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ON AN OPEN PROBLEM OF AMBROSETTI, BREZIS AND CERAMI

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(Submitted by: Haim Brezis)

Dedicated to Professor Matts Essén on occasion of his 65th birthday

Abstract. In this paper we study the structure of all solutions to the boundary value problem, which is the open problem D of Ambrosetti, Brezis and Cerami [1]

$$-u'' = \lambda |u|^{q-1}u + |u|^{p-1}u, \quad t \in [a, b], \quad u(a) = u(b) = 0,$$

where $0 < q < 1 < p$, $\lambda > 0$. We obtain a complete characterization of its solutions and the bifurcation graph. By perturbation, we show also instablity of the structure of the solutions for the above problem (see Figure 3).

1. Introduction

Let $0 < q < 1 < p$, $a < b$ be given constants and $\lambda > 0$ be a parameter. The purpose of this paper is to answer an open question posed in [1], i.e., to study in detail the structure of all solutions of the following two nonlinear boundary value problems

$$-u'' = \lambda u^q + u^p, \quad u(t) > 0, \quad t \in (a, b), \quad u(a) = u(b) = 0 \tag{1}$$

$$-u'' = \lambda |u|^{q-1}u + |u|^{p-1}u, \quad t \in [a, b], \quad u(a) = u(b) = 0. \tag{2}$$

Our main results are

Theorem 1. *There is a constant $\Lambda > 0$, which depends only on $p, q, b - a$, such that problem (1) has exactly two ordered solutions $u^-(t) < u^+(t)$, $t \in (a, b)$, if $\lambda \in (0, \Lambda)$; and has a unique solution $u(t)$ when $\lambda = \Lambda$; and has no solution in case $\lambda > \Lambda$. Moreover, the solution is given by*

$$u(t) = \begin{cases} G_\alpha^{-1}(t-a), & t \in [a, (a+b)/2]; \\ G_\alpha^{-1}(b-t), & t \in [(a+b)/2, b]; \end{cases}$$

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where $G_\alpha(s) = \int_0^s \frac{du}{\sqrt{F_\lambda(\alpha)-F_\lambda(u)}}$, *and* $\alpha > 0$ *is determined by* $G_\alpha(\alpha) = \frac{b-a}{2}$.

We should remark here that the existence part of Theorem 1 has already been established in [1], the non-trivial part of Theorem 1 is the sharpness. The first solution $u^-(t)$ is the minimizer of the corresponding energy functional and the second solution $u^+(t)$ is a mountain pass type critical point of the energy functional [9].

Concerning the solutions which change sign, let $m(\lambda)$ be the maximum of the function $\int_0^\alpha \frac{du}{\sqrt{F_\lambda(\alpha)-F_\lambda(u)}}$, $\alpha > 0$, then we have:

Theorem 2. *For any given integer* $r \geq 1$ (2) *has 'exactly'* (*up to a minus sign*) *two ordered solutions having* $r-1$ *common nodal points, if* $2rm(\lambda) > b-a$; (2) *has a 'unique' solution, which admits* $r-1$ *nodal points, when* $2rm(\lambda) = b-a$; *and* (2) *has no solution which has less than or equal to* $r-1$ *nodal points, in case* $2rm(\lambda) < b-a$. *Conversely, any solution of* (2) *has at most finitely many nodal points. If* $u(t)$ *is a solution of* (2) *with* r *nodal points, then up to a minus sign* u *is an odd periodic extension of* $u = u_r(t)$, *where* $u_r(t) > 0$ *solves*

$$u'' + \lambda u^q + u^p = 0, \quad t \in (0, \tfrac{b-a}{1+r})$$

with the boundary condition $u(0) = u(\frac{b-a}{1+r}) = 0$.

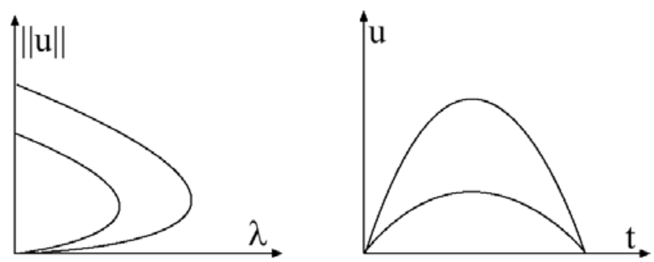

Figure 1.

