Differential and Integral Equations Volume 15, Number 9, September 2002, Pages 1073–1083

THE GLOBAL CAUCHY PROBLEM AND SCATTERING OF SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATIONS IN H^s

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(Submitted by: J.L. Bona)

Abstract. In this paper, we shall prove that the scattering operator for nonlinear Schrödinger equations $iu_t + (-\Delta)^m u = \lambda_1 |u|^{p_1} u + \lambda_2 |u|^{p_2} u$ carries a band $\dot{B}(p_1, p_2, \delta)$ in H^s into H^s for some $\delta > 0$, where $\dot{B}(p_1, p_2, \delta) = \{\varphi \in H^s : \|\varphi\|_{\dot{H}^{s(p_1)} \cap \dot{H}^{s(p_2)}} \leq \delta\}$, $s(p_i) = n/2 - 2m/p_i$, $s(p_2) \leq s \leq s(p_1) + 1$, $4m/n \leq p_1 \leq p_2 \leq 4m/(n-2s)$.

1. Introduction

The subject of this paper is the existence and scattering theory for nonlinear Schrödinger equations

$$iu_t + (-\Delta)^m u + f(u) = 0, \tag{1.1}$$

$$u(0, x) = \varphi(x), \tag{1.2}$$

where $u(t, x)$ defined on $\mathbb{R} \times \mathbb{R}^n$ is a complex valued function, Δ denotes the Laplace operator on $\mathbb{R}^n$, $m \geq 1$ is an integer. f is a scalar nonlinear function. For given $p_i > 0$ $(i = 1, 2)$, f is assumed to be a C^K function satisfying

$$|f^{(k)}(u)| \leq C(|u|^{p_1+1-k} + |u|^{p_2+1-k}), \quad k = 0, 1, ..., K \tag{1.3}$$

for some natural number K.

[1]Supported by the NSF of China, grant 19901007.
Accepted for publication: October 2000.
AMS Subject Classifications: 35Q20, 35Q55, 35P25.

If $m = 1$, it is well-known that (1.1) is the nonlinear second order Schrödinger equation. A large amount of work has been devoted to the study of the existence and scattering theory of (1.1) and (1.2) in the case $m = 1$. It is known in particular that the global existence of solutions in L^2, H^1 and H^2 was finally proved by Kato in [5]. The classical global solutions was shown by Hayashi and Tsutsumi in [4] if the space dimensions $n \leq 11$. Moreover, the scattering theory was also discussed by Strauss in [8] and by Ginibre and Velo in [3]. In the above work, the nonlinearity $f(u)$ is always assumed to be H^1-subcritical, i.e., $|f'(u)| \leq C(1 + |u|^p)$, $p < 4/(n-2)$. Recently, Cazenave and Weissler in [2] considered the H^s-critical and H^s-subcritical powers $p \leq 4/(n-2s)$ ($f(u) = \lambda |u|^p u$, $0 \leq s < n/2$) and proved the local existence of solutions in H^s. Also, they showed these solutions are global if p is H^s-critical and $\|(-\Delta)^{n/2-2/p}\varphi\|_{L^2}$ is sufficiently small. In the meantime, they pointed out that if an H^s subcritical power occurs in the same equation with the H^s critical power, namely, $f(u) = \pm|u|^{p_1}u \pm |u|^{p_2}u$, $0 < p_1 < 4/(n-2s)$, $p_2 = 4/(n-2s)$, the global existence of solutions with small initial data can not be obtained by using the same way as in [2]. However, the nonlinearities which are the sum of several different power terms, for example $f(u) = \pm|u|^2 u \pm |u|^4 u$, seem to be important in the latest development in the theory of plasma physics.

In the case $m \geq 1$, Pecher and Wahl in [7] proved the existence of classical global solutions of (1.1) and (1.2) for the space dimensions $n \leq 7m$. Recently, Miao in [6] generalized the work of Kato and proved the global existence of solutions in H^{2m}, where he assumed $f(u) = \lambda |u|^p u$ and $n \leq \frac{2m(m+1)}{m-1}$.

