Differential and Integral Equations Volume 15, Number 9, September 2002, Pages 1139–1152

UNBOUNDED SOLUTIONS OF FORCED ISOCHRONOUS OSCILLATORS AT RESONANCE

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Abstract. We study the behavior of large amplitude solutions for forced nonlinear oscillators at resonance, the nonlinearity being a bounded perturbation of a function deriving from an isochronous potential, i.e., a potential leading to free oscillations that all have the same period.

1. Introduction

In this note, we study the behavior of large amplitude solutions of the equation

$$x'' + V'(x) + g(x) = p(t), \tag{1.1}$$

in situations of resonance. The following hypotheses, where $k \in \mathbb{N}^*$ and $a \in [-\infty, 0)$, are assumed to hold throughout:

($\mathcal{H}_k$) $V : (a, +\infty) \to \mathbb{R}$ is a $2\pi/k$-isochronous, strictly convex potential whose derivative is locally lipschitzian; $g : (a, +\infty) \to \mathbb{R}$ is bounded and locally lipschitzian; p belongs to $L^1_{loc}(\mathbb{R})$ and is 2π-periodic.

By convention, we suppose that the minimum of V is reached at 0, so that $V'(0) = 0$. By a $2\pi/k$-isochronous potential, we mean that all nontrivial solutions of $x'' + V'(x) = 0$ are of (minimal) period $2\pi/k$. We also assume that V satisfies either

$$(\mathcal{S}) \qquad \lim_{x\to+\infty} \frac{V'(x)}{x} = \frac{k^2}{4} \quad \text{and} \quad \lim_{x\to a_+} \frac{V'(x)}{x} = +\infty,$$

where $a \in [-\infty, 0)$, or

$$(\mathcal{NS}) \qquad \lim_{x\to+\infty} \frac{V'(x)}{x} = \alpha > 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{V'(x)}{x} = \beta > 0$$

Accepted for publication: November 2001.
AMS Subject Classifications: 70K30, 34B15, 34C25, 34C11.

(in which case $a = -\infty$).

The first case is referred to as the singular case, because when $a \neq -\infty$, it corresponds to a repulsive singularity which, by convention, has been placed here on the negative side. In the second case, which is referred to as the non-singular case, V' is asymptotic to a so-called jumping or asymmetric nonlinearity $\alpha x^+ - \beta x^-$, where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$. The isochronism assumption implies that

$$\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2}{k}. \tag{1.2}$$

We refer to [1] and [2] for examples of isochronous potentials. It is shown in [2] (see Lemma 8 and Corollary 3) that perturbations of functions deriving from an isochronous potential cover a large class of nonlinearities.

In a recent paper [13], it is shown that for any isochronous potential, it is possible to find periodic forcings leading to the unboundedness of all the solutions of the forced equation. In the case where V is not defined on the whole line or V is not sufficiently smooth, impulse forcing terms have to be used. In this paper, we present conditions on the forcing term that implies the unboundedness of the large amplitude solutions, i.e., solutions with large initial conditions.

Consider equation (1.1) where V, g, p satisfy hypothesis $(\mathcal{H}_k)$ for some $k \in \mathbb{N}^*$, and V satisfies either $(\mathcal{S})$ or $(\mathcal{NS})$. Let us define the function Φ by

$$\Phi(\theta) = \int_0^{2\pi} p(t)\psi(t+\theta)\, dt\,, \tag{1.3}$$

where ψ denotes either $|\cos(kt/2)|$ in case $(\mathcal{S})$ holds, or the solution of

$$x'' + \alpha x^+ - \beta x^- = 0, \quad x(0) = 1, \quad x'(0) = 0, \tag{1.4}$$

when $(\mathcal{NS})$ holds. Notice that Φ, as ψ, is of period $2\pi/k$. Let us also define

$$G(\rho) = \int_0^{2\pi} g(\rho\psi(t))\psi(t)\, dt\,, \tag{1.5}$$

and its limits

$$G_+ = \liminf_{\rho\to+\infty} G(\rho) \quad \text{and} \quad G^+ = \limsup_{\rho\to+\infty} G(\rho)\,. \tag{1.6}$$

It is proved in [2] that, when $G^+ = G_+$, the existence of at least one 2π-periodic solution of equation (1.1) depends on the number of zeros of the function $\Phi - G^+$ in $[0, 2\pi)$. We want to show that the function Φ also plays a key role in the problem of boundedness of the solutions of equation (1.1).

For the simpler equation

$$x'' + \alpha x^+ - \beta x^- = p(t), \tag{1.7}$$

with p smooth, it has been shown by Liu in [9] that all the solutions are bounded if Φ is of constant sign. This is also true for nonlinearities deriving from regular isochronous potentials, as proved in [3].

By contrast, it follows from results of Fabry and Mawhin [8] that, if the function Φ has zeros, all being simple, then the large amplitude solutions of (1.7) are unbounded either in the past or in the future (see also [4]). On the other hand, it has also been shown in [2] that the equation

$$x'' + V'(x) = \varepsilon \sin t, \tag{1.8}$$

where V is a smooth potential satisfying hypothesis $(\mathcal{H}_k)$ with $k = 1$, and either $(\mathcal{S})$ or $(\mathcal{NS})$, has no 2π-periodic solution, at least for ε small. By a result of Massera [10], all the solutions of equation (1.8) are then unbounded. It is easy to check that for the forcing term $p(t) = \sin t$, the function Φ has exactly two (simple) zeros in $[0, 2\pi)$.