2. Structure analysis of the solutions

Let $u(t)$ be a solution of (1), then it solves the following initial value problem for some $k > 0$

$$-u'' = \lambda u^q + u^p, \quad t \in (a,b), \quad u(a) = 0,\ u'(a) = k. \tag{3}$$

Multiply the equation by $2u'(t)$ and then integrate, we obtain

$$-(u'(t))^2 + k^2 = \frac{2\lambda}{1+q}u^{1+q} + \frac{2}{1+p}u^{p+1} \overset{\text{def}}{=} F_\lambda(u). \tag{4}$$

Since $u''(t) < 0$, we deduce that u has only one maximum point at $t_0 \in (a, b)$ and $u'(t) > 0$, $t \in (a, t_0)$. Therefore, we have

$$k^2 = F_\lambda(u_0), \quad u_0 = u(t_0) = \max_t u(t).$$

By the symmetry result in [8], we see that $u(t)$ is symmetric with respect to the point $t = \frac{a+b}{2}$ and hence $t_0 = \frac{b+a}{2}$. From (4) we get that $u(t)$ satisfies

$$u'(t) = \sqrt{k^2 - F_\lambda(u)} = \sqrt{F_\lambda(u_0) - F_\lambda(u)}, \quad t \in (a, t_0).$$

Consequently,

$$u(t) = G_{u_0}^{-1}(t - a), \quad t \in [a, (a+b)/2], \quad G_{u_0}(s) = \int_0^s \frac{du}{\sqrt{F_\lambda(u_0) - F_\lambda(u)}}.$$

To determine the value of u_0, let $t = t_0$, we derive from (4) and the above formula that u_0 solves the following equation

$$H(u_0, \lambda) \stackrel{\text{def}}{=} \int_0^{u_0} \frac{du}{\sqrt{F_\lambda(u_0) - F_\lambda(u)}} = \frac{b-a}{2}.$$

Thus,

$$u(t) = \begin{cases} G_{u_0}^{-1}(t-a), & t \in [a, (a+b)/2]; \\ G_{u_0}^{-1}(b-t), & t \in [(a+b)/2, b]; \end{cases}$$

where $u_0 > 0$ is determined by $H(u_0, \lambda) = (b-a)/2$. Therefore, we get a characterization of the solvability of (1): (1) has a solution if and only if $m(\lambda) = \max_{u_0>0} H(u_0, \lambda) \geq (b-a)/2$. To get the exact number of all solutions of (1), we need to analyze the function $H(u_0, \lambda)$, which will be done in next section.

In the previous analysis, the solvability criterion is given in terms of the maximum of the solution $u(t)$. But it can also be characterized by using the derivative $k = u'(a)$ of the solution. To do so, we define

$$J(k, \lambda) = \int_0^{F_\lambda^{-1}(k^2)} \frac{du}{\sqrt{k^2 - F_\lambda(u)}}, \quad k > 0.$$

Then it follows that

$$J(k, \lambda) = H(u_0, \lambda), \quad \text{if} \quad k^2 = F_\lambda(u_0)$$

and $u(t)$ is a solution of (1) if and only if

$$J(k, \lambda) = \frac{b-a}{2}, \quad k = u'(a),$$

and hence (1) has a solution if and only if $\max_{k>0} J(k, \lambda) \geq (b-a)/2$.

3. Proof of Theorem 2

Let $u(t)$ be a solution of (2) with $1 \le r \le +\infty$ interior nodal points $a < t_1 < \cdots < t_r < b$, then $-u(t)$ is also a solution of (2). So we may assume $u(t) > 0$, $t \in (a, t_1)$. Therefore, $u(t)$ solves

$$-u'' = \lambda u^q + u^p, u(t) > 0, \quad t \in (a, t_1), \quad u(a) = u(t_1) = 0.$$

By the previous analysis we get

$$J(k, \lambda) = \frac{t_1 - a}{2}, \quad k = u'(a).$$

Further, $v = -u(t)$ solves the following boundary value problem

$$-v'' = \lambda v^q + v^p, v(t) > 0 \quad t \in (t_1, t_2), \quad v(t_1) = v(t_2) = 0,$$

analogously,

$$J(k_1, \lambda) = \frac{t_2 - t_1}{2}, \quad k_1 = v'(t_1).$$

But by the symmetry $u'(a) = -u'(t_1-) = -u'(t_1+) = v'(t_1)$, consequently $k = k_1, t_2 - t_1 = t_1 - a$. Repeating the above argument, we get $b - t_r = t_r - t_{r-1} = \cdots = t_1 - a$, which implies that $r < \infty$ and $t_i = a + ih_r$, $h_r = \frac{b-a}{r+1}, i = 1, \ldots, r$. Thus, $u(t)$ is an odd periodic extension of $U(t)$, where $U(t)$ solves

$$\begin{cases} -U'' = \lambda U^q + U^p, U(t) > 0 \quad t \in (a, a + h_r) \\ U(a) = U(a + h_r) = 0 \end{cases} \tag{r}$$

which proves the second part in Theorem 2.

