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NODAL SOLUTIONS TO SEMILINEAR ELLIPTIC EQUATIONS IN A BALL *

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Abstract. In this paper we are concerned with the existence and multiplicity of nodal solutions to the Dirichlet problem associated to the elliptic equation $\Delta u + q(|x|)g(u) = 0$ in the unit ball in $\mathbf{R}^N$. The nonlinearity g has a linear growth at infinity and zero, while the weight function q is nonnegative in $[0,1]$ and strictly positive in some interval $[r_1, r_2] \subset [0,1]$. By means of a topological degree approach, we are able to prove the existence of solutions with prescribed nodal properties, depending on the behaviour of the ratio $g(u)/u$ at infinity and zero.

1. Introduction

In this paper we deal with the existence and multiplicity of radial solutions to the Dirichlet problem

$$\begin{cases} \Delta u + q(|x|)g(u) = 0 & \text{if } x \in \Omega \\ u(x) = 0 & \text{if } x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is the unit ball in $\mathbf{R}^N$, $N \geq 1$. We assume that $g : \mathbf{R} \longrightarrow \mathbf{R}$ is locally Lipschitz and that $q : [0,1] \longrightarrow \mathbf{R}$ is continuous. This kind of problem has been faced by many authors, with different methods and techniques; without seek of completeness, we refer for instance to the papers [1, 2, 6, 8, 12, 15, 16, 25].

We are concerned with the case when g has a linear growth both in zero and at infinity; a precise mathematical formulation of this condition is given in (3.3) and (3.4). Here, for simplicity, we assume that there exist $g_0 > 0$

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and $g_\infty > 0$ such that

$$\lim_{u\to 0}\frac{g(u)}{u} = g_0 \tag{1.2}$$

and

$$\lim_{|u|\to+\infty}\frac{g(u)}{u} = g_\infty. \tag{1.3}$$

As far as the function q is concerned, we assume that $q(r) \geq 0$, for every $r \in [0,1]$, and that there exist $q_0 > 0$ and $[r_1, r_2] \subset (0,1]$ such that

$$q(r) \geq q_0, \quad \forall\, r \in [r_1, r_2]. \tag{1.4}$$

Moreover, we set $Q = \max\{q(r) : r \in [0,1]\}$. As corollaries of our main results (Theorems 3.1 and 3.2), we will prove the following:

Theorem A. *Under the previous conditions, let j be the smallest integer such that*

$$(Qg_0)^{(N-1)/2}(\sqrt{Qg_0} - 1) < \pi\big(j + \frac{1}{2}\big) \tag{1.5}$$

and let l be the largest integer such that

$$\pi\big(l + \frac{3}{2}\big) < r_1^{N-1}(r_2 - r_1)\sqrt{q_0 g_\infty}. \tag{1.6}$$

If $j+1 \leq l$, then for every integer $n \in [j+1, l]$ there exist two radial solutions u_n and v_n to (1.1) *such that $v_n(0) < 0 < u_n(0)$. Moreover, u_n and v_n have exactly n zeros in $(0,1)$.*

Theorem B. *Under the previous conditions, let j' be the largest integer such that*

$$\pi\big(j' + \frac{3}{2}\big) < r_1^{N-1}(r_2 - r_1)\sqrt{q_0 g_0} \tag{1.7}$$

and let l' be the smallest integer such that

$$(Qg_\infty)^{(N-1)/2}(\sqrt{Qg_\infty} - 1) < \pi\big(l' + \frac{1}{2}\big). \tag{1.8}$$

If $l' + 1 \leq j'$, then for every integer $n \in [l'+1, j']$ there exist two radial solutions u_n and v_n to (1.1) *such that $v_n(0) < 0 < u_n(0)$. Moreover, u_n and v_n have exactly n zeros in $(0,1)$.*

Roughly speaking, Theorems A and B ensure the existence of radial solutions to (1.1), provided there is a sufficiently large gap between g_0 and g_∞. In turns, this is equivalent to require that the asymptotic behaviours of g at zero and infinity are different.

In the literature there are several papers concerning boundary value problems when the nonlinearity is asymptotically linear [3, 4, 9, 11, 15, 17, 18, 20, 21, 24]. In [9, 18, 24] the authors study equations like $u'' + f(t,u) = p(t,u,u')$, (then $N = 1$), when f is asymptotically linear only at infinity; they give existence and multiplicity results depending on the function p.

A completely different approach is considered in [11]: according to the celebrated conjecture in [21], the authors prove multiplicity results for (1.1), with $N = 1$, assuming conditions like (1.2)-(1.3) and q strictly positive in $[0, 1]$. On the same lines, results comparing the behaviour of g at zero and infinity are obtained in [17] for autonomous equations involving p-laplacian like operators. As for the case $N > 1$, we refer to the paper [15], where a result similar to Theorems A and B is given in the autonomous case. We also mention the very recent article [4], which generalizes to systems of second order equations many of the previous quoted results.

Finally, we refer to the papers [3, 20] for the search of multiple solutions to (1.1) in the non-radial case.

To the best of our knowledge, the study of asymptotically linear problems when q has zeros in $[0, 1]$, as in the present article, is new. Indeed, there are few papers dealing with non-negative functions q having zeros in $[0, 1]$. In [19] some qualitative and oscillatory results are given; in [13, 14] the authors study the existence of multiple positive solutions to (1.1), in dimension $N = 1$, assuming that the limits in (1.2) and (1.3) are zero or infinity.

In order to prove Theorems A and B, we recall that the search of radial solutions to (1.1) can be performed through the study of the boundary value problem

$$(r^{N-1}u')' + r^{N-1}q(r)g(u) = 0, \quad u'(0) = 0 = u(1), \tag{1.9}$$

where $r = |x|$ and $u = u(r)$. Therefore, classical methods of ordinary differential equations can be applied. In particular, we combine some phase-plane analysis with a topological degree approach; indeed, we introduce the homotopy

$$(r^{s(N-1)}u')' + r^{s(N-1)}q_s(r)g(u) = 0, \quad u'(0) = 0 = u(1), \tag{1.10}$$

where $q_s(r) = sq(r) + (1 - s)q_0$, $s \in [0, 1]$, which carries (1.9) into an autonomous non-singular problem (corresponding to $s = 0$). Then, in order to study (1.10), we apply a Leray-Schauder type continuation theorem (see Theorem 3.9 and [7]) for an abstract fixed point equation equivalent to (1.10).

To this end, we observe that a compactness argument (see (2.34) and the related discussion) shows that nontrivial solutions u to (1.10) have a finite number $\mathbf{n}(u)$ of (simple) zeros in $(0, 1)$. The continuation theorem will give the existence of solutions to (1.10) with a certain number of zeros, provided some admissibility condition is fulfilled (see (3.23)). This leads to the study of the asymptotic properties of $\mathbf{n}(u)$ as a function of initial data of u; the needed estimates (see Propositions 2.11, 2.15, 2.18 and 2.21) are obtained by a phase-plane analysis which involves the growth conditions of g and q. We also notice that results analogous to Theorems A and B can be obtained when the laplacian Δ in (1.1) is replaced by the p-laplacian (see also Remark 3.11). The author wishes to thank Professor Anna Capietto for useful suggestions.

2. Qualitative results and asymptotic behaviour of the rotation number

In this section we are concerned with the differential equation

$$(r^{s(N-1)}u')' + r^{s(N-1)}q_s(r)g(u) = 0, \tag{2.1}$$

where $N \geq 1$, $q_s \in C([0,1], \mathbf{R}^+)$, for every $s \in [0,1]$, and $g : \mathbf{R} \longrightarrow \mathbf{R}$ is a locally Lipschitz function. Moreover, the function $s \mapsto q_s$ is continuous. For every $u \in \mathbf{R}$, we set $G(u) = \int_0^u g(v)\,dv$. We will prove some results concerning the behaviour of solutions to initial value problems or boundary value problems associated to (2.1).

To this aim, we will assume the following conditions on g and q_s:

$$g(u)u > 0, \quad \forall\, u \in \mathbf{R},\ u \neq 0, \tag{2.2}$$

$$\lim_{|u|\to+\infty} G(u) = +\infty, \tag{2.3}$$

$$\exists\, a' > 0 : \quad |g(u)| \leq a'|u|, \quad \forall\, u \in \mathbf{R}, \tag{2.4}$$

$$\exists\, C_{g,G} > 0 : \quad |g(u)|^2 \leq C_{g,G}G(u), \quad \forall\, u \in \mathbf{R}, \tag{2.5}$$

$$q_s(r) \geq 0, \quad \forall\, r \in [0,1],\ s \in [0,1]. \tag{2.6}$$

Throughout the paper, we will denote by (H) the set of assumptions (2.2)-(2.3)-(2.4)-(2.5)-(2.6). Moreover, for every function u defined in $[0,1]$, we denote

$$||u||_1 = \max_{r\in[0,1]} \sqrt{u(r)^2 + u'(r)^2}.$$

Existence and uniqueness results for initial value problems associated to (2.1). We first give a global existence and continuous dependence result for (2.1); in the proof we will adapt some arguments introduced in [6]. In opposition to [6], in the present case the function q_s could have zeros in $[0,1]$.

