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THE INITIAL VALUE PROBLEM FOR A GENERALIZED BOUSSINESQ EQUATION

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Abstract. We prove a result on the existence and uniqueness of a local solution to a generalized Boussinesq equation for initial data of low regularity. We also discuss the existence of global solutions and the occurrence of blow-up phenomena. Our results are applicable to several physically relevant equations that are obtained as special cases of our model equation.

1. Introduction

Let $F \in C^\infty(\mathbb{R})$ satisfy $F(0) = 0$. We consider the well-posedness of the problem

$$\begin{aligned} &u_{tt} = [F(u)]_{xx} + u_{xxtt}, \quad x \in \mathbb{R},\ t \geq 0, \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}, \\ &u_t(x,0) = u_1(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1.1}$$

in the Sobolev spaces $H^s(\mathbb{R})$ with $s \geq 0$.

Our interest in the problem (1.1) is motivated by the fact that particularizing the function F, one obtains equations that occur in a wide variety of physical systems. For example,

$$u_{tt} = u_{xx} - (u^2)_{xx} + u_{xxtt}, \quad x \in \mathbb{R},\ t \geq 0, \tag{1.2}$$

is a model for nonlinear waves in weakly dispersive media cf. [11], whereas

$$u_{tt} = u_{xx} + (u^3)_{xx} + u_{xxtt}, \quad x \in \mathbb{R},\ t \geq 0, \tag{1.3}$$

is relevant in the study of the properties of non-linear Alfvén waves cf. [7]. The equation

$$u_{tt} = u_{xx} + \tfrac{1}{5}\,(u^5)_{xx} + u_{xxtt}, \quad x \in \mathbb{R},\ t \geq 0, \tag{1.4}$$

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was derived in [2] to analyze the propagation of longitudinal deformation waves in elastic rods.

In Section 3, we prove the local well-posedness of (1.1) for initial data with quite low regularity. The last section is devoted to a discussion of the problem of existence of global solutions to (1.1). Regarding the question of gain of regularity for (1.1), note that for $F \equiv 0$ and initial data $u_0 \equiv 0$, $u_t = f$, we have the global solution $u(t,x) = t\,f(x)$. In conclusion, there is no gain in differentiability no matter how smooth the function F is.

The Cauchy problem (1.1) for more regular initial data was recently investigated in [5], [12]. Our study of the local well-posedness for (1.1) and of the global existence extends the results obtained in [5], [12].

2. Preliminaries

In this section we present some useful results from nonlinear microlocal analysis regarding the composition of C^∞-functions with Sobolev functions.

We have the following:

Lemma 1. ([3]) *Let $F \in C^\infty(\mathbb{R})$ be a function vanishing at zero. If $s > \frac{1}{2}$, then, for all $f \in H^s(\mathbb{R})$, the function $F(f)$ is also in $H^s(\mathbb{R})$. If $s = \frac{1}{2}$ and the derivative F' of F is bounded, then we still have $F(f) \in H^s(\mathbb{R})$ when $f \in H^s(\mathbb{R})$.*

The proof of Lemma 1 given in [3] is based on the Littlewood-Paley decomposition. Below we present an alternative simple proof that can also be used in dealing with the case $s \in [0, \frac{1}{2})$. More precisely, we have:

Lemma 2. *Let $F \in C^\infty(\mathbb{R})$ be a function vanishing at zero. If $s \in [0, \frac{1}{2}]$ and $f \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then $F(f) \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$.*

Proof of Lemmas 1 and 2. We first show that if $F \in C^\infty(\mathbb{R})$ is vanishing at zero, then $F(f) \in L^2(\mathbb{R})$ for all $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Indeed, as $F(0) = 0$, we have for $f : \mathbb{R} \to \mathbb{R}$ that

$$F(f(x)) = f(x) \int_0^1 F'\big(t\,f(x)\big)\,dt, \quad x \in \mathbb{R}.$$

If $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we infer

$$|F(f(x))| \le \sup_{|y| \le \|f\|_{L^\infty}} \{|F'(y)|\}\ |f(x)|, \quad x \in \mathbb{R},$$

thus, $F(f) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Let us now deal with the case $s \in (0,1)$. Recall that a function $f \in L^2(\mathbb{R})$, $s \in (0,1)$, belongs to $H^s(\mathbb{R})$ if and only if the integral