Conversely, for any integer $r \ge 1$ by the previous analysis for the problem (1), we have that (r) has no solution at all, in case $2rm(\lambda) < b - a$. We may assume that $2rm(\lambda) \ge b - a$, then (r) has precisely two solutions $U_r^1(t) < U_r^2(t)$, if $2rm(\lambda) > b-a$; and it has a unique solution $U_r(t)$, if $2rm(\lambda) = b-a$. The symmetry of the solution(s) of (r) enables us to extend it to an odd differentiable periodic function on the whole interval $[a, b]$, namely

$$u_r(t) = (-1)^{i-1} U_r(t - (i-1)h_r), \quad t \in (a + (i-1)h_r, a + ih_r), \; i = 1, \cdots, r,$$

and similarly for $U_r^1(t), U_r^2(t)$. Then one can easily check that $\pm u_r(t)$ and respectively $\pm u_r^1(t)$, $\pm u_r^2(t)$ solve (2), which gives a proof of the first part in Theorem 2.

4. Proof of Theorem 1

By a change of variable, we see

$$H(u_0,\lambda)=\int_0^1 \frac{u_0^{(1-q)/2}}{\sqrt{\frac{2\lambda}{1+q}(1-\theta^{1+q})+\frac{2}{1+p}(1-\theta^{1+p})u_0^{p-q}}}\,d\theta.$$

Clearly, H is well defined and is continuously differentiable on the domain $\mathcal{D}=\{(u_0,\lambda);u_0\geq 0,\lambda\geq 0,u_0+\lambda>0\}$. An easy calculation shows that H satisfies

$$H(u_0,\lambda)\leq c_1 u_0^{\frac{1-p}{2}},\quad u_0>0;\quad H(u_0,\lambda)\leq c_2\lambda^{\frac{1-p}{p-q}},\quad \lambda>0,$$

for some $c_1,c_2>0$ (formulas for c_1,c_2 can explicitly be written). Moreover, for any fixed $\lambda>0$, we have

$$\lim_{u_0\to 0^+} H(u_0,\lambda)=0;\qquad \lim_{u_0\to 0^+}\lim_{\lambda\to 0^+} H(u_0,\lambda)=+\infty.$$

Thus, $H(u_0,\lambda)=\frac{b-a}{2}$, $(u_0,\lambda)\in\mathcal{D}$ defines a bounded smooth curve $\mathcal{C}$, which is also a graph of a function $\lambda=\lambda(u_0)$ by the implicit function theorem, since H is strictly decreasing in the variable λ.

It is easy to see that the function $\lambda(u_0)$ has the definition interval $I=(0,\alpha)$, and has the asymptotic $\lambda(u_0)\approx c_0 u_0^{1-q}$, as $u_0\to 0^+$, where

$$\alpha^{(p-1)/2}=\frac{\sqrt{2}}{b-a}\int_0^1\sqrt{\frac{1+p}{1-\theta^{1+p}}}\,d\theta \quad\text{and}\quad \sqrt{c_0}=\frac{\sqrt{2}}{b-a}\int_0^1\sqrt{\frac{1+q}{1-\theta^{1+q}}}\,d\theta.$$