In the present paper we will study the existence and scattering theory of (1.1) and (1.2) and generalized the work of Cazenave and Weissler to all $m \geq 1$ and $f(u) = \sum_{i=1}^{N} \lambda_i |u|^{p_i} u$, $p_i \geq 4m/n$. First, we state our main results. For any $r \in [2, \infty)$ and $p \in [4m/n, \infty)$, we define

$$s(p) = \frac{n}{2} - \frac{2m}{p} \tag{1.4}$$

$$\frac{2}{\gamma(r)} = \frac{n}{m}\Big(\frac{1}{2} - \frac{1}{r}\Big) \tag{1.5}$$

$$\alpha(n) = \begin{cases} 2n/(n-2m) & \text{if} \quad m < n/2, \\ \infty & \text{if} \quad m > n/2. \end{cases} \tag{1.6}$$

Theorem 1.1. *Let* $4m/n \leq p_1 \leq p_2 < \infty$, *and let* $s(p_2) \leq s \leq s(p_1) + 1$. *Suppose that* $f(u)$ *satisfies* (1.3) *with* $K = [s] + 1, [s] \leq p_1$. *Let* $\varphi \in H^s$. *Then there exists* $T^* > 0$ *such that* (1.1) *and* (1.2) *have a unique solution*

$u \in \big(\cap_{2<r<\alpha(n)} L^{\gamma(r)}_{loc}(0,T^*;B^s_{r,2})\big) \cap C_{loc}(0,T^*;H^s)$. *Moreover, there exists* $\delta > 0$ *such that if* $\|\varphi\|_{\dot H^{s(p_1)}\cap \dot H^{s(p_2)}} < \delta$, *then* $T^* = \infty$, *i.e.* $u \in C(0,\infty;H^s)\cap L^{\gamma(r)}(0,\infty;B^s_{r,2})$ *for any* $r \in [2,\alpha(n))$.

Theorem 1.2. *Let* $4m/n \le p_1 \le p_2 < \infty$, *and let* $s(p_2) \le s \le s(p_1)+1$. *Suppose that* $f(u)$ *satisfies* (1.3) *with* $K = [s]+1, [s] \le p_1$. *Let* $S(t) = e^{it(-\Delta)^m}$. *There exists* $\delta > 0$ *with the following property: if* $\varphi^- \in H^s$, $\|\varphi^-\|_{\dot H^{s(p_1)}\cap \dot H^{s(p_2)}} < \delta$, *then there exists a unique solution* $u(t)$ *of the integral equation*

$$u(t) = S(t)\varphi^- + i\int_{-\infty}^{t} S(t-\tau)f(u(\tau))d\tau \tag{1.7}$$

such that $u \in C(\mathbb{R},H^s)\cap L^{\gamma(r)}(\mathbb{R},B^s_{r,2})$ $(2 \le r < \alpha(n))$ *and*

$$\lim_{t\to-\infty} \|u(t) - S(t)\varphi^-\|_{H^s} = 0.$$

Moreover, there exists a unique $\phi^+ \in H^s$ *such that*

$$\lim_{t\to+\infty} \|u(t) - S(t)\varphi^+\|_{H^s} = 0.$$

In addition,

$$u(t) = S(t)\varphi^+ - i\int_{t}^{\infty} S(t-\tau)f(u(\tau))d\tau. \tag{1.8}$$

Thus, the scattering operator $S : \varphi^- \to \varphi^+$ *carries a band* $\{\varphi \in H^s : \|\varphi\|_{\dot H^{s(p_1)}\cap \dot H^{s(p_2)}} \le \delta\}$ *into* H^s.

Theorems 1.1 and 1.2 call for several remarks. First, we point out that Theorems 1.1 and 1.2 contain the nonlinearities $f(u) = \sum_{i=1}^n \lambda_i |u|^{\alpha_i} u$ as special cases. In fact, let $p_1 = \min_{1\le i\le n} \alpha_i$ and $p_2 = \max_{1\le i\le n} \alpha_i$. It is easy to see that $|f^{(k)}(u)| \le C(|u|^{p_1+1-k} + |u|^{p_2+2-k})$ if p_1 and p_2 satisfy the conditions of Theorem 1.1. Secondly, if $f(u) = \sum_{i=1}^n \lambda_i |u|^{\alpha_i} u$ and α_i $(1 \le i \le n)$ are even integers, then we see that $f \in C^\infty$ and therefore, condition $[s] \le p_1 = \min_{1\le i\le n} \alpha_i$ is superfluous. Finally, if s is an integer, we can use Sobolev space $H^{s,r}$ instead of Besov space $B^s_{r,2}$ in Theorems 1.1 and 1.2. Moreover, condition $[s] \le p_1$ can be slightly improved by condition $s \le p_1 + 1$.

