B_\mu z_1) - \partial\varphi_\lambda^t(B_\mu z_2), w) \geq 0 \textit{ for some } w \in \partial\theta(z_1 - z_2),$$

where $\partial\theta$ *is the subdifferential of* θ *in* H.

This lemma is quite standard, so we omit the proof (cf. Chapter 3 in [10]). Here, we give a proof of Proposition 6.1.

Proof of Proposition 6.1. First, we consider the following Cauchy problem for evolution equations

$$\frac{du_{\lambda,\mu}(t)}{dt} + \partial\varphi_\lambda^t(B_\mu u_{\lambda,\mu}(t)) = f(t) \quad \text{in } H \text{ for a.e. } t \in [0, T],$$
$$u_{\lambda,\mu}(0) = u_{0,\mu}.$$

Then Lemmas 6.1 and 6.2 allow us to apply [10, Theorems 2.8.1 and 2.8.2] to the above problem, and we infer that the problem has a unique solution $u_{\lambda,\mu}$ in $W^{1,2}(0, T; H)$ such that $t \to \varphi_\lambda^t(B_\mu u_{\lambda,\mu}(t))$ is bounded on $[0, T]$. Hence, $u_{\lambda,\mu}$ satisfies the required conditions in the proposition. □

Next, the uniform estimates are obtained by some routine works.

Proposition 6.2. *Suppose the same assumptions as in Proposition* 6.1. *For* $\lambda \in (0, 1]$, $\mu \in (0, \mu_0]$ $(\mu_0 = \min\{\frac{M_\beta}{2}, 1\})$ *let* $u_{\lambda,\mu}$ *be a solution of (SP)*$_{\lambda,\mu}$ *in the sense of Proposition* 6.1. *Then there exists a positive constant* K_0 *depending only on* C_β, M_β, m_β, $|g|_{H^1(Q(T))}$, $|f|_{L^2(Q(T))}$, γ_i, $i = 0, 1$ *(independent of* λ *and* μ*) such that*

$$|u_{\lambda,\mu}(t)|_H \leq K_0 \quad \textit{for any } t \in [0, T], \tag{6.1}$$

$$|\beta_\mu(u_{\lambda,\mu})|_{L^2(0,T;X)} \leq K_0, \tag{6.2}$$

$$\int_{\Sigma_i(T)} \gamma_{i\lambda}(\beta_\mu(u_{\lambda,\mu})(t) - g(t))d\Gamma dt \leq K_0 \quad \textit{for } i = 0, 1. \tag{6.3}$$

Proof. First, it is obvious that

$$\frac{1}{4}M_\beta r^2 - m_\beta^2 \leq \hat{\beta}_\mu(r) \quad \text{for any } r \in \mathbb{R} \text{ and } \mu \in (0, 1]. \tag{6.4}$$

Also, putting $\mu_0 = \min\{\frac{M_\beta}{2}, 1\}$ we have $\{u_{0,\mu}; 0 < \mu \le \mu_0\}$ is bounded in $L^\infty(\Omega)$. In fact, by definition $\beta_\mu(u_0) = v_0$, that is, with help of the assumption (A1-1) we see that

$$M_\beta |u_{0,\mu}| - m_\beta \le |v_0| + \mu |u_{0,\mu}| \quad \text{a.e. on } \Omega.$$

It follows that

$$|u_{0,\mu}(x)| \le \frac{2}{M_\beta}(|v_0(x)| + m_\beta) \quad \text{for a.e. } x \in \Omega \text{ and } 0 < \mu \le \mu_0. \tag{6.5}$$

For simplicity we write u for $u_{\lambda,\mu}$. Let c_i be a number with $c_i \in D(\gamma_i)$ for $i = 0, 1$. Now, we can take a function $V \in X$ such that $V = c_i$ on $\Gamma_i (i = 0, 1)$. By using this function V, we see that

$$\begin{aligned} &\int_\Omega u_t(t)(\beta_\mu(u)(t) - g(t) - V)dx \\ &= \frac{d}{dt}\int_\Omega \hat{\beta}_\mu(u)(t)dx - \frac{d}{dt}\int_\Omega u(t)(g(t) + V)dx + \int_\Omega u(t) g_t(t)dx \end{aligned} \tag{6.6}$$

for a.e. $t \in [0, T]$. On the other hand, we have

$$\begin{aligned} & (u_t, \beta_\mu(u) - g - V) \\ = \;& -A(\beta_\mu(u), \beta_\mu(u) - g - V) - \sum_{i=0}^{1}(\partial\gamma_{i\lambda}(\beta_\mu(u) - g), \beta_\mu(u) - g - V)_i \\ & +(f, \beta_\mu(u) - g - V) \\ \le \;& -\frac{1}{2}\int_\Omega |\nabla\beta_\mu(u)|^2 dx + \int_\Omega |\nabla g|^2 dx + \int_\Omega |\nabla V|^2 dx \\ & + \sum_{i=0}^{1}\int_{\Gamma_i}(\gamma_{i\lambda}(V) - \gamma_{i\lambda}(\beta_\mu(u) - g))d\Gamma + 2\int_\Omega (|f|^2 + |g|^2 + |V|^2)dx \\ & +(C_\beta + 1)^2 \int_\Omega |u|^2 dx \quad \text{a.e. on } [0, T]. \end{aligned} \tag{6.7}$$

By (6.6) and (6.7) we conclude that

$$\begin{aligned} & \frac{d}{dt}\int_\Omega \hat{\beta}_\mu(u)(t)dx - \frac{d}{dt}\int_\Omega u(t)(g(t) + V(t))dx \\ & + \sum_{i=0}^{1}\int_{\Gamma_i}\gamma_{i\lambda}(\beta_\mu(u)(t) - g(t))d\Gamma + \frac{1}{2}\int_\Omega |\nabla\beta_\mu(u)(t)|^2 dx \\ \le \;& C_1(|u(t)|_H^2 + |g_t(t)|_H^2 + |g(t)|_X^2 + |V|_X^2 + |f(t)|_H^2) \end{aligned} \tag{6.8}$$

$$+\sum_{i=0}^{1}\int_{\Gamma_i}\gamma_{i\lambda}(c_i)d\Gamma \quad \text{for a.e. } t\in[0,T],$$

where C_1 is some positive constant depending only on C_β.

Here, we note that by the nonlinear operator theory (cf. [10, Lemma 1.2.1]) there exists a positive constant κ such that

$$\gamma_{i\lambda}(r)+\kappa|r|+\kappa\geq 0 \quad \text{for } r\in\mathbb{R}, i=0,1 \text{ and } \lambda\in(0,1].$$

This implies

$$\begin{aligned}
&\sum_{i=0}^{1}\int_{\Gamma_i}\gamma_{i\lambda}(\beta_\mu(u)(t)-g(t))d\Gamma\\
\geq\ & -\sum_{i=0}^{1}\int_{\Gamma_i}\kappa(|\beta_\mu(u)(t)-g(t)|+1)d\Gamma \qquad (6.9)\\
\geq\ & -\frac{1}{4}\int_\Omega|\nabla\beta_\mu(u)(t)|^2dx+C_2(|u(t)|_H+|g(t)|_X+1) \quad \text{for a.e. } t\in[0,T],
\end{aligned}$$

where C_2 is a positive constant depending only on Ω, κ and C_β.