To prove Theorem 1, we need to show that in the λu_0-plan the straight line $\lambda=$ constant intersects $\mathcal{C}$ two times if $\lambda<\Lambda$; meets $\mathcal{C}$ only once if $\lambda=\Lambda$; and does not touch $\mathcal{C}$ at all when $\lambda>\Lambda$. To do so, we introduce the following nonlinear transformation

$$\tau=\frac{2\lambda u_0^{q-1}}{1+q},\quad v=\frac{2u_0^{p-1}}{1+p}, \tag{5}$$

then

$$H(u_0,\lambda)=P(\tau,v)\overset{\text{def}}{=}\int_0^1\frac{d\theta}{\sqrt{\theta_1\tau+\theta_2 v}},\quad u_0>0,\ \lambda\geq 0,$$

where $\theta_1=1-\theta^{1+q}$, $\theta_2=1-\theta^{1+p}$. Under the transformation (5) the straight line $\lambda=\lambda_0$ and curve $\mathcal{C}$ are mapped respectively into $C_\mu:\tau=\mu v^{-\varepsilon_0}$, $\mathcal{C}_0:P(\tau,v)=(b-a)/2$, $\tau,v\geq 0$, (see Figure 2 below) where $\mu=\frac{2}{1+q}(\frac{2}{1+p})^{-\varepsilon_0}\lambda_0,\varepsilon_0=\frac{1-q}{p-1}>0$.

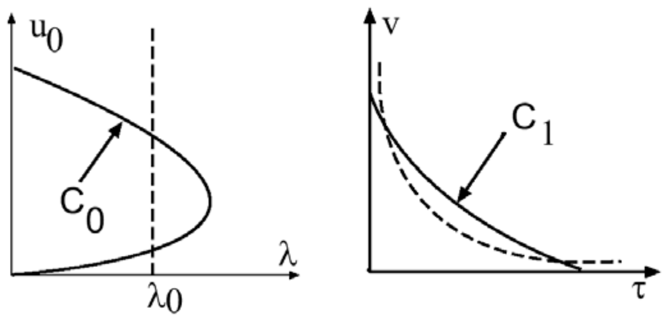

Figure 2.

By the implicit function theorem, the equation $P(\tau, v) = \frac{b-a}{2}, \tau, v \geq 0$ defines a function $\tau = \tau(v)$ on $[0, \alpha_0]$, where $\sqrt{\alpha}_0 = \int_0^1 \frac{2(1-\theta^{1+p})^{-\frac{1}{2}}}{b-a}\,d\theta$. It is clear that $\tau(v)$ is strictly decreasing with $\sqrt{\tau(0)} = \frac{2}{b-a}\int_0^1 (1-\theta^{1+q})^{-1/2}\,d\theta$ and $\tau(\alpha_0) = 0$. Further, by a direct differentiation we derive that

$$P_\tau \tau' + P_v = 0, \quad P_{\tau\tau}P_v^2 - 2P_{\tau v}P_v P_\tau + P_{vv}P_\tau^2 + P_\tau^3 \tau'' = 0.$$

By the Cauchy-Schwarz inequality, we deduce that $P_{\tau\tau}P_{vv} - (P_{\tau v})^2 > 0$ and thus

$$P_{\tau\tau}P_v^2 - 2P_{\tau v}P_v P_\tau + P_{vv}P_\tau^2 > 0.$$

Using the fact that $P_\tau < 0$, we see that $\tau(v)$ is also strictly convex on $[0, \alpha_0]$.

To investigate the intersections of $\mathcal{C}_\mu$ and $\mathcal{C}_0$, we consider the half-line $\ell_k : \tau = kv,\ v > 0$, where $k > 0$ is a parameter. It is easy to see that the intersections of ℓ_k with $\mathcal{C}_\mu$ and $\mathcal{C}_0$ are (v_{1k}, kv_{1k}) and (v_{2k}, kv_{2k}) where $v_{1k} = \mu^{2\varepsilon}k^{-2\varepsilon}$, $\sqrt{v_{2k}} = \frac{2}{b-a}\int_0^1 \frac{d\theta}{\sqrt{\theta_1 k + \theta_2}}$, $2\varepsilon = \frac{p-1}{p-q} < 1$. $\mathcal{C}_\mu$ and C_0 intersect if and only if $v_{1k} = v_{2k}$ for some $k > 0$. We introduce the following function

$$v = v(k) = \int_0^1 \frac{k^\varepsilon}{\sqrt{\theta_1 k + \theta_2}}\,d\theta, \quad k > 0.$$

Note that we have a relation $v(k)\frac{2\mu^{-\varepsilon}}{b-a} = \sqrt{\frac{v_{2k}}{v_{1k}}}$. Clearly, $v(k)$ is a continuously differentiable function on $(0, \infty)$ and has properties $v(k) > 0 = v(0)$, $k > 0$ and $v(k) \to 0$, as $k \to \infty$, due to the assumption $\varepsilon \in (0, 1/2)$.

