Differential and Integral Equations Volume 15, Number 9, September 2002, Pages 1045–1060

SOME REMARKS ABOUT THE FUCIK SPECTRUM AND APPLICATION TO EQUATIONS WITH JUMPING NONLINEARITIES

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(Submitted by: Haim Brezis)

Abstract. Let L be a selfadjoint operator with compact resolvent and λ an eigenvalue of L. When λ is simple, it is well known that the Fucik spectrum Σ near λ consists of two nonincreasing curves. In this paper, we show that when λ is not simple, Σ contains two nonincreasing curves such that all points above or under both curves are not in Σ. After that, we give some existence results of solutions of the equation $Lu = \alpha u^+ - \beta u^- + g(.,u)$ where $u^\pm = \max(0, \pm u)$.

1. Introduction

Let Ω be an open domain in $\mathbb{R}^n$ and let $L : D(L) \subset L^2(\Omega) \to L^2(\Omega)$ be a selfadjoint operator with compact resolvent. Hence, L is closed and the set of eigenfunctions of L is an orthogonal basis of $L^2(\Omega)$. We denote by $\langle u, v\rangle = \int_\Omega uv$, $\|u\|^2 = \langle u, u\rangle$ and $\|u\|^2_{D(L)} = \|u\|^2 + \|Lu\|^2$. Let λ be an eigenvalue of L, $\lambda_p = \max\{\tau \in \sigma(L)/\tau < \lambda\}$ and $\lambda_g = \min\{\tau \in \sigma(L)/\tau > \lambda\}$, where $\sigma(L)$ is the spectrum of L. We denote by m the multiplicity of λ and $I = (\lambda_p, \lambda_g)$.

It is shown by T. Gallouët and O. Kavian [1], B. Ruf [8] that when $m = 1$, the Fuçik spectrum Σ (i.e., the set of $(\alpha, \beta) \in \mathbb{R}^2$ such that the equation $Lu = \alpha u^+ - \beta u^-$ has a nontrivial solution) consists in $I \times I$ of two nonincreasing curves which may coincide.

When $m \geq 2$, N.P. Cac [2] has shown that for the Laplacian operator with Dirichlet boundary conditions, Σ contains two curves in $I \times I$. M. Schechter [3] and O. Kavian [4] obtain the same result for any operator L such that $\sigma(L) \subset \mathbb{R}^{+*}$.

Accepted for publication: July 1999.
AMS Subject Classifications: 35J65, 47H15.

In the third section of this paper, we show that when $\sigma(L)$ is not limited at left and $m \geq 2$, the Fuçik spectrum in $I \times I$ contains two nonincreasing curves Γ_1 and Γ_2 passing through the point (λ, λ) such that all points above or under Γ_1 and Γ_2 are not in Σ. The curves Γ_1 and Γ_2 are given by

$$\Gamma_1 = \{(\alpha, \beta) \in I \times I \text{ such that } \sigma(\alpha, \beta) = 0\}$$

$$\Gamma_2 = \{(\alpha, \beta) \in I \times I \text{ such that } \tau(\alpha, \beta) = 0\}$$

where

$$\sigma(\alpha, \beta) = \min_{\Sigma_{i=1}^{i=m} t_i^2 = 1} \sum_{i=1}^{i=m} C_i(\alpha, \beta, t) t_i$$

$$\tau(\alpha, \beta) = \max_{\Sigma_{i=1}^{i=m} t_i^2 = 1} \sum_{i=1}^{i=m} C_i(\alpha, \beta, t) t_i$$

and the coefficients $C_i(\alpha, \beta, t)$ verify

$$\begin{cases} Lw = \alpha w^+ - \beta w^- + \sum_{i=1}^{i=m} C_i(\alpha, \beta, t)\varphi_i \\ \int w\varphi_i = t_i \text{ for } i = 1, ..., m \end{cases}$$

$(\varphi_i)_{1 \leq i \leq m}$ are the eigenfunctions corresponding to λ.

In the fourth section, we study the equation

$$Lu = \alpha u^+ - \beta u^- + g(x, u) \tag{E}$$

in different cases of g and different positions of (α, β) toward the Fuçik spectrum. First we show that when (α, β) is above or under Γ_1 and Γ_2 and g verifies $|g(x, s)| \leq h^{1-\zeta}(x) |s|^\zeta + k(x)$, where $(h, k) \in L^2(\Omega) \times L^2(\Omega)$ and $0 \leq \zeta < 1$, then (E) has at least one solution.

After that, we prove that if $(\alpha_0, \beta_0) \in \Gamma_1$, $\lambda_p + \eta \leq a(x) \leq \alpha_0$ and $\lambda_p + \eta \leq b(x) \leq \beta_0$ a.e. in Ω for some $\eta > 0$ and

$$\int (\alpha_0 - a(x))(w^+)^2 + \int (\beta_0 - b(x))(w^-)^2 > 0$$

for each nontrivial solution of $Lw = \alpha_0 w^+ - \beta_0 w^-$, then the equation $Lu = au^+ - bu^-$ has only the trivial solution.

We use the last result for proving that if $(\alpha_0, \beta_0) \in \Gamma_1$, $g(x, s) = sh(x, s) + k(x, s)$,

$$\lambda_p + \eta \leq \alpha + \inf_{s \geq 0} h(x, s) \leq \alpha + \sup_{s \geq 0} h(x, s) < \alpha_0$$

$$\lambda_p + \eta \leq \beta + \inf_{s \leq 0} h(x, s) \leq \beta + \sup_{s \leq 0} h(x, s) < \beta_0$$

a.e. in Ω for some $\eta > 0$ and $\sup_{s\in\mathbb{R}} |k(x,s)| \in L^2(\Omega)$, then (E) has at least one solution.

Finally, we study the equation (E) when (α,β) is on Γ_1 or Γ_2. For that, we suppose that for each element φ of $N(L-\lambda I)$, we have $\|\varphi^+\|\,\|\varphi^-\| \neq 0$. If, in addition, $\sup_{s\in\mathbb{R}} |g(x,s)| \in L^2(\Omega)$, $\lim_{s\to\pm\infty} g(x,s) = g_\pm(x)$ and for each nontrivial solution of $Lw = \alpha w^+ - \beta w^-$, we have:

- $meas\,\{x \in \Omega \,/\, w(x) = 0\} = 0$

- $\int g_+ w^+ - g_- w^- < 0$ if $(\alpha,\beta) \in \Gamma_1$ and $\int g_+ w^+ - g_- w^- > 0$ if $(\alpha,\beta) \in \Gamma_2$,

then (E) has at least one solution.

2. Preliminaries

We denote $V = N(L-\lambda I)$, $V_1 = \oplus_{r<\lambda} N(L-rI) \cap D(L)$ and $V_2 = \oplus_{r>\lambda} N(L-rI) \cap D(L)$.

