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# ON A REGULARITY CRITERION FOR THE SOLUTIONS TO THE 3D NAVIER-STOKES EQUATIONS

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**Abstract.** In this paper we give a simple proof of the regularity of a class of solutions to the 3D Navier-Stokes equations for a fluid filling any smooth three-dimensional domain. The regularity in the same class was proved by Beirão da Veiga in reference [3], for the Cauchy problem in $\mathbf{R}^n$.

## 1. Introduction

Let $\Omega \subseteq \mathbf{R}^3$ be an open set with smooth boundary $\partial\Omega$. We consider the Navier-Stokes equations in $\Omega \times (0,T)$, for an arbitrarily fixed $0 < T < \infty$ :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)\, u + \nabla p = 0 & \text{in } \Omega \times (0,T) \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u_{|t=0} = u_0(x) & \text{in } \Omega \end{cases} \tag{N-S}$$

and in addition $u(x,t) \to 0$ as $|x| \to \infty$, if $\Omega$ is unbounded. Hereafter $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p = p(x,t)$ denote the unknown velocity and pressure, respectively. We denote by $L^2_\sigma(\Omega)$ and $H^1_{0,\sigma}$ the closure of $\mathcal{D}(\Omega)$ (the space of compactly-supported divergence-free vector-fields) with respect to the norm of $L^2(\Omega)$ and $H^1_0(\Omega)$, respectively. The spaces $H^s_0(\Omega)$, for $s \geq 0$, are the customary Sobolev spaces, and we denote with the same symbol both scalar and vector function spaces.

It is well-known (see Leray [16] and Hopf [12]) that for every $u_0 \in L^2_\sigma(\Omega)$ there exists at least a weak solution, i.e., a function

$$u \in C_w(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_{0,\sigma}(\Omega)).$$

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such that

$$\int_0^T \int_\Omega \Big[u\frac{\partial\phi}{\partial t} - \nu\,\nabla u\cdot\nabla\phi - (u\cdot\nabla)\,u\,\phi\Big]dx\,dt = \int_\Omega [u(T)\phi(T) - u_0\phi(0)]\;dx,$$

for all $\phi \in L^2(0,T;H^2(\Omega)\cap H^1_{0,\sigma}(\Omega))$, with $\partial\phi/\partial t \in L^2(0,T;L^2_\sigma(\Omega))$. A weak solution $u$ satisfies also the *energy inequality*

$$\frac{1}{2}\|u(t)\|^2 + \nu\int_0^t \|\nabla u(\tau)\|^2\,d\tau \le \frac{1}{2}\|u_0\|^2 \quad \forall\, t\in[0,T],$$

where $\|\,.\,\|$ denotes the norm of $L^2(\Omega)$. On the other hand, if the initial datum is more regular, say $u_0 \in H^1_{0,\sigma}(\Omega)$, then there exists $0 < T_* \le T$ and a unique *strong solution* in $(0,T_*)$, i.e., a weak solution such that

$$u \in L^\infty(0,T_*;H^1_{0,\sigma})\cap L^2(0,T_*;\mathcal{D}(A)) \quad \text{and} \quad \frac{\partial u}{\partial t} \in L^2(0,T_*;L^2_\sigma(\Omega)),$$

where $A:\mathcal{D}(A)\to L^2_\sigma(\Omega)$ is the well-known Stokes operator. We recall that for smooth domains $\mathcal{D}(A) = H^2(\Omega)\cap H^1_{0,\sigma}(\Omega)$, for an elementary proof see Beirão da Veiga [4].

The basic problem is that, in the general case, the uniqueness and regularity of weak solutions have not been known.

One of the most studied classes of weak solutions that are unique and regular (say in $C^k(\epsilon,T;H^2(\Omega))$, for each $\epsilon > 0$ and $k \in \mathbf{N}$) are the weak solutions such that

$$u \in L^r(0,T;L^s(\Omega)) \quad \text{for} \quad \frac{2}{r}+\frac{3}{s} = 1, \quad 2\le r<\infty,\; 3<s\le\infty, \qquad (1.1)$$

see Leray [16], Prodi [17], Serrin [18], and Sohr [19]. The regularity in the limit case $r=\infty$ and $s=3$ is only conjectured, see Beirão da Veiga [5] and Berselli [6] for some recent results. In particular, we observe that strong solutions satisfy (1.1) since they belong, for instance, to $L^4(0,T;L^6(\Omega))$.