In the following we shall show that $v(k)$ has exactly one maximum point $k = k_0 > 0$ and is strictly increasing on $[0, k_0]$ and is strictly decreasing on $[k_0, \infty)$. Let $\sigma = \frac{2\varepsilon}{1-2\varepsilon} = \frac{p-1}{1-q}$, $\theta_0 = \frac{\theta_2}{\theta_1} = \frac{1-\theta^{1+p}}{1-\theta^{1+q}}$, then

$$v(k) = \int_0^1 \frac{k^\varepsilon}{\sqrt{k + \theta_0}}\frac{d\theta}{\sqrt{\theta_1}},$$

and a direct differentiation yields

$$v'(k) = \int_0^1 \frac{\varepsilon\theta_0 k^{\varepsilon-1} - (\frac{1}{2}-\varepsilon)k^{\varepsilon}}{(k+\theta_0)^{3/2}} \frac{d\theta}{\sqrt{\theta_1}} = (\frac{1}{2}-\varepsilon)k^{\varepsilon-1}\int_0^1 \frac{\sigma\theta_0 - k}{(k+\theta_0)^{3/2}}\frac{d\theta}{\sqrt{\theta_1}}.$$

Since $\theta_0 \in [1, \frac{1+p}{1+q}]$, we see that

$$v'(k) > 0, \quad \text{if } k \in (0,\sigma]; \quad v'(k) < 0, \quad \text{if } k \geq \frac{\sigma(1+p)}{1+q}.$$

To show that $v'(k)$ vanishes exactly once on $(\sigma, \sigma\frac{1+p}{1+q})$, we consider

$$g(k) = \int_0^1 \frac{\sigma\theta_0 k^{\frac{1}{2}} - k^{\frac{3}{2}}}{(k+\theta_0)^{3/2}}\frac{d\theta}{\sqrt{\theta_1}}, \quad k > 0,$$

which is related to v' by $v'(k) = (\frac{1}{2}-\varepsilon)k^{\varepsilon-\frac{3}{2}}g(k)$. It is easy to see that

$$g'(k) = \frac{1}{2\sqrt{k}}\int_0^1 \frac{\sigma\theta_0 - (2\sigma+3)k}{(k+\theta_0)^{5/2}}\frac{\theta_0}{\sqrt{\theta_1}}\,d\theta.$$

Note that $\theta_0 \leq \frac{1+p}{1+q}$ for all $\theta \in (0,1)$ and if $k \in (\sigma, \sigma\frac{1+p}{1+q})$, then

$$\sigma\theta_0 - (2\sigma+3)k \leq \sigma(\frac{1+p}{1+q} - 2\sigma - 3) = \frac{(1+3q)(q-p)}{1-q^2} < 0.$$

Thus, we have $g'(k) < 0$, $k \in (\sigma, \sigma\frac{1+p}{1+q})$. Therefore, $g(k)$ vanishes only once on $(\sigma, \sigma\frac{1+p}{1+q})$ and consequently, $v'(k)$ vanishes exactly once on $(0,\infty)$.

Let $v_0 = \max_{k>0} v(k) > 0$, then $\mathcal{C}_\mu$ and $\mathcal{C}_0$ intersect exactly two times, if $\frac{2\mu^{-\varepsilon}}{b-a}v_0 > 1$; $\mathcal{C}_\mu$ and $\mathcal{C}_0$ intersect at only one point if $\frac{2\mu^{-\varepsilon}}{b-a}v_0 = 1$; and they do not touch each other in case $\frac{2\mu^{-\varepsilon}}{b-a}v_0 < 1$. Going back to the λu_0-plan, if $\Lambda = \frac{1+q}{2}(\frac{2}{1+p})^{\varepsilon_0}(\frac{2v_0}{b-a})^{1/\varepsilon}$, we have that the straight line $\lambda = \lambda_0$ intersects $\mathcal{C}$ exactly twice, only one time, and zero time , if $\lambda < \Lambda$, $\lambda = \Lambda$, $\lambda > \Lambda$ respectively. The proof of Theorem 1 is complete.