Let $h \in L^2(\Omega)$, $\phi \in V$ and $(a,b) \in L^\infty(\Omega) \times L^\infty(\Omega)$ be such that $\lambda_p + \eta \leq a(x), b(x) \leq \lambda_g - \eta$ a.e. in Ω for some $\eta > 0$. Consider the function $J_h(a,b,.) : D(L) \to \mathbb{R}$ such that

$$J_h(a,b,u) = \frac{1}{2}\langle Lu,u\rangle - \frac{1}{2}\int a(u^+)^2 - \frac{1}{2}\int b(u^-)^2 - \int hu$$

and the function $K_{h,\phi}(a,b,.,.) : V_1 \times V_2 \to \mathbb{R}$ such that $K_{h,\phi}(a,b,v_1,v_2) = J_h(a,b,v_1+\phi+v_2)$.

Lemma 1. ([5]) *The function $J_h(a,b,.)$ belongs to $C^1(D(L),\mathbb{R})$ and its differential is given by*

$$(DJ_h(a,b,u),v) = \langle Lu,v\rangle - \int au^+v + \int bu^-v - \int hv.$$

For each $v_2 \in V_2$, the function $K_{h,\phi}(a,b,.,v_2)$ is strictly concave on V_1. For each $v_1 \in V_1$, the function $K_{h,\phi}(a,b,v_1,.)$ is strictly convex on V_2.

Theorem 1. ([1]) *There exists a unique $v \in V_1 \oplus V_2$ such that*

$$Lv = P_{V^\perp}\left[a(v+\phi)^+ - b(v+\phi)^- + h\right]. \tag{2.1}$$

Remark 1. ([1]) a) There exists a constant $c > 0$ depending on $a, b, \lambda_p, \lambda, \lambda_g, \eta$ such that $\|v\|_{D(L)} \leq c(\|\phi\| + \|h\|)$, where v is a solution of (2.1).

b) If we write v as $v = v_1^0 + v_2^0$, where $v_1^0 \in V_1$ and $v_2^0 \in V_2$, then

$$K_{h,\phi}(a,b,v_1^0,v_2^0) = \max_{v_1\in V_1}\min_{v_2\in V_2} K_{h,\phi}(a,b,v_1,v_2) = \min_{v_2\in V_2}\max_{v_1\in V_1} K_{h,\phi}(a,b,v_1,v_2)$$

and for $v_1 \neq v_1^0, v_2 \neq v_2^0$ we have

$$K_{h,\phi}(a,b,v_1,v_2^0) < K_{h,\phi}(a,b,v_1^0,v_2^0) < K_{h,\phi}(a,b,v_1^0,v_2).$$

Remark 2. a) For each $z_2 \in V \oplus V_2$, there exists a unique $v_1^0 \in V_1$ such that $J_h(a,b,v_1^0+z_2) = \max_{v_1 \in V_1} J_h(a,b,v_1+z_2)$. The function v_1^0 verifies

$$\langle Lv_1^0, v_1 \rangle = \int a(v_1^0+z_2)^+ v_1 - \int b(v_1^0+z_2)^- v_1 + \int hv_1 \qquad \forall v_1 \in V_1$$

and there exists $c_1 > 0$ such that $\|v_1^0\|_{D(L)} \leq c_1(\|z_2\| + \|h\|)$.

b) For each $z_1 \in V_1 \oplus V$, there exists a unique $v_2^0 \in V_2$ such that $J_h(a,b,z_1+v_2^0) = \min_{v_2 \in V_2} J_h(a,b,z_1+v_2)$. Then v_2^0 verifies

$$\langle Lv_2^0, v_2 \rangle = \int a(z_1+v_2^0)^+ v_2 - \int b(z_1+v_2^0)^- v_2 + \int hv_2 \qquad \forall v_2 \in V_2$$

and there exists $c_2 > 0$ such that $\|v_2^0\|_{D(L)} \leq c_2(\|z_1\| + \|h\|)$.

Let the function $\theta_1 : V \oplus V_2 \to V_1$ defined by $J_0(a,b,\theta_1(z_2)+z_2) = \max_{v_1 \in V_1} J_0(a,b,v_1+z_2)$ and the function $\theta_2 : V_1 \oplus V \to V_2$ defined by $J_0(a,b,z_1+\theta_2(z_1)) = \min_{v_2 \in V_2} J_0(a,b,z_1+v_2)$. θ_1 and θ_2 are positively homogeneous and there exists $c_1 > 0$, $c_2 > 0$ such that $\|\theta_1(z_2)\|_{D(L)} \leq c_1 \|z_2\|$ and $\|\theta_2(z_1)\|_{D(L)} \leq c_2 \|z_1\|$. Finally, if $(z_{2n})_{n\geq 0}$ (resp. $(z_{1n})_{n\geq 0}$) converges weakly in $D(L)$ to z_2 (resp. z_1), then $(\theta_1(z_{2n}))_{n\geq 0}$ (resp. $(\theta_2(z_{1n}))_{n\geq 0}$) converges weakly in $D(L)$ to $\theta_1(z_2)$ (resp. $\theta_2(z_1)$).

3. The Fuçik spectrum

We suppose, without loosing of generality, that $m = 2$.

Theorem 2. *For each* $(\alpha,\beta) \in I \times I$ *and* $(t_1,t_2) \in \mathbb{R} \times R$, *there exists a unique* $(u,d_1,d_2) \in D(L) \times R \times R$ *such that*

$$\begin{cases} Lu = \alpha u^+ - \beta u^- + d_1\varphi_1 + d_2\varphi_2 \\ \int u\varphi_1 = t_1 \ , \ \int u\varphi_2 = t_2. \end{cases} \tag{3.1}$$

Proof. Let $u = v + t_1\varphi_1 + t_2\varphi_2$, then (3.1) is equivalent to

$$\begin{cases} Lv = P_{V^\perp}\left[\alpha(v+t_1\varphi_1+t_2\varphi_2)^+ - \beta(v+t_1\varphi_1+t_2\varphi_2)^-\right] \\ d_1 = \lambda t_1 - \int\left[\alpha(v+t_1\varphi_1+t_2\varphi_2)^+ - \beta(v+t_1\varphi_1+t_2\varphi_2)^-\right]\varphi_1 \\ d_2 = \lambda t_2 - \int\left[\alpha(v+t_1\varphi_1+t_2\varphi_2)^+ - \beta(v+t_1\varphi_1+t_2\varphi_2)^-\right]\varphi_2. \end{cases}$$

We conclude by Theorem 1 that v exists and it is unique. So (3.1) has a unique solution (u,d_1,d_2) in $D(L) \times R \times R$. □

Remark 3. There exists a constant $c > 0$ depending on $\min(\alpha,\beta) - \lambda_p$ and $\lambda_g - \max(\alpha,\beta)$ such that $\|u\|_{D(L)} \leq c\|t_1\varphi_1 + t_2\varphi_2\|$ and we deduce from

Remark 1.b that

$$t_1d_1 + t_2d_2 = 2 \max_{v_1 \in V_1} \min_{v_2 \in V_2} J_0(\alpha, \beta, v_1 + t_1\varphi_1 + t_2\varphi_2 + v_2)$$
$$= 2 \min_{v_2 \in V_2} \max_{v_1 \in V_1} J_0(\alpha, \beta, v_1 + t_1\varphi_1 + t_2\varphi_2 + v_2) = 2J_0(\alpha, \beta, u).$$

For $i = 1, 2$, we denote by $D_i : I \times I \times \mathbb{R} \times \mathbb{R} \to R$ the function such that $D_i(\alpha, \beta, t_1, t_2) = d_i$.