Proposition 2.1. *Assume that* (2.4) *and* (2.5) *hold. Then, for every* $\epsilon > 0$ *there exists* $d_\epsilon \in (0,\epsilon]$ *such that if* $u_{d,s}$ *is the solution to*

$$\begin{cases} (r^{s(N-1)}u')' + r^{s(N-1)}q_s(r)g(u) = 0 \\ u(0) = d,\ u'(0) = 0, \end{cases} \tag{2.7}$$

with $|d| \le d_\epsilon$*, then* $u_{d,s}$ *can be defined on* $[0,1]$ *and* $||u_{d,s}||_1 \le \epsilon$*.*

Remark 2.2. 1. We observe that the existence and uniqueness of the solution to (2.7) follow from standard fixed point arguments (see e.g. [23]).

2. We stress the fact that the constant d_ϵ given in Proposition 2.1 is independent from $s \in [0,1]$.

Proposition 2.1 is based on some estimates on the function E_s defined by

$$E_s(r,d) = \frac{1}{2}u'_{d,s}(r)^2 + G(u_{d,s}(r)). \tag{2.8}$$

Proof. Let $u_{d,s}$ be the solution of (2.7) and let $[0,\rho] \subset [0,1]$ be the maximal interval of definition of $u_{d,s}$; we will show that, when $|d|$ is sufficiently small, $u_{d,s}$ and $u'_{d,s}$ are uniformly bounded in $[0,\rho]$.

For simplicity, we will denote the function $u_{d,s}$ by u and we will set

$$v(r) = E_s(r,d), \quad \forall\ r \in [0,\rho]. \tag{2.9}$$

Using the fact that u is a solution to (2.1), we simply deduce

$$\begin{aligned} v'(r) &= u'(r)u''(r) + g(u(r))u'(r) \\ &= -\frac{s(N-1)}{r}u'(r)^2 + (1-q_s(r))\, g(u(r))u'(r) \\ &\le (1-q_s(r))\, g(u(r))u'(r), \quad \forall\ r \in [0,\rho]; \end{aligned} \tag{2.10}$$

therefore, from assumption (2.5), we obtain

$$\begin{aligned} v'(r) &\le (1-q_s(r))\, g(u(r))u'(r) \le \frac{1}{2}\left((1-q_s(r))^2 g(u(r))^2 + u'(r)^2\right) \\ &\le \frac{1}{2}\left(CC_{g,G}G(u(r)) + u'(r)^2\right) \le \frac{2+CC_{g,G}}{2}v(r), \quad \forall\ r \in [0,\rho], \end{aligned} \tag{2.11}$$

where $C = \max\{(1-q_s(r))^2 : r \in [0,1],\ s \in [0,1]\}$. Now, integrating (2.11) on $(0,r)$, we infer that

$$v(r) \le v(0)e^{(2+CC_{g,G})r/2} \le HG(d), \tag{2.12}$$

where $H = \exp((2+CC_{g,G})/2)$. Next, we fix $(a_1,a_2) \in (0,1)^2$ such that $a_1 + a_2 \le 1/2$ and $2a_2 < 1$; moreover, let us consider $d_\epsilon \in (0, a_1\epsilon)$ such that

$$HG(d) \le \frac{a_2^2\epsilon^2}{2}, \quad \forall\ |d| \le d_\epsilon. \tag{2.13}$$

From (2.12), we then deduce that

$$|u'(r)| \le \sqrt{2v(r)} \le a_2\epsilon \le \frac{\epsilon}{2}, \quad \forall\ r \in [0,\rho]$$

and

$$|u(r)| \le d + \int_0^r |u'(t)|\, dt \le (a_1 + a_2)\epsilon \le \frac{\epsilon}{2}, \quad \forall\, r \in [0, \rho].$$

This implies that u can be extended to $[0, 1]$ and that

$$||u||_1 = \max_{r\in[0,1]} \sqrt{|u(r)|^2 + |u'(r)|^2} \le \epsilon. \quad \square$$

In the sequel, we will also be interested in solutions u of (2.1) satisfying $u(r_0) = 0 = u'(r_0)$, for some $r_0 \in (0, 1]$. For these solutions, arguing as in the proof of Proposition 2.1, it is possible to show the validity of the following:

Proposition 2.3. *Assume that conditions* (2.4) *and* (2.5) *hold. Let u be a solution of*

$$\begin{cases} (r^{s(N-1)}u')' + r^{s(N-1)}q_s(r)g(u) = 0 \\ u(r_0) = 0 = u'(r_0), \ \ r_0 \in (0, 1]. \end{cases} \tag{2.14}$$

Then, $u \equiv 0$ in $[0, 1]$.

Global behaviour of solutions with large initial values. In what follows, we will consider solutions to (2.1) with $u'(0) = 0$ and $u(1) = 0$; we assume that $u(0) = d > 0$. The case when $u(0) < 0$ can be treated in a similar way. Let us first define

$$\hat{g}(x) = \begin{cases} \sup_{v\in[0,x]} g(v) & \text{if } x \ge 0 \\ \sup_{v\in[x,0]} g(v) & \text{if } x < 0 \end{cases} \tag{2.15}$$

and let us set

$$Q = \max\{q_s(r) : \ r \in [0, 1], \ s \in [0, 1]\}. \tag{2.16}$$

For every $d > 0$, let us consider the solution $u = u_{d,s}$ to (2.1) such that $u(0) = d$ and $u'(0) = 0$; from (2.2) and (2.6), we deduce that

$$(r^{s(N-1)}u')' = -r^{s(N-1)}q_s(r)g(u(r)) \le 0,$$

for every $r \in [0, 1]$ such that $u(r) > 0$. Since $u(1) = 0$, for every $\theta \in [0, 1)$ we can consider the first number $r_0(d)$ such that

$$u(r_0(d)) = \theta d. \tag{2.17}$$

We now prove an estimate on r_0, depending on q_s and g; the technique based on the study of the asymptotic properties of r_0 has been introduced in [8] and it is now standard in developing shooting argument for solutions to (2.1) (see also [6, 16]):

Lemma 2.4. *Assume that conditions* (2.2) *and* (2.6) *hold. Moreover, let* $\hat{g}$*,* Q *and* r_0 *as in* (2.15), (2.16) *and* (2.17), *respectively. Then, for every* $d > 0$ *and for every* $\theta \in (0,1)$*, we have*

$$r_0(d) \geq \sqrt{\frac{2(1-\theta)}{Q}}\sqrt{\frac{d}{\hat{g}(d)}}. \tag{2.18}$$

Remark 2.5. An analogous statement holds true when $d < 0$.

Proof. Let us consider $d > 0$, $\theta \in (0,1)$ and r_0 as in (2.17). For every $r \in (0, r_0(d))$, we integrate (2.1) from 0 to r to obtain

$$r^{s(N-1)}u'(r) = -\int_0^r t^{s(N-1)}q_s(t)g(u(t))\,dt, \quad \forall\, r \in (0, r_0(d));$$

hence, recalling (2.16), we deduce that

$$-r^{s(N-1)}u'(r) \leq Q\hat{g}(d)\frac{r^{s(N-1)+1}}{s(N-1)+1}, \quad \forall\, r \in (0, r_0(d)),$$

i.e.,

$$u'(r) \geq -\frac{Q\hat{g}(d)}{s(N-1)+1}r, \quad \forall\, r \in (0, r_0(d)).$$

By integrating this relation from 0 to $r_0(d)$, recalling (2.17), we get

$$(1-\theta)d \leq \frac{Q\hat{g}(d)}{2(s(N-1)+1)}r_0(d)^2$$

and so

$$r_0(d) \geq \sqrt{\frac{2(s(N-1)+1)(1-\theta)}{Q}}\sqrt{\frac{d}{\hat{g}(d)}}.$$

The result is proved. □

By means of (2.18) we are able to give an important estimate on the function E_s defined in (2.8). Indeed, we can prove the following:

Lemma 2.6. *Assume that condition (H) holds true. Then, we have*

$$\lim_{d \to +\infty} E_s(r,d) = +\infty,$$

uniformly in $r \in [0,1]$ *and* $s \in [0,1]$.

Remark 2.7. As before, a completely analogous result holds true when $d \to -\infty$.