The class (1.1) is important from the point of view of *scaling invariance.* If a pair $\{u,p\}$ solves (N-S) in $\Omega$, then so does in $\Omega_\lambda = \{x/\lambda \text{ with } x\in\Omega\}$ the family $\{u_\lambda,p_\lambda\}_{\lambda>0}$ defined by:

$$u_\lambda := \lambda\,u(\lambda x,\lambda^2 t) \qquad p_\lambda := \lambda^2\,p(\lambda x,\lambda^2 t).$$

The scaling invariance means that $\|u_\lambda\|_{L^r(0,T;L^s(\Omega_\lambda))} = \|u\|_{L^r(0,T;L^s(\Omega))}$ if and only if $r$ and $s$ satisfy (1.1). It is interesting to study if it is possible to find other criteria, similar to (1.1), by weakening the summability in the time variable but correspondingly increasing the regularity in the space variable beyond the space $L^\infty(\Omega)$. The results obtained in Beirão da Veiga [2, 3] state, for the Cauchy problem ($\Omega = \mathbf{R}^3$), the regularity of weak-solutions such that

$$\nabla u \in L^\alpha(0,T;L^\beta(\Omega)) \quad \text{for} \quad \frac{2}{\alpha} + \frac{3}{\beta} = 2, \quad 1 \le \alpha \le 2, \ 3 \le \beta \le \infty. \quad (1.2)$$

More precisely, in references [2, 3] it is considered the case $\Omega = \mathbf{R}^n$, for $n \ge 3$, with $2/\alpha + n/\beta = 2$ and $1 \le \alpha \le 2$, $n \le \beta \le \infty$.

As "historical" remark, observe that the first criterion involving the derivatives of $u$ was just introduced by Leray. In fact, in reference [16], pp. 227, Leray proved the following result involving the derivatives of the velocity: the condition

$$\nabla u \in L^4(0,T;L^2(\mathbf{R}^3)) \quad (1.3)$$

implies that there is no *epochs of irregularity*. In the above condition the exponent satisfy formally (1.2) (here $\alpha = 4 > 2$ and $\beta = 2 < 3$, see also Remark 2.3). Furthermore, (1.3) is a stronger condition (with respect to (1.2)): due to a Sobolev embedding theorem (see also Remark 1.3 below) if $u$ satisfies (1.3), then it also belong to $L^4(0,T;L^6(\mathbf{R}^3))$, namely $u$ satisfies condition (1.1).

For partial results in this direction, concerning the problem in a smooth bounded domain, see the results announced in Section 5 of Galdi [9], and see also He [11].

We now recall the following definition, for the reader's convenience.

**Definition.** We say that $\Omega$ is *uniformly of class* $C^k$, $k \ge 0$, if $\Omega$ lies on one part of its boundary $\partial\Omega$ and, for each $y_0 \in \partial\Omega$, there exists a ball $B$ centered at $y_0$ and of radius independent of $y_0$, such that $\partial\Omega \cap B$ admits a Cartesian representation of the form $y_3 = \gamma(y_1, y_2)$. Here $\gamma$ is a function of class $C^k$ in its domain, with derivatives up to order $k$ inclusive uniformly bounded by the same constants, independently of $y_0$. If $\Omega$ is uniformly of class $C^k$, for all $k \ge 0$, we say that $\Omega$ is uniformly of class $C^\infty$.

We do not consider the maximal generality in the number of space dimensions, but we limit to the three-dimensional case. We improve the known results by considering arbitrary smooth open sets of $\mathbf{R}^3$. Furthermore, our proof works also in the case $\Omega = \mathbf{R}^3$, and also in this case the technique we use is completely different from that one in reference [3].

The main result we will prove is the following.

**Theorem 1.1.** *Let $\Omega \subseteq \mathbf{R}^3$ be either the whole space or a domain uniformly of class $C^2$, and let $u_0 \in H^1_{0,\sigma}(\Omega)$. Then the solution to* (N-S) *with $u_0$ as initial datum and satisfying* (1.2) *is strong in* $[0,T]$.

By recalling some results of regularity, see for instance Galdi and Maremonti [10], if the boundary $\partial\Omega$ is uniformly of class $C^\infty$ and if $u$ is strong solution, then $u \in C^\infty(\overline{\Omega} \times (0,T])$. With this observation we can state the following corollary.