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x+y) - f(x)|^2 \frac{dy}{|y|^{1+2s}} \, dx \tag{2.1}$$

is finite, the integral being equal to a multiple of $\int_{\mathbb{R}} |\hat{f}(\xi)|^2 |\xi|^{2s} \, d\xi$, cf. [10], while the norm of $H^s(\mathbb{R})$ is given by

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi, \quad f \in H^s(\mathbb{R}).$$

It is now easy to see that if $f \in L^\infty(\mathbb{R})$, then

$$\begin{aligned} &|F(f(x+y)) - F(f(x))| \\ &= \Big| [f(x+y) - f(x)] \int_0^1 F'\Big(f(x) + t\,[f(x+y) - f(x)] \Big) \, dt \Big| \\ &\leq \sup_{|r| \leq 2\|f\|_{L^\infty}} \{|F'(r)|\} \, |f(x+y) - f(x)| = M|f(x+y) - f(x)|, \quad x, y \in \mathbb{R}. \end{aligned}$$

The above relation shows that the integral (2.1), computed for $F(f)$ instead of f, is finite. We know already that $F(f) \in L^2(\mathbb{R})$. Therefore, $F(f) \in H^s(\mathbb{R})$. If $s \in (\frac{1}{2}, 1)$ we know that $H^s(\mathbb{R}) \subset L^\infty(\mathbb{R})$ so there are no changes.

In order to deal with $s \in [1,2)$, note that if $1 > \alpha \geq \beta > 0$ satisfy $\alpha + \beta > 1$, then the multiplication is a continuous operation from $H^\alpha(\mathbb{R}) \times H^\beta(\mathbb{R})$ into $H^\beta(\mathbb{R})$, cf. [1]. The case $s = 1$ can be easily dealt with using the chain rule, as $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$. For $f \in H^s(\mathbb{R})$ with $s \in (1,2)$, note that $\partial_x F(f) = F'(f) \, f_x$. As $f \in H^s(\mathbb{R})$ and $[F'(\cdot) - F'(0)] \in C^\infty(\mathbb{R})$ is vanishing at zero, the first part of the proof shows that $[F'(f(\cdot)) - F'(0)] \in H^{1-\varepsilon}(\mathbb{R})$ for all $\varepsilon > 0$. As $f_x \in H^{s-1}(\mathbb{R})$ and $F'(0) \, f_x \in H^{s-1}(\mathbb{R})$, we deduce by the above multiplication theorem that $\partial_x F(f) \in H^{s-1}(\mathbb{R})$. Therefore, $F(f) \in H^s(\mathbb{R})$.

We proceed similarly with all cases $s \in [n, n+1)$ for all $n \geq 2$. □

Remark. It is natural to impose some additional conditions if $s \in [0, \frac{1}{2}]$ as in this case $H^s(\mathbb{R})$ is not an algebra - the function $F(x) = x^2$, $x \in \mathbb{R}$, is not admissible without further restrictions. A careful analysis of the above proof shows that instead of imposing $f \in L^\infty(\mathbb{R})$ we could ask for the derivative F' to be bounded on $\mathbb{R}$.

For $s \geq 0$, consider the Banach space $X_s = H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$ endowed with the norm

$$\|f\|_s = \|f\|_{H^s} + \|f\|_{L^\infty}, \quad f \in X_s.$$

The next result is very useful for our approach.

Lemma 3. *Let $F \in C^\infty(\mathbb{R})$. If $f, g \in X_s$, $s \geq 0$, then*

$$\|F(f) - F(g)\|_s \leq K\,\|f - g\|_s, \tag{2.2}$$

where K depends on $\|f\|_s$ and $\|g\|_s$.