Throughout this paper, we always assume that $S(t) = e^{it(-\Delta)^m}$; C will denote a constant which can be changed from line to line; r' will denote the dual number of $r \in [1,\infty)$. We will have occasions to use a variety of function spaces, Lebesgue space $L^r = L^r(\mathbb{R}^n)$; Bessel potential space $H^{s,r} = H^{s,r}(\mathbb{R}^n), H^s = H^{s,2}$; Riesz potential space $\dot H^{s,r} = \dot H^{s,r}(\mathbb{R}^n), \dot H^s = \dot H^{s,2}$;

Besov space $B^s_r = B^s_{r,2}(\mathbb{R}^n)$ and homogeneous Besov space $\dot{B}^s_r = \dot{B}^s_{r,2}(\mathbb{R}^n)$. The definitions of these spaces allow $1 < r < \infty, s \geq 0$. If $s > 0$, then $B^s_r = L^r \cap \dot{B}^s_r$, $H^{s,r} = L^r \cap \dot{H}^{s,r}$. An equivalent form of the norm on $\dot{B}^s_r$ is that

$$\|u\|_{\dot{B}^s_r} = \Big(\int_0^\infty t^{-2(s-[s])} \sup_{|h|\leq t} \sum_{|\alpha|=[s]} \|\Delta_h D^\alpha u\|^2_{L^r} \frac{dt}{t} \Big)^{1/2}, \tag{1.9}$$

where $[s]$ denotes the largest integer less than or equal to s, $\Delta_h u(\cdot) = u(\cdot + h) - u(\cdot)$. For additional information on Besov spaces we refer to [1].

The fundamental tools used in this paper are time-space $L^p - L^{p'}$ estimates usually named Strichartz inequalities. The basic method is a contraction mapping argument which has been developed by Cazenave and Weissler, Kato, Ginibre and Velo and Tsutsumi (cf. [2]). For $m = 1$, the generalized Strichartz inequalities on Lebesgue-Besov spaces can be found in Cazenave and Weissler's [2]. Analogously, these inequalities hold also true for all $m \geq 1$ (cf. [10]). We have

Theorem 1.3. *Let $s \in \mathbb{R}$, $2 \leq r, \rho < \alpha(n)$ and let $\gamma(\cdot)$ satisfy* (1.5).

(i) *If $\varphi \in \dot{H}^s$, then $S(t)\varphi \in L^{\gamma(r)}(\mathbb{R}, \dot{B}^s_r)$; there exists a constant $C > 0$ such that*

$$\|S(t)\varphi\|_{L^{\gamma(r)}(\mathbb{R}, \dot{B}^s_r)} \leq C\|\varphi\|_{\dot{H}^s}$$

holds for all $\varphi \in \dot{H}^s$.

(ii) *If $f \in L^{\gamma(r)'}(\mathbb{R}, \dot{B}^s_{r'})$, then $\int_0^t S(t-\tau)f(\tau)d\tau \in L^{\gamma(\rho)}(\mathbb{R}, \dot{B}^s_\rho)$; there exists a constant $C > 0$ such that*

$$\Big\| \int_0^t S(t-\tau)f(\tau)d\tau \Big\|_{L^{\gamma(\rho)}(\mathbb{R}, \dot{B}^s_\rho)} \leq C\|f\|_{L^{\gamma(r)'}(\mathbb{R}, \dot{B}^s_{r'})}$$

holds for all $f \in L^{\gamma(r)'}(\mathbb{R}, \dot{B}^s_{r'})$.

Remark 1.4. By (ii) of Proposition 1.3 we see that

$$\Big\| \int_0^t S(t-\tau)(f_1(\tau) + f_2(\tau))d\tau \Big\|_{L^{\gamma(\rho)}(\mathbb{R}, \dot{B}^s_\rho)} \leq C \sum_{i=1,2} \|f_i\|_{L^{\gamma(r_i)'}(\mathbb{R}, \dot{B}^s_{r_i'})}$$

if $2 \leq r_i, \rho < \alpha(n)$, $f_i \in L^{\gamma(r_i)'}(\mathbb{R}, \dot{B}^s_{r_i'})$.

2. PROOF OF THEOREM 1.2

Let

$$4m/n \leq p_1 \leq p_2 < \infty, \tag{2.1}$$

$$s(p_2) \leq s \leq s(p_1) + 1. \tag{2.2}$$

Denote

$$r_i = \frac{2n(2+p_i)}{n(2+p_i)-4m}, \quad i=1,2. \tag{2.3}$$

It is easy to see that

$$\gamma(r_i) = 2+p_i, \quad i=1,2. \tag{2.4}$$

Let $M, \sigma > 0$. Put

$$\mathcal{D} = \left\{ u \in L^{\gamma(r_i)}(\mathbb{R}, B^s_{r_i}) : \begin{array}{l} \|u\|_{L^{\gamma(r_i)}(\mathbb{R}, B^s_{r_i})} \le M, \\ \|u\|_{L^{\gamma(r_i)}(\mathbb{R}, \dot{B}^{s(p_j)}_{r_i})} \le \sigma, \quad i,j=1,2 \end{array} \right\}, \tag{2.5}$$

$$d(u,v) = \|u-v\|_{L^{\gamma(r_1)}(\mathbb{R},L^{r_1}) \cap L^{\gamma(r_2)}(\mathbb{R},L^{r_2})}. \tag{2.6}$$

One can easily verify that $(\mathcal{D}, d)$ is a complete metric space. Considering the mapping

$$\mathcal{T} : u \to S(t)\varphi^- + i\int_{-\infty}^t S(t-\tau) f(u(\tau))d\tau, \tag{2.7}$$

we shall prove that $\mathcal{T}$ is a contraction mapping on $(\mathcal{D}, d)$ if $\|\varphi\|_{\dot{H}^{s(p_1)} \cap \dot{H}^{s(p_2)}}$ is sufficiently small.