Theorem 3. a) *For $r \geq 0$ and $i = 0, 1$, we have $D_i(\alpha, \beta, rt_1, rt_2) = rD_i(\alpha, \beta, t_1, t_2)$.*

b) *For $i = 0, 1$, we have $D_i(\alpha, \alpha, t_1, t_2) = (\lambda - \alpha)t_i$.*

c) *D_1 and D_2 are continuous on $I \times I \times \mathbb{R} \times \mathbb{R}$.*

Proof. For the proof of a) we use the definition of D_i. For b) we let $u = t_1\varphi_1 + t_2\varphi_2$ so

$$Lu = \lambda u = \alpha u + (\lambda - \alpha)t_1\varphi_1 + (\lambda - \alpha)t_2\varphi_2.$$

For c) we consider a sequence $(\alpha_n, \beta_n, t_{1n}, t_{2n})_{n\geq 0}$ which converges to $(\alpha, \beta, t_1, t_2)$. Let

$$d_{1n} = D_1(\alpha_n, \beta_n, t_{1n}, t_{2n})$$
$$= \lambda t_{1n} - \int \left[\alpha_n(v_n + t_{1n}\varphi_1 + t_{2n}\varphi_2)^+ - \beta_n(v_n + t_{1n}\varphi_1 + t_{2n}\varphi_2)^-\right]\varphi_1,$$
$$d_{2n} = D_2(\alpha_n, \beta_n, t_{1n}, t_{2n})$$
$$= \lambda t_{2n} - \int \left[\alpha_n(v_n + t_{1n}\varphi_1 + t_{2n}\varphi_2)^+ - \beta_n(v_n + t_{1n}\varphi_1 + t_{2n}\varphi_2)^-\right]\varphi_2,$$

where $v_n \in V_1 \oplus V_2$ is the unique solution of the equation

$$Lv_n = P_{V^\perp}\left[\alpha_n(v_n + t_{1n}\varphi_1 + t_{2n}\varphi_2)^+ - \beta_n(v_n + t_{1n}\varphi_1 + t_{2n}\varphi_2)^-\right].$$

The sequence $(v_n)_{n\geq 0}$ is bounded because

$$\|v_n\|_{D(L)} \leq c_n \|t_{1n}\varphi_1 + t_{2n}\varphi_2\|$$

and $(c_n)_{n\geq 0}$ is bounded (Remark 1.3). We can then extract from $(v_n)_{n\geq 0}$ a subsequence (v_{n_k}) which converges weakly in $D(L)$ to v such that

$$Lv = P_{V^\perp}\left[\alpha(v + t_1\varphi_1 + t_2\varphi_2)^+ - \beta(v + t_1\varphi_1 + t_2\varphi_2)^-\right]$$

hence, (d_{1n_k}) tends to

$$\lambda t_1 - \int \left[\alpha(v + t_1\varphi_1 + t_2\varphi_2)^+ - \beta(v + t_1\varphi_1 + t_2\varphi_2)^-\right]\varphi_1 = D_1(\alpha, \beta, t_1, t_2)$$

and (d_{2n_k}) tends to

$$\lambda t_2 - \int \left[\alpha(v + t_1\varphi_1 + t_2\varphi_2)^+ - \beta(v + t_1\varphi_1 + t_2\varphi_2)^-\right] \varphi_2 = D_2(\alpha, \beta, t_1, t_2).$$

The last limits do not depend on the subsequence considered, so $(d_{1n})_{n\geq 0}$ (resp. $(d_{2n})_{n\geq 0}$) converges to $D_1(\alpha, \beta, t_1, t_2)$ (resp. $D_2(\alpha, \beta, t_1, t_2)$). We have then proved the continuity of D_1 and D_2. □

We introduce now the functions C_1, $C_2 : I \times I \times [0, 2\Pi] \to \mathbb{R}$ defined by

$$C_1(\alpha, \beta, \theta) = D_1(\alpha, \beta, \cos(\theta), \sin(\theta)),$$
$$C_2(\alpha, \beta, \theta) = D_2(\alpha, \beta, \cos(\theta), \sin(\theta)).$$

Assumption Θ. We suppose that

$$\left\|[\cos(\theta)\varphi_1 + \sin(\theta)\varphi_2]^-\right\| \neq 0 \quad \text{and} \quad \left\|[\cos(\theta)\varphi_1 + \sin(\theta)\varphi_2]^+\right\| \neq 0$$

for each $\theta \in [0, 2\Pi]$.

For the case $\sigma(L) \subset \mathbb{R}^{+*}$, M. Schechter [3] supposed that the first eigenvalue of L is simple and that the corresponding eigenfunction is positive. By orthogonality between eigenfunctions, we deduce that Θ is verified.

Theorem 4. *If Θ is verified, then for each $\theta \in [0, 2\Pi]$, the function*

$$C(\alpha, \beta, \theta) = C_1(\alpha, \beta, \theta)\cos(\theta) + C_2(\alpha, \beta, \theta)\sin(\theta)$$

is decreasing in each variable α and β.

Proof. Let $\beta_1 > \beta$ and $(u_1, u_2) \in D(L) \times D(L)$ be such that

$$\begin{cases} Lu_1 = \alpha u_1^+ - \beta u_1^- + C_1(\alpha, \beta, \theta)\varphi_1 + C_2(\alpha, \beta, \theta)\varphi_2 \\ \int u_1\varphi_1 = \cos(\theta)\ ,\ \int u_1\varphi_2 = \sin(\theta) \end{cases}$$

$$\begin{cases} Lu_2 = \alpha u_2^+ - \beta_1 u_2^- + C_1(\alpha, \beta_1, \theta)\varphi_1 + C_2(\alpha, \beta_1, \theta)\varphi_2 \\ \int u_2\varphi_1 = \cos(\theta)\ ,\ \int u_2\varphi_2 = \sin(\theta). \end{cases}$$

We have

a) $\|u_1^-\| * \|u_1^+\| \neq 0$ and $\|u_2^-\| * \|u_2^+\| \neq 0$. Suppose that $\|u_1^-\| = 0$, if we write u_1 as $u_1 = v + \cos(\theta)\varphi_1 + \sin(\theta)\varphi_2$, where $v \in V_1 \oplus V_2$, then $Lv = \alpha v$. Hence, $v = 0$ because $\alpha \in I$. So $u_1 = \cos(\theta)\varphi_1 + \sin(\theta)\varphi_2 \geq 0$, which is in contradiction with Θ.