Proof. For simplicity, we will write u instead of $u_{d,s}$. We fix $\theta \in (0,1)$; with r_0 as in (2.17), it is easy to see that for every $d > 0$ we have

$$E_s(r,d) \geq G(\theta d), \quad \forall\, r \in [0, r_0(d)],\ s \in [0,1]. \tag{2.19}$$

Therefore, by (2.3), for every $M > 0$ there exists $d'_M > 0$, independent from s, such that for every $d \geq d'_M$ we have

$$E_s(r,d) \geq M, \quad \forall\, r \in [0, r_0(d)],\ s \in [0,1]. \tag{2.20}$$

Let us now fix

$$\gamma = \max\{2N-1, C_{g,G}(1+Q)^2/2\}, \tag{2.21}$$

where $C_{g,G}$ and Q are given in (2.5) and (2.16), respectively. For every $r > 0$ we have

$$\begin{aligned} &E'_s(r,d) + \frac{\gamma}{r}E_s(r,d) \\ &= -\frac{s(N-1)}{r}u'(r)^2 + (1-q_s(r))g(u(r))u'(r) + \frac{\gamma}{2r}u'(r)^2 + \frac{\gamma}{r}G(u(r)); \end{aligned} \tag{2.22}$$

now, from (2.5) and recalling (2.16), we obtain

$$|(1-q_s(r))g(u(r))u'(r)| \leq \frac{1}{2}(1+Q)^2 g(u(r))^2 + \frac{1}{2}u'(r)^2, \tag{2.23}$$

for every $r \in [0,1]$ and $s \in [0,1]$. Therefore, from (2.22) and (2.23), we deduce that

$$\begin{aligned} E'_s(r,d) + \frac{\gamma}{r}E_s(r,d) \geq \big(\big(\frac{\gamma}{2} - s(N-1)\big)\frac{1}{r} - \frac{1}{2}\big)u'(r)^2 \\ + \frac{\gamma}{r}G(u(r)) - \frac{(1+Q)^2}{2}g(u(r))^2, \end{aligned} \tag{2.24}$$

for every $r > 0$. Recalling the choice of γ in (2.21) and using (2.5), we finally obtain

$$E'_s(r,d) + \frac{\gamma}{r}E_s(r,d) \geq \big(\frac{\gamma}{C_{g,G}}\frac{1}{r} - \frac{(1+Q)^2}{2}\big)g(u(r))^2 \geq 0, \tag{2.25}$$

for every $r > 0$. Now, let us multiply (2.25) by $r_0(d)^\gamma$ and let us integrate from $r_0(d)$ to r: we obtain

$$E_s(r,d) \geq \frac{r_0^\gamma}{r^\gamma}E_s(r_0,d) \geq G(\theta d)r_0^\gamma, \tag{2.26}$$

for every $r > r_0$, $s \in [0,1]$. Using (2.18), we deduce that there exists $K > 0$ such that

$$E_s(r,d) \geq KG(\theta d)\big(\frac{d}{\hat{g}(d)}\big)^{\gamma/2}, \tag{2.27}$$

for every $r > r_0$, $s \in [0,1]$. Now, from conditions (2.4) and the definition of $\hat{g}$, we deduce that there exists $C' > 0$ such that

$$\frac{d}{\hat{g}(d)} \geq C', \quad \forall\, d \geq 0; \tag{2.28}$$

from (2.3) and (2.28) we obtain

$$\lim_{d\to+\infty} G(\theta d)\Big(\frac{d}{\hat{g}(d)}\Big)^{\gamma/2} = +\infty. \tag{2.29}$$

Therefore, from (2.27) and (2.29) we infer that for every $M > 0$ there exists $d''_M > 0$, independent from s, such that for every $d \geq d''_M$ we have

$$E_s(r,d) \geq M, \quad \forall\, r \in [r_0(d), 1],\ s \in [0,1]. \tag{2.30}$$

The relations (2.20) and (2.30) prove the result. □

Now, it is easy to deduce from Lemma 2.6 the following result:

Lemma 2.8. *Assume condition* (H). *Then, the following statements hold true:*

1. *For every* $R_1 > 0$ *there exists* $R_2 = R_2(R_1) \geq R_1$ *such that for every solution* $u = u_{d,s}$ *to* (2.7) *with* $u(1) = 0$ *we have*
$$|d| \leq R_1 \quad\Rightarrow\quad u(r)^2 + u'(r)^2 \leq R_2, \quad \forall\, r \in [0,R].$$
2. *For every* $R_1 > 0$ *there exists* $R_2 = R_2(R_1) \geq R_1$ *such that for every solution* $u = u_{d,s}$ *to* (2.7) *with* $u(1) = 0$, *we have*
$$|d| \geq R_2 \quad\Rightarrow\quad u(r)^2 + u'(r)^2 \geq R_1, \quad \forall\, r \in [0,R].$$

Remark 2.9. Lemma 2.8 is usually referred as the 'elastic property'; it is crucial in proving existence results for boundary value problems associated to (2.1), especially when g has a superlinear growth at infinity. In previous works (see e.g. [7, 16]) it has been proved in the case when $q_s > 0$.

On the number of zeros of solutions to (2.1). We pass to the study of the oscillatory properties of solutions to the boundary value problem

$$(r^{s(N-1)}u')' + r^{s(N-1)}q_s(r)g(u) = 0, \quad u'(0) = 0 = u(1). \tag{2.31}$$

Beside condition (H), we will suppose that there exist $q_0 > 0$ and $[r_1, r_2] \subset (0,1]$ such that

$$q_s(r) \geq q_0, \quad \forall\, r \in [r_1, r_2],\ s \in [0,1]. \tag{2.32}$$

Recalling Proposition 2.3, by means of a compactness argument, we deduce that every nontrivial solution u to (2.31) has a finite number of (simple) zeros in $[0,1]$. Moreover, from Proposition 2.1 we infer that for every $d>0$ there exists $\mu_d>0$ such that for every solution u to (2.31) we have

$$|u(0)|\geq d \quad \Rightarrow \quad u(r)^2+u'(r)^2\geq \mu_d, \quad \forall\, r\in[0,R]. \tag{2.33}$$

Next, for every $d>0$, let us denote by Σ_d the set of pairs (u,s), where u is a solution to (2.31) such that $|u(0)|\geq d$; by the previous remark, we can define the map

$$\begin{array}{rccc} \mathbf{n}: & \Sigma_d & \longrightarrow & \mathbf{N} \\ & (u,s) & \longmapsto & \mathbf{n}(u), \end{array} \tag{2.34}$$

where $\mathbf{n}(u)$ is the number of zeros of u in $[0,1)$. Moreover, from [16, Lemma 3.1] and (2.33), we immediately deduce the validity of the following:

Lemma 2.10. *Assume conditions* (2.4) *and* (2.5). *Then, the application* $\mathbf{n}$ *is continuous in* Σ_d*, for every* $d>0$.

Our aim is to study the asymptotic behaviour of $\mathbf{n}$ when $|u(0)|\to+\infty$ and $|u(0)|\to 0$. We will be able to give such estimates on $\mathbf{n}$ depending on the behaviour of g at infinity and zero, respectively. We give our first result:

Proposition 2.11. *Assume conditions* (H) *and* (2.32). *Moreover, suppose that there exist* $Z_\infty>0$ *and* $g_\infty^->0$ *such that*

$$\frac{g(u)}{u}\geq g_\infty^-, \quad \forall\ |u|\geq Z_\infty. \tag{2.35}$$

Then, for every $\chi>0$ *there exists* $d_\chi>0$ *such that for every solution* u *to* (2.31)*, we have*

$$|u(0)|\geq d_\chi \quad \Rightarrow \quad \mathbf{n}(u)\geq \sqrt{q_0 g_\infty^-}\,\frac{r_1^{N-1}(r_2-r_1)}{\pi}-1-\chi. \tag{2.36}$$

Proof. We first take $\eta=\eta(\chi)>0$ such that

$$\sqrt{q_0 g_\infty^-}\,\frac{r_1^{N-1}(r_2-r_1)}{\pi}-1-\chi=\frac{r_2-r_1-2\eta-\dfrac{\pi}{\sqrt{r_1^{2(N-1)}q_0 g_\infty^-}}}{2\eta+\dfrac{\pi}{\sqrt{r_1^{2(N-1)}q_0 g_\infty^-}}}. \tag{2.37}$$

With this choice of η, we are led to prove that there exists $d_\chi>0$ such that

$$|u(0)|\geq d_\chi \quad \Rightarrow \quad \mathbf{n}(u)\geq \frac{r_2-r_1-2\eta-\dfrac{\pi}{\sqrt{r_1^{2(N-1)}q_0 g_\infty^-}}}{2\eta+\dfrac{\pi}{\sqrt{r_1^{2(N-1)}q_0 g_\infty^-}}}. \tag{2.38}$$

It is easy to see that

$$1 \le r^{-s(N-1)} \le r_1^{-(N-1)}, \quad \forall\, r \in [r_1, r_2],\ s \in [0,1]. \tag{2.39}$$

Moreover, from assumption (2.32) and (2.35) we deduce that

$$r^{s(N-1)} q_s(r) g(u) u \ge q_0 r_1^{N-1} g_\infty^- u^2, \quad \forall\, r \in [r_1, r_2],\ |u| \ge Z_\infty,\ s \in [0,1]. \tag{2.40}$$

An application of Proposition 2.8, Statement 2, with $R_1 = (1 + 4/\eta(\xi)^2) Z_\infty^2$ gives the existence of $d_\chi = R_2(R_1) > 0$ such that for every solution u to (2.31), we have

$$|u(0)| \ge d_\chi \quad \Rightarrow \quad u(r)^2 + u'(r)^2 \ge \big(1 + \frac{4}{\eta^2}\big) Z_\infty^2,\ \forall\, r \in [r_1, r_2]. \tag{2.41}$$

From now on, we will consider a solution u satisfying $|u(0)| \ge d_\chi$; in order to prove (2.38), it is sufficient to show that

$$|u(0)| \ge d_\chi \quad \Rightarrow \quad \mathbf{n}^*(u) \ge \frac{r_2 - r_1 - 2\eta - \dfrac{\pi}{\sqrt{r_1^{2(N-1)} q_0 g_\infty^-}}}{2\eta + \dfrac{\pi}{\sqrt{r_1^{2(N-1)} q_0 g_\infty^-}}}, \tag{2.42}$$

where $\mathbf{n}^*(u)$ denotes the number of zeros of u in $[r_1, r_2)$. To this aim, we will estimate the length of the intervals where $|u(r)| \le Z_\infty$ or $|u(r)| \ge Z_\infty$, respectively.