Lemma 2.1. *Suppose that $f(u)$ is a k-times continuously differentiable function. Then we have*

$$\sum_{|\alpha|=k} D^\alpha f(u) = \sum_{q=1}^k \sum_{\Lambda_k^q} C(\alpha_1, \cdots, \alpha_q) f^{(q)}(u) \prod_{i=1}^q D^{\alpha_i} u, \tag{2.8}$$

$$\begin{aligned} \Big| \sum_{|\alpha|=k} D^\alpha (f(u)-f(v)) \Big| &\le C \sum_{q=1}^k \sum_{\Lambda_k^q} \Big| f^{(q)}(u) - f^{(q)}(v) \Big| \prod_{i=1}^q \Big| D^{\alpha_i} u \Big| \\ &+ C \sum_{q=1}^k \sum_{\Lambda_k^q} \Big| f^{(q)}(v) \Big| \sum_{i=1}^q \Big| \prod_{j=1}^{i-1} D^{\alpha_j} v \prod_{j=i+1}^q D^{\alpha_j} u D^{\alpha_i}(u-v) \Big| \\ &= C \sum_{q=1}^k \sum_{\Lambda_k^q} (I_q + II_q), \end{aligned} \tag{2.9}$$

where we assume that $\prod_{j=1}^0 = \prod_{j=q+1}^q = 1$, $C(\alpha_1, \cdots, \alpha_q)$ denotes a constant only depending on $\alpha_1, \cdots, \alpha_q$ and $\Lambda_k^q = \left(\begin{array}{l} |\alpha_1| + \cdots + |\alpha_q| = k \\ |\alpha_1|, \cdots, |\alpha_q| \ge 1 \end{array} \right)$.

Proof. See [11].

Lemma 2.2. *Let* $f \in C^{[s]+1}$ *and* $|f^{(k)}(u)| \le C|u|^{p+1-k}$, $k = 0, 1, \cdots, [s]+1$. *Suppose that* $\max(4m/n, [s]) \le p$ *and* $0 \le s \le s(p)+1$. *Let* $\rho = \frac{2n(2+p)}{n(2+p)-4m}$. *Then we have*

$$\|f(u)\|_{\dot{B}^s_{\rho'}} \le C\|u\|^p_{\dot{B}^{s(p)}_\rho}\|u\|_{\dot{B}^s_\rho}. \tag{2.10}$$

Proof. Step 1. We consider the case $s \ge 1$. By (1.9) we see that

$$\|f(u)\|_{\dot{B}^s_{\rho'}} = \Big(\int_0^\infty t^{-2(s-[s])} \sup_{|h|\le t} \sum_{|\alpha|=[s]} \|\triangle_h D^\alpha f(u)\|^2_{L^{\rho'}} \frac{dt}{t}\Big)^{1/2}. \tag{2.11}$$

Letting $v = u_h$ in (2.9), we have

$$\|\triangle_h D^\alpha f(u)\|_{L^{\rho'}} \le C \sum_{q=1}^{[s]} \sum_{\Lambda^q_{[s]}} (\|I_q\|_{L^{\rho'}} + \|II_q\|_{L^{\rho'}}). \tag{2.12}$$

Noticing that

$$|f^{(q)}(u_h) - f^{(q)}(u)| \le C(|u_h|^{p-q} + |u|^{p-q})|u_h - u|, \tag{2.13}$$

we have

$$\|I_q\|_{L^{\rho'}} \le C\Big\|(|u_h|^{p-q} + |u|^{p-q}) \prod_{i=1}^q D^{\alpha_i} u|u_h - u|\Big\|_{L^{\rho'}}. \tag{2.14}$$

If $[s] = [s(p)]+1$, let $a_0 = (p-q)(\frac{1}{\rho} - \frac{s(p)}{n})$, $a_1 = \frac{1}{\rho} - \frac{s-|\alpha_1|}{n}$, $a_i = \frac{1}{\rho} - \frac{s(p)-|\alpha_i|}{n}$, $i = 2, \cdots, q$, $a_{q+1} = \frac{1}{\rho} - \frac{s(p)-(s-[s])}{n}$.