b) If we write u_1 and u_2 as $u_1 = v_1 + \cos(\theta)\varphi_1 + \sin(\theta)\varphi_2 + v_2$ and $u_2 = w_1 + \cos(\theta)\varphi_1 + \sin(\theta)\varphi_2 + w_2$, where $(v_1, v_2) \in V_1 \times V_2$ and $(w_1, w_2) \in V_1 \times V_2$, then we have $v_1 + v_2 \neq w_1 + w_2$. Suppose that $v_1 + v_2 = w_1 + w_2$, then $u_1 = u_2$ and

$$Lu_1 = \alpha u_1^+ - \beta u_1^- + C_1(\alpha, \beta, \theta)\varphi_1 + C_2(\alpha, \beta, \theta)\varphi_2$$

$$= \alpha u_1^+ - \beta_1 u_1^- + C_1(\alpha,\beta_1,\theta)\varphi_1 + C_2(\alpha,\beta_1,\theta)\varphi_2$$

hence,

$$0 \le (\beta_1-\beta)u_1^- = (C_1(\alpha,\beta_1,\theta)-C_1(\alpha,\beta,\theta))\varphi_1+(C_2(\alpha,\beta_1,\theta)-C_2(\alpha,\beta,\theta))\varphi_2$$

which is in contradiction with Θ.

c) Since $v_1+v_2 \neq w_1+w_2$, we suppose for example that $v_1 \neq w_1$, then

$$\begin{aligned} C(\alpha,\beta,\theta) &= C_1(\alpha,\beta,\theta)\cos(\theta) + C_2(\alpha,\beta,\theta)\sin(\theta) \\ &= 2J_0(\alpha,\beta,v_1+\cos(\theta)\varphi_1+\sin(\theta)\varphi_2+v_2) \\ &> 2J_0(\alpha,\beta,w_1+\cos(\theta)\varphi_1+\sin(\theta)\varphi_2+v_2) \end{aligned}$$

and

$$\begin{aligned} C(\alpha,\beta_1,\theta) &= C_1(\alpha,\beta_1,\theta)\cos(\theta) + C_2(\alpha,\beta_1,\theta)\sin(\theta) \\ &= 2J_0(\alpha,\beta_1,w_1+\cos(\theta)\varphi_1+\sin(\theta)\varphi_2+w_2) \\ &\le 2J_0(\alpha,\beta_1,w_1+\cos(\theta)\varphi_1+\sin(\theta)\varphi_2+v_2) \\ &\le 2J_0(\alpha,\beta,w_1+\cos(\theta)\varphi_1+\sin(\theta)\varphi_2+v_2) \end{aligned}$$

so $C(\alpha,\beta,\theta) > C(\alpha,\beta_1,\theta)$.

Remark 4. If Θ is not verified, then for each $\theta \in [0,2\Pi]$, the function $C(\alpha,\beta,\theta)$ is nonincreasing in each variable α,β.

Let σ, $\tau : I \times I \to \mathbb{R}$ be such that

$$\sigma(\alpha,\beta) = \min_{\theta\in[0,2\Pi]} C(\alpha,\beta,\theta), \quad \tau(\alpha,\beta) = \max_{\theta\in[0,2\Pi]} C(\alpha,\beta,\theta).$$

Remark 5. a) $\sigma(\alpha,\alpha) = \tau(\alpha,\alpha) = \lambda - \alpha$.

b) σ and τ are continuous on $I \times I$.

c) If Θ is verified, then σ and τ are decreasing in each variable α and β. Otherwise, σ and τ are nonincreasing in each variable α and β.

d) If $\sigma(\alpha,\beta) > 0$ or $\tau(\alpha,\beta) < 0$, then $(\alpha,\beta) \notin \Sigma$.

e) If $\sigma(\alpha,\beta) = 0$, then $\sigma(\beta,\alpha) = 0$. If $\tau(\alpha,\beta) = 0$, then $\tau(\beta,\alpha) = 0$.

Theorem 5. a) *If* $\sigma(\alpha,\beta) = 0$*, then* $(\alpha,\beta) \in \Sigma$. b) *If* $\tau(\alpha,\beta) = 0$*, then* $(\alpha,\beta) \in \Sigma$.

Proof. Suppose that $\sigma(\alpha,\beta) = 0$. Since the function $C(\alpha,\beta,\theta)$ is continuous on $[0,2\Pi]$, there exists $\theta_0 \in [0,2\Pi]$ such that $C(\alpha,\beta,\theta_0) = \sigma(\alpha,\beta) = 0$. Let $u_0 \in D(L)$ be such that

$$\begin{cases} Lu_0 = \alpha u_0^+ - \beta u_0^- + C_1(\alpha,\beta,\theta_0)\varphi_1 + C_2(\alpha,\beta,\theta_0)\varphi_2 \\ \int u_0\varphi_1 = \cos(\theta_0)\ ,\ \int u_0\varphi_2 = \sin(\theta_0). \end{cases} \tag{3.2}$$

Let $u_0 = v_1^0 + z_2^0$, where $v_1^0 \in V_1$ and $z_2^0 = \cos(\theta_0)\varphi_1 + \sin(\theta_0)\varphi_2 + v_2^0$ with $v_2^0 \in V_2$. We conclude from (3.2) and Remark 2 that $v_1^0 = \theta_1(z_2^0)$ and from Remark 3 that

$$\begin{aligned} 0 &= C_1(\alpha,\beta,\theta_0)\cos(\theta_0) + C_2(\alpha,\beta,\theta_0)\sin(\theta_0) \\ &= 2 \min_{v_2\in V_2} \max_{v_1\in V_1} J_0(\alpha,\beta, v_1 + \cos(\theta_0)\varphi_1 + \sin(\theta_0)\varphi_2 + v_2) = 2J_0(\alpha,\beta,u_0). \end{aligned}$$

Since $\sigma(\alpha,\beta) = 0$, we have

$$\min_{(t_1,t_2)\in\mathbb{R}} \min_{v_2\in V_2} \max_{v_1\in V_1} J_0(\alpha,\beta,v_1 + t_1\varphi_1 + t_2\varphi_2 + v_2) = 0$$

so

$$\min_{z_2\in V\oplus V_2} \max_{v_1\in V_1} J_0(v_1 + z_2) = 0.$$

Let $t > 0$ and $z_2 \in V \oplus V_2$, then

$$\frac{1}{t}\left[J_0(\alpha,\beta,\theta_1(z_2^0 + tz_2) + z_2^0 + tz_2) - J_0(\alpha,\beta,\theta_1(z_2^0) + z_2^0)\right] \geq 0$$

since $J_0(\alpha,\beta,\theta_1(z_2^0) + z_2^0) \geq J_0(\alpha,\beta,\theta_1(z_2^0 + tz_2) + z_2^0)$, we obtain

$$\frac{1}{t}\left[J_0(\alpha,\beta,\theta_1(z_2^0 + tz_2) + z_2^0 + tz_2) - J_0(\alpha,\beta,\theta_1(z_2^0 + tz_2) + z_2^0)\right] \geq 0$$

so

$$\frac{1}{t}\int_0^1 \big(DJ_0(\alpha,\beta,\theta_1(z_2^0 + tz_2) + z_2^0 + \xi tz_2), tz_2\big)\, d\xi \geq 0$$

which is equivalent to

$$\begin{aligned} \int_0^1 \Big[\big\langle L(z_2^0 + \xi tz_2), z_2\big\rangle &- \alpha \int (\theta_1(z_2^0 + tz_2) + z_2^0 + \xi tz_2)^+ z_2 \\ &+ \beta \int (\theta_1(z_2^0 + tz_2) + z_2^0 + \xi tz_2)^- z_2\Big] d\xi \geq 0, \end{aligned}$$

when t goes to 0, we conclude by using Lebesgue theorem that

$$\big\langle Lz_2^0, z_2\big\rangle - \alpha \int (\theta_1(z_2^0) + z_2^0)^+ z_2 + \beta \int (\theta_1(z_2^0) + z_2^0)^- z_2 \geq 0.$$