First, let us consider an interval $[r', r''] \subset [r_1, r_2]$ where $|u(r)| \le Z_\infty$. By (2.41), we necessarily have

$$|u'(r)| \ge \frac{2Z_\infty}{\eta}, \quad \forall\, r \in [r', r'']. \tag{2.43}$$

Assume for instance that $u'(r) \ge 2Z_\infty/\eta$ (the other case is similar). We infer that

$$u(r'') = u(r') + \int_{r'}^{r''} u'(r)\, dr \ge u(r') + \frac{2Z_\infty}{\eta}(r'' - r'); \tag{2.44}$$

recalling that $|u(r)| \le Z_\infty$ in $[r', r'']$, we finally get

$$r'' - r' \le \eta. \tag{2.45}$$

Now, we assume that $[r', r'']$ is such that $|u(r)| \ge Z_\infty$, for every $r \in [r', r'']$, and $|u(r')| = |u(r'')| = Z_\infty$. Again, we suppose that $u(r) \ge Z_\infty$ in $[r', r'']$

(the other case can be treated in a similar way). We recall that the equation in (2.31) is equivalent to the first order system

$$\begin{cases} u' = r^{-s(N-1)}y \\ y' = -r^{s(N-1)}q_s(r)g(u). \end{cases} \tag{2.46}$$

From the second equation in (2.46) and condition (2.2) we deduce that y is decreasing in $[r', r'']$. Moreover, since u has a maximum in $[r', r'']$, there exists $r^* \in [r', r'']$ such that $y(r^*) = 0$; this point r^* is unique and so we can conclude that

$$\begin{array}{l} y(r) \geq 0 \text{ for every } r \in [r', r^*] \\ y(r) \leq 0 \text{ for every } r \in [r^*, r''] \end{array} \tag{2.47}$$

and

$$\begin{array}{l} u \text{ is increasing in } [r', r^*] \\ u \text{ is decreasing in } [r^*, r'']. \end{array} \tag{2.48}$$

Now, we consider the interval $[r', r^*]$. From (2.39)-(2.40) and (2.46) we deduce that

$$\begin{array}{l} u'(r) \leq r_1^{-(N-1)} y(r) \\ y'(r) \leq -q_0 r_1^{N-1} g_\infty^- u(r), \end{array} \tag{2.49}$$

for every $r \in [r', r^*]$. We then multiply by $q_0 r_1^{N-1} g_\infty^- u(r)$ the first inequality in (2.49) and by $r_1^{-(N-1)} y(r)$ the second one; adding up, we obtain

$$r_1^{2(N-1)} q_0 g_\infty^- u(r) u'(r) + y(r) y'(r) \leq 0, \quad \forall\, r \in [r', r^*]. \tag{2.50}$$

This implies that the function

$$\frac{1}{2} y(r)^2 + \frac{r_1^{2(N-1)} q_0 g_\infty^-}{2} u(r)^2 \tag{2.51}$$

is decreasing in $[r', r^*]$; therefore, we deduce that

$$y(r^*)^2 + r_1^{2(N-1)} q_0 g_\infty^- u(r^*)^2 \leq u'(r)^2 + r_1^{2(N-1)} q_0 g_\infty^- u(r)^2, \tag{2.52}$$

for every $r \in [r', r^*]$. This implies that

$$r_1^{2(N-1)} q_0 g_\infty^- \left[u(r^*)^2 - u(r)^2 \right] \leq u'(r)^2, \tag{2.53}$$

for every $r \in [r', r^*]$. Solving with respect to $u'(r)$ and integrating on (r', r^*), we obtain

$$r^* - r' \leq \frac{1}{\sqrt{r_1^{2(N-1)} q_0 g_\infty^-}} \int_{Z_\infty}^{u(r^*)} \frac{dz}{\sqrt{u(r^*)^2 - z^2}} \leq \frac{\pi}{2} \frac{1}{\sqrt{r_1^{2(N-1)} q_0 g_\infty^-}}. \tag{2.54}$$

In an analogous way, we are able to prove that

$$r'' - r^* \le \frac{\pi}{2} \frac{1}{\sqrt{r_1^{2(N-1)} q_0 g_\infty^-}}. \tag{2.55}$$

From (2.54) and (2.55) we conclude that

$$r'' - r' \le \frac{\pi}{\sqrt{r_1^{2(N-1)} q_0 g_\infty^-}}. \tag{2.56}$$

Now, using (2.45) and (2.56), a simple computation shows that

$$r_2 - r_1 \le (\mathbf{n}^*(u) + 1)\Big(2\eta + \frac{\pi}{\sqrt{r_1^{2(N-1)} q_0 g_\infty^-}}\Big).$$

We then deduce that

$$\mathbf{n}^*(u) \ge \frac{r_2 - r_1 - 2\eta - \dfrac{\pi}{\sqrt{r_1^{2(N-1)} q_0 g_\infty^-}}}{2\eta + \dfrac{\pi}{\sqrt{r_1^{2(N-1)} q_0 g_\infty^-}}};$$

hence, (2.42) is fulfilled and the result is proved. □

Remark 2.12. When $q_s(r) \ge q_0$ for every $r \in [0, r_2]$, the result of Proposition 2.11 holds true for every $r_1 \in [0, r_2]$. It is useful to observe that the natural choice $r_1 = 0$ leads to a very poor estimate (due to the presence of the factor r_1^{N-1} in (2.36)); henceforth, the optimal choice is the value r_1 which maximizes the quantity $r_1^{N-1}(r_2 - r_1)$. It is easy to see that this value is $r_1 = r_2(1 - 1/N)$; according to this observation, we can restate the result as follows:

Proposition 2.13. *Assume conditions* (H) *and* (2.35). *Moreover, suppose that there exist* $q_0 > 0$ *and* $[0, r_2] \subset [0, 1]$ *such that*

$$q_s(r) \ge q_0, \quad \forall\ r \in [0, r_2],\ s \in [0, 1].$$

Then, for every $\chi > 0$ *there exists* $d_\chi > 0$ *such that for every solution* u *to* (2.31), *we have*

$$|u(0)| \ge d_\chi \quad \Rightarrow \quad \mathbf{n}(u) \ge \sqrt{q_0 g_\infty^-}\, \frac{r_2^N (N-1)^{N-1}}{N^N \pi} - 1 - \chi. \tag{2.57}$$

Remark 2.14. In the case when $N = 1$ and $q(r) \geq q_0$ on $[0,1]$, it is possible to refine the proof and to show that

$$|u(0)| \geq d_\chi \quad \Rightarrow \quad \mathbf{n}(u) \geq \frac{\sqrt{q_0 g_\infty^-}}{\pi} - \frac{1}{2} - \chi. \tag{2.58}$$

We observe that, when dealing with solutions u satisfying the boundary conditions $u'(0) = 0 = u(1)$, the presence of the term $1/2$ in (2.58) is standard. Indeed, as it is clear for instance from [10, Sect. 3], the estimate (2.58) is sharp.

Proposition 2.15. *Assume conditions* (H). *Moreover, suppose that there exist* $Z_\infty > 0$ *and* $g_\infty^+ > 0$ *such that*

$$\frac{g(u)}{u} \leq g_\infty^+, \quad \forall\ |u| \geq Z_\infty. \tag{2.59}$$

Finally, let Q *as in* (2.16). *Then, for every* $\chi > 0$ *there exists* $d'_\chi > 0$ *such that for every solution* u *to* (2.31), *we have*

$$|u(0)| \geq d'_\chi \quad \Rightarrow \quad \mathbf{n}(u) \leq \frac{(Qg_\infty^+)^{(N-1)/2}}{\pi}(\sqrt{Qg_\infty^+} - 1) + \chi. \tag{2.60}$$

Remark 2.16. It is obvious that, when $Qg_\infty^+ = 1$, the estimate (2.60) means that $\mathbf{n}(u) = 0$, i.e., u is positive in $(0,1)$. Moreover, when $Qg_\infty^+ < 1$, (2.60) simply means that (3.1) has no solutions u with $|u(0)| \geq d'_\chi$.

An analogous observation holds for (2.97) in Proposition 2.21.