The question of boundedness of solutions has also been examined by Ortega [12] for an oscillator with obstacle, which can be considered as a limit case of (1.7), with $\beta = +\infty$.

All the above results suggest that, for equation (1.1), the boundedness of the solutions depends on whether Φ vanishes at some point or not, at least when $G^+ = G_+ = 0$. We show in this paper how to adapt the condition to the general case of arbitrary values of G^+ and G_+. We will prove the following result, which improves Theorem 2 of [8].

Theorem 1. *Let Φ, G_+, G^+ be defined as above. Suppose that $\max \Phi > G^+$, $\min \Phi < G_+$ and that $[G_+, G^+]$ does not contain any critical value of Φ. Then, there exists $R > 0$ such that all the solutions $x(t)$ of (1.1) satisfying $(x(0))^2 + (x'(0))^2 > R$ are unbounded, either in the future or in the past.*

Notice that Theorem 1 does apply to the particular case $G_+ = G^+$, which holds for a large class of functions g. For instance, $G^+ = G_+ = 0$ when g has a sublinear primitive. On the other hand, it is easy to see, that if g has limits $g(\pm\infty)$ at $\pm\infty$, we have

$$G^+ = G_+ = g(+\infty)\int_{\psi>0} \psi + g(-\infty)\int_{\psi<0} \psi = 2k\sqrt{\alpha}\Big(\frac{g(+\infty)}{\alpha} - \frac{g(-\infty)}{\beta}\Big),$$

with β considered as $+\infty$ in the singular case. When $G^+ = G_+$, the assumption of Theorem 1 amounts to ask that $\Phi - G^+$ vanishes at some point, the zeros being simple. Hence, if one considers, for instance, the equation

$$x'' + \alpha x^+ - \beta x^- + \sin(x) = p(t)\,, \tag{1.9}$$

where α, β satisfy (1.2), with $k = 1$ or the equation

$$x'' - \frac{1}{4(x+1)^3} + \frac{(x+1)}{4} + \sin(x) = p(t)\,, \tag{1.10}$$

for both of which $g(x) = \sin(x)$, it results from Theorem 1 that if the function Φ defined by (1.3) vanishes at some point in $[0, 2\pi)$ (the zeros are supposed to be simple), then the large amplitude solutions of these equations are unbounded either in the past or in the future. In the particular case where $p(t) = a + b\cos t$, it is easily computed that unbounded solutions are present when $3|a| < |b|$. Equation (1.9) is already covered by the results of [8] but, when $G^+ \neq G_+$, the result of Theorem 1 is new, even in the case of a harmonic oscillator. We refer to Proposition 1 of [2] for examples where the limits G^+ and G_+ are different and where these can be easily computed from the limits of g.

The paper is organized as follows. In Section 2, we investigate the behavior of the solutions of (1.1) with large initial conditions, assuming that the perturbation term g has limits at $\pm\infty$. This last hypothesis allows us to use the same arguments as in [8] and to prove a particular case of Theorem 1. Section 3 is devoted to the proof of Theorem 1 itself.

2. Behavior of Large amplitude solutions

Consider the equation

$$x'' + V'(x) + g(x) = p(t), \tag{2.1}$$

where V, g, p satisfy hypothesis $(\mathcal{H}_k)$ for some $k \in \mathbb{N}^*$, and V satisfies either $(\mathcal{S})$ or $(\mathcal{NS})$. Moreover, assume that g has limits at $\pm\infty$ (only at $+\infty$ in the singular case).

For $\rho \geq 0$, we denote by $\varphi(t, \rho)$ the solution of the Cauchy problem

$$x''(t) + V'(x(t)) = 0, \quad x(0) = \rho,\ x'(0) = 0. \tag{2.2}$$

If x is a solution of (2.1) with $|x(0)| + |x'(0)|$ large enough, then $|x(t)| + |x'(t)| \neq 0$ for all $t \in [0, 2\pi]$ and functions $\rho(\cdot) > 0$, $\theta(\cdot)$ can be defined such that

$$x(t) = \varphi(t + \theta(t), \rho(t)), \quad x'(t) = \frac{\partial \varphi}{\partial t}(t + \theta(t), \rho(t)). \tag{2.3}$$

The function $\theta(\cdot)$ can be built to be continuous, with $\theta(0) \in [0, 2\pi)$; $\rho(\cdot)$ and $\theta(\cdot)$ are then differentiable and it is shown in [2] that the equation (2.1) in these new variables becomes

$$\rho' = -\frac{1}{V'(\rho)}\, g(\varphi(t+\theta,\rho))\, \frac{\partial\varphi}{\partial t}(t+\theta,\rho) + \frac{1}{V'(\rho)}\, p(t)\, \frac{\partial\varphi}{\partial t}(t+\theta,\rho), \quad (2.4)$$

$$\theta' = \frac{1}{V'(\rho)}\, g(\varphi(t+\theta,\rho))\, \frac{\partial\varphi}{\partial\rho}(t+\theta,\rho) - \frac{1}{V'(\rho)}\, p(t)\, \frac{\partial\varphi}{\partial\rho}(t+\theta,\rho)\,. \quad (2.5)$$