Taking $-z_2$ instead of z_2, we deduce that

$$\big\langle Lz_2^0, z_2\big\rangle = \alpha \int (\theta_1(z_2^0) + z_2^0)^+ z_2 - \beta \int (\theta_1(z_2^0) + z_2^0)^- z_2 \quad \forall z_2 \in V + V_2.$$

Since

$$\big\langle L\theta_1(z_2^0), v_1\big\rangle = \alpha \int (\theta_1(z_2^0) + z_2^0)^+ v_1 - \beta \int (\theta_1(z_2^0) + z_2^0)^- v_1 \quad \forall v_1 \in V_1$$

we conclude that

$$\langle Lu_0, v\rangle = \alpha \int u_0^+ v - \beta \int u_0^- v$$

for each $v \in D(L)$. So $Lu_0 = \alpha u_0^+ - \beta u_0^-$, because $D(L)$ is dense in $L^2(\Omega)$. Hence, $(\alpha, \beta) \in \Sigma$. For the proof of the second part of Theorem 6, we proceed as for the first part.

Remark 6. Let

$$\Gamma_1 = \{(\alpha, \beta) \in I \times I \text{ such that } \sigma(\alpha, \beta) = 0\},$$

$$\Gamma_2 = \{(\alpha, \beta) \in I \times I \text{ such that } \tau(\alpha, \beta) = 0\}.$$

We have proved that $\Gamma_1 \cup \Gamma_2 \subset \Sigma$. If $\Gamma_1 \neq \Gamma_2$, then Γ_2 is above Γ_1 and if $\Gamma_1 = \Gamma_2$, then the Fuçik spectrum consists in $I \times I$ of one curve.

The problem of determining completely the Fuçik spectrum remains open.

4. Existence results

In this section, we study the equation

$$Lu = \alpha u^+ - \beta u^- + g(\cdot, u), \tag{4.1}$$

where $g : \Omega \times \mathbb{R} \to R$ is a caratheodory function and $(\alpha, \beta) \in I \times I$. Let $(\mu, \nu) \in \mathbb{R} \times R$ be such that $\lambda_p < \alpha + \mu, \beta + \nu < \lambda_g$. For each $s \in [0, 1]$, we consider the equation

$$Lu = \alpha u^+ - \beta u^- + sg(\cdot, u) + (1 - s)(\mu u^+ - \nu u^-). \tag{4.2}$$

A function u is a solution of (4.2) if and only if $u = T(s, u)$, where $T : [0, 1] \times L^2(\Omega) \to L^2(\Omega)$ is the completely continuous operator defined by

$$T(s, u) = (L - \gamma I)^{-1} \left[\alpha u^+ - \beta u^- + sg(., u) + (1 - s)(\mu u^+ - \nu u^-) - \gamma u\right]$$

the scalar γ is such that $(L - \gamma I)^{-1}$ is compact.

Theorem 6. *Suppose that $|g(x, s)| \leq h^{1-\zeta}(x)|s|^\zeta + k(x)$, where h and k are in $L^2(\Omega)$ and $0 \leq \zeta < 1$. If $\sigma(\alpha, \beta) > 0$ or $\tau(\alpha, \beta) < 0$, then (4.1) has at least one solution.*

Proposition 1. *Suppose that $\sigma(\alpha, \beta) > 0$ and g as in Theorem 6. Consider $\mu = \rho - \alpha$, $\nu = \rho - \beta$, where $\rho \in (\lambda_p, \min(\alpha, \beta))$, then there exists $R > 0$ such that for each $s \in [0, 1]$ and u a solution of (4.2), we have $\|u\|_{D(L)} < R$.*

Proof. (by contradiction) $\forall n > 0$, $\exists s_n \in [0,1]$, $\exists u_n \in D(L)$ such that $\alpha_n = \|u_n\|_{D(L)} \geq n$ and

$$Lu_n = \alpha u_n^+ - \beta u_n^- + s_n g(.,u_n) + (1-s_n)(\mu u_n^+ - \nu u_n^-).$$

Since $(s_n)_{n\geq 0}$ and $(v_n = u_n/\alpha_n)_{n\geq 0}$ are bounded, we can extract from them subsequences denoted in the same manner such that $(s_n)_{n\geq 0}$ tends to s in $\mathbb{R}$ and $(v_n)_{n\geq 0}$ tends to v weakly in $D(L)$. Since $\|g(\cdot,u_n)\| \leq \|h\|^{1-\zeta}\|u_n\|^{\zeta}+\|k\|$, then v satisfies

$$Lv = (\alpha + (1-s)\mu)v^+ - (\beta + (1-s)\nu)v^-$$

and $(v_n)_{n\geq 0}$ converges to v in $D(L)$. We conclude that $v = 0$ because $\sigma(\alpha + (1-s)\mu, \beta + (1-s)\nu) \geq \sigma(\alpha,\beta) > 0$. In contradiction with $\|v\|_{D(L)} = 1$.

Proof of Theorem 6. Since $u \neq T(s,u)$ for each $s \in [0,1]$ and $u \in \partial B(R)$ (R is given by Proposition 4.1), the invariance by homotopy of topological degree of Leray-Schauder allows us to conclude that $D(I-T(1,\cdot), B(R), 0) = D(I-T(0,\cdot), B(R), 0) = \pm 1$. Hence, (4.1) has at least one solution. □

Now, we give some remarks about the nonexistence of nontrivial solutions of the equation $Lu = a(x)u^+ - b(x)u^-$.

Theorem 7. *Suppose that $\sigma(\alpha_0,\beta_0) = 0$, $\lambda_p + \eta \leq a(x) \leq \alpha_0$ and $\lambda_p + \eta \leq b(x) \leq \beta_0$ a.e. in Ω for some $\eta > 0$. If in addition*

$$\int (\alpha_0 - a(x))(w^+)^2 + \int (\beta_0 - b(x))(w^-)^2 > 0$$

for each nontrivial solution of $Lw = \alpha_0 w^+ - \beta_0 w^-$, then the equation $Lu = au^+ - bu^-$ has only the trivial solution.