Proof. We will deal only with the case $s \in (0,1)$ - the other cases can be dealt with by using the same approach as in Lemma 2. Throughout this proof K stands for a constant depending on $\|f\|_s$ and $\|g\|_s$. Note that

$$\begin{aligned}|F(f(x)) - F(g(x))| &= \Big|[f(x) - g(x)] \int_0^1 F'\Big(g(x) + t[f(x) - g(x)]\Big)\,dt\\ &\leq \sup_{|r| \leq \|f\|_{L^\infty} + \|g\|_{L^\infty}} \{|F'(r)|\}\; |f(x) - g(x)| = K|f(x) - g(x)|, \quad x \in \mathbb{R}.\end{aligned}$$

This ensures

$$\|F(f) - F(g)\|_0 \leq K\|f - g\|_0, \quad f, g \in X_s. \tag{2.3}$$

To show that F is locally Lipschitz on $H^s(\mathbb{R})$, all we have to do now is to evaluate appropriately the integral (2.1) for $[F(f) - F(g)]$. Observe that

$$\begin{aligned}&F(f(x+y)) - F((g(x+y)) - F(f(x)) + F(g(x))\\ &= [f(x+y) - g(x+y)] \int_0^1 F'\Big(g(x+y) + t\,[f(x+y) - g(x+y)]\Big)\,dt\\ &- [f(x) - g(x)] \int_0^1 F'\Big(g(x) + t\,[f(x) - g(x)]\Big)\,dt \qquad (2.4)\\ &= \alpha(x+y)\,[f(x+y) - g(x+y)] - \alpha(x)\,[f(x) - g(x)]\\ &= \alpha(x+y)\,[f(x+y) - g(x+y) - f(x) + g(x)]\\ &+ [\alpha(x+y) - \alpha(x)]\,[f(x) - g(x)], \qquad x, y \in \mathbb{R},\end{aligned}$$

where

$$\alpha(z) := \int_0^1 F'\Big(g(z) + t\,[f(z) - g(z)]\Big)\,dt, \quad z \in \mathbb{R}.$$

On the other hand, denoting

$$A(x,y) := f(x+y) - g(x+y) - f(x) + g(x), \quad x, y \in \mathbb{R},$$

we have for all $t \in [0,1]$ that

$$\begin{aligned}&F'\Big(g(x+y) + t\,[f(x+y) - g(x+y)]\Big) - F'\Big(g(x) + t\,[f(x) - g(x)]\Big)\\ &= [g(x+y) - g(x) + tA(x,y)] \int_0^1 F''\Big(g(x) + t\,[f(x) - g(x)]\end{aligned}$$

$$+ r\,[g(x+y) - g(x) + t\,A(x,y)]\Big)\,dr.$$

Since $f, g \in L^\infty(\mathbb{R})$, we obtain from the previous relation that for all $x, y \in \mathbb{R}$ and $t \in [0,1]$,

$$\begin{aligned}|F'\big(g(x+y) + t\,[f(x+y) - g(x+y)]\big) - F'\big(g(x) + t\,[f(x) - g(x)]\big)|\\ \leq K\,|A(x,y)| + K\,|g(x+y) - g(x)|.\end{aligned}$$

This on its turn implies that

$$|\alpha(x+y) - \alpha(x)| \leq K\,|A(x,y)| + K\,|g(x+y) - g(x)|, \quad x, y \in \mathbb{R}.$$

Using the above relation back in (2.4), we infer

$$\begin{aligned}&|\,F(f(x+y)) - F((g(x+y)) - F(f(x)) + F(g(x))\,|\\ &\leq K\,|A(x,y)| + K\,|g(x+y) - g(x)|\,|f(x) - g(x)|\\ &\leq K\,|A(x,y)| + K\,|g(x+y) - g(x)|\,\|f - g\|_{L^\infty}, \quad x, y \in \mathbb{R}.\end{aligned}$$