Since we are interested in solutions of large amplitude, we set $\rho = \rho_\varepsilon/\varepsilon$, where $\varepsilon > 0$ is a small parameter and $\rho_\varepsilon = O(1)$ for $\varepsilon \to 0_+$. Then, (2.4) and (2.5) become

$$\rho_\varepsilon' = -\frac{\varepsilon}{V'(\frac{\rho_\varepsilon}{\varepsilon})}\Big[g\left(\varphi\left(t+\theta,\frac{\rho_\varepsilon}{\varepsilon}\right)\right)\frac{\partial\varphi}{\partial t}\left(t+\theta,\frac{\rho_\varepsilon}{\varepsilon}\right) - p(t)\frac{\partial\varphi}{\partial t}\left(t+\theta,\frac{\rho_\varepsilon}{\varepsilon}\right)\Big], \quad (2.6)$$

$$\theta' = \frac{1}{V'(\frac{\rho_\varepsilon}{\varepsilon})}\Big[g\left(\varphi\left(t+\theta,\frac{\rho_\varepsilon}{\varepsilon}\right)\right)\frac{\partial\varphi}{\partial \rho}\left(t+\theta,\frac{\rho_\varepsilon}{\varepsilon}\right) - p(t)\frac{\partial\varphi}{\partial \rho}\left(t+\theta,\frac{\rho_\varepsilon}{\varepsilon}\right)\Big]. \quad (2.7)$$

According to Lemmas 2 and 4 of [2], we have as $\varepsilon \to 0_+$,

$$\frac{\varepsilon}{\rho_\varepsilon}\varphi(t+\theta,\frac{\rho_\varepsilon}{\varepsilon}) \to \psi(t+\theta), \quad \text{in } L^\infty(0,2\pi) \quad (2.8)$$

$$\frac{\varepsilon}{\rho_\varepsilon}\frac{\partial\varphi(t+\theta,\frac{\rho_\varepsilon}{\varepsilon})}{\partial t} \to \psi'(t+\theta), \quad * - \text{weakly in } L^\infty(0,2\pi) \quad (2.9)$$

and

$$\frac{\partial\varphi(t+\theta,\frac{\rho_\varepsilon}{\varepsilon})}{\partial\rho} \to \psi(t+\theta), \quad \text{in } L^\infty(0,2\pi), \quad (2.10)$$

where ψ is defined as above. For ε small, we can approximate $g(\varphi(t+\theta,\rho_\varepsilon/\varepsilon))$ by $g(\rho_\varepsilon\psi(t+\theta)/\varepsilon)$ which in turn can be approximated by $g(+\infty)$ when $\psi(t+\theta) > 0$ and by $g(-\infty)$ when $\psi(t+\theta) < 0$. On the other hand, Lemma 6 of [2] implies that for ε small, the integral

$$\int_0^{2\pi} \frac{\varepsilon}{V'(\frac{\rho_\varepsilon}{\varepsilon})}\, g\left(\varphi\left(t+\theta,\frac{\rho_\varepsilon}{\varepsilon}\right)\right)\, \frac{\partial\varphi}{\partial t}\left(t+\theta,\frac{\rho_\varepsilon}{\varepsilon}\right)\, dt$$

is small. We can now approximate the equations (2.6) and (2.7) by

$$\tilde\rho_\varepsilon' = \frac{\varepsilon}{\alpha}\, p(t)\, \psi'(t+\tilde\theta(0)), \quad (2.11)$$

$$\tilde{\theta}' = \frac{\varepsilon}{\alpha\,\tilde{\rho}_\varepsilon}\Big(g(+\infty)\psi^+(t+\tilde{\theta}(0)) - g(-\infty)\psi^-(t+\tilde{\theta}(0)) - p(t)\psi(t+\tilde{\theta}(0))\Big), \tag{2.12}$$

where $\alpha = \lim_{\rho\to+\infty} V'(\rho)/\rho$, using also the fact that the variation of θ is small. Of course, in the singular case $\psi^-(t) = 0$ for all t, and $\alpha = k^2/4$.

The system (2.11), (2.12) containing a small parameter ε, we can apply the averaging method to obtain approximations of the solutions which remain close to the solutions on time intervals of the order of $1/\varepsilon$. Averaging the right-hand sides leads to the following equations:

$$\sigma' = \frac{\varepsilon}{2\pi\,\alpha}\,\Phi'(\tau) \tag{2.13}$$

$$\tau' = \frac{\varepsilon}{\sigma}\frac{1}{2\pi\,\alpha}\Big[2k\sqrt{\alpha}\Big(\frac{g(+\infty)}{\alpha} - \frac{g(-\infty)}{\beta}\Big) - \Phi(\tau)\Big], \tag{2.14}$$

where β has to be considered as $+\infty$ in the singular case. Define the function

$$\zeta(\tau) = \frac{1}{2\pi\,\alpha}\Big[2k\sqrt{\alpha}\Big(\frac{g(+\infty)}{\alpha} - \frac{g(-\infty)}{\beta}\Big) - \Phi(\tau)\Big]. \tag{2.15}$$