Proof. (by contradiction) Let $u \in D(L)$ be such that $\|u\| \neq 0$ and $Lu = au^+ - bu^-$.

a) $\int (u\varphi_1)^2 + (u\varphi_2)^2 \neq 0$ because if not, the function u is then orthogonal to V and verifies $Lu = P_{V^\perp}(au^+ - bu^-)$. By Theorem 1, we deduce that $u = 0$.

b) Let $\xi \geq 0$ be such that $\xi^2 = \int (u\varphi_1)^2 + (u\varphi_2)^2$ and put $v = u/\xi$, $\cos(\theta_0) = \int u\varphi_1/\xi$ and $\sin(\theta_0) = \int u\varphi_2/\xi$. The function v verifies

$$\begin{cases} Lv = av^+ - bv^- \\ \int v\varphi_1 = \cos(\theta_0)\ ,\ \int v\varphi_2 = \sin(\theta_0). \end{cases}$$

If we write v as $v = v_1^0 + \phi_0 + v_2^0$, where $\phi_0 = \cos(\theta_0)\varphi_1 + \sin(\theta_0)\varphi_2$, then we have

$$0 = K_{0,\phi_0}(a,b,v_1^0,v_2^0) = \max_{v_1 \in V1} K_{0,\phi_0}(a,b,v_1,v_2^0)$$

$$\geq \max_{v_1 \in V1} K_{0,\phi_0}(\alpha_0, \beta_0, v_1, v_2^0) = K_{0,\phi_0}(\alpha_0, \beta_0, v_1^1, v_2^0)$$
$$\geq \min_{v_2 \in V_2} \max_{v_1 \in V1} K_{0,\phi_0}(\alpha_0, \beta_0, v_1, v_2)$$
$$\geq \frac{1}{2}\left[C_1(\alpha_0, \beta_0, \theta_0)\cos(\theta_0) + C_2(\alpha_0, \beta_0, \theta_0)\sin(\theta_0)\right] \geq \frac{1}{2}\sigma(\alpha_0, \beta_0) = 0$$

so $C_1(\alpha_0, \beta_0, \theta_0) = 0$, $C_2(\alpha_0, \beta_0, \theta_0) = 0$ and

$$\min_{v_2 \in V_2} \max_{v_1 \in V1} K_{0,\phi_0}(\alpha_0, \beta_0, v_1, v_2) = \max_{v_1 \in V1} \min_{v_2 \in V_2} K_{0,\phi_0}(\alpha_0, \beta_0, v_1, v_2)$$
$$= K_{0,\phi_0}(\alpha_0, \beta_0, v_1^1, v_2^0) = 0.$$

Let $w = v_1^1 + \phi_0 + v_2^0$, this function satisfies

$$\begin{cases} Lw = \alpha_0 w^+ - \beta_0 w^- \\ \int w\varphi_1 = \cos(\theta_0)\ ,\ \int w\varphi_2 = \sin(\theta_0). \end{cases}$$

c) $v_1^1 \neq v_1^0$ because if not, $v = w$. So $Lw = \alpha_0 w^+ - \beta_0 w^- = aw^+ - bw^-$ which implies that

$$\int (\alpha_0 - a(x))(w^+)^2 + \int (\beta_0 - b(x))(w^-)^2 = 0$$

which is in contradiction with assumptions. Since $v_1^1 \neq v_1^0$, we have

$$0 = K_{0,\phi_0}(a, b, v_1^0, v_2^0) = \max_{v_1 \in V1} K_{0,\phi_0}(a, b, v_1, v_2^0)$$
$$> K_{0,\phi_0}(a, b, v_1^1, v_2^0) > K_{0,\phi_0}(\alpha_0, \beta_0, v_1^1, v_2^0) = 0$$

which is impossible. □

Remark 7. Suppose that $\tau(\alpha_0, \beta_0) = 0$, $\alpha_0 \leq a(x) \leq \lambda_g - \eta$ and $\beta_0 \leq b(x) \leq \lambda_g - \eta$ a.e. in Ω for some $\eta > 0$. If in addition

$$\int (a(x) - \alpha_0)(w^+)^2 + \int (b(x) - \beta_0)(w^-)^2 > 0$$

for each nontrivial solution of $Lw = \alpha_0 w^+ - \beta_0 w^-$, then the equation $Lu = au^+ - bu^-$ has only the trivial solution.

The next result is a consequence of Theorem 7.

Theorem 8. *Suppose that $g(x,s) = sh(x,s) + k(x,s)$ and Ω is bounded. If $\sigma(\alpha_0, \beta_0) = 0$ and*

$$\lambda_p + \eta \leq \alpha + \inf_{s \geq 0} h(x,s) \leq \alpha + \sup_{s \geq 0} h(x,s) < \alpha_0$$
$$\lambda_p + \eta \leq \beta + \inf_{s \leq 0} h(x,s) \leq \beta + \sup_{s \leq 0} h(x,s) < \beta_0$$

a.e. in Ω *for some* $\eta > 0$

$$\sup_{s\in\mathbb{R}} |k(x,s)| \in L^2(\Omega),$$

then (4.1) *has at least one solution.*

Proposition 2. *Let* $\mu = \rho - \alpha$ *and* $\nu = \rho - \beta$*, where* $\rho \in (\lambda_p, \min(\alpha_0, \beta_0))$*. Under the assumptions of Theorem* 8*, there exists* $R > 0$ *such that for each* $s \in [0,1]$ *and* u *a solution of* (4.2)*, we have* $|u\|_{D(L)} < R$*.*

Proof. (by contradiction) $\forall n > 0$, $\exists s_n \in [0,1]$, $\exists u_n \in D(L)$ such that $\alpha_n = \|u_n\|_{D(L)} \geq n$ and

$$Lu_n = \alpha u_n^+ - \beta u_n^- + s_n g(\cdot, u_n) + (1-s_n)(\mu u_n^+ - \nu u_n^-)$$

which is equivalent to

$$\begin{aligned} Lu_n = &\left[\alpha + s_n h(\cdot, u_n^+) + (1-s_n)\mu\right] u_n^+ \\ &- \left[\beta + s_n h(\cdot, -u_n^-) + (1-s_n)\nu\right] u_n^- + s_n k(\cdot, u_n). \end{aligned}$$

We can extract from $(s_n)_{n\geq 0}$, $(v_n = u_n/\alpha_n)_{n\geq 0}$, $(h(\cdot, u_n^+))_{n\geq 0}$, $(h(\cdot, -u_n^-))_{n\geq 0}$ subsequences denoted in the same manner such that $(s_n)_{n\geq 0}$ tends to s in $\mathbb{R}$, $(v_n)_{n\geq 0}$ tends to v weakly in $D(L)$, $(h(\cdot, u_n^+))_{n\geq 0}$ (resp. $(h(\cdot, -u_n^-))_{n\geq 0}$) tends to h_*^+ (resp. h_*^-) weakly star in $L^\infty(\Omega)$ (and weakly in $L^2(\Omega)$). The function v is then a solution of the equation

$$Lv = (\alpha + sh_*^+ + (1-s)\mu)v^+ - (\beta + sh_*^- + (1-s)\nu)v^-.$$

Since

$$\begin{aligned} \min(\lambda_p + \eta, \rho) &\leq \alpha + sh_*^+ + (1-s)\mu < \alpha_0 \\ \min(\lambda_p + \eta, \rho) &\leq \beta + sh_*^- + (1-s)\nu < \beta_0 \end{aligned}$$

we conclude by Theorem 7 that $v = 0$. So $(v_n)_{n\geq 0}$ tends to v in $D(L)$ and $\|v\|_{D(L)} = 1$. We have then a contradiction.