Proof. We first consider the case $Qg_\infty^+ > 1$. We take $\eta = \eta(\chi) > 0$ such that

$$\frac{(Qg_\infty^+)^{(N-1)/2}}{\pi}(\sqrt{Qg_\infty^+} - 1) + \chi = \frac{(Qg_\infty^+)^{(N-1)/2}}{\pi - 2\eta}(\sqrt{Qg_\infty^+} - 1). \tag{2.61}$$

With this choice, we are led to prove that there exists $d'_\chi > 0$ such that

$$|u(0)| \geq d'_\chi \quad \Rightarrow \quad \mathbf{n}(u) \leq \frac{(Qg_\infty^+)^{(N-1)/2}}{\pi - 2\eta}(\sqrt{Qg_\infty^+} - 1). \tag{2.62}$$

Now, let $\eta_* \in (0,1)$ be such that

$$|x| \leq \eta_* \quad \Rightarrow \quad |\arcsin x| \leq \eta. \tag{2.63}$$

From (2.16) and (2.59) we deduce that

$$r^{s(N-1)} q_s(r) g(u) u \leq Qg_\infty^+ u^2, \quad \forall\ r \in [0,1],\ |u| \geq Z_\infty,\ s \in [0,1]. \tag{2.64}$$

Using (2.59) and recalling the definition of $\hat{g}$ (see (2.15), it is easy to deduce that there exists $d'' > 0$ such that for every $|d| \geq d''$ we have

$$\frac{\sqrt{2}}{\sqrt{Q}}\sqrt{\frac{d}{\hat{g}(d)}} \geq \frac{1}{\sqrt{Qg_\infty^+}}. \tag{2.65}$$

Now, we apply Proposition 2.8, Statement 2, with $R_1 = Z_\infty^2/{\eta_*}^2$: we infer that there exists $d_\chi^* = R_2(R_1) \geq Z_\infty > 0$ such that for every solution u to (2.31) we have

$$|u(0)| \geq d_\chi^* \quad \Rightarrow \quad u(r)^2 + u'(r)^2 \geq \frac{Z_\infty^2}{\eta_*^2},\ \forall\ r \in [0,1]. \tag{2.66}$$

From now on, we will consider a solution u satisfying $|u(0)| \geq d'_\chi$, with $d'_\chi = \max(d'', d_\chi^*)$. We estimate the length of an interval $[r', r''] \subset (0,1)$ such that $|u(r)| \geq Z_\infty$, for every $r \in [r', r'']$, and $|u(r')| = |u(r'')| = Z_\infty$. We suppose that $u(r) \geq Z_\infty$ in $[r', r'']$ (the other case can be treated in a similar way); arguing as in the proof of Proposition 2.11, we deduce that y is decreasing in $[r', r'']$ and there exists $r^* \in [r', r'']$ such that

$$\begin{aligned} &y(r) \geq 0 \text{ for every } r \in [r', r^*] \\ &y(r) \leq 0 \text{ for every } r \in [r^*, r''] \end{aligned} \tag{2.67}$$

and

$$\begin{aligned} &u \text{ is increasing in } [r', r^*] \\ &u \text{ is decreasing in } [r^*, r'']. \end{aligned} \tag{2.68}$$

Moreover, since $u'(r^*) = 0$, from (2.66) we deduce that $u(r^*) \geq Z_\infty/\eta_*$; therefore, from (2.63) we get

$$\arcsin \frac{Z_\infty}{u(r^*)} \leq \eta. \tag{2.69}$$

Now, we consider the interval $[r', r^*]$. From (2.39) and (2.64) we deduce that

$$\begin{aligned} &u'(r) \geq y(r) \\ &y'(r) \geq -Qg_\infty^+ u(r), \end{aligned} \tag{2.70}$$

for every $r \in [r', r^*]$. We then multiply by $Qg_\infty^+ u(r)$ the first inequality in (2.70) and by $y(r)$ the second one; adding up, we obtain

$$Qg_\infty^+ u(r)u'(r) + y(r)y'(r) \geq 0, \quad \forall\ r \in [r', r^*]. \tag{2.71}$$

This implies that the function

$$\frac{1}{2}y(r)^2 + \frac{Qg_\infty^+}{2}u(r)^2 \tag{2.72}$$

is increasing in $[r', r^*]$; therefore, we deduce that

$$y(r^*)^2 + Qg_\infty^+ u(r^*)^2 \geq {r'}^{2(N-1)} u'(r)^2 + Qg_\infty^+ u(r)^2, \tag{2.73}$$

for every $r \in [r', r^*]$. This implies that

$$Qg_\infty^+ \left[u(r^*)^2 - u(r)^2\right] \geq {r'}^{2(N-1)} u'(r)^2, \tag{2.74}$$

for every $r \in [r', r^*]$. Solving with respect to $u'(r)$ and integrating on (r', r^*), taking into account (2.69), we obtain

$$r^* - r' \geq \frac{{r'}^{N-1}}{\sqrt{Qg_\infty^+}} \int_{Z_\infty}^{u(r^*)} \frac{dz}{\sqrt{u(r^*)^2 - z^2}} \geq \frac{{r'}^{N-1}}{\sqrt{Qg_\infty^+}} \left(\frac{\pi}{2} - \eta\right). \tag{2.75}$$

In an analogous way, we are able to prove that

$$r'' - r^* \geq \frac{{r'}^{N-1}}{\sqrt{Qg_\infty^+}} \left(\frac{\pi}{2} - \eta\right). \tag{2.76}$$

From (2.75) and (2.76) we conclude that

$$r'' - r' \geq \frac{{r'}^{N-1}}{\sqrt{Qg_\infty^+}} (\pi - 2\eta). \tag{2.77}$$

Now, let us denote by r_0 the first zero of u; r_0 is the point defined by (2.17) with $\theta = 0$. Therefore, from (2.18) and (2.65), we have

$$r_0 \geq \frac{1}{\sqrt{Qg_\infty^+}}. \tag{2.78}$$

Finally, using (2.77) and (2.78), a simple computation shows that

$$1 \geq \mathbf{n}(u)\,(\pi - 2\eta) \left(\frac{1}{\sqrt{Qg_\infty^+}}\right)^N + \frac{1}{\sqrt{Qg_\infty^+}}.$$

We then deduce that

$$\mathbf{n}(u) \leq \frac{(\sqrt{Qg_\infty^+})^N - (\sqrt{Qg_\infty^+})^{(N-1)}}{\pi - 2\eta}$$

and this proves the result. When $Qg_\infty^+ = 1$, from (2.78) we immediately deduce that for large initial values any solution u to (3.1) is positive in $(0, 1)$. Analogously, estimate (2.78) implies that (3.1) has no solutions with large initial values if $Qg_\infty^+ < 1$. □

Remark 2.17. In the case when $N = 1$ and $q(r) \geq q_0$ on $[0, 1]$, it is possible to improve the result and to obtain the sharp estimate

$$|u(0)| \geq d'_\chi \quad \Rightarrow \quad \mathbf{n}(u) \leq \frac{\sqrt{Qg_\infty^+}}{\pi} - \frac{1}{2} + \chi. \tag{2.79}$$

Proposition 2.18. *Assume conditions* (H) *and* (2.32)*. Moreover, suppose that there exist* $Z_0 > 0$ *and* $g_0^- > 0$ *such that*

$$\frac{g(u)}{u} \geq g_0^-, \quad \forall\ |u| \leq Z_0,\ u \neq 0. \tag{2.80}$$

Then, there exists $d_0 > 0$ *such that for every solution* u *to* (2.31)*, we have*

$$|u(0)| \leq d_0 \quad \Rightarrow \quad \mathbf{n}(u) \geq \sqrt{q_0 g_0^-}\,\frac{r_1^{N-1}(r_2 - r_1)}{\pi} - 1. \tag{2.81}$$

Proof. We first apply Proposition 2.1 with $\epsilon = Z_0$. We get the existence of $d_0 = d_{Z_0}$ such that for every solution u to (2.31) we have

$$|u(0)| \leq d_0 \quad \Rightarrow \quad |u(r)| \leq Z_0, \quad \forall\ r \in [0,1]. \tag{2.82}$$

From now on, we will consider a solution u satisfying $|u(0)| \leq d_0$. From assumptions (2.32) and (2.80), recalling (2.82), we deduce that

$$r^{s(N-1)} q_s(r) g(u(r)) u(r) \geq q_0 r_1^{N-1} g_0^- u(r)^2, \quad \forall\ r \in [r_1, r_2],\ s \in [0,1]. \tag{2.83}$$

We now estimate the length of an interval $[r', r''] \subset [r_1, r_2]$, where r' and r'' are two consecutive zeros of u. Without loss of generality, we assume that $u(r) > 0$, for every $r \in (r', r'')$. Arguing as in the proof of Proposition 2.11, it is easy to see that there exists $r^* \in (r', r'')$ such that

$$\begin{aligned} &y(r) \geq 0 \text{ for every } r \in [r', r^*] \\ &y(r) \leq 0 \text{ for every } r \in [r^*, r''] \end{aligned} \tag{2.84}$$

and

$$\begin{aligned} &u \text{ is increasing in } [r', r^*] \\ &u \text{ is decreasing in } [r^*, r'']. \end{aligned} \tag{2.85}$$

Now, we consider the interval $[r', r^*]$. From (2.39)-(2.40) and (2.83) we deduce that

$$\begin{aligned} &u'(r) \leq r_1^{-(N-1)} y(r) \\ &y'(r) \leq -q_0 r_1^{N-1} g_0^- u(r), \end{aligned} \tag{2.86}$$

for every $r \in [r', r^*]$. We then multiply by $q_0 r_1^{N-1} g_0^- u(r)$ the first inequality in (2.86) and by $r_1^{-(N-1)} y(r)$ the second one; adding up, we obtain

$$r_1^{2(N-1)} q_0 g_0^- u(r) u'(r) + y(r) y'(r) \leq 0, \quad \forall\ r \in [r', r^*]. \tag{2.87}$$