Therefore, it follows from (6.8) and (6.9) with help of (6.4) that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\hat{\beta}_\mu(u)(t)dx-\frac{d}{dt}\int_\Omega u(t)(g(t)+V)dx+\frac{1}{4}\int_\Omega|\nabla\beta_\mu(u)(t)|^2dx\\
\leq\ & C_3\int_\Omega\hat{\beta}_\mu(u)(t)dx+G_1(t) \quad \text{for a.e. } t\in[0,T], \qquad (6.10)
\end{aligned}$$

where $G_1(t)=C_3(|g(t)|_X^2+|g_t(t)|_H^2+|V|_X^2+|f(t)|_H^2+\sum_{i=0}^{1}\int_{\Gamma_i}\gamma_{i\lambda}(c_i)d\Gamma+1)$ for $t\in[0,T]$ and C_3 is a positive constant which depends on M_β, m_β, C_β, κ and Ω. Hence, it holds that

$$\frac{d}{dt}(e^{-C_3t}\int_\Omega\hat{\beta}_\mu(u)(t)dx)\leq e^{-C_3t}\frac{d}{dt}\int_\Omega u(t)(g(t)+V)dx+G_1(t)$$

for a.e. $t\in[0,T]$. This together with integration by parts yields

$$\begin{aligned}
&e^{-C_3t}\int_\Omega\hat{\beta}_\mu(u(t))dx\\
\leq\ & \int_\Omega\hat{\beta}_\mu(u_{0,\mu})dx+\int_0^t G_1(\tau)d\tau+\int_0^t e^{-C_3\tau}\frac{d}{d\tau}\int_\Omega u(\tau)(g(\tau)+V)dxd\tau\\
\leq\ & \int_\Omega\hat{\beta}_\mu(u_{0,\mu})dx+\int_0^t G_1(\tau)d\tau+C_3\int_0^t e^{-C_3\tau}\int_\Omega u(\tau)(g(\tau)+V)dxd\tau
\end{aligned}$$

$$
\begin{aligned}
&+|\int_\Omega u(t)(g(t)+V)dx| + |\int_\Omega u_{0,\mu}(g(0)+V)dx| \\
\leq\ & \int_\Omega \hat{\beta}_\mu(u_{0,\mu})dx + \frac{1}{2}\int_\Omega \hat{\beta}_\mu(u)(t)dx + \int_0^t \int_\Omega \hat{\beta}_\mu(u)(\tau)dxd\tau \\
&+C_4(|u_{0,\mu}|_H^2 + \int_0^t G_1(\tau)d\tau + |g(t)|_H^2 + |V|_H^2) \\
&+C_4\int_0^t (|g(\tau)|_H^2 + |V|_H^2)d\tau) \quad \text{for any } 0 \leq s \leq t \leq T,
\end{aligned}
$$

where C_4 is a suitable positive constant.

Then by applying Gronwall's inequality with help of (6.5) there exists a positive constant C_5 independent of λ and μ such that

$$\int_\Omega \hat{\beta}_\mu(u)(t)dx \leq C_5 \quad \text{for any } 0 \leq t \leq T.$$

Thus, we get (6.1). Also, (6.2) and (6.3) are direct consequences of (6.10) and (6.8). □

At the end of this section we show other uniform estimates.

Proposition 6.3. *Suppose that* $(H1)$ *holds. For* $\lambda \in (0,1]$, $\mu \in (0,\mu_0]$ *let* $u_{\lambda,\mu}$ *be a solution of* $(SP)_{\lambda,\mu}$ *in the sense of Proposition* 6.1. *Then there exists a positive constant* K_1 *independent of* $\lambda \in (0,1]$ *and* $\mu \in (0,\mu_0]$ *such that*

$$
\begin{aligned}
&\int_0^T |\beta_\mu(u_{\lambda,\mu})_t(t)|_H dt \leq K_1, \\
&\int_\Omega |\nabla \beta_\mu(u_{\lambda,\mu})(t)|^2 dx \leq K_1 \quad \text{for } t \in [0,T], \\
&\int_{\Gamma_i} \gamma_{i\lambda}(\beta_\mu(u_{\lambda,\mu})(t) - g(t))d\Gamma \leq K_1 \quad \text{for } t \in [0,T] \text{ and } i = 0,1.
\end{aligned}
$$

Proof. For simplicity we put $u = u_{\lambda,\mu}$. First, by (A1) we have

$$
\begin{aligned}
&\int_\Omega u_t(\beta_\mu(u)_t - g_t)dx \geq \frac{1}{C_\beta+1}\int_\Omega |\beta_\mu(u)_t|^2 dx - \frac{1}{C_\beta+1}|\beta_\mu(u)_t|_H |g_t|_H \\
\geq\ & \kappa_1|\beta_\mu(u)_t|_H^2 - C_6|g_t|_H^2 \quad \text{a.e. on } [0,T], \qquad (6.11)
\end{aligned}
$$

where $\kappa_1 = \frac{1}{2(C_\beta+1)}$ and $C_6 = \frac{(C_\beta+1)}{2}$

Similarly to the proof of Lemma 3.3 it holds that

$$\int_\Omega u_t(\beta_\mu(u)_t - g_t)dx = -\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\beta_\mu(u)|^2 dx + \int_\Omega \nabla\beta_\mu(u)\cdot\nabla g_t dx \quad (6.12)$$

$$-\sum_{i=0}^{1}\frac{d}{dt}\int_{\Gamma_i}\gamma_{i\lambda}(\beta_\mu(u)-g)d\Gamma+\int_\Omega f(\beta_\mu(u)_t-g_t)dx \quad \text{a.e. on } [0,T].$$

Then (6.11) and (6.12) yield

$$\frac{\kappa_1}{2}|\beta_\mu(u)_\tau(\tau)|_H^2+\tfrac{1}{2}\frac{d}{d\tau}\int_\Omega|\nabla\beta_\mu(u)(\tau)|^2dx+\sum_{i=0}^{1}\frac{d}{d\tau}\int_{\Gamma_i}\gamma_{i\lambda}(\beta_\mu(u)(\tau)-g(\tau))d\Gamma$$
$$\le C_6|g_\tau(\tau)|_H^2+|\beta_\mu(u)(\tau)|_X|g_\tau(\tau)|_X+\frac{1}{2\kappa_1}|f(\tau)|_H^2+|f(\tau)|_H|g_\tau(\tau)|_H$$
$$=: G_{2\mu}(\tau) \quad \text{for a.e. } \tau\in[0,T]. \tag{6.13}$$

Integrating (6.13) over $[0,t]$, $0\le t\le T$, we infer that

$$\frac{\kappa_1}{2}\int_0^t|\beta_\mu(u)_\tau(\tau)|_H^2d\tau+\frac{1}{2}\int_\Omega|\nabla\beta_\mu(u)(t)|^2dx+\sum_{i=0}^{1}\int_{\Gamma_i}\gamma_{i\lambda}(\beta_\mu(u)(t)-g(t))d\Gamma$$
$$\le\kappa_1\int_0^T G_{2\mu}(\tau)d\tau+\frac{1}{2}\int_\Omega|\nabla\beta_\mu(u_{0,\mu})|^2dx+\sum_{i=0}^{1}\int_{\Gamma_i}\gamma_{i\lambda}(\beta_\mu(u_{0,\mu})-g(0))d\Gamma$$
$$\le\kappa_1\int_0^T G_{2\mu}(\tau)d\tau+\frac{1}{2}\int_\Omega|\nabla v_0|^2dx+\sum_{i=0}^{1}\int_{\Gamma_i}\gamma_i(v_0-g(0))d\Gamma$$

for any $t\in[0,T]$. From Proposition 6.2 it follows that $\int_0^T G_{2\mu}(t)dt$ is bounded with respect to $\mu\in(0,\mu_0]$. Therefore, the assertion of this proposition is valid.