Notice that

$$G^+ = G_+ = 2k\sqrt{\alpha}\Big(\frac{g(+\infty)}{\alpha} - \frac{g(-\infty)}{\beta}\Big).$$

As mentioned in [8], the system (2.13), (2.14) is easy to solve if we assume that $\zeta(\tau(0))$ is different from 0. The solution for the initial conditions $(1, \tau(0))$ is given by

$$\sigma(t) = \frac{\zeta(\tau(0))}{\zeta(\tau(t))} \qquad \text{and} \qquad \tau(t) = P^{-1}\Big[\frac{\varepsilon t}{\zeta(\tau(0))}\Big]$$

where $P(u) = \int_u^{\tau(0)} \frac{1}{\zeta^2(s)}\,ds$.

The following theorem give a precise relation between the solutions of (2.1) and the solutions of the system (2.13),(2.14). Since the proof uses the same arguments as in [8], we skip it. We just mention that the needed estimations have been proved in [2].

Theorem 2. *Let V, g, p satisfy hypothesis $(\mathcal{H}_k)$ for some $k \in \mathbb{N}^*$, V satisfy $(\mathcal{S})$ or $(\mathcal{NS})$ and assume that g has limits at $\pm\infty$. Given $T > 0$, $\eta > 0$, there exists $R > 0$ such that, for $\rho(0) \geq R$, the solutions of* (2.1) *can be written in the form*

$$x(t) = \varphi(t + \theta(t), \rho(t)), \quad x'(t) = \frac{\partial\varphi}{\partial t}(t + \theta(t), \rho(t))$$

with

$$\Big|\rho(t)-\rho(0)\frac{\zeta(\theta(0))}{\zeta(\tau(t))}\Big|\le\eta\rho(0)\qquad|\theta(t)-\tau(t)|\le\eta$$

for $t\in[0,T\rho(0)]$, *where* ζ *is defined by* (2.15), *and* $\tau(t)$ *is defined by* (2.13) *and* (2.14) *with* $\sigma(0)=\rho(0)$, $\tau(0)=\theta(0)$.

Theorem 2 allow us to investigate the behavior of solutions of (2.1) with large initial conditions.

We can distinguish two different behaviors depending on the properties of the function

$$\Phi(\cdot)-G^{+}=\Phi(\cdot)-G_{+}=\Phi(\cdot)-2k\sqrt{\alpha}\Big(\frac{g(+\infty)}{\alpha}-\frac{g(-\infty)}{\beta}\Big).$$

Suppose first that this function is of constant sign (positive, for instance). In this case, the function $t\to\zeta(\tau(t))$ will oscillate between

$$\zeta_{\min}=\min\{\zeta(t)\mid t\in[0,2\pi]\}\quad\text{and}\quad\zeta_{\max}=\max\{\zeta(t)\mid t\in[0,2\pi]\}$$

and this approximation holds on intervals of the order of $\rho(0)$. This estimation makes us believe that all the solutions of (2.1) are bounded in this case. As mentioned in the introduction, this was proved in [9] for the case $V'(x)=\alpha x^{+}-\beta x^{-}$, without perturbation terms, and assuming further regularity on the forcing term.

Suppose now that $\Phi(\cdot)-G^{+}$ takes both signs and that the zeros are simple. In this case, we claim that all the solutions of (2.1) with sufficiently large initial condition are unbounded either in the past or in the future.

Assume for example that $\Phi(\tau(0))-G^{+}=-2\pi\alpha\zeta(\tau(0))$ is different from zero, for instance positive. We have already seen that $\sigma(t)\zeta(\tau(t))=\zeta(\tau(0))$ is a first integral for the system (2.13), (2.14) for the initial condition $\sigma(0)=1$. As $\zeta(\tau(t))$ cannot change sign, there exists τ^* such that $\lim_{t\to+\infty}\tau(t)=\tau^*$. Since $\tau'(t)$ is negative, τ^* will be the nearest zero of ζ at the left of $\tau(0)$. But then we must have $\lim_{t\to+\infty}\sigma(t)=+\infty$ and we infer from the differential equation (2.13) that the growth of σ is asymptotically linear as τ^* is a simple zero of ζ. Now the same conclusion holds for $\rho(t)/\rho(0)$ as by Theorem 2, the function $\rho(t)/\rho(0)$ is close to $\sigma(t)$ on large intervals. So, we are able to proof a particular case of Theorem 1.