If $[s] = [s(p)]$, let $b_0 = (p-q)(\frac{1}{\rho} - \frac{s(p)}{n})$, $b_i = \frac{1}{\rho} - \frac{s(p)-|\alpha_j|}{n}$, $j = 1, \cdots, q$, $b_{q+1} = \frac{1}{\rho} - \frac{[s]}{n}$.

In any case, it is easy to see that a_j, $b_j > 0$ $(j = 0, \cdots, q+1)$. Moreover, $\sum_{i=1}^q |\alpha_i| = [s]$ and $s(p) = n/2 - 2m/p$ imply that $\sum_{i=0}^{q+1} a_i = 1/\rho'$ and $\sum_{i=0}^{q+1} b_i = 1/\rho'$. In view of the Hölder inequality, we have

$$\|I_q\|_{L^{\rho'}} \le C\|u\|^{p-1}_{\dot{B}^{s(p)}_\rho}\|u\|_{\dot{B}^s_\rho}\|u_h - u\|_{L^{1/a_{q+1}}}, \quad \text{if} \quad [s] = [s(p)]+1; \tag{2.15}$$

$$\|I_q\|_{L^{\rho'}} \le C\|u\|^p_{\dot{B}^{s(p)}_\rho}\|u_h - u\|_{L^{1/b_{q+1}}}, \quad \text{if} \quad [s] \le [s(p)]. \tag{2.15a}$$

If $[s] = [s(p)] + 1$, then we have $a_{q+1} - \frac{s-[s]}{n} = \frac{1}{\rho} - \frac{s(p)}{n}$, which implies that $\dot{B}^{s(p)}_{\rho} \subset \dot{B}^{s-[s]}_{1/a_q+1}$. Thus

$$\Big(\int_0^\infty t^{-2(s-[s])} \sup_{|h|\le t} \sum_{|\alpha|=[s]} \|I_q\|^2_{L^{\rho'}} \frac{dt}{t}\Big)^{1/2} \le C\|u\|^{p-1}_{\dot{B}^{s(p)}_{\rho}} \|u\|_{\dot{B}^s_{\rho}} \|u\|_{\dot{B}^{s-[s]}_{1/a_q+1}} \le C\|u\|^{p}_{\dot{B}^{s(p)}_{\rho}} \|u\|_{\dot{B}^s_{\rho}}. \tag{2.16}$$

If $[s] \le [s(p)]$, we can also obtain (2.16) by using (2.15a). We now estimate $\|II_q\|_{L^{\rho'}}$. We have

$$\|II_q\|_{L^{\rho'}} \le C \sum_{i=1}^{q} \Big\| |u_h|^{p+1-q} \prod_{j=1}^{i-1} D^{\alpha_j} u_h \prod_{j=i+1}^{q} D^{\alpha_j} u D^{\alpha_i}(u_h - u)\Big\|_{L^{\rho'}}. \tag{2.17}$$

If $[s] \ge [s(p)]$, let $a_0 = (p+1-q)(\frac{1}{\rho} - \frac{s(p)}{n})$, $a_j = \frac{1}{\rho} - \frac{s(p)-|\alpha_j|}{n}$, $j \ne i$, $j = 1, \cdots, q$, $a_i = \frac{1}{\rho} - \frac{[s]-|\alpha_i|}{n}$.

One can easily verify that $a_j > 0$ $(j = 0, \cdots, q)$ and $\sum_{j=0}^{q} a_j = 1/\rho'$. By the Hölder inequality we have

$$\|II_q\|_{L^{\rho'}} \le C \sum_{i=1}^{q} \|u\|^p_{\dot{B}^{s(p)}_{\rho}} \|D^{\alpha_i}(u_h - u)\|_{L^{1/a_i}}. \tag{2.18}$$

Since $a_i - \frac{s-[s]}{n} = \frac{1}{\rho} - \frac{s-|\alpha_i|}{n}$, we have $\dot{B}^s_{\rho} \subset \dot{B}^{|\alpha_i|+s-[s]}_{1/a_i}$. So we have

$$\Big(\int_0^\infty t^{-2(s-[s])} \sup_{|h|\le t} \sum_{|\alpha|=[s]} \|II_q\|^2_{L^{\rho'}} \frac{dt}{t}\Big)^{1/2} \le C\|u\|^p_{\dot{B}^{s(p)}_{\rho}} \|u\|_{\dot{B}^s_{\rho}}. \tag{2.19}$$

If $[s] < [s(p)]$, let $a_0 = (p-q)(\frac{1}{\rho} - \frac{s(p)}{n})$, $a_0' = \frac{1}{\rho} - \frac{s}{n}$, $a_j = \frac{1}{\rho} - \frac{s(p)-|\alpha_j|}{n}$, $j \ne i$, $j = 1, \cdots, q$, $a_i = \frac{1}{\rho} - \frac{s(p)-(s-[s])-|\alpha_i|}{n}$.