For the proof of Theorem 8, we use the invariance of topological degree of Leray-Schauder and Proposition 2.

Remark 8. Suppose that $g(x,s) = sh(x,s) + k(x,s)$ and Ω is bounded. If $\tau(\alpha_0, \beta_0) = 0$ and

$$\alpha_0 < \alpha + \inf_{s\geq 0} h(x,s) \leq \alpha + \sup_{s\geq 0} h(x,s) \leq \lambda_g - \eta$$

$$\beta_0 < \beta + \inf_{s\leq 0} h(x,s) \leq \beta + \sup_{s\leq 0} h(x,s) \leq \lambda_g - \eta$$

$a.e.$ in Ω for some $\eta > 0$

$$\sup_{s\in\mathbb{R}} |k(x,s)| \in L^2(\Omega),$$

then (4.1) has at least one solution.

Now, we give an existence result of solutions of (4.1) when $(\alpha,\beta) \in \Gamma_1 \cup \Gamma_2$.

Theorem 9. *Suppose that* Θ *is verified,* $(\alpha,\beta) \in \Gamma_1 \cup \Gamma_2$*,* $\sup_{s\in\mathbb{R}} |g(x,s)| \in L^2(\Omega)$ *and* $\lim_{s\to\pm\infty} g(x,s) = g_\pm(x)$*. If, in addition, for each nontrivial solution of* $Lw = \alpha w^+ - \beta w^-$*, we have*
meas $\{x \in \Omega \,/\, w(x) = 0\} = 0$
$\int g_+ w^+ - g_- w^- < 0$ *if* $\sigma(\alpha,\beta) = 0$ *and* $\int g_+ w^+ - g_- w^- > 0$ *if* $\tau(\alpha,\beta) = 0$*, then* (4.1) *has at least one solution.*

For each $\theta \in [0, 2\pi]$, we denote by u_θ the function such that

$$\begin{cases} Lu_\theta = \alpha u_\theta^+ - \beta u_\theta^- + C_1(\alpha,\beta,\theta)\varphi_1 + C_2(\alpha,\beta,\theta)\varphi_2 \\ \int u_\theta \varphi_1 = \cos(\theta)\ ,\ \int u_\theta\varphi_2 = \sin(\theta). \end{cases} \tag{4.3}$$

Proposition 3. *Suppose that* $\sigma(\alpha,\beta) = 0$ *and consider* $\mu = \rho - \alpha$*,* $\nu = \rho - \beta$*, where* $\rho \in (\lambda_p, \min(\alpha,\beta))$*. Under the assumptions of Theorem* 9*, there exists* $R > 0$ *such that for each* $s \in [0,1]$ *and* u *a solution of* (4.2)*, we have* $\|u\|_{D(L)} < R$*.*

Proof. (by contradiction) $\forall n > 0$, $\exists s_n \in [0,1]$, $\exists u_n \in D(L)$ such that $\alpha_n = \|u_n\|_{D(L)} \geq n$ and

$$Lu_n = \alpha u_n^+ - \beta u_n^- + s_n g(.,u_n) + (1-s_n)(\mu u_n^+ - \nu u_n^-). \tag{4.4}$$

We can extract from $(s_n)_{n\geq 0}$ and $(v_n = u_n/\alpha_n)_{n\geq 0}$ subsequences denoted in the same manner such that $(s_n)_{n\geq 0}$ tends to s in $\mathbb{R}$ and $(v_n)_{n\geq 0}$ tends to v weakly in $D(L)$. The function v is then a solution of the equation

$$Lv = (\alpha + (1-s)\mu)v^+ - (\beta + (1-s)\nu)v^-$$

and it is easy to prove that $(v_n)_{n\geq 0}$ converges to v in $D(L)$. If $s \neq 1$, then

$$\sigma(\alpha + (1-s)\mu, \beta + (1-s)\nu) > \sigma(\alpha,\beta) = 0,$$

so $v = 0$ which is in contradiction with $\|v\|_{D(L)} = 1$. Hence, v is a nontrivial solution of $Lv = \alpha v^+ - \beta v^-$. Let a_n and a be nonnegative scalars such that

$$a_n^2 = \int (v_n\varphi_1)^2 + \int (v_n\varphi_2)^2 \quad \text{and} \quad a^2 = \int (v\varphi_1)^2 + \int (v\varphi_2)^2.$$

The constant a is positive because if $a = 0$, then $v = 0$ (Theorem 1). Since $(a_n)_{n\geq 0}$ converges to a, there exists $N > 0$ such that for $n \geq N$, the constant

$a_n > 0$. For $n \geq N$, we define $\theta_n \in [0, 2\Pi]$ by $\cos(\theta_n) = \int (v_n \varphi_1)/a_n$ and $\sin(\theta_n) = \int (v_n \varphi_2)/a_n$. The sequence $(u_{\theta_n})_{n\geq 0}$ converges to u_{θ_0} where $v = au_{\theta_0}$. Let $z_n = v_n - a_n u_{\theta_n}$, then z_n is orthogonal to V and $(\|z_n\|_{D(L)})_{n\geq 0}$ tends to 0. Multiplying (4.4) by u_{θ_n}, we obtain

$$\begin{aligned}\langle Lu_n, u_{\theta_n}\rangle &= \int (\alpha u_n^+ - \beta u_n^-) u_{\theta_n} + s_n \int g(\cdot, u_n) u_{\theta_n} + (1 - s_n) \int (\mu u_n^+ - \nu u_n^-) u_{\theta_n} \\ &= \langle u_n, Lu_{\theta_n}\rangle \\ &= \alpha_n a_n (C_1(\alpha, \beta, \theta_n)\cos(\theta_n) + C_2(\alpha, \beta, \theta_n)\sin(\theta_n)) + \int (\alpha u_{\theta_n}^+ - \beta u_{\theta_n}^-) u_n,\end{aligned}$$

then

$$\begin{aligned}&s_n \int g(\cdot, u_n) u_{\theta_n} + (1 - s_n) \int (\mu u_n^+ - \nu u_n^-) u_{\theta_n} \\ &-\alpha_n a_n (C_1(\alpha, \beta, \theta_n)\cos(\theta_n) + C_2(\alpha, \beta, \theta_n)\sin(\theta_n)) = \alpha_n E_n,\end{aligned} \tag{4.5}$$

where

$$E_n = (\alpha - \beta) \int (v_n^+ u_{\theta_n}^- - v_n^- u_{\theta_n}^+).$$

Let $n \geq N$, if $v_n(x) \geq 0$ and $u_{\theta_n}(x) \leq 0$, then $v_n(x) \leq z_n(x)$ and $u_{\theta_n}(x) \geq -z_n(x)/a_n$, so $\int v_n^+ u_{\theta_n}^- \leq \|z_n\|^2 / a_n$. Hence, $|E_n| \leq |\alpha - \beta| \, (2/a_n) \, \|z_n\|_{D(L)}^2$. We claim that

$$\text{there exists } M > 0 \text{ such that for each } n \in \mathbb{N}, \text{ we have } \alpha_n |z_n|_{D(L)} \leq M \tag{4.6}$$