This last inequality guarantees that the integral (2.1) for $[F(f) - F(g)]$ can be estimated from above by

$$\begin{aligned}K\int_{\mathbb{R}}\int_{\mathbb{R}} |A(x,y)|^2 \frac{dy}{|y|^{1+2s}} dx + K\|f-g\|_{L^\infty}^2 \int_{\mathbb{R}}\int_{\mathbb{R}} |g(x+y) - g(x)|^2 \frac{dy}{|y|^{1+2s}} dx\\ \leq K\|f-g\|_{H^s}^2 + K\,\|f-g\|_{L^\infty}^2 \|g\|_{H^s}^2 \leq K\|f-g\|_s^2.\end{aligned}$$

Taking into account (2.3) we now obtain the statement. □

Remark. To ensure in Lemma 3 that $F(f) \in H^s(\mathbb{R})$ for $f \in H^s(\mathbb{R})$, $s \geq 0$, one needs the additional assumption $F(0) = 0$, cf. Lemma 2.

Proposition. *Let $F \in C^\infty(\mathbb{R})$ with $F(0) = 0$. If $f \in X_s$, $s \geq 0$, then*

$$\|F(f)\|_s \leq K\,\|f\|_s,$$

where K depends only on $\|f\|_{L^\infty}$.

Proof. Immediate after analyzing carefully the steps of the proof of Lemma 3 with $g \equiv 0$. □

3. The Cauchy problem

We consider now the problem of local existence and uniqueness of solutions to (1.1).

Theorem 1. *For $u_0, u_1 \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $s \geq 0$, there exists a unique solution $u \in C^2([0,T); H^s(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ of (1.1) defined for some maximal*

$T > 0$. The solution depends continuously on the initial data. If $T < \infty$, then

$$\limsup_{t\uparrow T}\big(\|u(t,\cdot)\|_{H^s}+\|u_t(t,\cdot)\|_{H^s}+\|u(t,\cdot)\|_{L^\infty}+\|u_t(t,\cdot)\|_{L^\infty}\big)=\infty.$$

For the proof we need:

Lemma 4. *The operator $L := (1-\partial_x^2)^{-1}\,\partial_x^2$ is bounded on $H^s(\mathbb{R})$ and on $H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for all $s \geq 0$.*

Proof. For $f \in H^s(\mathbb{R})$, $s \geq 0$, we have

$$\|Lf\|_{H^s}^2 = \int_{\mathbb{R}} (1+\xi^2)^s |\hat{Lf}(\xi)|^2\,d\xi = \int_{\mathbb{R}} (1+\xi^2)^s \frac{\xi^4}{(1+\xi^2)^2} |\hat{f}(\xi)|^2\,d\xi \leq \|f\|_{H^s}^2.$$

On the other hand, note that

$$(1-\partial_x^2)^{-1} f = p * f, \quad f \in L^2(\mathbb{R}),$$

where $p(x) := \frac{1}{2}\,e^{-|x|}$, $x \in \mathbb{R}$. It is then easy to see that

$$\partial_x^2\,[p*f] = p*f - f, \quad f \in L^2(\mathbb{R}). \tag{3.1}$$

From the above identities we infer

$$Lf = p*f - f, \quad f \in L^2(\mathbb{R}). \tag{3.2}$$

From Young's inequality we obtain now that

$$\|Lf\|_{L^\infty} \leq 2\,\|f\|_{L^\infty}, \quad f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}). \qquad \square$$

Proof of Theorem 1. We can write the equation from (1.1) as

$$u_{tt} = L[F(u)], \quad t > 0,$$

and regard the problem as an ODE-system in the Banach space $X_s = H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$,

$$\begin{aligned} u_t &= v, \\ v_t &= L[F(u)], \end{aligned} \tag{3.3}$$

with initial data $u(0) = u_0$ and $v(0) = v_0$.