It is easy to see that $a_0' > 0$, $a_j > 0$ $(j = 0, \cdots, q)$. Also, we have $a_0' + \sum_{j=0}^{q} a_j = 1/\rho'$. It follows that

$$\|II_q\|_{L^{\rho'}} \le C \sum_{i=1}^{q} \|u\|^{p-1}_{\dot{B}^{s(p)}_{\rho}} \|u\|_{\dot{B}^s_{\rho}} \|D^{\alpha_i}(u_h - u)\|_{L^{1/a_i}}.$$

Similarly as in the above procedure, we have (2.19). (2.11), (2.16) and (2.19) imply that (2.10) holds.

Step 2. $0 \le s < 1$. In view of (1.8) we have

$$\|f(u)\|_{\dot{B}^s_{\rho'}} = \Big(\int_0^\infty t^{-2s} \sup_{|h|\le t} \|\triangle_h f(u)\|^2_{L^{\rho'}} \frac{dt}{t} \Big)^{1/2}. \tag{2.11a}$$

It follows from (2.13) that

$$\|\triangle_h f(u)\|_{L^{\rho'}} \le C\|u\|^p_{\dot{B}^{s(p)}_\rho} \|u_h - u\|_{L^\rho}. \tag{2.15b}$$

Combining (2.11a) with (2.15b) , we have (2.10).

Corollary 2.3. *Let the conditions of Lemma* 2.2 *be satisfied. Then we have*

$$\|f(u)\|_{L^{\gamma(\rho)'}(I,\dot{B}^s_{\rho'})} \le C\|u\|^p_{L^{\gamma(\rho)}(I,\dot{B}^{s(p)}_\rho)} \|u\|_{L^{\gamma(\rho)}(I,\dot{B}^s_\rho)},$$

where I is an arbitrary measurable set of $\mathbb{R}$.

Lemma 2.4. *Let the conditions of Lemma* 2.2 *be satisfied. Then we have*

$$\begin{aligned} &\|f(u) - f(v)\|_{L^{\gamma(\rho)'}(I,L^{\rho'})} \\ &\le C(\|u\|^p_{L^{\gamma(\rho)}(I,\dot{B}^{s(p)}_\rho)} + \|v\|^p_{L^{\gamma(\rho)}(I,\dot{B}^{s(p)}_\rho)}) \|u - v\|_{L^{\gamma(\rho)}(I,L^\rho)}, \end{aligned} \tag{2.20}$$

where I is an arbitrary measurable set of $\mathbb{R}$.

Proof. In virtue of Lemma 2.2, we have

$$\|f(u) - f(v)\|_{L^{\rho'}} \le C\|(|u|^p + |v|^p)|u - v|\|_{L^{\rho'}}. \tag{2.21}$$

Let $a_0 = p(\frac{1}{\rho} - \frac{s(p)}{n})$, $a_1 = \frac{1}{\rho}$. It is easy to see that a_0, $a_1 > 0$ and $a_0 + a_1 = 1/\rho'$. By the Hölder inequality we have

$$\|(|u|^p + |v|^p)|u - v|\|_{L^{\rho'}} \le C(\|u\|^p_{\dot{B}^{s(p)}_\rho} + \|v\|^p_{\dot{B}^{s(p)}_\rho}) \|u - v\|_{L^\rho}. \tag{2.22}$$

Note $\frac{1}{\gamma(\rho)'} = \frac{p+1}{\gamma(\rho)}$. In view of the Hölder inequality about time variable, (2.21) and (2.22) imply (2.13). For convenience, we denote

$$\begin{aligned} &\dot{V}(s,\rho) = L^{\gamma(\rho)}(\mathbb{R}, \dot{B}^s_\rho), \quad \dot{V}'(s,\rho) = L^{\gamma(\rho)'}(\mathbb{R}, \dot{B}^s_{\rho'}); \\ &V(s,\rho) = L^{\gamma(\rho)}(\mathbb{R}, B^s_\rho), \quad V'(s,\rho) = L^{\gamma(\rho)'}(\mathbb{R}, B^s_{\rho'}); \\ &V(\rho) = L^{\gamma(\rho)}(\mathbb{R}, L^\rho), \quad V'(\rho) = L^{\gamma(\rho)'}(\mathbb{R}, L^{\rho'}). \end{aligned}$$

Let $\rho(u)$ be a smooth cut-off function, for instance

$$\rho(u) = \begin{cases} 1 & \text{if} \quad 0 \le |u| \le 1, \\ \text{smooth} & \text{if} \quad 1 < |u| < 2, \\ 0 & \text{if} \quad |u| \ge 2. \end{cases}$$