7. Comparison principal and elliptic problems

In this section we give propositions concerned with comparison principle and L^∞-estimate for solutions of $(\mathrm{SP})_{\lambda,\mu}$. In order to describe the comparison principle we recall a notion of order for convex functions on $\mathbb{R}$.

Definition 7.1. Given two proper, l.s.c. and convex functions ρ_1 and ρ_2, on $\mathbb{R}$ the relation "$\rho_1\le^*\rho_2$" means that

$$\rho_1(r_1\wedge r_2)+\rho_2(r_1\vee r_2)\le\rho_1(r_1)+\rho_2(r_2) \quad \text{for } r_1,r_2\in\mathbb{R},$$

where $r_1\wedge r_2=\min\{r_1,r_2\}$ and $r_1\vee r_2=\max\{r_1,r_2\}$.

It is easy to prove the following lemma.

Lemma 7.1. *Let ρ_1 and ρ_2 be proper, l.s.c. and convex functions on $\mathbb{R}$. Then the following three conditions are equivalent.*

(1) $\rho_1 \le^* \rho_2$.

(2) $(r_1' - r_2')(r_1 - r_2)^+ \ge 0$ *for any* $r_i \in D(\rho_i)$ *and* $r_i' \in \partial\rho_i(r_i)$, $i = 1, 2$.

(3) *For each* $\lambda > 0$ $\rho_{1\lambda} \le^* \rho_{2\lambda}$, *where* $\rho_{1\lambda}$ *and* $\rho_{2\lambda}$ *are Yosida approximations for* ρ_1 *and* ρ_2, *respectively.*

Now, we recall a proposition concerned with the comparison principle for solutions of $(\mathrm{SP})_{\lambda,\mu}$

Proposition 7.1. (cf. [11, Proposition 3.4]) *Let* β *be a function satisfying* (A1) *and suppose that the data sets* $\{f, \gamma_0, \gamma_1, g_0, g_1, u_0\}$ *and* $\{\bar f, \bar\gamma_0, \bar\gamma_1, \bar g_0, \bar g_1, \bar u_0\}$ *satisfy all the assumptions* (A2)–(A6). *For* $\lambda > 0$ *and* $\mu > 0$ *we denote by* u *and* $\bar u$ *solutions of* $(SP)_{\lambda,\mu}$ *in the sense of Proposition* 6.1 *corresponding to* $\{f, \gamma_0, \gamma_1, g_0, g_1, u_0\}$ *and* $\{\bar f, \bar\gamma_0, \bar\gamma_1, \bar g_0, \bar g_1, \bar u_0\}$, *respectively. If* $u_0 \le \bar u_0$ *a.e. on* Ω, $g_i \le \bar g_i$ *a.e. on* $\Sigma_i(T)$ *and* $\gamma_i \le^* \bar\gamma_i$, $i = 0, 1$, *then* $u \le \bar u$ *a.e. on* $Q(T)$.

Next, we show Lemma 7.2 concerned with the existence of functions $U^0_{\lambda,\mu}$ and $U^0_{\lambda,\mu}$ satisfying (2.4) $\sim$ (2.6). To do so we define a proper, l.s.c. and convex function q on $\mathbb{R}$ in the following way:

$$q(r) = \begin{cases} 0 & \text{if } r = 0, \\ \infty & \text{otherwise.} \end{cases}$$

Clearly, if γ_0 and γ_1 satisfies (A7), then $\gamma_1 \le^* q$ and $q \le^* \gamma_0$, and $\gamma_{1\lambda} \le^* q_\lambda$ and $q_\lambda \le^* \gamma_{0\lambda}$ for $\lambda > 0$ where q_λ is Yosida-approximation of q.

Lemma 7.2. *Assume that* (A1) *and* (A1-1) *hold. Let* $u_0 \in L^\infty(\Omega)$, γ_0 *and* γ_1 *be functions satisfying the condition* (A7). *Then there exists a positive constant* λ_0 *such that*

$$U^{(0)}_{\lambda,\mu} \le u_{0,\mu} \le U^{(1)}_{\lambda,\mu} \quad \text{a.e. on } \Omega \text{ for } 0 < \lambda \le \lambda_0 \text{ and } 0 < \mu \le 1,$$

where $U^{(i)}_{\lambda,\mu}$ *is a solution of* $(EP)^{(i)}_{\lambda,\mu}$ *defined by* (2.4)–(2.6) *for* $i = 0, 1$, $0 < \lambda \le \lambda_0$ *and* $0 < \mu \le 1$.

Proof. Let $V^{(i)}$ be a function in the assumption (A7) for $i = 0, 1$, and set $\bar V^{(0)} = V^{(0)} - \frac{m_*}{2}$ and $\bar V^{(1)} = V^{(1)} + \frac{m_*}{2}$. It is obvious that

$$\bar V^{(0)} < \bar V^{(0)} + \frac{m_*}{2} \le v_0 \le \bar V^{(1)} - \frac{m_*}{2} < \bar V^{(1)} \quad \text{a.e. on } \Omega, \tag{7.1}$$

and

$$(\mathrm{EP})^{(0)}_* : \begin{cases} -\Delta \bar V^{(0)} = b_0 \quad \text{in } \Omega, \\ \bar V^{(0)} = \dfrac{m_*}{2}, \text{ i.e., } -\dfrac{\partial \bar V^{(0)}}{\partial \nu} \in \partial q(\bar V^{(0)} - \dfrac{m_*}{2}) \quad \text{on } \Gamma_0, \\ -\dfrac{\partial \bar V^{(0)}}{\partial \nu} \in \partial\gamma_1(\bar V^{(0)} + M_*' + \dfrac{m_*}{2}) \qquad \text{on } \Gamma_1, \end{cases}$$

$$(\mathrm{EP})_*^{(1)}:\begin{cases}-\Delta\hat{V}^{(1)}=b_1 & \text{in } \Omega,\\ \hat{V}^{(1)}=-\dfrac{m_*}{2}, \text{ i.e., } -\dfrac{\partial\bar{V}^{(1)}}{\partial\nu}\in\partial q(\bar{V}^{(1)}+\dfrac{m_*}{2}) & \text{on } \Gamma_1,\\ -\dfrac{\partial\bar{V}^{(1)}}{\partial\nu}\in\partial\gamma_0(\bar{V}^{(1)}-M'_*-\dfrac{m_*}{2}) & \text{on } \Gamma_0.\end{cases}$$