This implies that the function

$$\frac{1}{2} y(r)^2 + \frac{r_1^{2(N-1)} q_0 g_0^-}{2} u(r)^2 \tag{2.88}$$

is decreasing in $[r', r^*]$; therefore, we deduce that

$$y(r^*)^2 + r_1^{2(N-1)} q_0 g_0^- u(r^*)^2 \leq u'(r)^2 + r_1^{2(N-1)} q_0 g_0^- u(r)^2, \tag{2.89}$$

for every $r \in [r', r^*]$. This implies that

$$r_1^{2(N-1)} q_0 g_0^- \left[u(r^*)^2 - u(r)^2\right] \leq u'(r)^2, \tag{2.90}$$

for every $r \in [r', r^*]$. Solving with respect to $u'(r)$ and integrating on (r', r^*), we obtain

$$r^* - r' \leq \frac{1}{\sqrt{r_1^{2(N-1)} q_0 g_0^-}} \int_0^{u(r^*)} \frac{dz}{\sqrt{u(r^*)^2 - z^2}} = \frac{\pi}{2} \frac{1}{\sqrt{r_1^{2(N-1)} q_0 g_0^-}}. \tag{2.91}$$

In an analogous way, we are able to prove that

$$r'' - r^* \leq \frac{\pi}{2} \frac{1}{\sqrt{r_1^{2(N-1)} q_0 g_0^-}}. \tag{2.92}$$

From (2.91) and (2.92) we conclude that

$$r'' - r' \leq \frac{\pi}{\sqrt{r_1^{2(N-1)} q_0 g_0^-}}. \tag{2.93}$$

Now, a simple computation shows that

$$r_2 - r_1 \leq (\mathbf{n}^*(u) + 1) \frac{\pi}{\sqrt{r_1^{2(N-1)} q_0 g_0^-}},$$

where $\mathbf{n}^*(u)$ denotes the number of zeros of u in $[r_1, r_2)$. We then deduce that

$$\mathbf{n}^*(u) \geq \sqrt{q_0 g_0^-} \frac{r_1^{N-1}(r_2 - r_1)}{\pi} - 1$$

and the result is proved. □

Arguing as in Remark 2.12, it is possible to prove the following:

Proposition 2.19. *Assume conditions* (H) *and* (2.80). *Moreover, suppose that there exist* $q_0 > 0$ *and* $[0, r_2] \subset [0, 1]$ *such that*

$$q_s(r) \geq q_0, \quad \forall\, r \in [0, r_2],\ s \in [0, 1].$$

Then, there exists $d_0 > 0$ *such that for every solution* u *to* (2.31) *we have*

$$|u(0)| \leq d_0 \quad \Rightarrow \quad \mathbf{n}(u) \geq \sqrt{q_0 g_0^-} \frac{r_2^N (N-1)^{N-1}}{N^N \pi} - 1. \tag{2.94}$$

Remark 2.20. As in Proposition 2.11, when $N = 1$ and $q(r) \geq q_0$ on $[0, 1]$, it is possible to refine the proof and to show that

$$|u(0)| \leq d_0 \quad \Rightarrow \quad \mathbf{n}(u) \geq \frac{\sqrt{q_0 g_0^-}}{\pi} - \frac{1}{2}. \tag{2.95}$$

Proposition 2.21. *Assume conditions* (H). *Moreover, suppose that there exist* $Z_0 > 0$ *and* $g_0^+ > 0$ *such that*

$$\frac{g(u)}{u} \leq g_0^+, \quad \forall\ |u| \leq Z_0,\ u \neq 0. \tag{2.96}$$

Finally, let Q *as in* (2.16). *Then, there exists* $d_0' > 0$ *such that for every solution* u *to* (2.31)*, we have*

$$|u(0)| \leq d_0' \quad \Rightarrow \quad \mathbf{n}(u) \leq \frac{(Qg_0^+)^{(N-1)/2}}{\pi}(\sqrt{Qg_0^+} - 1). \tag{2.97}$$

Proof. We first apply Proposition 2.1 with $\epsilon = Z_0$: we get the existence of $d_0' = d_{Z_0}'$ such that for every solution u to (2.31) we have

$$|u(0)| \leq d_0' \quad \Rightarrow \quad |u(r)| \leq Z_0, \quad \forall\ r \in [0, 1]. \tag{2.98}$$

From now on, we will consider a solution u satisfying $|u(0)| \leq d_0'$. From assumption (2.96), recalling (2.82), we deduce that

$$r^{s(N-1)} q_s(r) g(u(r)) u(r) \leq Q r_1^{N-1} g_0^+ u(r)^2, \quad \forall\ r \in [0, 1],\ s \in [0, 1]. \tag{2.99}$$

We now estimate the length of an interval $[r', r''] \subset [r_1, r_2]$, where r' and r'' are two consecutive zeros of u. Without loss of generality, we assume that $u(r) > 0$, for every $r \in (r', r'')$. Arguing as in the proof of Proposition 2.11, it is easy to see that there exists $r^* \in (r', r'')$ such that

$$\begin{aligned} &y(r) \geq 0 \text{ for every } r \in [r', r^*] \\ &y(r) \leq 0 \text{ for every } r \in [r^*, r''] \end{aligned} \tag{2.100}$$

and

$$\begin{aligned} &u \text{ is increasing in } [r', r^*] \\ &u \text{ is decreasing in } [r^*, r'']. \end{aligned} \tag{2.101}$$

Now, we consider the interval $[r', r^*]$. From (2.99) we deduce that

$$\begin{aligned} &u'(r) \geq y(r) \\ &y'(r) \geq -Qg_0^+ u(r), \end{aligned} \tag{2.102}$$

for every $r \in [r', r^*]$. We then multiply by $Qg_0^+ u(r)$ the first inequality in (2.102) and by $y(r)$ the second one; adding up, we obtain

$$Qg_0^+ u(r) u'(r) + y(r) y'(r) \geq 0, \quad \forall\ r \in [r', r^*]. \tag{2.103}$$

This implies that the function

$$\frac{1}{2}y(r)^2 + \frac{Qg_0^+}{2}u(r)^2 \tag{2.104}$$

is increasing in $[r', r^*]$; therefore, we deduce that

$$y(r^*)^2 + Qg_0^+ u(r^*)^2 \geq r'^{2(N-1)}u'(r)^2 + Qg_0^- u(r)^2, \tag{2.105}$$

for every $r \in [r', r^*]$. This implies that

$$Qg_0^+ \left[u(r^*)^2 - u(r)^2\right] \geq r'^{2(N-1)}u'(r)^2, \tag{2.106}$$

for every $r \in [r', r^*]$. Solving with respect to $u'(r)$ and integrating on (r', r^*), we obtain

$$r^* - r' \geq \frac{r'^{N-1}}{\sqrt{Qg_0^+}} \int_0^{u(r^*)} \frac{dz}{\sqrt{u(r^*)^2 - z^2}} = \frac{\pi}{2}\frac{r'^{N-1}}{\sqrt{Qg_0^+}}. \tag{2.107}$$

In an analogous way, we are able to prove that

$$r'' - r^* \geq \frac{\pi}{2}\frac{r'^{N-1}}{\sqrt{Qg_0^+}}. \tag{2.108}$$

From (2.107) and (2.108) we conclude that

$$r'' - r' \geq r'^{N-1}\frac{\pi}{\sqrt{Qg_0^+}}. \tag{2.109}$$

Now, let us denote by r_0 the first zero of u; r_0 is the point defined by (2.17) with $\theta = 0$. Therefore, from (2.18) and (2.96), recalling the definition of (2.15), we can conclude that

$$r_0 \geq \frac{1}{\sqrt{Qg_0^+}}. \tag{2.110}$$

Now, using (2.109) and (2.110), a simple computation shows that

$$1 \geq \mathbf{n}(u)\frac{\pi}{(Qg_0^+)^{N/2}} + \frac{1}{\sqrt{Qg_0^+}}.$$

We then deduce that

$$\mathbf{n}(u) \leq \frac{(Qg_0^+)^{N/2} - (Qg_0^+)^{(N-1)/2}}{\pi}$$

and the result is proved. □

Remark 2.22. As in the previous situations, in the case when $N = 1$ and $q(r) \geq q_0$ on $[0, 1]$, it is possible to improve the result and to obtain the sharp estimate

$$|u(0)| \leq d_0' \quad \Rightarrow \quad \mathbf{n}(u) \leq \frac{\sqrt{Qg_0^+}}{\pi} - \frac{1}{2}. \tag{2.111}$$