**Corollary 1.2.** *Let $u$ be a solution to the 3D Navier-Stokes equations in $\Omega$, with $\partial\Omega$ uniformly of class $C^\infty$. If $u_0 \in H^1_{0,\sigma}$ and if condition* (1.2) *is satisfied, then $u \in C^\infty(\overline{\Omega} \times (0,T])$.*

**Remark 1.3.** The fact that the exponents should satisfy condition (1.2) can be understood, from the point of view of scaling invariance, by observing that $\nabla u_\lambda = \lambda^2 \nabla u(\lambda x, \lambda^2 t)$. It is also interesting to consider such exponents in the light of the the Sobolev embedding theorems. In fact, if $D_0^{1,\beta}(\Omega)$ denotes the closure of smooth, compactly-supported functions, with respect to the norm

$$\|\phi\|_{D_0^{1,\beta}(\Omega)} := \Big[ \int_\Omega |\nabla \phi|^\beta \, dx \Big]^{1/\beta},$$

then

$$D_0^{1,\beta}(\Omega) \subset L^{\beta^*}(\Omega), \quad \text{for} \quad \frac{1}{\beta^*} = \frac{1}{\beta} - \frac{1}{3},$$

with algebraic and topological inclusion. If $u$ belongs to the class (1.2), then $u \in L^\alpha(0,T; L^{\beta^*}(\Omega))$ and

$$\frac{2}{\alpha} + \frac{3}{\beta^*} = \frac{2}{\alpha} + \frac{3}{\beta} - \frac{3}{3} = 1,$$

which is (formally, because now $1 \leq \alpha \leq 2$) the class (1.1). Observe that the limit case $\alpha = 2$, $\beta = 3$ implies that $L^2(0,T; W^{1,3}(\Omega))$ is a regularity class. This case overlaps with (1.1), that gives $u \in L^2(0,T; L^\infty(\Omega))$, and $W^{1,3}(\Omega)$ is not a subset of $L^\infty(\Omega)$. If $\Omega = \mathbf{R}^3$ this result has been recently improved by Kozono and Taniuchi [13], who showed that $L^2(0,T; BMO)$ is a regularity class. In reference [15] Kozono, Ogawa, and Taniuchi consider conditions similar to (1.2), but involving Besov spaces.

On the other hand, the limit case $\alpha = 1$ gives $\nabla u \in L^1(0,T; L^\infty(\Omega))$. Note that, except the use of the logarithmic estimate (1.6) involving $\omega = \nabla \times u$ and $\nabla u$, this is a regularity-class for the 3D Euler equations, see Beale, Kato, and Majda [1] for $\Omega = \mathbf{R}^3$ and Ferrari [8] for the equations in a smooth bounded domain.

From Theorem 1.1 it is immediate to infer the following result.

**Theorem 1.4.** *Let* $\Omega$ *be as in Theorem* 1.1 *and let* $u_0 \in H^1_{0,\sigma}(\Omega)$. *If* $u$ *is a solution with* $u_0$ *as initial datum and if* $\omega = \nabla \times u$ *satisfies*

$$\omega \in L^\alpha(0,T;L^\beta(\Omega)) \quad \text{for} \quad \frac{2}{\alpha}+\frac{3}{\beta}=2, \quad 1<\alpha\leq 2,\ 3\leq\beta<\infty, \tag{1.4}$$

*then* $u$ *is a strong solution on* $[0,T]$.

**Proof of Theorem 1.4.** For the reader convenience we give the sketch of the proof. We observe that $\nabla\times(\nabla\times u)=\nabla(\nabla\cdot u)-\Delta u$ and consequently, since $\nabla\cdot u=0$, we have

$$-\Delta u = \nabla\times\omega \quad \text{in} \quad \Omega. \tag{1.5}$$

If $1/p+1/q=1$, by multiplying (1.5) by a smooth function $\psi$, vanishing on the boundary $\partial\Omega$, and integrating by parts we obtain

$$\int_\Omega \nabla u\,\nabla\psi\,dx = \int_\Omega \omega\,\nabla\times\psi\,dx$$

and then

$$\|\nabla u\|_p = \sup_{0\neq\nabla\psi\in L^q}\frac{\left|\int_\Omega \nabla u\,\nabla\psi\,dx\right|}{\|\nabla\psi\|_q} = \sup_{0\neq\nabla\psi\in L^q}\frac{\left|\int_\Omega \omega\ \nabla\times\psi\,dx\right|}{\|\nabla\psi\|_q}$$

$$\leq \sup_{0\neq\nabla\psi\in L^q}\frac{\|\omega\|_p\,\|\nabla\times\psi\|_q}{\|\nabla\psi\|_q} \leq \sup_{0\neq\nabla\psi\in L^q}\frac{\|\omega\|_p\,\|\nabla\psi\|_q}{\|\nabla\psi\|_q} = \|\omega\|_p,$$

where $\|\,.\,\|_p$ denotes the norm of the Lebesgue space $L^p(\Omega)$. With this estimate we can conclude by using Theorem 1.1, since if $\omega\in L^p(\Omega)$, for $1<p<\infty$, then $\nabla u\in L^p(\Omega)$. □