Next, we consider the following elliptic problems: For $\lambda>0$

$$(\mathrm{EP})_\lambda^{(0)}:\begin{cases}-\Delta\bar{V}_\lambda^{(0)}=b_0 & \text{in } \Omega,\\ -\dfrac{\partial\bar{V}_\lambda^{(0)}}{\partial\nu}=\partial q_\lambda(\bar{V}_\lambda^{(0)}-\dfrac{m_*}{2}) & \text{on } \Gamma_0,\\ -\dfrac{\partial\bar{V}_\lambda^{(0)}}{\partial\nu}=\partial\gamma_{1\lambda}(\bar{V}_\lambda^{(0)}+M'_*+\dfrac{m_*}{2}) & \text{on } \Gamma_1,\end{cases}$$

$$(\mathrm{EP})_\lambda^{(1)}:\begin{cases}-\Delta\bar{V}_\lambda^{(1)}=b_1 & \text{in } \Omega,\\ -\dfrac{\partial\bar{V}_\lambda^{(1)}}{\partial\nu}=\partial q_\lambda(\bar{V}^{(1)}+\dfrac{m_*}{2}) & \text{on } \Gamma_1,\\ -\dfrac{\partial\bar{V}_\lambda^{(1)}}{\partial\nu}=\partial\gamma_{0\lambda}(\bar{V}_\lambda^{(1)}-M'_*-\dfrac{m_*}{2}) & \text{on } \Gamma_0.\end{cases}$$

For each $\lambda>0$ and $i=0,1$ the problem $(\mathrm{EP})_\lambda^{(i)}$ has a unique solution $\bar{V}_\lambda^{(i)}\in H^2(\Omega)\cap C(\overline{\Omega})$ because of the boundary conditions, and we have

$$\bar{V}_\lambda^{(i)}\to\bar{V}^{(i)}\quad\text{uniformly on } \overline{\Omega} \text{ as } \lambda\downarrow 0, \text{ for } i=0,1, \tag{7.2}$$

by a direct application of the results in [6, Chapter 1] since $(\mathrm{EP})_*^{(i)}$ admits at most one solution for $i=0,1$. It follows that there exists a positive constant $\lambda_0\in(0,1]$ such that

$$|\bar{V}_\lambda^{(i)}(x)-\bar{V}^{(i)}(x)|\le\frac{m_*}{4}\quad\text{for } x\in\overline{\Omega}, \lambda\in(0,\lambda_0] \text{ and } i=0,1$$

so that on account of (7.1) it holds that

$$\bar{V}_\lambda^{(0)}(x)+\frac{m_*}{4}\le\beta(u_0(x))\le\bar{V}_\lambda^{(1)}(x)-\frac{m_*}{4}\quad\text{for } x\in\Omega \text{ and } \lambda\in(0,\lambda_0].$$

Now, putting $U_{\lambda,\mu}^{(i)}=\beta_\mu^{-1}(\overline{V}_\lambda^{(i)})$ by the definition of $u_{0,\mu}$ it is clear that $U_{\lambda,\mu}^{(0)}\le u_{0,\mu}\le U_{\lambda,\mu}^{(1)}$ a.e. on Ω. □

Proposition 7.2. *For $\lambda\in(0,\lambda_0]$ and $\mu\in(0,\mu_0]$ let $U_{\lambda,\mu}^{(i)}$ be a function defined by Lemma 7.2 for $i=0,1$, and suppose that the condition $(H1)$*

holds. Denote by $u_{\lambda,\mu}$ a solution of $(SP)_{\lambda,\mu}$ in the sense of Proposition 6.1. *Then it holds that*

$$U^{(0)}_{\lambda,\mu}(x) \le u_{\lambda,\mu}(t,x) \le U^{(1)}_{\lambda,\mu}(x) \quad \textit{for a.e. } (t,x) \in Q(T). \tag{7.3}$$

Moreover, the set

$$\{u_{\lambda,\mu}; 0 < \lambda \le \lambda_0, 0 < \mu \le \mu_0\} \textit{ is bounded in } L^\infty(Q(T)), \tag{7.4}$$

and there exist positive numbers δ_0 and λ_1 such that

$$\beta_\mu(u_{\lambda,\mu}) \ge \frac{m_*}{8} \quad \textit{on} \quad Q_{0,\delta_0} \quad \textit{and} \quad \beta_\mu(u_{\lambda,\mu}) \le -\frac{m_*}{8} \quad \textit{on} \quad Q_{1,\delta_0} \tag{7.5}$$

for $0 < \mu \le 1$, $0 < \lambda \le \lambda_0$, where $\Omega_{i,\delta_0} = \{x \in \Omega; dist(x,\Gamma_i) < \delta_0\}$ and $Q_{i,\delta_0} = (0,T) \times \Omega_{i,\delta}$ for $i = 0,1$.

Hence, there exist positive constants $\mu_1 \in (0,\mu_0]$ and m'_* such that

$$u_{\lambda,\mu} \ge 1 + m'_* \quad \textit{on} \quad Q_{0,\delta_0} \quad \textit{and} \quad u_{\lambda,\mu} \le -m'_* \quad \textit{on} \quad Q_{1,\delta_0} \tag{7.6}$$

for $0 < \mu \le \mu_1$, $0 < \lambda \le \lambda_1$.

Proof. Clearly,

$$\begin{aligned}
&U^{(0)}_{\lambda,\mu t} - \Delta\beta_\mu(U^{(0)}_{\lambda,\mu}) = b_0 \quad \text{in } Q(T),\\
&-\frac{\partial\beta_\mu(U^{(0)}_{\lambda,\mu})}{\partial\nu} = \partial q_\lambda(\beta_\mu(U^{(0)}_{\lambda,\mu}) - \frac{m_*}{2}) \quad \text{on } \Gamma_0,\\
&-\frac{\partial\beta_\mu(U^{(0)}_{\lambda,\mu})}{\partial\nu} = \partial\gamma_{1\lambda}(\beta_\mu(U^{(0)}_{\lambda,\mu}) + M'_* + \frac{m_*}{2}) \quad \text{on } \Gamma_1.
\end{aligned}$$

According to assumptions (A4) and (A7) and Lemma 7.2, we get $b_0 \le f$ a.e. on $Q(T)$, $q_\lambda \le^* \gamma_{0\lambda}$, $\frac{m_*}{2} \le g_0$ on $\Sigma_0(T)$, $-M'_* - \frac{m_*}{2} \le g_1$ on $\Sigma_1(T)$ and $U^{(0)}_{\lambda,\mu} \le u_{0,\mu}$ a.e. on Ω. Therefore, (7.3) is easily obtained from Proposition 7.1.

By (7.2), the set $\{\bar{V}^{(i)}_\lambda; 0 < \lambda \le \lambda_0\}$ is bounded in $L^\infty(\Omega)$ for $i = 0,1$. Also, we have $|\beta_\mu^{-1}(r)| \le \frac{|r|+m_\beta}{M_\beta}$ for $r \in \mathbb{R}$ and $\mu \in (0,\mu_0]$. Then (7.3) implies (7.4). Furthermore, by using (7.2), again, (7.5) holds since $\bar{V}^{(0)}$(resp. $\bar{V}^{(1)}$) is strictly positive (resp. negative) near the boundary Γ_0(resp. Γ_1). Hence, (7.4) yields (7.6) because β is bi-Lipschitz continuous on both $[1,\infty)$ and $(-\infty,0]$. □

8. Uniform estimate for heat flux on the boundary

Throughout this section we assume that the condition (H1) holds and use the same notations as in the previous section.