3. The main results

In this section we present our main result. We are concerned with the existence and multiplicity of radial solutions to the Dirichlet problem

$$\begin{cases} \Delta u + q(|x|)g(u) = 0 & \text{if } x \in \Omega \\ u(x) = 0 & \text{if } x \in \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is the unit ball in $\mathbf{R}^N$, $N \geq 1$. We assume that $g : \mathbf{R} \longrightarrow \mathbf{R}$ is locally Lipschitz and that $q : [0, 1] \longrightarrow \mathbf{R}$ is continuous; moreover, let us suppose that

$$g(u)u > 0, \quad \forall\ u \in \mathbf{R},\ u \neq 0, \tag{3.2}$$

and that there exist $0 < Z_0 \leq Z_\infty$ and $g_0^\pm > 0$, $g_\infty^\pm > 0$ such that

$$g_0^- \leq \frac{g(u)}{u} \leq g_0^+, \quad \forall\ |u| \leq Z_0,\ u \neq 0, \tag{3.3}$$

and

$$g_\infty^- \leq \frac{g(u)}{u} \leq g_\infty^+, \quad \forall\ |u| \geq Z_\infty. \tag{3.4}$$

On the function q, we assume that

$$q(r) \geq 0, \quad \forall\ r \in [0, 1], \tag{3.5}$$

and that there exist $[r_1, r_2] \subset (0, 1]$ and $q_0 > 0$ such that

$$q(r) \geq q_0, \quad \forall\ r \in [r_1, r_2]. \tag{3.6}$$

Moreover, we set $Q = \max\{q(r) : \ r \in [0, 1]\}$. We will prove the following results:

Theorem 3.1. *Assume conditions* (3.2), (3.3), (3.4), (3.5) *and* (3.6). *Moreover, let j be the smallest integer such that*

$$(Qg_0^+)^{(N-1)/2}(\sqrt{Qg_0^+} - 1) < \pi\big(j + \frac{1}{2}\big) \tag{3.7}$$

and let l be the largest integer such that

$$\pi\big(l + \frac{3}{2}\big) < r_1^{N-1}(r_2 - r_1)\sqrt{q_0 g_\infty^-}. \tag{3.8}$$

If $j+1 \leq l$, then for every integer $n \in [j+1, l]$ there exist two radial solutions u_n and v_n to (3.1) *such that $v_n(0) < 0 < u_n(0)$. Moreover, u_n and v_n have exactly n zeros in $(0,1)$.*

Theorem 3.2. *Assume conditions* (3.2), (3.3), (3.4), (3.5) *and* (3.6). *Moreover, let j' be the largest integer such that*

$$\pi\big(j' + \frac{3}{2}\big) < r_1^{N-1}(r_2 - r_1)\sqrt{q_0 g_0^-} \tag{3.9}$$

and let l' be the smallest integer such that

$$(Qg_\infty^+)^{(N-1)/2}(\sqrt{Qg_\infty^+} - 1) < \pi\big(l' + \frac{1}{2}\big). \tag{3.10}$$

If $l' + 1 \leq j'$, then for every integer $n \in [l'+1, j']$ there exist two radial solutions u_n and v_n to (3.1) *such that $v_n(0) < 0 < u_n(0)$. Moreover, u_n and v_n have exactly n zeros in $(0,1)$.*

Remark 3.3. It is interesting to observe the role played by the dimension of the space $\mathbf{R}^N$ in Theorems 3.1 and 3.2. Since $r^{N-1} \to 0$ when $N \to +\infty$, it is clear that, for large values of N, there are not integers l or j' satisfying (3.8) or (3.9), respectively. This implies that, when N is big, we are not able to give existence and multiplicity results for (3.1); however, we are confident in the fact that this is not due to the technique of the proof and that, in general, for N large, (3.1) could not have solutions.

On the other hand, the inequalities (3.7) and (3.8) suggest another feature of the problem: given N, as large as we want, and two integers j and l (with $j+1 \leq l$), in order (3.7) and (3.8) to be fulfilled it is sufficient to take g_0^+ small enough and g_∞^- large enough. This means that, in high dimension, multiplicity results can be obtained when the behaviour of g in zero and infinity is sensitively different (i.e. when the difference $g_\infty^- - g_0^+$ is large). An analogous remark holds in the dual case ($g_0^- - g_\infty^+$ large) for Theorem 3.2.

Remark 3.4. In the case when $N = 1$ and $q(r) \geq q_0 > 0$ in $[0,1]$, taking into account Remarks 2.14, 2.17, 2.20 and 2.22, it is possible to see that Theorems 3.1 and 3.2 reduce to the results contained in [11] and in [17] (for the case of p-laplacian like operators). See also the forthcoming paper [4].

Remark 3.5. In the situation when $q(r) \geq q_0$ in some interval $[0, r_2] \subset [0,1]$, recalling Remark 2.12 and Propositions 2.13 and 2.19, it is possible to prove results analogous to Theorems 3.1 and 3.2. Indeed, it is sufficient to replace (3.8) and (3.9) with

$$\pi\big(l + \frac{3}{2}\big) < \frac{r_2^N (N-1)^{N-1}}{N^N}\sqrt{q_0 g_\infty^-} \tag{3.11}$$

and

$$\pi\big(j' + \frac{3}{2}\big) < \frac{r_2^N (N-1)^{N-1}}{N^N} \sqrt{q_0 g_0^-}, \tag{3.12}$$

respectively.

In the particular case when the ratio $g(u)/u$ admits limits at infinity and zero, we can restate our results in the following way:

Corollary 3.6. *Assume* (3.2)-(3.5) *and* (3.6). *Moreover, suppose that there exist* $g_0 > 0$ *and* $g_\infty > 0$ *such that*

$$\lim_{u \to 0} \frac{g(u)}{u} = g_0, \qquad \lim_{|u| \to +\infty} \frac{g(u)}{u} = g_\infty. \tag{3.13}$$

Let j *and* j' *be the smallest and the largest integer such that*

$$(Qg_0)^{(N-1)/2}(\sqrt{Qg_0} - 1) < \pi\big(j + \frac{1}{2}\big) \tag{3.14}$$

and

$$\pi\big(j' + \frac{3}{2}\big) < r_1^{N-1}(r_2 - r_1)\sqrt{q_0 g_0}, \tag{3.15}$$

respectively. Let also l *and* l' *be the largest and the smallest integer such that*

$$\pi\big(l + \frac{3}{2}\big) < r_1^{N-1}(r_2 - r_1)\sqrt{q_0 g_\infty} \tag{3.16}$$

and

$$(Qg_\infty)^{(N-1)/2}(\sqrt{Qg_\infty} - 1) < \pi\big(l' + \frac{1}{2}\big), \tag{3.17}$$

respectively. If $j+1 \le l$ (*or if* $l'+1 \le j'$), *then for every integer* $n \in [j+1, l]$ (*or* $n \in [l'+1, j']$, *respectively*) *there exist two radial solutions* u_n *and* v_n *to* (3.1) *such that* $v_n(0) < 0 < u_n(0)$. *Moreover,* u_n *and* v_n *have exactly* n *zeros in* $(0,1)$.

The following result gives sufficient conditions for the existence of radial solutions to (3.1) with a prescribed number of zeros; for simplicity, we consider the situation when the limits of $g(u)/u$ at infinity and zero exist:

Corollary 3.7. *Assume* (3.2)-(3.5) *and* (3.6). *Moreover, suppose that there exist* $g_0 > 0$ *and* $g_\infty > 0$ *such that*

$$\lim_{u \to 0} \frac{g(u)}{u} = g_0, \qquad \lim_{|u| \to +\infty} \frac{g(u)}{u} = g_\infty. \tag{3.18}$$

For every integer $n \ge 1$, *there exist* $0 < b_n \le c_n$ *such that if*

$$(g_0, g_\infty) \in (0, b_n) \times (c_n, +\infty) \bigcup (c_n, +\infty) \times (0, b_n) \tag{3.19}$$

then there exist at least two radial solutions u_n and v_n to (3.1), such that $v_n(0) < 0 < u_n(0)$, having exactly n zeros in $(0,1)$.

Remark 3.8. According to Remarks 2.14-2.17-2.20 and 2.22, it is possible to show that, when $N = 1$, $[r_1, r_2] = [0,1]$ and $q_0 = q(r) = Q$, for every $r \in [0,1]$, the numbers b_n and c_n are given by

$$b_n = c_n = \frac{\pi^2(n+1/2)^2}{Q}.$$

Therefore, condition (3.19) reduces to the expected

$$\sqrt{Qg_0} < \pi\big(n+\frac{1}{2}\big) < \sqrt{Qg_\infty} \quad \text{or} \quad \sqrt{Qg_\infty} < \pi\big(n+\frac{1}{2}\big) < \sqrt{Qg_0}.$$

Proof. By Corollary 3.6, we can prove the existence of radial solutions with n zeros in $(0,1)$ if

$$(Qg_0)^{(N-1)/2}(\sqrt{Qg_0}-1) < \pi\big(n-\frac{1}{2}\big), \quad \pi\big(n+\frac{3}{2}\big) < r_1^{N-1}(r_2-r_1)\sqrt{q_0g_\infty} \tag{3.20}$$

or if

$$(Qg_\infty)^{(N-1)/2}(\sqrt{Qg_\infty}-1) < \pi\big(n-\frac{1}{2}\big), \quad \pi\big(n+\frac{3}{2}\big) < r_1^{N-1}(r_2-r_1)\sqrt{q_0g_0}. \tag{3.21}$$

For every $n \geq 1$, let us denote by ϕ_n the (unique) point such that

$$\phi_n^N - \phi_n^{N-1} = \pi\big(n-\frac{1}{2}\big);$$

setting

$$b_n = \frac{\phi_n^2}{Q}, \quad c_n = \frac{\pi^2}{r_1^{2(N-1)}(r_2-r_1)^2}\frac{(n+3/2)^2}{q_0},$$

it is easy to see that (3.20)-(3.21) are equivalent to (3.19). □

The proofs of Theorem 3.1 and Theorem 3.2 are based on a topological degree approach. To this aim, we need to introduce a continuation theorem and to develop an homotopy between problem (3.1) and a suitable autonomous problem.