Theorem 3. *Let* V,g,p *satisfy hypothesis* $(\mathcal{H}_k)$ *for some* $k\in\mathbb{N}^*$, V *satisfy either* $(\mathcal{S})$ *or* $(\mathcal{NS})$, *and assume moreover that* g *has limits* $g(\pm\infty)$ *at* $\pm\infty$. *Assume that* $\Phi-G^{+}=\Phi(\cdot)-2k\sqrt{\alpha}\,(g(+\infty)/\alpha-g(-\infty)/\beta)$ *has zeros, all* *being simple. Then, all the solutions of* (2.1), *written as*

$$x(t) = \varphi(t + \theta(t), \rho(t)), \quad x'(t) = \frac{\partial \varphi}{\partial t}(t + \theta(t), \rho(t))$$

with $\rho(0)$ *large enough, are unbounded either in the past or in the future.*

The proof can be easily adapted from the proof of Theorem 2 [8]. □

We have assumed that g has a limit at $\pm\infty$ in order to be able to use the averaging method. When g does not have limits at $\pm\infty$, we have to adapt the arguments. Although the above theorem will be generalized below, we have considered the presentation of its proof worthwhile, since it provides some qualitative information on the behavior of the solutions of large amplitude. We also mention that Theorem 2 provides information that can be used for other purposes than the study of the boundedness of the solutions. For example, it is easy to adapt the arguments of [8] to prove the existence of subharmonic solutions of (2.1) when ζ does not change sign. More precisely, we are able to prove the following theorem.

Theorem 4. *Let* V, g, p *satisfy hypothesis* $(\mathcal{H}_k)$ *for some* $k \in \mathbb{N}^*$, V *satisfy either* $(\mathcal{S})$ *or* $(\mathcal{NS})$, *and assume moreover that* g *has limits* $g(\pm\infty)$ *at* $\pm\infty$. *Assume that* $\Phi - G^+ = \Phi(\cdot) - 2k\sqrt{\alpha}\,(g(+\infty)/\alpha - g(-\infty)/\beta)$ *is of constant sign. Then, for any positive integer* m, *equation* (2.1) *has at least two solutions* $x_m,\ y_m$ *of minimal period* $2m\pi$ *having exactly* $2(m-1)$ *zeros in* $[0, 2\pi m)$. *Moreover, the amplitudes of the solutions* $x_m,\ y_m$ *go to infinity as* m *goes to infinity.*

The proof of this result makes use of the Poincaré-Birkhoff theorem in a generalized version due to Ding [5]. More precisely, it is sufficient to use the particular version of Rebelo and Zanolin [14], see also Theorem 3 of [8].

As an example, we deduce from Theorem 4 and [2] that for all positive integer m, there exist at least two solutions $x_m,\ y_m$ of minimal period $2m\pi$ having exactly $2(m-1)$ zeros in $[0, 2\pi m)$ for the equation

$$x'' - \frac{1}{(x+1)^\nu} + \frac{1}{4}(x+1) = a + b\cos t,$$

with $3|a| > |b|$ and $\nu \geq 1$.

3. Proof of the main result

In this section, we still consider the equation (2.1) assuming that V, g, p satisfy hypothesis $(\mathcal{H}_k)$ for some $k \in \mathbb{N}^*$, and V satisfies either $(\mathcal{S})$ or $(\mathcal{NS})$

but we do not impose g to have limits at $\pm\infty$. We first begin with an auxiliary result.

Lemma 1. *Let Φ, G_+, G^+ be defined as above. Suppose that there exist two consecutive critical points θ^*, $\theta^{**} \in [0, 2\pi/k)$ of Φ such that $\Phi(\theta^*) < G_+$, $\Phi(\theta^{**}) > G^+$ and $\Phi'(\theta) > 0$ for all $\theta \in (\theta^*, \theta^{**})$. For any $\eta > 0$ such that $\Phi(\theta^* + \eta) < G_+$ and $\Phi(\theta^{**} - \eta) > G^+$, there exists $K > 0$ such that for $\rho_0 > K$ and for all $\theta_0 \in [\theta^* + \eta, \theta^{**} - \eta]$, any solution $x(t)$ of (2.1) with initial position $x(0) = \varphi(\theta_0, \rho_0)$ and initial speed $x'(0) = \partial\varphi/\partial t(\theta_0, \rho_0)$ is unbounded in the future.*

Proof. Let $x(\cdot)$ be a solution of (2.1) with $|x(0)| + |x'(0)|$ sufficiently large. As in Section 2, we define differentiable functions $\rho(\cdot) > 0, \theta(\cdot)$ by

$$x(t) = \varphi(t + \theta(t), \rho(t)), \quad x'(t) = \frac{\partial\varphi}{\partial t}(t + \theta(t), \rho(t)). \tag{3.1}$$

The functions $\rho(\cdot)$ and $\theta(\cdot)$ satisfy (2.4) and (2.5).

Using estimates like (2.9), (2.10) proved in [2], it can then be shown that

$$\rho(2\pi) - \rho(0) = \frac{1}{\alpha}\, \Phi'(\theta(0)) + o(1) \text{ for } \rho(0) \to +\infty\,, \tag{3.2}$$

$$(\theta(2\pi) - \theta(0))\rho(0) = \frac{1}{\alpha}\,[-\Phi(\theta(0)) + G(\rho(0))] + o(1) \text{ for } \rho(0) \to +\infty. \tag{3.3}$$

Let us denote by Φ^{-1} the inverse function of the restriction of Φ to (θ^*, θ^{**}). Since Φ' is continuous and positive on (θ^*, θ^{**}), there exists $\varepsilon > 0$ such that $\Phi'(\theta)/\alpha \geq \varepsilon$ for all $\theta \in [\theta^* + \eta, \theta^{**} - \eta]$.