Lemma 8.1. *For $\lambda \in (0, \lambda_1]$ and $\mu \in (0, \mu_1]$ let $u_{\lambda,\mu}$ be a solution of $(SP)_{\lambda,\mu}$ in the sense of Proposition 6.1. Then there exists a positive constant R_0 such that*

$$\int_{Q_{i,\delta_1}} |u_{\lambda,\mu t}|^2 dxdt \le R_0 \quad \text{for } i = 0, 1, \tag{8.1}$$

where $\delta_1 = \delta_0/2$.

Proof. Let $\lambda \in (0, \lambda_1]$, $\mu \in (0, \mu_1]$, and $\eta \in C^\infty(\overline{\Omega})$ with $0 \le \eta \le 1$ on Ω, $\eta = 1$ on Ω_{0,δ_1} and $\eta = 0$ on $\Omega - \Omega_{0,\delta_0}$. For simplicity we put $u = u_{\lambda,\mu}$. Since by Propositions 6.1 and 7.2 $u_t \in L^2(Q(T))$ and $u \ge m'_* + 1$ on Q_{0,δ_0}, the assumption (A1-2) implies that

$$\begin{aligned}
&\int_\Omega \eta^2 u_t(\beta_\mu(u)_t - g_t)dx = \int_{\Omega_{0,\delta_0}} \eta^2 u_t(\beta_\mu(u)_t - g_t)dx \\
&= M'_\beta \int_{\Omega_{0,\delta_0}} \eta^2 |u_t|^2 dx - \int_{\Omega_{0,\delta_0}} \eta^2 u_t g_t dx \\
&\ge \frac{M'_\beta}{2} \int_{\Omega_{0,\delta_0}} \eta^2 |u_t|^2 dx - \frac{1}{2M'_\beta} \int_{\Omega_{0,\delta_0}} \eta^2 |g_t|^2 dx \quad \text{a.e. on } [0, T].
\end{aligned}$$

Also, similarly to Proposition 6.3, we have

$$\begin{aligned}
&\int_\Omega \eta^2 u_t(\beta_\mu(u)_t - g_t)dx = \int_\Omega \eta^2 (\Delta\beta_\mu(u) + f)(\beta_\mu(u)_t - g_t)dx \\
&= -\frac{1}{2}\frac{d}{dt} \int_\Omega \eta^2 |\nabla\beta_\mu(u)|^2 dx + \int_\Omega \eta^2 \nabla\beta_\mu(u) \cdot \nabla g_t dx \\
&- 2\int_\Omega \eta\beta_\mu(u)_t \nabla\beta_\mu(u) \cdot \nabla\eta dx + 2\int_\Omega \eta^2 \nabla\beta_\mu(u) \cdot \nabla g_t dx \\
&- \frac{d}{dt} \int_{\Gamma_0} \gamma_{0\lambda}(\beta_\mu(u) - g_0) d\Gamma + \int_\Omega \eta^2 f(\beta_\mu(u)_t - g_t)dx \\
&\le -\frac{1}{2}\frac{d}{dt} \int_\Omega \eta^2 |\nabla\beta_\mu(u)|^2 dx - \frac{d}{dt} \int_{\Gamma_0} \gamma_{0\lambda}(\beta_\mu(u) - g_0) d\Gamma \qquad (8.2) \\
&+ C_7(|\beta_\mu(u)|_X + |f|_H^2 + |g_t|_X^2) \quad \text{a.e. on } [0, T].
\end{aligned}$$

Integrating (8.2) over $[0, T]$ and adding to Proposition 6.2 this lemma is easily obtained. □

The following proposition means the uniform estimates for heat flux on the boundary.

Proposition 8.1. *There exists a positive constant* R_1 *such that*

$$\int_{\Sigma_i(T)} |\partial\gamma_{i\lambda}(\beta_\mu(u_{\lambda,\mu})(t) - g_i(t))|^2 d\Gamma dt \le R_1$$

for any $\lambda \in (0, \lambda_1]$, $\mu \in (0, \mu_1]$ *and* $i = 0, 1$.

Proof. Let $\delta_2 = \delta_1/2$, $\lambda \in (0, \lambda_1]$, $\mu \in (0, \mu_1]$ and $\eta \in C^\infty(\overline{\Omega})$ with $0 \le \eta \le 1$ on Ω, $\eta = 1$ on Ω_{0,δ_2} and $\eta = 0$ on $\Omega - \Omega_{0,\delta_1}$, and we write $\eta\beta_\mu(u_{\lambda,\mu})$ as v. Then v satisfies

$$\begin{cases} \Delta v(t) = \eta(u_{\lambda,\mu t}(t) - f(t)) + 2\nabla\beta_\mu(u_{\lambda,\mu})(t)\cdot\nabla\eta + \beta_\mu(u_{\lambda,\mu})(t)\Delta\eta & \text{in } \Omega_{0,\delta_1}, \\ v(t) = 0 \quad \text{on } \partial\Omega_{0,\delta_1} - \Gamma_0, \\ -\dfrac{\partial v(t)}{\partial \nu} = \partial\gamma_{0\lambda}(v(t) - g_0(t)) \quad \text{on } \Gamma_0 \end{cases}$$

for a.e. $t \in [0, T]$. By the application of the theory for elliptic problems (cf. [6, Chapter 1]) we conclude that for some positive constant R_2 independent of $\lambda \in (0, \lambda_1]$

$$|v(t)|_{H^2(\Omega_{0,\delta_1})} \le R_2(|u_{\lambda,\mu t}(t)|_{L^2(\Omega_{0,\delta_1})} + |\beta_\mu(u_{\lambda,\mu})(t)|_X + |f(t)|_H) \tag{8.3}$$

for a.e. $t \in [0, T]$. Hence, Proposition 6.2 and Lemma 8.1 together with (8.3) imply the conclusion of this proposition for $i = 0$. Similarly, we can get the same estimate in case $i = 1$. □

9. Proof of the existence

First, we give a proposition concerned with the existence of solutions as letting $\mu \downarrow 0$.