5. A generalization

In this part we study a class of more general nonlinear equations, namely

$$\Delta_p u = f(u), \quad t \in (a,b), \quad u(a) = u(b) = 0, \tag{6}$$

where $\Delta_p u = -(|u'(t)|^{p-2}u'(t))'$, $p > 1$ is a constant. The crucial point in the generalization is that the differential equation admits also a first integral

$$\frac{p-1}{p}|u'(t)|^p + F(u) = C, \quad F'(u) = f(u). \tag{7}$$

If $u(t)$ is a **positive** solution of (6) and has **only one** maximum point, say, at $t = t_0 \in (a, b)$ (this is certainly true when either $f(u) > 0$, $u > 0$ or $f(u)$ vanishes only once on $(0, \infty)$, like $\alpha u^r - \beta u^q, r > q$), then we see that

$$u'(t) = c_p \sqrt[p]{F(u_0) - F(u)},\ t \in [a, t_0];\quad u'(t) = -c_p \sqrt[p]{F(u_0) - F(u)},\ t \in [t_0, b],$$

where $c_p = \sqrt[p]{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$, $u_0 = \max u(t)$. A direct integration yields

$$\int_0^{u(t)} \frac{ds}{\sqrt[p]{F(u_0) - F(s)}} = c_p(t - a), \quad t \in [a, t_0];$$

$$\int_0^{u(t)} \frac{ds}{\sqrt[p]{F(u_0) - F(s)}} = c_p(b - t), \quad t \in [t_0, b].$$

Let $t = t_0$, we get

$$\int_0^{u_0} \frac{ds}{\sqrt[p]{F(u_0) - F(s)}} = c_p(t_0 - a); \quad \int_0^{u_0} \frac{ds}{\sqrt[p]{F(u_0) - F(s)}} = c_p(b - t_0).$$

It follows that $t_0 = (b+a)/2$ and $u(t)$ is symmetric with respect to the middle point $t_0 = (b + a)/2$ and moreover, $u(t)$ has the following representation

$$u(t) = \begin{cases} P_\alpha^{-1}(c_p(t - a)), & t \in [a, (a + b)/2]; \\ P_\alpha^{-1}(c_p(b - t)), & t \in [(a + b)/2, b]; \end{cases}$$

where $P_\alpha(s) = \int_0^s (F(\alpha) - F(u))^{-1/p}\, du$ and the positive constant α solves the following equation

$$\int_0^1 \frac{\alpha d\theta}{\sqrt[p]{F(\alpha) - F(\theta\alpha)}} = \frac{c_p(b - a)}{2}. \tag{8}$$

To show the existence of a positive solution of (6) and analyzing the structure of all its positive solutions, we need to study the solvability of (8) and its roots distribution.

The existence of at least one or two roots of (8) can be established whenever the asymptotics of $f(u)$ at $u = 0$ and $u = \infty$ are known. Concerning the uniqueness, we see by rewriting the difference as integral, then (8) is equivalent to

$$\int_0^1 \frac{d\theta}{\sqrt[p]{\int_0^1 \frac{f((\theta+(1-\theta)t)\alpha)}{((\theta+(1-\theta)t)\alpha)^{p-1}} (\theta + (1 - \theta)t)^{p-1}(1 - \theta)\, dt}} = \frac{c_p(b - a)}{2}.$$

Thus, we have the uniqueness whenever $f(u)/u^{p-1}$ is monotone for $u > 0$. If this is violated, then multiple solutions are created in generic. However, it can be very difficult to get a complete analysis of (8) for some cases. Anyhow under a very mild assumption that $F(u)$ is strictly increasing for $u > 0$, then all positive solutions of (6) are **ordered** even if it has finitely or infinitely many solutions.

To show the claim, given two distinct solutions $u_1(t)$, $u_2(t)$ of (6) and let $M_1 = \max_{a<t<b} u_1(t)$, $M_2 = \max_{a<t<b} u_2(t)$, then it follows from (7) that

$$\begin{aligned} \frac{p-1}{p}|u_1'(t)|^p &= F(M_1) - F(u_1), \\ \frac{p-1}{p}|u_2'(t)|^p &= F(M_2) - F(u_2), \end{aligned} \quad t \in (a,b). \tag{9}$$

Without loss of generality, we may assume $M_1 \leq M_2$. If $M_1 = M_2$, then $u_1 = u_2$ by the representation formulas. This is a contradiction and thus $M_1 < M_2$. Since F is strictly increasing, we get from (9) that $u_1'(a) < u_2'(a)$. If there were some $\tilde{t} \in (0, \frac{a+b}{2})$ such that $u_1(\tilde{t}) = u_2(\tilde{t})$, then for $\hat{t} = \sup\{t; u_1(s) < u_2(s), s \in (a,s)\}$, $u_1(\hat{t}) = u_2(\hat{t})$, $u_1'(\hat{t}) \geq u_2'(\hat{t}) > 0$. But this yields a contradiction, since F is strictly increasing and it should be $u_1'(\hat{t}) < u_2'(\hat{t})$ by (9). Thus, $u_1(t) < u_2(t), t \in (a,b)$.