Claim - There exists $K > 0$ such that, if $\rho(0) > K$ and $\theta(0) \in [\theta^*+\eta, \theta^{**}-\eta]$, then

$$\rho(2\pi) - \rho(0) > \frac{\varepsilon}{2}, \tag{3.4}$$

$$\theta(2\pi) \in [\theta^* + \eta, \theta^{**} - \eta]. \tag{3.5}$$

1st CASE: Take $\eta' > \eta$ such that $\Phi(\theta^* + \eta') < G_+$ and $\Phi(\theta^{**} - \eta') > G^+$. Suppose that $\theta(0) \in [\theta^* + (\eta + \eta')/2, \theta^{**} - (\eta + \eta')/2]$. We infer from (3.3) that there exists $L_1 > 0$ such that if $\rho(0) > L_1$, then

$$|\theta(2\pi) - \theta(0)| \leq \frac{\eta' - \eta}{2}, \tag{3.6}$$

which implies (3.5). On the other hand, for some $L_2 \geq L_1$, the estimate (3.2) implies that if $\rho(0) > L_2$, then

$$\rho(2\pi) - \rho(0) > \frac{\varepsilon}{2}.$$

Observe that the choice of L_2 does not depend on $\theta(0) \in [\theta^* + \eta, \theta^{**} - \eta]$.

2nd case: Suppose that $\theta(0) \in [\theta^* + \eta, \theta^* + (\eta + \eta')/2)$. Let $L_3 \geq L_2$ be such that $G(\rho) \in (\Phi(\theta^* + \eta'), \Phi(\theta^{**} - \eta'))$ if $\rho \geq L_3$. It follows from (3.3) that

$$\theta(2\pi) - \theta(0) > \frac{1}{\rho(0)\alpha} \left[\Phi(\theta^* + \eta') - \Phi(\theta^* + (\eta + \eta')/2)\right] + o\left(\frac{1}{\rho(0)}\right)$$

and then for some $L_4 \geq L_3$, we obtain that $\theta(2\pi) > \theta(0)$ if $\rho(0) \geq L_4$. Since $|\theta(2\pi) - \theta(0)|$ is small when L_4 is large enough, we conclude that $\theta(2\pi) \in [\theta^* + \eta, \theta^{**} - \eta]$ and, as $L_4 \geq L_2$, we also have

$$\rho(2\pi) - \rho(0) > \frac{\varepsilon}{2}.$$

3rd case: The case $\theta(0) \in (\theta^{**} - (\eta + \eta')/2, \theta^{**} - \eta]$ is treated in an analogous way.

Now, as $\theta(0) \in [\theta^* + \eta, \theta^{**} - \eta]$, we conclude by induction that if $\rho(0) > K$ with sufficiently large K , then $\theta(2\pi m) \in [\theta^* + \eta, \theta^{**} - \eta]$ for all $m \in \mathbb{N}$ and it follows from the repeated application of the claim that the solution $x(t)$ of (2.1) is unbounded in the future. □

A similar result can be written in the reverse situation, namely when Φ is decreasing between two critical points, the critical values bracketing the interval $[G_+, G^+]$.

Lemma 2. *Let θ^*, $\theta^{**} \in [0, 2\pi/k)$ be two consecutive critical points of Φ, such that $\Phi(\theta^*) > G^+$, $\Phi(\theta^{**}) < G_+$ and $\Phi'(\theta) < 0$ for all $\theta \in (\theta^*, \theta^{**})$. For any $\eta > 0$ such that $\Phi(\theta^* + \eta) > G^+$ and $\Phi(\theta^{**} - \eta) < G_+$, there exists $K > 0$ such that for $\rho_0 > K$ and for all $\theta_0 \in [\theta^* + \eta, \theta^{**} - \eta]$, any solution x of (2.1) with initial position $x(0) = \varphi(\theta_0, \rho_0)$ and initial speed $x'(0) = \partial\varphi/\partial t(\theta_0, \rho_0)$ is unbounded in the past.*

Notice that Lemmas 1 and 2 do also apply in the case where $G_+ = G^+$ and $\Phi - G_+$ has no double zeros.

Let us assume from now on the hypotheses of Theorem 1.

We fix $\eta > 0$ and $\varepsilon > 0$ such that for all $\theta \in [0, 2\pi/k)$ satisfying

$$G_+ - \eta \leq \Phi(\theta) \leq G^+ + \eta, \tag{3.7}$$

one has

$$|\Phi'(\theta)| \geq \varepsilon. \tag{3.8}$$

The following notation is used in the sequel: sets $I_1,\ I_2,\ I_3$ and I_4 are defined by

$$
\begin{aligned}
&I_1 = \{\theta \in [0, 2\pi/k) \mid \Phi'(\theta) > 0 \text{ and } G_+ - \eta \le \Phi(\theta) \le G^+ + \eta\}\,,\\
&I_2 = \{\theta \in [0, 2\pi/k) \mid \Phi(\theta) > G^+ + \eta\},\\
&I_3 = \{\theta \in [0, 2\pi/k) \mid \Phi'(\theta) < 0 \text{ and } G_+ - \eta \le \Phi(\theta) \le G^+ + \eta\},\\
&I_4 = \{\theta \in [0, 2\pi/k) \mid \Phi(\theta) < G_+ - \eta\}.
\end{aligned}
$$