If $\beta=\infty$ we cannot prove the same result of Theorem 1.4. In this case we cannot estimate all the derivatives in the *sup-norm* with the same norm of the curl, which is the anti-symmetric part of the matrix of derivatives. In this case an available estimate is the following one

$$\|\nabla u\|_\infty \leq C\left(1+\|\omega\|_\infty(1+\log^+\|u\|_{H^s})+\|\omega\|_2\right), \quad \text{for} \quad s\geq\tfrac{5}{2} \tag{1.6}$$

where $\log^+(x)=\log(x)$ if $x\geq 1$, 0 otherwise. For the proof of (1.6) (and some variants), the application to study of the breakdown of smooth solutions to the 3D Euler and Navier-Stokes equations and some recent improvements, see references [1, 8, 13, 14].

Concerning Theorem 1.4, note also that in the whole space $\mathbf{R}^3$ it is sufficient to assume the above condition (1.4) just for two of the three components of the vorticity, see Chae and Choe [7].

## 2. Proof of Theorem 1.1

Let $u$ be a strong solution corresponding to the initial datum $u_0$. Note that if the initial datum $u_0$ belongs to $H^1_{0,\sigma}(\Omega)$, then there is a unique strong solution in an interval $[0,\delta)$, where $\delta > 0$ can be bounded from below by a constant depending only on the viscosity $\nu$, on the $C^2$ regularity of $\Omega$, and on the norm $\|\nabla u_0\|$ (see, for instance, [9] pp. 45). Let us suppose, *per absurdum*, that the maximal interval of existence of such $u$ is $[0,T^*)$, with $T^* < T$. Then $T^*$ is an *epoch of irregularity*. By using a standard continuation argument (see [16], pp. 245-245), together with the following Lemma 2.1, it is possible to show that $u$ is strong up to $T^*$, hence on the whole time interval $[0,T]$.

**Lemma 2.1.** *Let $\Omega \subseteq \mathbf{R}^3$ be either the whole space or a domain uniformly of class $C^2$. Let be given $0 < b < T$, and let $u$ be a weak solution to* (N-S) *in* $[0,b]$, *as well as a strong solution in* $[0,t']$, *for each $t' < b$. If condition* (1.2) *is satisfied for $3 \leq \beta \leq \infty$, then $u$ is strong on* $[0,b]$.

**Proof.** We start by observing that classical regularity results (see for instance reference [10]) imply that, for each $k \in \mathbf{N}$

$$\frac{\partial^k u}{\partial t^k} \in L^2(\epsilon, t'; H^2(\Omega)) \quad \text{for each } \epsilon \in (0,t'].$$

By using the previous observation, it follows that all the calculation we are going to do over the time interval $(\epsilon, t')$ are not formal, but completely justified, since each term is well-defined and the boundary integrals (arising in the integration by parts) vanish.

We derive with respect to the time the Navier-Stokes equations (hereafter for each function $F$ we write $F_t = \partial F/\partial t$), we multiply by $u_t$ and we integrate over $\Omega$ to obtain:

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \nu\|\nabla u_t\|^2 \leq \Big|\int_\Omega (u\cdot\nabla)\, u_t\, u_t\, dx\Big| + \Big|\int_\Omega (u_t\cdot\nabla)\, u\, u_t\, dx\Big|.$$

The first integral on the right hand side vanishes, since $u$ is divergence-free. The second one can be estimated as follows by the Hölder inequality:

$$\Big|\int_\Omega (u_t\cdot\nabla) u\, u_t\, dx\Big| \leq \|u_t\|^2_{2p}\|\nabla u\|_q \quad \text{for} \quad \frac{1}{p}+\frac{1}{q} = 1, \text{ with } 3 < q \leq \infty,$$

and recall that $p = 1$, if $q = \infty$.