In the sequel, we shall show through examples the complexity of the structure of solutions of (6). Our first one is a nice analogy of Theorem 1 for the following problem

$$\begin{cases} \Delta_p u = \lambda u^{q-1} + u^{r-1}, u(t) > 0, \quad t \in (a,b) \\ u(a) = u(b) = 0, \end{cases} \tag{10}$$

where $\lambda > 0, r > p > q > 0$ are constants, the proof, which is almost identical to that for Theorem 1, is omitted. Note that this equation is singular when $q < 1$ (see [3] for the multi-dimensional case).

Theorem 3. *There is a constant $\Lambda > 0$, which depends only on $p, q, r, b-a$, such that problem* (10) *has exactly two ordered solutions $u^-(t) < u^+(t), t \in (a,b)$, if $\lambda \in (0,\Lambda)$; and has a unique solution $u(t)$ when $\lambda = \Lambda$; and has no solutions in case $\lambda > \Lambda$. Moreover, the solution is given by*

$$u(t) = \begin{cases} P_\alpha^{-1}(t-a), & t \in [a,(a+b)/2]; \\ P_\alpha^{-1}(b-t), & t \in [(a+b)/2, b]; \end{cases}$$

where $P_\alpha(s) = \int_0^s (E_\lambda(\alpha) - E_\lambda(u))^{-1/p}\, du, E = \frac{p}{p-1}(\frac{\lambda}{q}u^q + \frac{1}{r}u^r)$, and $\alpha > 0$ is determined by $P_\alpha(\alpha) = (b-a)/2$.

Remark. We conjecture that the result in Theorem 3 remains valid even for $q \leq 0$. But our method breaks down when $q \leq 0$. Note that if $q \leq 0$, then for any solution of (10) the derivative blows up at $t = a$ and $t = b$.

To get an example with a polynomial like nonlinearity which has more than two positive solutions, we consider the following boundary value problem,

$$\begin{cases} \Delta_p u = \lambda u^\alpha (u^\alpha + 1)(u^{2\alpha} - 2\tau u^\alpha + \tau 2 + \tau 3), u(t) > 0, \quad t \in (a,b) \\ u(a) = u(b) = 0, \end{cases} \tag{11}$$

where λ, α, τ are positive constants.

Remark. The equation in (11) is a perturbation of $\Delta_p u = \lambda(u^{3\alpha} + u^{4\alpha})$, which can be transformed under a change of variable $v = \lambda^{\frac{1}{4\alpha+1-p}} u$ into an equation of type (10).

Theorem 4. *If $\alpha \in (\frac{p-1}{4}, \frac{p-1}{3})$ and τ is small, then there exist positive constants $\Lambda_1 \le \Lambda_2 < \Lambda_3 \le \Lambda_4$, which depend only on $\alpha, \tau, p, b-a$ such that problem* (11) *has exactly two positive solutions if $\lambda \in (0, \Lambda_1)$; has at least four ordered solutions if $\lambda \in (\Lambda_2, \Lambda_3)$; and has no solutions when $\lambda > \Lambda_4$.*

Proof. In view of the previous analysis, we need to study the solvability of the equation (8). By the definition of $f(u)$, we have

$$F(u) = \lambda(\frac{1}{1+4\alpha} u^{1+4\alpha} + \frac{1-2\tau}{1+3\alpha} u^{1+3\alpha} + \frac{\tau 3 + \tau 2 - 2\tau}{1+2\alpha} u^{1+2\alpha} + \frac{\tau 3 + \tau 2}{1+\alpha} u^{1+\alpha}).$$