Proposition 9.1. *Assume that* $(H1)$ *holds. For each* $\lambda \in (0, \lambda_1]$ *there exists one and only one function* $u_\lambda \in L^\infty(Q(T)) \cap W^{1,2}(0, T; X^*)$ *with* $\beta(u_\lambda) \in L^\infty(0, T; X) \cap W^{1,2}(0, T; H)$ *satisfying*

$$\begin{aligned} &-\int_0^T (u_\lambda(t), \eta_t(t))dt + \int_0^T A(\beta(u_\lambda)(t), \eta(t))dt \\ &+ \sum_{i=0}^{1} \int_0^T (\partial\gamma_{i\lambda}(\beta(u_\lambda)(t) - g(t)), \eta(t))_i dt \\ &= \int_0^T (f(t), \eta(t))dt + (u_0, \eta(0)) \quad \textit{for any } \eta \in \mathcal{L}(Q(T)). \end{aligned} \tag{9.1}$$

Moreover, there is a positive constant L_1 *independent of* $\lambda \in (0, \lambda_1]$ *such that*

$$-L_1 \le u_\lambda(t,x) \le L_1 \quad \text{for a.e. } (t,x) \in Q(T), \tag{9.2}$$

$$\int_0^T |\beta(u_\lambda)_t(t)|_H^2 dt \le L_1, \tag{9.3}$$

$$|\beta(u_\lambda)(t)|_X \le L_1 \quad \text{for any } t \in [0,T], \tag{9.4}$$

$$\int_0^T |u_{\lambda t}(t)|_{X^*}^2 dt \le L_1, \tag{9.5}$$

$$\int_{\Gamma_i} \gamma_{i\lambda}(\beta(u_\lambda)(t) - g(t)) d\Gamma \le L_1 \quad \text{for any } t \in [0,T] \text{ and } i = 0,1, \tag{9.6}$$

$$\int_{\Sigma_i(T)} |\partial\gamma_{i\lambda}(\beta(u_\lambda)(t) - g(t))|^2 d\Gamma dt \le L_1 \quad \text{for } i = 0,1. \tag{9.7}$$

Before the proof of Proposition 9.1 we denote by β_μ^* (resp. β^*) the primitive of β_μ^{-1} (resp. β^{-1}) with $\beta_\mu^*(0) = \beta^*(0) = 0$ and give a lemma concerned with the elementary property of β_μ.

Lemma 9.1. *Assume that* (A1) *and* (A1-1) *hold. If* $z_\mu \to z$ *in* H *as* $\mu \downarrow 0$, *then* $\int_\Omega \beta_\mu^*(z_\mu)dx \to \int_\Omega \beta^*(z)dx$ *as* $\mu \downarrow 0$.

This lemma is easily proved, so we omit its proof.

Proof of Proposition 9.1. The uniqueness is due to [16]. First, we show that

$$u_{0,\mu} \to u_0 \quad \text{weakly in } H \text{ as } \mu \downarrow 0. \tag{9.8}$$

In fact, by (6.5) there is a subsequence $\{\mu_n\}$ such that $u_{0,\mu_n} \to \tilde{u}_0$ weakly in H as $n \to \infty$. On the other hand, $\beta(u_{0,\mu}) + \mu u_{0,\mu} = v_0$ a.e. on Ω. Then $\beta(u_{0,\mu}) \to v_0 = \beta(u_0)$ in H as $\mu \downarrow 0$. Since β is Lipschitz continuous and monotone increasing, we have $\beta(u_0) = \beta(\tilde{u}_0)$ a.e. on Ω. Now, by the assumption (A7) $u_0(x) > 1$ or $u_0(x) < 0$ for a.e. $x \in \Omega$. Noting that β^{-1} is single-valued on $(0,\infty)$ and $(-\infty, 0)$, $u_0 = \tilde{u}_0$ a.e. on Ω. Hence, (9.8) holds.

For $\lambda \in (0, \lambda_1]$ and $\mu \in (0, \mu_1]$ let $u_{\lambda,\mu}$ be a solution of $(\mathrm{SP})_{\lambda,\mu}$ in the sense of Proposition 6.1. According to (7.3), there exists a positive constant L_2 such that

$$|u_{\lambda,\mu}|_{L^\infty(Q(T))} \le L_2 \quad \text{for any } \lambda \in (0, \lambda_1] \text{ and } \mu \in (0, \mu_1]. \tag{9.9}$$

Also, Propositions 6.2, 6.3 and 8.1 imply the existence of a positive constant L_3 independent of λ and μ such that

$$\int_0^T |\beta_\mu(u_{\lambda,\mu})_t(t)|_H^2 dt \le L_3, \tag{9.10}$$

$$|\beta_\mu(u_{\lambda,\mu})(t)|_X \le L_3 \quad \text{for any } t \in [0,T], \tag{9.11}$$

$$\int_{\Gamma_i} \gamma_{i\lambda}(\beta_\mu(u_{\lambda,\mu})(t) - g(t))d\Gamma \le L_3 \quad \text{for any } t \in [0,T] \text{ and } i = 0, 1, \tag{9.12}$$

$$\int_{\Sigma_i(T)} |\partial\gamma_{i\lambda}(\beta_\mu(u_{\lambda,\mu})(t) - g(t))|^2 d\Gamma dt \le L_3 \quad \text{for } i = 0, 1. \tag{9.13}$$

Moreover, using integration by parts it holds that

$$- \int_0^T (u_{\lambda,\mu}, \eta_t)dt + \int_0^T A(\beta_\mu(u_{\lambda,\mu}), \eta)dt + \sum_{i=0}^{1} \int_0^T (\partial\gamma_{i\lambda}(\beta(u_{\lambda,\mu}) - g), \eta)_i dt$$
$$= \int_0^T (f, \eta)dt + (u_{0,\mu}, \eta(0)) \quad \text{for any } \eta \in \mathcal{L}(Q(T)). \tag{9.14}$$

This shows that

$$|\int_0^T (u_{\lambda,\mu}(t), \eta_t(t))dt| \le \int_0^T |\beta_\mu(u_{\lambda,\mu})(t)|_X |\eta(t)|_X dt$$
$$+ C_\Omega \sum_{i=0}^{1} \int_0^T |\partial\gamma_{i\lambda}(\beta_\mu(u_{\lambda,\mu})(t) - g(t))|_{L^2(\Gamma_i)} |\eta(t)|_X dt$$
$$+ \int_0^T |f(t)|_H |\eta(t)|_X dt \quad \text{for any } \eta \in \mathcal{L}(Q(T)) \text{ with } \eta(0) = 0,$$

where C_Ω is a positive constant depending only on Ω, that is, for some positive constant L_4 we have

$$\int_0^T |u_{\lambda,\mu t}(t)|^2_{X^*} dt \le L_4.$$

Hence, there exist a subsequence of $\{\mu_k\} \subset \{\mu\}$, $u_\lambda \in L^\infty(Q(T)) \cap W^{1,2}(0,T; X^*)$ and $V_\lambda \in L^\infty(0,T;X) \cap W^{1,2}(0,T;H)$ satisfying

$$u_{\lambda,k} := u_{\lambda,\mu_k} \to u_\lambda \quad \text{weakly* in } L^\infty(Q(T)) \text{ and weakly in } W^{1,2}(0,T;X^*), \tag{9.15}$$

$$\beta_k(u_{\lambda,k}) \to V_\lambda \quad \text{weakly* in } L^\infty(0,T;X) \text{ and weakly in } W^{1,2}(0,T;H), \tag{9.16}$$

as $k \to \infty$ where $\beta_k := \beta_{\mu_k}$ for each k. Thus, we see that

$$\beta_k(u_{\lambda,k}) \to V_\lambda \quad \text{in } C([0,T];H) \text{ as } k \to \infty. \tag{9.17}$$

It yields

$$\beta(u_\lambda) = V_\lambda \quad \text{a.e. on } Q(T). \tag{9.18}$$

In fact, let $w \in L^2(0,T;H)$. Then by using (9.15), (9.16) and the definition of β_μ we obtain that

$$\int_0^T (u_\lambda(t) - w(t), V_\lambda(t) - \beta(w)(t))dt \geq 0.$$

Since β is maximal monotone on $L^2(0,T;H)$, (9.18) is derived.