By Lemma 2 and Lemma 3 we know that the map $f \mapsto F(f)$ is locally Lipschitz on $X_s(\mathbb{R})$. As L is a bounded linear operator on X_s in view of Lemma 4, we have that

$$\|L[F(u)] - L[F(v)]\|_s \leq |||L|||_{\mathcal{L}(X_s)} \|u-v\|_s, \quad u, v \in X_s.$$

We deduce that the right-hand side of the above ODE is locally Lipschitz. The statement is now a consequence of the classical Picard iteration procedure for ODE's on Banach spaces (see [6]). $\square$

Since $H^s(\mathbb{R})$ is imbedded in $L^\infty(\mathbb{R})$ for $s > \frac{1}{2}$, we have the following:

Corollary. ([5]) *For $u_0, u_1 \in H^s(\mathbb{R})$, $s > \frac{1}{2}$, there exists $T > 0$ such that* (1.1) *has a unique solution $u \in C^2([0,T); H^s(\mathbb{R}))$.*

4. Existence of global solutions

In this section we discuss the existence of global solutions to (1.1) for initial data in the spaces $H^s(\mathbb{R})$ with $s > \frac{1}{2}$.

Theorem 2. *Let $u_0, v_0 \in H^s(\mathbb{R})$, $s > \frac{1}{2}$, and let $T > 0$ be the maximal existence time of the corresponding solution $u(t)$ to* (1.1). *Then $T < \infty$ if and only if*

$$\limsup_{t\uparrow T} \|u(t,\cdot)\|_{L^\infty} = \infty.$$

Proof. One implication is obvious in view of Theorem 1. Let us prove that if

$$\sup_{t\in[0,T)} \|u(t,\cdot)\|_{L^\infty} = K < \infty, \tag{4.1}$$

then $T = \infty$. Using (3.3), we obtain for $t \in (0,T)$ that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|u\|_{H^s}^2 + \|v\|_{H^s}^2\right) &= (u,v)_{H^s} + (v, p * F(u) - F(u))_{H^s} \\ &\le \|u\|_{H^s}\|v\|_{H^s} + \|v\|_{H^s}\big(\|p * F(u)\|_{H^s} + \|F(u)\|_{H^s}\big).\end{aligned}$$

Using the Fourier transform, it is easy to check that

$$\|p * h\|_{H^s} \le \|h\|_{H^s}, \quad h \in H^s(\mathbb{R}).$$

It follows that

$$\frac{d}{dt}(\|u\|_{H^s}^2 + \|v\|_{H^s}^2) \le (\|u\|_{H^s}^2 + \|v\|_{H^s}^2) + 4\|v\|_{H^s}\,\|F(u)\|_{H^s}. \tag{4.2}$$

Since $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ for $s > \frac{1}{2}$, we deduce from (4.1) and the Proposition that

$$\|F(u)\|_{H^s} \le C\|u\|_{H^s} \tag{4.3}$$

for some constant $C > 0$. Substituting (4.3) in (4.2), one obtains

$$\frac{d}{dt}\left(\|u\|_{H^s}^2 + \|v\|_{H^s}^2\right) \le (1+2C)\left(\|u\|_{H^s}^2 + \|v\|_{H^s}^2\right), \quad t \in (0,T).$$

The previous relation shows by Gronwall's lemma that the $H^s(\mathbb{R})$-norms of $u(t)$ and $v(t)$ do not blow-up in finite time. Then $T = \infty$ by Theorem 1. □

Looking for conditions that ensure the global existence of solutions to (1.1) for initial data in $H^s(\mathbb{R})$, $s \geq 1$, let us introduce the class

$$\Re = \{w : \mathbb{R}_+ \to (0, \infty),\ w \text{ nondecreasing and } \int_1^\infty \frac{ds}{sw(s)} = \infty\}.$$

Besides bounded functions, we note that functions with a sublogarithmic growth, i.e., satisfying $w(r) \leq M\,[\ln(1+r)+1]$ for $r \geq 0$, belong also to the class $\Re$.

Theorem 3. *Let $F \in C^\infty(\mathbb{R})$ be such that $F(0) = 0$ and*

$$|F'(x)|^2 \leq w(x^2), \qquad x \in \mathbb{R}, \tag{4.4}$$

for some $w \in \Re$. Then for all $u_0, v_0 \in H^s(\mathbb{R})$, $s \geq 1$, the corresponding solution to (1.1) *exists globally in time.*

Proof. Fix $u_0, v_0 \in H^s(\mathbb{R})$, $s \geq 1$, and let $u \in C^2([0,T); H^s(\mathbb{R}))$ be the solution of (1.1) with initial data (u_0, v_0), defined on the maximal interval of existence $[0,T)$ with $T > 0$, cf. Theorem 1.