Then we use the classical convex-interpolation inequality that holds for $f \in L^2(\Omega)\cap L^6(\Omega)$, (observe that $1 \leq p < 3$, and if $p = 1$ there is nothing to do)

$$\|f\|_{2p} = \|f\|_{2q/(q-1)} \le \|f\|_2^{\theta} \|f\|_6^{1-\theta}, \quad \text{for} \quad \theta = \frac{2q-3}{2q}.$$

By using the Sobolev embedding $H^1(\Omega) \subset L^6(\Omega)$, we obtain

$$\Big| \int_\Omega (u_t \cdot \nabla) u \, u_t \, dx \Big| \le c_1 \|u_t\|^{\frac{2q-3}{q}} \|\nabla u_t\|^{\frac{3}{q}} \|\nabla u\|_q.$$

We use Young's inequality with exponents $x = 2q/3$, $x' = 2q/(2q-3)$ and we finally obtain

$$\Big| \int_\Omega (u_t \cdot \nabla) \, u \, u_t \, dx \Big| \le \frac{\nu}{2} \|\nabla u_t\|^2 + c \|\nabla u\|_q^{\frac{2q}{2q-3}} \|u_t\|^2.$$

By using Gronwall lemma, from

$$\frac{d}{dt} \|u_t\|^2 + \nu \|\nabla u_t\|^2 \le 2c \|\nabla u\|_q^{\frac{2q}{2q-3}} \|u_t\|^2$$

we can infer that

$$u_t \in L^\infty(\epsilon, b; L^2) \cap L^2(\epsilon, b; H^1),$$

provided condition (1.2) is satisfied. By multiplying (N-S) by $u$ and integrating over $\Omega$, we have, $\forall \, \tau \in [\epsilon, b]$,

$$\nu \int_\Omega |\nabla u(\tau)|^2 \, dx = \Big| \int_\Omega u_t(\tau) \, u(\tau) \, dx \Big| \le \|u_t(\tau)\| \, \|u(\tau)\| \le M < \infty.$$

Furthermore, $\nabla u$ is bounded in $L^\infty(0, \epsilon; L^2)$, since $u$ is a strong solution in $[0, t']$. Hence we have

$$u \in L^\infty(0, b; H^1). \tag{2.1}$$

The borderline case $q = 3$ can be treated more simply. By using the classical manipulations introduced in [17], we multiply (N-S) by $Au$ and we integrate over $\Omega$ to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \nu \|Au\|^2 &\le \int_\Omega |u| |\nabla u| |Au| \, dx \le \|u\|_6 \|\nabla u\|_3 \|Au\| \\ &\le \frac{\nu}{2} \|Au\|^2 + c \|\nabla u\|_3^2 \|\nabla u\|^2, \end{aligned}$$

and the Gronwall lemma implies (2.1).

As pointed out before, the *life-span* of the unique strong solution $\overline{u}$ starting from $u(t_k)$ (that in its interval of existence $[t_k, t_k + \delta_M)$ coincides with $u$) depends, apart on the data of the problem, on the Dirichlet-norm $\|\nabla u(t_k)\|$ of $u$ at $t = t_k$. We have

$$\sup_{0<t<b} \|\nabla u(t)\| \le M, \quad \text{for some finite } M.$$

We consider a sequence $t_k \nearrow b$, and we denote by $\delta_M > 0$ the lower bound for the life-span of a strong solution starting with an initial datum of Dirichlet-norm less or equal than $M$. It is now enough to choose a $t_k$ such that $b - t_k < \delta_M$ to conclude the proof of Lemma 2.1. □

**Remark 2.2.** If the initial datum $u_0$ does not belong to $H^1_{0,\sigma}(\Omega)$, but just to $L^2_\sigma(\Omega)$, then condition (1.2) (or (1.4)) implies the regularity on $[\bar{t}, T]$, for each $\bar{t} > 0$.

**Remark 2.3.** By using the same arguments, it is possible to show that Theorem 1.1 holds also for $\nabla u$, or indifferently $\omega$ (see Theorem 1.4), belonging to

$$L^\alpha(0,T;L^\beta(\Omega)) \quad \text{for} \quad \frac{2}{\alpha}+\frac{3}{\beta}=2, \quad 2<\alpha<\infty,\ \frac{3}{2}<\beta<3.$$

This do not improve the known results, since (thanks to the Sobolev embedding theorems) for $3/2 < \beta < 3$ the above class is contained in (1.1).

**Remark 2.4.** It is straightforward to prove that Theorems 1.1-1.4 still hold if we add to (N-S) a smooth enough external force, say $f \in W^{1,2}(0,T;L^2_\sigma(\Omega))$.

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