Next, we prove that

$$\beta_k(u_{\lambda,k}) \to \beta(u_\lambda) \quad \text{in } L^2(0,T;X) \text{ as } k \to \infty. \tag{9.19}$$

Firstly, we note that

$$\begin{aligned}&\int_0^T \int_\Omega |\nabla\beta_k(u_{\lambda,k}) - \nabla\beta(u_\lambda)|^2 dxdt \\ &= \int_0^T A(\beta_k(u_{\lambda,k}), \beta_k(u_{\lambda,k}) - \beta(u_\lambda))dt - \int_0^T A(\beta(u_\lambda), \beta_k(u_{\lambda,k}) - \beta(u_\lambda))dt.\end{aligned}$$

It follows from (9.16) and (9.18) that

$$\lim_{k\to\infty} \int_0^T A(\beta(u_\lambda)(t), \beta_k(u_{\lambda,k})(t) - \beta(u_\lambda)(t))dt = 0.$$

Since $u_{\lambda,\mu}$ is the solution of $(\mathrm{SP})_{\lambda,\mu}$ we have

$$\begin{aligned}&\int_0^T A(\beta_k(u_{\lambda,k})(t), \beta_k(u_{\lambda,k})(t) - \beta(u_\lambda)(t))dt \\ = \quad & -\int_0^T (u_{\lambda,kt}(t), \beta_k(u_{\lambda,k})(t) - \beta(u_\lambda)(t))dt \\ & -\sum_{i=0}^{1} \int_0^T (\partial\gamma_{i\lambda}(\beta_k(u_{\lambda,k})(t) - g(t)), \beta_k(u_{\lambda,k})(t) - \beta(u_\lambda)(t))_i dt \\ & + \int_0^T (f(t), \beta_k(u_{\lambda,k})(t) - \beta(u_\lambda)(t))dt =: I_{1,k} + I_{2,k} + I_{3,k}.\end{aligned}$$

It is clear that $\lim_{k\to\infty} I_{3,k} = 0$. Also, by putting $V_{\lambda,k} = \beta_k(u_{\lambda,k})$ we infer from integration by parts that

$$\begin{aligned}I_{1,k} \quad = \quad & \int_0^T (u_{\lambda,k}(t), \beta_k(u_{\lambda,k})_t(t) - \beta(u_\lambda)_t(t))dt \\ & -(u_{\lambda,k}(T), V_{\lambda,k}(T) - V_\lambda(T)) + (u_{\lambda,k}(0), V_{\lambda,k}(0) - V_\lambda(0)).\end{aligned}$$

Moreover, it follows from Lemma 9.1 and (9.17) that

$$\begin{aligned}
&\int_0^T (u_{\lambda,k}(t), \beta_k(u_{\lambda,k})_t(t) - \beta(u_\lambda)_t(t))dt \\
&= \int_0^T \frac{d}{dt}\int_\Omega \beta_k^*(V_{\lambda,k})(t)dxdt - \int_0^T (u_{\lambda,k}(t), \beta(u_\lambda)_t(t))dt \\
&= \int_\Omega \beta_k^*(V_{\lambda,k})(T)dx - \int_\Omega \beta_k^*(V_{\lambda,k})(0)dx - \int_0^T (u_{\lambda_k}(t), \beta(u_\lambda)_t(t))dt \\
&\to \int_\Omega \beta^*(V_\lambda)(T)dx - \int_\Omega \beta^*(V_\lambda)(0)dx - \int_0^T (u_\lambda(t), \beta(u_\lambda)_t(t))dt \text{ (as } k \to \infty) \\
&= 0,
\end{aligned}$$

where $\beta_k^* = \beta_{\mu_k}^*$. Using (9.17), again, it guarantees that

$$\lim_{k\to\infty}\{-(u_{\lambda,k}(T), V_{\lambda,k}(T) - V_\lambda(T)) + (u_{\lambda,k}(0), V_{\lambda,k}(0) - V_\lambda(0))\} = 0.$$

Furthermore, on account of monotonicity of subdifferentials we obtain that

$$I_{2,k} \le -\sum_{i=0}^{1}\int_0^T (\partial\gamma_{i\lambda}(\beta(u_\lambda)(t) - g(t)), \beta_k(u_{\lambda,k})(t) - \beta(u_\lambda)(t))_i dt$$

so that $\limsup_{k\to\infty} I_{2,k} \le 0$ since $\beta_k(u_{\lambda,k}) \to \beta(u_\lambda)$ weakly in $L^2(\Gamma_i)$ as $k \to \infty$, $i = 0, 1$. From the above argument we conclude that

$$\limsup_{k\to\infty}\int_0^T\int_\Omega |\nabla\beta_k(u_{\lambda,k})(t) - \nabla\beta(u_\lambda)(t)|^2 dxdt \le 0.$$

Thus, we get (9.19). Because of Lipschitz continuity of $\partial\gamma_{i\lambda}$, $i = 0, 1$, the following convergences are true:

$$\partial\gamma_{i\lambda}(\beta_k(u_{\lambda,k}) - g) \to \partial\gamma_{i\lambda}(\beta(u_\lambda) - g) \quad \text{in } L^2(\Sigma_i(T)) \text{ as } k \to \infty \text{ for } i = 0, 1.$$

By using (9.8) $\sim$ (9.16) it is easy to see that (9.1) $\sim$ (9.7) hold. □

In order to accomplish the proof of Theorem 2.1 we prove the following lemma concerned with Yosida-approximation.

Lemma 9.2. *Let W be a real Hilbert space and ϕ be a proper, l.s.c. and convex function on W and denote by ϕ_λ the Yosida approximation of ϕ for $\lambda > 0$. Then $\phi_\lambda \to \phi$ on W in the sense of Mosco as $\lambda \downarrow 0$. In particular, let $\{\lambda_k\}$ be any subsequence of $\{\lambda\}$ and $\{z_k\}$ be a sequence in W such that $z_k \to z$ weakly in W as $k \to \infty$, then*

$$\liminf_{k\to\infty}\phi_{\lambda_k}(z_k) \ge \phi(z).$$

Proof. Here, we give a brief proof of this lemma. By [3, Theorem 1.2] it is sufficient to show that $(\phi_\lambda)_\mu(z) \to \phi_\mu(z)$ as $\lambda \downarrow 0$ for $\mu > 0$ and $z \in W$, where $(\phi_\lambda)_\mu$ is the Yosida-approximation of ϕ_λ.