Using (3.3), we have for $t \in (0,T)$ that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}(u^2+u_x^2)\,dx &= 2\int_{\mathbb{R}} uv\,dx + 2\int_{\mathbb{R}} u_x v_x\,dx\\ &\leq \int_{\mathbb{R}}(u^2+u_x^2)\,dx + \int_{\mathbb{R}}(v^2+v_x^2)\,dx,\end{aligned}$$

whereas

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}(v^2+v_x^2)\,dx &= 2\int_{\mathbb{R}} v\,L[F(u)]\,dx + 2\int_{\mathbb{R}} v_x\,\partial_x L[F(u)]\,dx\\ &= 2\int_{\mathbb{R}} v\,[p*F(u)]\,dx - 2\int_{\mathbb{R}} vF(u)\,dx\\ &\quad + 2\int_{\mathbb{R}} v_x\,\partial_x[p*F(u)]\,dx - 2\int_{\mathbb{R}} v_x\,\partial_x F(u)\,dx\end{aligned}$$

if we take into account the formula (3.2). Integration by parts in the third term on the right-hand side of the above identity leads in view of (3.1) to

$$\frac{d}{dt}\int_{\mathbb{R}}(v^2+v_x^2)\,dx = -2\int_{\mathbb{R}} v_x\,F'(u)u_x\,dx, \quad t \in (0,T).$$

Adding up, we see that

$$\frac{d}{dt}\int_{\mathbb{R}}(u^2+u_x^2+v^2+v_x^2)dx \leq \int_{\mathbb{R}}(u^2+u_x^2+v^2+v_x^2)dx + 2\int_{\mathbb{R}}|v_x||F'(u)u_x|dx \tag{4.5}$$

$$\leq \int_{\mathbb{R}} (u^2 + u_x^2 + v^2 + v_x^2)\,dx + \int_{\mathbb{R}} v_x^2\,dx + \int_{\mathbb{R}} |F'(u)u_x|^2\,dx, \quad t \in (0,T).$$

From (4.4), we infer, for $t \in (0,T)$, $x \in \mathbb{R}$,

$$|F'(u(t,x))|^2 \leq w(u^2(t,x)) \leq w(\|u(t,\cdot)\|_{L^\infty}^2) \leq w(\|u(t,\cdot)\|_{H^1}^2),$$

so that

$$\int_{\mathbb{R}} |F'(u)u_x|^2\,dx \leq w(\|u(t,\cdot)\|_{H^1}^2) \int_{\mathbb{R}} u_x^2\,dx \leq \|u(t,\cdot)\|_{H^1}^2\, w(\|u(t,\cdot)\|_{H^1}^2), \quad t \in (0,T). \tag{4.6}$$

Combining (4.5) and (4.6), we obtain

$$\frac{d}{dt}\left(\|u(t,\cdot)\|_{H^1}^2 + \|v(t,\cdot)\|_{H^1}^2\right) \leq 2\left(\|u(t,\cdot)\|_{H^1}^2 + \|v(t,\cdot)\|_{H^1}^2\right) + \left(\|u(t,\cdot)\|_{H^1}^2 + \|v(t,\cdot)\|_{H^1}^2\right) w(\|u(t,\cdot)\|_{H^1}^2 + \|v(t,\cdot)\|_{H^1}^2), \quad t \in (0,T).$$

For $w \in \Re$, we have that $G(\infty) = \infty$, where

$$G(r) := \int_0^r \frac{ds}{s + sw(s)} = \infty, \quad r \geq 0,$$

cf. [4]. From the previous differential inequality, we obtain

$$\|u(t,\cdot)\|_{H^1}^2 + \|v(t,\cdot)\|_{H^1}^2 \leq G^{-1}(2t + G(\|u_0\|_{H^1}^2 + \|v_0\|_{H^1}^2)), \quad t \in (0,T).$$

The above inequality ensures that the $H^1(\mathbb{R})$-norm of $u(t)$ does not blow-up in finite time. This prevents $\limsup_{t\uparrow T} \|u(t,\cdot)\|_{L^\infty} = \infty$ for $T < \infty$ so that the solution is global, cf. Theorem 2. □

Remark. Besides sublinear functions, we see that $F(x) = x\sqrt{\ln(1+x^2)}$, $x \in \mathbb{R}$, satisfies also the condition (4.4).