so

$$\begin{aligned}&s_n \int g(\cdot, u_n) u_{\theta_n} + (1 - s_n) \int (\mu u_n^+ - \nu u_n^-) u_{\theta_n} - \alpha_n a_n (C_1(\alpha, \beta, \theta_n)\cos(\theta_n) \\ &+ C_2(\alpha, \beta, \theta_n)\sin(\theta_n)) \geq - |\alpha - \beta| \, (2/a_n) M \, \|z_n\|_{D(L)}\end{aligned}$$

since $\lim \int (\mu v_n^+ - \nu v_n^-) u_{\theta_n} < 0$ because $\mu < 0$ and $\nu < 0$ and since

$$C_1(\alpha, \beta, \theta_n)\cos(\theta_n) + C_2(\alpha, \beta, \theta_n)\sin(\theta_n) \geq 0$$

because $\sigma(\alpha, \beta) = 0$, we conclude that $\lim s_n \int g(., u_n) u_{\theta_n} \geq 0$ and, by using Lebesgue theorem, that $\int g_+ u_{\theta_0}^+ - g_- u_{\theta_0}^- \geq 0$, which is in contradiction with assumptions. The proof will be complete once we have established (4.6), for that we proceed by contradiction. Suppose that $\lim \alpha_n \|z_n\|_{D(L)} = \infty$ and put $c_n = \|z_n\|_{D(L)}$ and $y_n = z_n / c_n$. We can extract from $(y_n)_{n\geq 0}$ a subsequence such that $(y_n)_{n\geq 0}$ tends to y weakly in $D(L)$ and there exists

$l \in L^2(\Omega)$ verifying $|y_n(x)| \le l(x)$ a.e. in Ω. Dividing (4.5) by $\alpha_n c_n$, we obtain

$$\frac{(1-s_n)}{c_n}\int(\mu v_n^+ - \nu v_n^-)u_{\theta_n} - \frac{a_n}{c_n}(C_1(\alpha,\beta,\theta_n)\cos(\theta_n) + C_2(\alpha,\beta,\theta_n)\sin(\theta_n))$$
$$= \frac{1}{c_n}E_n - \frac{s_n}{\alpha_n c_n}\int g(.,u_n)u_{\theta_n},$$

when n goes to ∞, we deduce that $\lim(1-s_n)/c_n = 0$ and that

$$\lim \frac{1}{c_n}(C_1(\alpha,\beta,\theta_n)\cos\theta_n + C_2(\alpha,\beta,\theta_n)\sin(\theta_n)) = 0$$

because $\sigma(\alpha,\beta) = 0$, $\mu < 0$ and $\nu < 0$. On the other hand, the function y_n verifies

$$\begin{aligned} Ly_n = {} & \alpha\Big[(y_n + \frac{a_n}{c_n}u_{\theta_n})^+ - \frac{a_n}{c_n}u_{\theta_n}^+\Big] - \beta\Big[(y_n + \frac{a_n}{c_n}u_{\theta_n})^- - \frac{a_n}{c_n}u_{\theta_n}^-\Big] \\ & + \frac{s_n}{\alpha_n c_n}g(.,u_n) + \frac{(1-s_n)}{c_n}(\mu v_n^+ - \nu v_n^-) \\ & - \frac{a_n}{c_n}(C_1(\alpha,\beta,\theta_n)\varphi_1 + C_2(\alpha,\beta,\theta_n)\varphi_2). \end{aligned}$$

Since y_n is orthogonal to V and L is a selfadjoint operator, y_n verifies also

$$\begin{aligned} Ly_n = P_{V^\perp}\Big\{ & \alpha\Big[(y_n + \frac{a_n}{c_n}u_{\theta_n})^+ - \frac{a_n}{c_n}u_{\theta_n}^+\Big] - \beta\Big[(y_n + \frac{a_n}{c_n}u_{\theta_n})^- - \frac{a_n}{c_n}u_{\theta_n}^-\Big] \\ & + \frac{s_n}{\alpha_n c_n}g(.,u_n) + \frac{(1-s_n)}{c_n}(\mu v_n^+ - \nu v_n^-)\Big\}. \end{aligned}$$

Put

$$\omega_n = \alpha\Big[(y_n + \frac{a_n}{c_n}u_{\theta_n})^+ - \frac{a_n}{c_n}u_{\theta_n}^+\Big] - \beta\Big[(y_n + \frac{a_n}{c_n}u_{\theta_n})^- - \frac{a_n}{c_n}u_{\theta_n}^-\Big]$$

by using the inequality $|(p+q)^\pm - p^\pm| \le |q|$ for each $(p,q) \in \mathbb{R}\times\mathbb{R}$, we deduce that $|\omega_n(x)| \le (|\alpha| + |\beta|)l(x)$ a.e. in Ω. Moreover, if $u_{\theta_0}(x) > 0$, then $\omega_n(x)$ tends to $\alpha y(x)$ and if $u_{\theta_0}(x) < 0$, then $\omega_n(x)$ tends to $\beta y(x)$. Since $meas\,\{x \in \Omega \,/\, u_{\theta_0}(x) = 0\} = 0$, we conclude by Lebesgue theorem that (ω_n) converges in $L^2(\Omega)$ to χy where

$$\chi(x) = \begin{cases} \alpha & \text{if } u_{\theta_0}(x) > 0 \\ \beta & \text{if } u_{\theta_0}(x) < 0. \end{cases}$$

In addition $\frac{s_n}{\alpha_n c_n}g(.,u_n) + \frac{(1-s_n)}{c_n}(\mu v_n^+ - \nu v_n^-)$ tends to 0 in $L^2(\Omega)$, so (Ly_n) converges in $L^2(\Omega)$ to $Ly = P_{V^\perp}(\chi y)$ and $\|y\|_{D(L)} = 1$. If we write y as

$y = y_2 + y_1$, where $y_2 \in V_2$ and $y_1 \in V_1$, then we have

$$\langle L(y_2 + y_1), y_2 - y_1 \rangle = \langle \chi(y_2 + y_1), y_2 - y_1 \rangle$$

which is equivalent to

$$\langle Ly_2, y_2 \rangle - \langle \chi y_2, y_2 \rangle = \langle Ly_1, y_1 \rangle - \langle \chi y_1, y_1 \rangle$$

so

$$0 \leq \langle (\lambda_g - \chi) y_2, y_2 \rangle \leq \langle (\lambda_p - \chi) y_1, y_1 \rangle \leq 0,$$

thus $y_1 = y_2 = 0$, which is in contradiction with $\|y\|_{D(L)} = 1$. □

For the proof of Theorem 9, we use the invariance by homotopy of the topological degree of Leray-Schauder.

Remark 9. When $\tau(\alpha, \beta) = 0$ and Θ is verified, if $\lim_{s \to \mp\infty} g(x, s)$ does not exist and if we suppose that each nontrivial solution of $Lw = \alpha w^+ - \beta w^-$ verifies

$meas\,\{x \in \Omega \,/\, w(x) = 0\} = 0$

$\int \liminf_{s \to \infty} g(., s) w^+ - \limsup_{s \to -\infty} g(., s) w^- > 0,$

then (4.1) has at least one solution.

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