Let $f_1(u) = \rho(u)f(u)$, $f_2(u) = (1-\rho(u))f(u)$. If $f(u)$ satisfies the conditions of Theorem 1.1, one can easily verify that

$$|f_i(u)| \le C|u|^{p_i+1-k}, \quad k = 0, 1, \cdots, [s] + 1. \tag{2.23}$$

Proof of Theorem 1.2. In view of (2.7), Proposition 1.3 and Corollary 2.3, we have

$$\begin{aligned}\|\mathcal{T}u\|_{\dot V(s,r_1)\cap \dot V(s,r_2)} &\le C(\|\varphi^-\|_{\dot H^s} + \|f_1(u)\|_{\dot V'(s,r_1)} + \|f_2(u)\|_{\dot V'(s,r_2)}) \\ &\le C\|\varphi^-\|_{\dot H^s} + C\|u\|^{p_1}_{\dot V(s(p_1),r_1)}\|u\|_{\dot V(s,r_1)} + C\|u\|^{p_2}_{\dot V(s(p_2),r_2)}\|u\|_{\dot V(s,r_2)}.\end{aligned} \tag{2.24}$$

In particular, if $s = s(p_i)$ $(i = 1, 2)$, then we have

$$\|\mathcal{T}u\|_{\dot V(s(p_i),r_1)\cap \dot V(s(p_i),r_2)} \le C(\|\varphi^-\|_{\dot H^{s(p_i)}} + \sum_{j=1}^{2} \|u\|^{p_j}_{\dot V(s(p_j),r_j)}\|u\|_{\dot V(s(p_i),r_j)}). \tag{2.25}$$

In virtue of Lemma 2.4, we have

$$\|\mathcal{T}u\|_{V(r_1)\cap V(r_2)} \le C(\|\varphi^-\|_{L^2} + \|u\|^{p_1}_{\dot V(s(p_1),r_1)}\|u\|_{V(r_1)} + \|u\|^{p_2}_{\dot V(s(p_2),r_2)}\|u\|_{V(r_2)}). \tag{2.26}$$

Since $L^{r_i} \cap \dot B^s_{r_i} = B^s_{r_i}$ $(i = 1, 2)$, from (2.24) and (2.26), we have

$$\begin{aligned}&\|\mathcal{T}u\|_{V(s,r_1)\cap V(s,r_2)} \\ &\le C\|\varphi^-\|_{H^s} + C(\|u\|^{p_1}_{\dot V(s(p_1),r_1)} + \|u\|^{p_2}_{\dot V(s(p_2),r_2)})\|u\|_{V(s,r_1)\cap V(s,r_2)}.\end{aligned} \tag{2.27}$$

Thus, if $u \in \mathcal{D}$ then we have

$$\|\mathcal{T}u\|_{V(s,r_1)\cap V(s,r_2)} \le C(\|\varphi^-\|_{H^s} + (\sigma^{p_1} + \sigma^{p_2})M), \tag{2.28}$$

$$\|\mathcal{T}u\|_{\dot V(s(p_i),r_1)\cap \dot V(s(p_i),r_2)} \le C(\|\varphi^-\|_{\dot H^{s(p_i)}} + \sigma^{p_1+1} + \sigma^{p_2+1}), \quad i = 1, 2. \tag{2.29}$$

Let $C(\sigma^{p_1} + \sigma^{p_2}) \le 1/2$, $C\|\varphi^-\|_{H^{s(p_i)}} = \sigma/2$ and let $M = 2C\|\varphi^-\|_{H^s}$. In view of (2.28) and (2.29) we have

$$\|\mathcal{T}u\|_{V(s,r_1)\cap V(s,r_2)} \le M, \quad \|\mathcal{T}u\|_{\dot V(s(p_i),r_1)\cap \dot V(s(p_i),r_2)} \le \sigma, \quad i = 1, 2. \tag{2.30}$$

For any $u, v \in \mathcal{D}$, it follows from Lemma 2.4 and Proposition 1.3 that

$$\begin{aligned}&\|\mathcal{T}u - \mathcal{T}v\|_{V(r_1)\cap V(r_2)} \le \|f_1(u) - f_1(v)\|_{V'(r_1)} + \|f_2(u) - f_2(v)\|_{V'(r_2)} \\ &\le C(\|u\|^{p_1}_{\dot V(s(p_1),r_1)} + \|v\|^{p_1}_{\dot V(s(p_1),r_1)})\|u - v\|_{V(r_1)} \\ &\quad + C(\|u\|^{p_2}_{\dot V(s(p_2),r_2)} + \|v\|^{p_2}_{\dot V(s(p_2),r_2)})\|u - v\|_{V(r_2)} \\ &\le C(\sigma^{p_1} + \sigma^{p_2})\|u - v\|_{V(r_1)\cap V(r_2)} \le \frac{1}{2}\|u - v\|_{V(r_1)\cap V(r_2)}.\end{aligned} \tag{2.31}$$