Regarding solutions with a finite life-span, we have:

Theorem 4. ([5]) *Let* $P(x) := \int_0^x F(s)\,ds$, $x \in \mathbb{R}$. *If there exists* $\alpha > 0$ *such that*

$$(2+\alpha)\,P(x) \geq xF(x), \quad x \in \mathbb{R},$$

and if $\int_{\mathbb{R}} P(u_0)\,dx < 0$ *for some* $u_0 \in H^s(\mathbb{R})$, $s > \frac{1}{2}$, *then the solution to* (1.1) *with initial data* $(u_0, 0)$ *blows-up in finite time.*

Example. In view of Theorem 4, occurrence of blow-up in the spaces $H^s(\mathbb{R})$, $s > \frac{1}{2}$, holds for equation (1.2) as well as for equation (1.1) with

(i) $F(x) = cx^{2p}$, $x \in \mathbb{R}$, with $c \in \mathbb{R}^*$ and $p \in \mathbb{N}$, $p \geq 1$; and

(ii) $F(x) = cx^{2p+1}$, $x \in \mathbb{R}$, with $c < 0$ and $p \in \mathbb{N}$, $p \geq 1$.

The next result shows that the negativity of the coefficient in (ii) of the above example is essential.

Theorem 5. *Let* $P(x) := \int_0^x F(s)\, ds$, $x \in \mathbb{R}$. *Assume that there exists a decomposition* $F = F_1 + F_2 + ... + F_n$ *such that*

$$|F_i(x)|^{q_i} \leq K\, P(x), \qquad i = 1, ..., n, \quad x \in \mathbb{R},$$

for some $q_i > 1$, $i = 1, ..., n$ *and* $K > 0$. *Then for all* $u_0, v_0 \in H^s(\mathbb{R})$, $s > \frac{1}{2}$, *the corresponding solution to* (1.1) *exists globally in time.*

Proof. According to Theorem 2, it is enough to ensure that the $L^\infty(\mathbb{R})$-norm of the solution $u(t)$ to (1.1) does not blow-up in finite time. We denote by $T > 0$ the maximal time of existence of this solution. Also, $K > 0$ stands here for a generic constant.

Let us note that it is enough to prove the result for initial data $u_0, v_0 \in H^s(\mathbb{R})$ such that both u_0 and v_0 are derivatives of $H^{s+1}(\mathbb{R})$-functions. Indeed, passing to Fourier transforms, one can easily see that the latter restricted space is dense in $H^s(\mathbb{R})$ (for details, see [8]). We then argue by the continuous dependence on the initial data of the solutions to (1.1) to obtain the statement in its full generality.

Since $u_0 = w_{0,x}$, $v_0 = z_{0,x}$ for some $w_0, z_0 \in H^{s+1}(\mathbb{R})$, note that for all $t \in [0, T)$, the solution $u(t)$ of the problem (1.1) is given by $u(t, x) = w_x(t, x)$, with (w, z) satisfying the system

$$\begin{aligned} w_t &= z, \\ z_t &= \partial_x\, [p * F(w_x)]. \end{aligned}$$

Indeed, if (w, z) is a solution of the above system, using (3.1), one can see that u defined by $u(t, x) = w_x(t, x)$ solves (1.1). Conversely, if u is a solution to (1.1), then

$$u(t, x) = u_0(x) + \int_0^t v(s, x)\, ds$$

where (u, v) solves the system (3.3). The term $u_0(x)$ is an x-derivative by hypothesis and

$$v(s, x) = v_0(x) + \int_0^s \partial_x^2 (1 - \partial_x^2)^{-1}\, [F(u)](r, x)\, dr$$

is also an x-derivative as $v_0 = z_{0,x}$. We deduce the existence of a function $w(t, x)$ with $u(t, x) = w_x(t, x)$. Now it is easy to obtain that

$$w_{tt} = \partial_x\, p * F(u)$$

and to write this second-order equation as the desired first-order system.