A Leray-Schauder type continuation theorem. Let us consider an abstract equation of the form

$$u = \mathcal{G}(u,s), \tag{3.22}$$

where X is a Banach space and $\mathcal{G} : \operatorname{dom}\mathcal{G} \subset X \times [0,1] \longrightarrow X$ is a completely continuous operator. Moreover, we shall consider an open set C such that $\bar{C} \subset \operatorname{dom}\mathcal{G}$.

Let Σ be the set of the solutions of (3.22), i.e., $\Sigma = \{(u, s) : u = \mathcal{G}(u, s)\}$ and, for any subset $D \subset X \times [0, 1]$, let us denote the section of D at $s \in [0, 1]$ by $D_s = \{x \in X : (x, s) \in D\}$; we also set $\mathcal{G}_s = \mathcal{G}(\cdot, s)$. We have the following theorem (see e.g. [5, 7]):

Theorem 3.9. ([5, Th. 3.4]) *Let $\mathbf{k} : \Sigma \cap \bar{C} \longrightarrow \mathbf{N}$ be a continuous function; suppose that there exists $n \in \mathbf{N}$ satisfying the following conditions:*

$$n \notin \mathbf{k}(\partial \bar{C}) \tag{3.23}$$

and

$$\mathbf{k}^{-1}(n) \quad \text{is bounded.} \tag{3.24}$$

Then, for an open set U_0^n such that $(\mathbf{k}^{-1}(n))_0 \subset U_0^n \subset \overline{U_0^n} \subset \bar{C}_0$ and $\Sigma_0 \cap U_0^n = (\mathbf{k}^{-1}(n))_0$, the Leray-Schauder degree $\deg(I - \mathcal{G}_0, U_0^n)$ is defined. If

$$\deg(I - \mathcal{G}_0, U_0^n) \neq 0, \tag{3.25}$$

then there is a continuum $C_n \subset \Sigma$ with

$$\{s \in [0, 1] : \exists u \in X : (u, s) \in C_n\} = [0, 1]$$

and such that $(u, s) \in C_n \Longrightarrow (u, s) \in C$ and $\mathbf{k}(u, s) = n$. In particular there is at least one solution $\tilde{u} \in C_1$ of the operator equation $u = \mathcal{G}(u, 1)$ with $\mathbf{k}(\tilde{u}, 1) = n$.

Proofs of the main results. We first observe that, if $u(x)$ is a radial solution to (3.1), then, setting $r = |x|$, $u(r)$ is a solution of the boundary value problem

$$(r^{N-1}u')' + r^{N-1}q(r)g(u) = 0, \quad u'(0) = 0 = u(1). \tag{3.26}$$

Thus, we are led to study (3.26); to this aim, let us consider the problem

$$(r^{s(N-1)}u')' + r^{s(N-1)}q_s(r)g(u) = 0, \quad u'(0) = 0 = u(1), \tag{3.27}$$

where $s \in [0, 1]$ and $q_s(r) = sq(r) + (1 - s)q_0$, with q_0 given in (3.6). It is clear that (3.26) correspond to (3.27) with $s = 1$.

We now observe that, from (3.2)-(3.3)-(3.4)-(3.5)-(3.6) and the definition of q_s, conditions (H) and (2.32) of Section 2 are fulfilled: indeed, it is easy to see that (2.4) holds true for some $a' \geq \max(g_0^+, g_\infty^+)$ and that (2.5) is satisfied with a constant $C_{g,G}$ depending on $g_0^\pm$ and $g_\infty^\pm$. Moreover, we also have $Q = \max\{q_s(r) : \ r \in [0, 1], \ s \in [0, 1]\}$. Therefore, we can apply to solutions of (3.27) all the results proved in Section 2.

Using standard arguments (see e.g. [22]), it is possible to see that (3.27) is equivalent to a fixed point equation of the form (3.22), for a suitable operator $\mathcal{G}$ defined on the space

$$X = \{u \in C^1([0, 1]) : \ u'(0) = 0 = u(1)\}.$$

Now, let us conclude the proof of Theorem 3.1; in Remark 3.10 we will give the (minor) changes needed to prove Theorem 3.2. Let us fix $\chi > 0$ such that

$$\sqrt{q_0 g_\infty^-}\frac{r_1^N(r_2 - r_1)}{\pi} - \chi > l + \frac{3}{2} \tag{3.28}$$

and let us consider the numbers $d_\infty = d_\chi$ and d_0' given in Proposition 2.11 and Proposition 2.21, respectively. Moreover, let us set

$$C = \{(u,s) \in X \times [0,1] :\ d_0' < u(0) < d_\infty\}. \tag{3.29}$$

Finally, for every $(u,s) \in \Sigma \cap \bar{C}$, let us define

$$\mathbf{k}(u,s) = \mathbf{n}(u),$$

where $\mathbf{n}(u)$ is the number of zeros of u in $[0,1)$. From Lemma 2.10, the function $\mathbf{k}$ is continuous in $\Sigma \cap \bar{C}$; therefore, in order to apply Theorem 3.9, we need to verify (3.23)-(3.24) and (3.25).

To this aim, let us fix an integer $n \in [j+1, l]$. Assume that there exists $(u,s) \in \Sigma \cap \partial C$ such that $\mathbf{k}(u,s) = n$; this implies that u is a solution to (3.27) such that $\mathbf{n}(u) = n$ and $u(0) = d_\infty$ or $u(0) = d_0'$. In the former case, from Proposition 2.11 and (3.8)-(3.28), we deduce that $\mathbf{n}(u) > l$ and this contradicts the choice of n; in the latter, from Proposition 2.21 and (3.7), we deduce that $\mathbf{n}(u) < j+1$, which is again a contradiction. Therefore, (3.23) is fulfilled.

Moreover, since $|u(0)| \le d_\infty$, for every $(u,s) \in \Sigma \cap \bar{C}$, an application of Lemma 2.8, Statement 1, with $R_1 = d_\infty$, gives the existence of $R_2 > 0$ such that $||(u,s)|| \le R_2$; henceforth, also (3.24) is satisfied.

As far as (3.25) is concerned, we observe that it requires that some local degree associated to solutions of the autonomous non-singular problem

$$u'' + q_0 g(u) = 0, \quad u'(0) = 0 = u(1) \tag{3.30}$$

is different from zero. The study of (3.30), based on the notion of 'time-map', has been widely developed (see e.g. [10]); in particular, from [10, Th. 2.3.10] and [10, Th. 2.4.4], we deduce that there exists U_0^n such that (3.25) holds true if

$$\sqrt{q_0 g_0^+} < \pi\Big(n + \frac{1}{2}\Big) < \sqrt{q_0 g_\infty^-} \tag{3.31}$$

(recall that the numbers $\pi^2(n+\frac{1}{2})^2$ are the eigenvalues of $-u''$ with boundary conditions $u'(0) = 0 = u(1)$.)

It is immediate to see that (3.8) and $n \le l$ ensure that

$$\pi\Big(n + \frac{1}{2}\Big) < \sqrt{q_0 g_\infty^-}.$$

Finally, by means of a careful computation, it is possible to show that (3.7) and $j \leq n$ guarantee the validity of the left inequality in (3.31).

Hence, an application of Theorem 3.9 gives the existence of a solution u_n to (3.26) such that $u_n(0) > 0$ and having exactly n zeros in $(0,1)$.

It is clear that, taking

$$C = \{(u,s) \in X \times [0,1] : \; -d_\infty < u(0) < -d_0'\},$$

we obtain the existence of the solution v_n such that $v_n(0) < 0$. Then, Theorem 3.1 is proved.

Remark 3.10. In order to prove Theorem 3.2, it is sufficient to take $\chi > 0$ such that

$$\frac{(Qg_\infty^+)^{(N-1)/2}}{\pi}\big(\sqrt{Qg_\infty^+} - 1\big) + \chi < \big(l' + \frac{1}{2}\big)$$

and to define

$$C = \{(u,s) \in X \times [0,1] : \; d_0 < u(0) < d_\infty'\},$$

where d_0 and $d_\infty' = d_\chi'$ are given in Proposition 2.18 and Proposition 2.15, respectively. The proof follows the same lines of the one developed above.

Remark 3.11. We underline the fact that existence and multiplicity results on the lines of Theorems 3.1 and 3.2 can be obtained also when the laplacian Δ in (3.1) is replaced by the p-laplacian (or even more general strongly nonlinear operators, see e.g. [6, 16]). In this case, we need to assume conditions like (3.3) and (3.4) relating the growth of g and $\phi_p(u) = u|u|^{p-2}$.

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