It follows from (2.30) and (2.31) that $\mathcal{T}$ is a contraction mapping on $(\mathcal{D}, d)$, so $\mathcal{T}$ has a unique fixed point $u \in (\mathcal{D}, d)$, namely

$$u(t) = S(t)\varphi^- - i\int_{-\infty}^t S(t-\tau)f(u(\tau))d\tau. \tag{2.32}$$

In view of (2.30) we can see that this solution enjoys the following property:

$$\|u\|_{V(s,r_1)\cap V(s,r_2)} \le 2C\|\varphi^-\|_{H^s},$$
$$\|u\|_{\dot V(s(p_i),r_1)\cap \dot V(s(p_i),r_2)} \le 2C\|\varphi\|_{\dot H^{s(p_i)}}, \quad i=1,2.$$

In virtue of (2.32) and Proposition 1.3, we have

$$\|u\|_{V(s,r)} \le C(\|\varphi^-\|_{H^s} + \|f_1(u)\|_{V'(s,r_1)} + \|f_2(u)\|_{V'(s,r_2)}).$$

Similarly to (2.27), we have

$$\|u\|_{V(s,r)} \le 2C\|\varphi^-\|_{H^s}, \tag{2.33}$$

$$\|u\|_{\dot V(s(p_i),r)} \le 2C\|\varphi^-\|_{\dot H^{s(p_i)}}, \quad i=1,2. \tag{2.34}$$

Now we show that the solution of (2.32) is unique. Assume for the contrary that there exist two solutions $u, v \in L^{\gamma(r_i)}(\mathbb{R}, B^s_{r_i})$ $(i=1,2)$ satisfying (2.32). Similarly to (2.31), we can choose a sufficiently negative T such that $v \equiv v$ in $(-\infty, T]$. We may repeat above procedure on $[T, T+\varepsilon]$, $[T+\varepsilon, T+2\varepsilon]$, $\cdots$ for some $\varepsilon > 0$. So we have $u \equiv v$ in $(-\infty, \infty)$.

Similarly as in (2.27), we have

$$\|u - S(t)\varphi\|_{L^\infty(-\infty,T;H^s)} \le C(\|u\|^{p_1}_{L^{\gamma(r_1)}(-\infty,T;\dot B^{s(p_1)}_{r_1})} + \|u\|^{p_2}_{L^{\gamma(r_2)}(-\infty,T;\dot B^{s(p_2)}_{r_2})})$$
$$\times \|u\|_{L^{\gamma(r_1)}(-\infty,T;B^s_{r_1})\cap L^{\gamma(r_2)}(-\infty,T;B^s_{r_2})},$$

which implies that

$$\lim_{t\to-\infty} \|u(t) - S(t)\varphi^-\|_{H^s} = 0.$$

We now define

$$u^+(t) = u(t) + i\int_t^\infty S(t-\tau)f(u(\tau))d\tau,$$

and let $u^+(0) = \varphi^+$. Then we show in exactly the same manner as above that

$$\lim_{t\to\infty} \|u(t) - u^+(t)\|_{L^{\gamma(r)}(t,\infty;B^s_r)} = 0$$

for all $r \in [2, \alpha(n))$. Moreover, $u^+(t)$ satisfies the estimates

$$\|u^+(t)\|_{L^{\gamma(r)}(\mathbb{R},B^s_r)} \le C\|\varphi^-\|_{H^s},$$

$$\|u^+(t)\|_{L^{\gamma(r)}(\mathbb{R},\dot{B}_r^{s(p_i)})} \le C\|\varphi^-\|_{\dot{H}^{s(p_i)}}, \quad i = 1, 2.$$

Thus, the scattering operator $S : \varphi^- \to \varphi^+$ is shown.

Proof of Theorem 1.1. If we replace the mapping $\mathcal{T}$ in (2.7) by

$$\mathcal{T}' : u \to S(t)\varphi + i\int_0^t S(t-\tau)f(u(\tau))d\tau,$$

then we can show in the same way as in Theorem 1.2 that $\mathcal{T}'$ has a unique fixed point. The details of proof are omitted. The proof of the local existence of solutions is standard, one can refer to [2].

Acknowledgement. The authors would like to give their great thanks to Professor F. B. Weissler for his valuable comments and remarks. The second author is supported by the NSFC, Grant 19901007.

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