Observe that

$$E(t) := \frac{1}{2}\int_{\mathbb{R}} (z^2 + z_x^2)\, dx + \int_{\mathbb{R}} P(u)\, dx, \quad t \in [0, T), \tag{4.7}$$

is conserved in time. Indeed,

$$\begin{aligned} \frac{d}{dt} E(t) &= \int_{\mathbb{R}} (z z_t + z_x z_{xt})\, dx + \int_{\mathbb{R}} F(u) u_t\, dx \\ &= \int_{\mathbb{R}} z\, \partial_x\, [p * F(u)]\, dx + \int_{\mathbb{R}} z_x\, \partial_x^2\, [p * F(u)]\, dx + \int_{\mathbb{R}} z_x F(u)\, dx \\ &= \int_{\mathbb{R}} z\, \partial_x\, [p * F(u)]\, dx + \int_{\mathbb{R}} z_x\, [p * F(u)]\, dx, \quad \text{a.e.} \quad t \in (0, T). \end{aligned}$$

From the second to the last line above we applied (3.1). Integration by parts shows that an overall cancellation holds and $E(t) = E(0)$ for $t \in [0, T)$.

On the other hand, note that $u, v \in C^2([0, T); L^2(\mathbb{R}))$. Thus, for a.e. $(t, x) \in (0, T) \times \mathbb{R}$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}[u^2 + v^2 + 2\, P(u)] &= uv + v[p * F(u)] \\ &\le \frac{1}{2}[u^2 + v^2 + 2\, P(u)] + v[p * \big(\sum_{i=1}^{n} F_i(u)\big)]. \end{aligned} \tag{4.8}$$

By Young's inequality we have

$$\begin{aligned} \Big| v[p * \big(\sum_{i=1}^{n} F_i(u)\big)] \Big| &\le \frac{n}{2} v^2 + \frac{1}{2} \sum_{i=1}^{n} \|p * F_i(u)\|_{L^\infty}^2 \\ &\le \frac{n}{2} v^2 + \sum_{i=1}^{n} \|p\|_{L^{r_i}}^2 \|F_i(u)\|_{L^{q_i}}^2 \le \frac{n}{2} v^2 + K \sum_{i=1}^{n} \Big(\int_{\mathbb{R}} P(u)\, dx \Big)^{\frac{2}{q_i}}, \end{aligned}$$

with $\frac{1}{r_i} + \frac{1}{q_i} = 1$, $i = 1, ..., n$. From (4.7) we infer

$$\int_{\mathbb{R}} P(u)\, dx \le E(0), \quad t \in [0, T).$$

Using the above estimates back in (4.8), we obtain that for a.e. $(t, x) \in (0, T) \times \mathbb{R}$ the following inequality holds

$$\frac{d}{dt}\, [u^2 + v^2 + 2\, P(u)] \le (n + 1)\, [u^2 + v^2 + 2\, P(u)] + K.$$

By Gronwall's inequality we conclude from the previous relation that $\|u(t, \cdot)\|_{L^\infty}$ does not blow-up in finite time and therefore $T = \infty$. □

Example. Theorem 5 shows that if $F(x) = c\,x^{2p+1}$, $x \in \mathbb{R}$, with $p \in \mathbb{N}^*$ and $c > 0$, then for all $u_0, v_0 \in H^s(\mathbb{R})$, $s \in \mathbb{N}^*$, the corresponding solution to (1.1) exists globally in time. Also, all solutions to (1.3) and (1.4) with initial data $u_0, v_0 \in H^s(\mathbb{R})$, $s > \frac{1}{2}$, are global in time.

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