By Lemmas 1 and 2, there exists $K > 0$ such that if $\rho_0 > K$ and $\theta_0 \in I_1$, then the solution $x(t)$ of (2.1) with initial position $x(0) = \varphi(\theta_0, \rho_0)$ and initial speed $x'(0) = \partial\varphi/\partial t(\theta_0, \rho_0)$ is unbounded in the future while if $\theta_0 \in I_2$, the solution $x(t)$ of (2.1) with initial position $x(0) = \varphi(\theta_0, \rho_0)$ and initial speed $x'(0) = \partial\varphi/\partial t(\theta_0, \rho_0)$ is unbounded in the past. In order to prove Theorem 1, we show that when $\rho(0)$ is sufficiently large, if $\theta(0) \in I_2 \cup I_4$, there always exists a finite time after which $\theta(t)$ enters I_1 in the future and a finite time before which $\theta(t)$ was in I_3 in the past. As a direct consequence, we will obtain that any solution with sufficiently large initial condition is unbounded either in the past or in the future.

We will investigate the behavior of large amplitude solutions by analyzing them, in the ρ, θ coordinates, at times $2\pi n$. For short, we set for all integer n, $\rho_n = \rho(2\pi n)$ and $\theta_n = \theta(2\pi n)$. We need to control the amplitude of these solutions to ensure the validity of the estimates derived from (3.2) and (3.3). An auxiliary estimation will show that the amplitude variation can be controlled while θ stays in the region I_2 or I_4. It makes use of the following elementary observation concerning a sequence $\{\rho_n\}$.

Lemma 3. *Given $M > 0$, there exists $N > 0$ such that, if $\rho_n \ge N$ and $|\rho_{n+1} - \rho_n| \le M$ for $n = 0, 1, \ldots, m-1$, then*

$$\rho_0 \exp\Big(-2M \sum_{n=0}^{m-1} \frac{1}{\rho_n}\Big) \le \rho_m \le \rho_0 \exp\Big(2M \sum_{n=0}^{m-1} \frac{1}{\rho_n}\Big). \tag{3.9}$$

Proof. With N large, ρ_{n+1}/ρ_n is close to 1 and we can therefore approximate $\ln(\rho_{n+1}/\rho_n)$ by $\rho_{n+1}/\rho_n - 1$. More precisely, we can choose $N > 0$ such that, if $\rho_n \ge N$ and $|\rho_{n+1} - \rho_n| \le M$, then

$$\Big|\ln\Big(\frac{\rho_{n+1}}{\rho_n}\Big)\Big| \le \frac{2M}{\rho_n}.$$

Summing these inequalities for $n = 0, 1, \ldots, m-1$, we conclude that

$$-2M \sum_{n=0}^{m-1} \frac{1}{\rho_n} \le \ln\Big(\frac{\rho_m}{\rho_0}\Big) \le 2M \sum_{n=0}^{m-1} \frac{1}{\rho_n}$$

and the proof is completed. □

For convenience, we define the set $C^+ = \{\theta \in \mathbb{R} : \Phi'(\theta) = 0, \Phi(\theta) > G^+\}$. Similarly, we define C_+ as the set of critical points of Φ corresponding to critical values strictly smaller than G_+.

Next, we show that any solution of (2.1) starting with an initial "angle" $\theta(0)$ in I_2 or I_4 and a sufficiently large initial amplitude $\rho(0)$ enters in I_1 after a finite time.

Lemma 4. *Under the assumptions of Theorem* 1, *there exist* $L > 0$ *and* $S > 0$ *such that if* $x(t)$ *is a solution of* (2.1) *with initial conditions* $\theta(0) \in I_2 \cup I_4$ *and* $\rho(0) > L$, *then* $\theta(2\pi m) \in I_1$ *and* $\rho(2\pi m) \geq S\rho(0)$ *for some* $m \in \mathbb{N}$.

Proof. We write the proof only for $\theta(0) \in I_2$, the case of $\theta(0) \in I_4$ being similar. Given $\theta(0) \in I_2$, define $\theta^* = \max\{\theta \in C_+ : \theta < \theta(0)\}$ and $\theta^{**} = \min\{\theta \in C^+ \mid \theta > \theta^*\}$. As before, we denote by Φ^{-1} the inverse function of the restriction of Φ to (θ^*, θ^{**}). We also define $\bar{\theta} = \Phi^{-1}(G^+ + \eta)$ where η is defined by (3.7), (3.8). We will show that there exists $L > 0$ such that if $\rho(0) > L$, then for some $m \in \mathbb{N}$, $\theta_m \leq \bar{\theta}$. Assuming that m is the first integer for which the inequality occurs, this will clearly imply $\theta_m \in I_1$, since $\theta_m - \theta_{m-1}$ will be small for $\rho(0)$ large.