Let $z \in W$, and let J_μ and $(J_\lambda)_\mu$ be the resolvents of $\partial\phi$ and $\partial\phi_\lambda$, respectively, that is, $J_\mu = (I + \mu\partial\phi)^{-1}$ and $(J_\lambda)_\mu = (I + \mu\partial\phi_\lambda)^{-1}$. Then we see that for $z \in W$

$$\phi_\mu(z) = \frac{1}{2\mu}|z - J_\mu z|_W^2 + \phi(J_\mu z),$$
$$(\phi_\lambda)_\mu(z) = \frac{1}{2\mu}|z - (J_\lambda)_\mu z|_W^2 + \phi_\lambda((J_\lambda)_\mu z).$$

Easily, we obtain that $(J_\lambda)_\mu z \to J_\mu z$ in W as $\lambda \downarrow 0$. Since $(\partial\phi_\lambda)_\mu = \partial\phi_{\lambda+\mu}$, the set $\{\partial\phi_\lambda((J_\lambda)_\mu z)\}$ is bounded in W. Hence, it follows from [10, Proposition 0.3.5] that $\phi_\lambda((J_\lambda)_\mu z) \to \phi(J_\mu z)$ as $\lambda \downarrow 0$. Thus, this lemma is true. We refer [5, Chapter 2] for the precise calculations in this proof. □

Proof of Theorem 2.1. First, Proposition 9.1 shows that for some subsequence $\{\lambda_k\}$ of $\{\lambda\}$ the following convergences hold;

$$\begin{aligned} &u_k := u_{\lambda_k} \to u \quad \text{weakly* in } L^\infty(Q(T)) \text{ and weakly in } W^{1,2}(0,T;X^*),\\ &\beta(u_k) \to V \quad \text{weakly* in } L^\infty(0,T;X) \text{ and weakly in } W^{1,2}(0,T;H),\\ &\partial\gamma_{i\lambda_k}(\beta(u_k) - g) \to \xi_i \quad \text{weakly in } L^2(\Sigma_i(T)) \text{ for } i = 0, 1, \end{aligned} \tag{9.20}$$

as $k \to \infty$ where $u \in L^\infty(Q(T)) \cap W^{1,2}(0,T;X^*)$, $V \in L^\infty(0,T;X) \cap W^{1,2}(0,T;H)$ and $\xi_i \in L^2(\Sigma_i(T))$, $i=0,1$. Clearly, $\beta(u_k) \to V$ in $C([0,T];H)$ as $k \to \infty$.

By the same argument for (9.18) we obtain $\beta(u) = V$ a.e. on $Q(T)$. Similarly to (9.19), we can prove:

$$\beta(u_k) \to \beta(u) \quad \text{in } L^2(0,T;X) \text{ as } k \to \infty. \tag{9.21}$$

Here, by Lemma 9.2 it is clear that

$$\begin{aligned} &\limsup_{k\to\infty}\{-\sum_{i=0}^{1}\int_0^T (\partial\gamma_{i\lambda_k}(\beta(u_k)(t) - g(t)), \beta(u_k)(t) - \beta(u)(t))_i dt\}\\ &\le \limsup_{k\to\infty}\sum_{i=0}^{1}\int_0^T\int_{\Gamma_i}\{\gamma_{i\lambda_k}(\beta(u)(t) - g(t)) - \gamma_{i\lambda_k}(\beta(u_k)(t) - g(t))\}d\Gamma dt\\ &\le \sum_{i=0}^{1}\int_0^T\int_{\Gamma_i}\gamma_i(\beta(u)(t) - g(t))d\Gamma dt \end{aligned}$$

$$-\liminf_{k\to\infty}\sum_{i=0}^{1}\int_0^T\int_{\Gamma_i}\gamma_{i\lambda_k}(\beta(u_k)(t)-g(t))d\Gamma dt=0.$$

According to the property of $\gamma_{i\lambda}(i=0,1)$, (9.20) and (9.21), we have $\xi_i\in\partial\gamma_i(\beta(u)-g)$ a.e. on $\Sigma_i(T)$ for $i=0,1$.

Now, Proposition 9.1 implies that

$$-\int_0^T(u_k,\eta_t)dt+\int_0^T A(\beta(u_k),\eta)dt+\sum_{i=0}^{1}\int_0^T(\partial\gamma_{i\lambda_k}(\beta(u_k)-g),\eta)_i dt$$

$$=\int_0^T(f,\eta)dt+(u_0,\eta(0))\quad\text{ for any }\eta\in\mathcal{L}(Q(T)).$$

Letting $k\to\infty$ it follows from the above argument that

$$-\int_0^T(u,\eta_t)dt+\int_0^T A(\beta(u),\eta)dt+\sum_{i=0}^{1}\int_0^T(\xi_i,\eta)_i dt$$

$$=\int_0^T(f,\eta)dt+(u_0,\eta(0))\quad\text{ for any }\eta\in\mathcal{L}(Q(T)).$$

Therefore, u is a weak solution of SP. Thus, we have proved Theorem 2.1.

References

[1] T. Aiki, *Multi-dimensional two-phase Stefan problems with nonlinear dynamic boundary conditions,* Nonlinear Analysis and Applications, Gakuto International Series, Mathematical Sciences and Applications, Vol. 7, pp. 1-25, 1995.

[2] T. Aiki, *Multi-dimensional Stefan problems with dynamic boundary conditions,* Applicable Analysis, 56 (1995), 71–94.

[3] H. Attouch, *Familles d'opérateurs maximaux monotones et measurabilité,* Ann. Mat. Pura Appl., 120 (1979), 35 – 111.

[4] V. Barbu, "Nonlinear Semigroups and Differential Equations in Banach Spaces," Noordhoff International Publishing, 1976.

[5] H. Brézis, "Opérateurs maximaux monotones et semi-groupes de contractions dans les spaces de Hilbert," North-Holland, Amesterdam, 1973.

[6] H. Brézis, *Problèmes unilatéraux,* J. Math. pures et appl., 51 (1972), 1–168.

[7] E. DiBenedetto and R. E. Showalter, *Implicit degenerate evolution equations and applications,* SIAM J. Math. Anal., 12 (1981), 731-751.

[8] O. Grange and F. Mignot, *Sur la resolution d'une equation et d'une inequation paraboliques non-lineares,* J. Funct. Anal., 11 (1972), 77-92.

[9] A. Haraux and N. Kenmochi, *Asymptotic behavior of solutions to some degenerate parabolic equations,* Funk. Ekvac., 34 (1991), 19–38.

[10] N. Kenmochi, *Solvability of nonlinear evolution equations with time-dependent constraints and applications,* Bull. Fac. Education, Chiba Univ., 30 (1981), 1–87.

[11] N. Kenmochi and I. Pawlow, *The vanishing viscosity method in two-phase Stefan problems with nonlinear heat flux conditions,* Recent Topics in Nolinear PDE IV, Kyoto, 1988, pp. 211–243, Lecture Notes in Num. Appl. Anal., Vol.10, 1989.

[12] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural'ceva, "Linear and Quasi-Linear Equations of Parabolic Type," Transl. Math. Monograph 23, Amer. Math. Soc., Providence R. I., 1968.

[13] J. L. Lions, "Quelques Methods de Resolutions des Problems aux Limites Non Lineares," Dunod, Paris, 1969.

[14] E. Magenes, C. Verdi, and A. Visintin, *Semigroup approach to the Stefan problem with nonlear heat flux,* atti Acc. Lincei Rend. fis, 75 (1983), 24 – 33.

[15] J. Necǎs, "Les méthodes directes en théorie des équations," Éditeurs Academia, Prague, 1967.

[16] M. Niezgódka and I. Pawlow, *A generalized Stefan problem in several space variables,* Applied Math. Opt., 9 (1983), 193–224.

[17] A. Visitntin, "Models of Phase Transitions," Birkhäuser, Boston-Basel-Berlin, 1996.