Let $L_1 > 0$ be such that if $\rho > L_1$, then $G(\rho) \in [G_+ - \eta/2, G^+ + \eta/2]$. As long as $\Phi(\theta_n) > \Phi(\bar{\theta}) = G^+ + \eta$ and $\rho_n > L_1$, we have $G(\rho_n) - \Phi(\theta_n) \leq -\eta/2$. It follows from (3.3) that for some $L_2 \geq L_1$, we have

$$\theta_{n+1} - \theta_n \leq \frac{-\eta}{4\alpha\rho_n}, \tag{3.10}$$

if $\rho_n \geq L_2$. Let us fix $\delta = \eta/4\alpha$. If ρ_n remains greater or equal to L_2 for $n = 0, 1, \ldots, m-1$, it follows from (3.10) that

$$\theta_m \leq \theta_0 - \delta \sum_{n=0}^{m-1} \frac{1}{\rho_n}. \tag{3.11}$$

On the other hand, by (3.2), for ρ_n sufficiently large, we will have that, for some constant M, $|\rho_{n+1} - \rho_n| \leq M$. Let N be the number associated to M by Lemma 3. Provided that $\rho_n \geq L_3 = \max(L_2, N)$ for $n = 0, 1 \ldots, m-1$, we will have, by (3.9) and (3.11), that

$$\rho_0 \exp\Big(\frac{2M}{\delta}(\theta_m - \theta_0)\Big) \leq \rho_m \leq \rho_0 \exp\Big(-\frac{2M}{\delta}(\theta_m - \theta_0)\Big). \tag{3.12}$$

It results from the above inequalities that, with

$$\rho_0 \geq L_3 \exp\Big(-\frac{2M}{\delta}(\bar{\theta} - \theta_0)\Big),$$

the numbers ρ_n will satisfy $\rho_n \geq L_3$, as long as θ_n remains above $\bar{\theta}$. But, the inequality (3.11) then implies that $\theta_m \leq \bar{\theta}$ for some m. Observe also that, since there exists some positive constant C such that

$$|\theta_{n+1} - \theta_n| \leq \frac{C}{\rho_n}$$

for sufficiently large ρ_n, we can take ρ_0 large enough in such a way that

$$G_+ - \eta \leq \Phi(\theta_m),$$

and thus $\theta_m \in I_1$. Since $\theta_0 - \theta_m \leq 2\pi/k$, we also deduce from (3.12) that

$$\rho_m \geq \rho_0 \exp\Big(-\frac{16M\pi\alpha}{k\eta}\Big). \qquad \square$$

As a direct consequence of the preceding lemma, we obtain that every solution of large amplitude starting from regions I_2 or I_4 is unbounded in the future.

Corollary 1. *Under the hypotheses of Theorem* 1, *there exists* $K' > 0$ *such that if* $\rho(0) > K'$ *and* $\theta(0) \in I_2 \cup I_4$, *then any solution* $x(t)$ *of* (2.1) *with initial position* $x(0) = \varphi(\theta(0), \rho(0))$ *and initial speed* $x'(0) = \partial\varphi/\partial t(\theta(0), \rho(0))$ *is unbounded in the future.*

Proof. By Lemma 4, if $\rho(0)$ is sufficiently large, there exists $m \in \mathbb{N}$ such that $\theta(2\pi m) \in I_1$ and $\rho(2\pi m) \geq S\rho(0)$ for some positive S. Now, we are in the situation of Lemma 1 with $\rho_0 = \rho(2\pi m)$ and $\theta_0 = \theta(2\pi m)$. When $\rho(0)$ is large enough, Lemma 1 can be applied and the result follows. $\square$

Next we give without proof the equivalent of Corollary 1, when observing the behavior of the solutions in the past.

Corollary 2. *Under the hypotheses of Theorem* 1, *there exists* $K' > 0$ *such that if* $\rho(0) > K'$ *and* $\theta(0) \in I_2 \cup I_4$, *then any solution* $x(t)$ *of* (2.1) *with initial position* $x(0) = \varphi(\theta(0), \rho(0))$ *and initial speed* $x'(0) = \partial\varphi/\partial t(\theta(0), \rho(0))$ *is unbounded in the past.*

Now the proof of Theorem 1 results from the preceding lemmas and corollaries. All the solutions starting with large amplitude from I_1 (respectively from I_3), will be unbounded in the future (respectively in the past) while solutions starting from I_2 or I_4 are unbounded in the future and in the past.

Proof of Theorem 1. Let $x(t)$ be a solution of (2.1) with initial conditions $(x(0), x'(0))$. Let $\rho(0)$ and $\theta(0)$ be the corresponding initial values in the ρ, θ coordinates. If $(x(0))^2+(x'(0))^2$ is sufficiently large, then $\rho(0) > \max(K, K')$ where K and K' are given by Corollary 1 and 2. From what precedes, it follows that if $\theta(0) \in I_1 \cup I_2 \cup I_3$ the amplitude $\rho(t)$ grows indefinitely, leading to the unboundedness of $x(t)$ in the future. If $\theta(0) \in I_2 \cup I_3 \cup I_4$, then if we reverse the time, the amplitude grows indefinitely, leading to the unboundedness of $x(t)$ in the past. □

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