Therefore, equation (8) becomes

$$\int_0^1 \frac{u^{\frac{p-1-\alpha}{p}} d\theta}{\sqrt[p]{\frac{1}{1+4\alpha}\theta_4 u^{3\alpha} + \frac{1-2\tau}{1+3\alpha}\theta_3 u^{2\alpha} + \frac{\tau 3+\tau 2-2\tau}{1+2\alpha}\theta_2 u^\alpha + \frac{\tau 3+\tau 2}{1+\alpha}\theta_1}} = \frac{c_p(b-a)\lambda^{1/p}}{2}, \tag{12}$$

where $\theta_i = 1 - \theta^{1+i\alpha}, i = 1,2,3,4$. Denote the left hand side function in (12) by $A(u)$, then $A(u)$ is continuous on $[0,\infty)$ and is differentiable for $u > 0$. We see also clearly that $A(u) > 0$, $u > 0$ and $A(u) \to 0$ as either $u \to 0+$ or $u \to +\infty$ and consequently $A(u)$ is bounded on $[0,\infty)$, which gives the existence of at least two solutions for small λ, and no solutions at all for large λ. To show the existence of at least four solutions, we need to study the further property of $A(u)$.

By direct differentiation, we deduce that

$$A'(u) = \frac{u^{-\frac{1+\alpha}{p}}}{p} \int_0^1 \frac{N(u,\theta,\tau,\alpha)\theta_1}{\sqrt[p]{\frac{1}{1+4\alpha}\theta_4 u^{3\alpha} + \frac{1-2\tau}{1+3\alpha}\theta_3 u^{2\alpha} + \frac{\tau 3+\tau 2-2\tau}{1+2\alpha}\theta_2 u^\alpha + \frac{\tau 3+\tau 2}{1+\alpha}\theta_1}} d\theta$$

$$= \frac{u^{-\frac{1+\alpha}{p}}}{p} \int_0^1 \frac{N(u,\theta,\tau,\alpha)}{\sqrt[p]{\frac{1}{1+4\alpha}\varphi_3 u^{3\alpha} + \frac{1-2\tau}{1+3\alpha}\varphi_2 u^{2\alpha} + \frac{\tau 3+\tau 2-2\tau}{1+2\alpha}\varphi_1 u^\alpha + \frac{\tau 3+\tau 2}{1+\alpha}}} d\mu(\theta),$$

where $\varphi_i = \frac{1-\theta^{1+(1+i)\alpha}}{1-\theta^{1+\alpha}}$, $i = 1, 2, 3$, $d\mu(\theta) = (1-\theta^{1+\alpha})^{-p} d\theta$ and

$$N(u,\theta,\tau,\alpha) = \frac{p-1-4\alpha}{1+4\alpha}\varphi_3 u^{3\alpha} + \frac{(1-2\tau)(p-1-3\alpha)}{1+3\alpha}\varphi_2 u^{2\alpha} + \frac{(\tau 3+\tau 2-2\tau)(p-1-2\alpha)}{1+2\alpha}\varphi_1 u^{\alpha} + \frac{(\tau 3+\tau 2)(p-1-\alpha)}{1+\alpha}.$$

Clearly, we have $N(0,\theta,\tau,\alpha) > 0$, $N(u,\theta,\tau,\alpha) < 0$ for large $u > 0$ due to the assumption $\alpha > (p-1)/4$. Further, we define

$$s = s_1^{1/\alpha}, \quad s_1 = s_1(\tau) = \frac{(p-1-2\alpha)(1+3\alpha)}{(p-1-3\alpha)(1+2\alpha)}\tau > 0,$$

then by an easy manipulation we get

$$N(s,\theta,\tau,\alpha) = \{\frac{(p-1-2\alpha)^2(1+3\alpha)}{(p-1-3\alpha)(1+2\alpha)^2}(\varphi_2 - 2\varphi_1) + \frac{p-1-\alpha}{1+\alpha}\}\tau 2 + o(\tau 2).$$

Furthermore, let $Q(\psi,\alpha) = -1 - \psi^{\frac{1+3\alpha}{1+\alpha}} + 2\psi^{\frac{1+2\alpha}{1+\alpha}}$, then by the definitions of φ_2, φ_1 we deduce that under the change of variables $\theta^{1+\alpha} = 1-\psi$,

$$Q(\psi,\alpha)/(1-\psi) = \varphi_2 - 2\varphi_1.$$

For the function Q, one can easily check that for any $\psi \in (0,1)$

$$\frac{\partial Q}{\partial \alpha} = \frac{2\ln\psi}{(1+\alpha)^2}(1-\psi^{\frac{\alpha}{1+\alpha}})\psi^{\frac{1+2\alpha}{1+\alpha}} < 0.$$

Therefore, $Q(\psi,\alpha) \le Q(\psi,0) = \psi - 1$ and thus $\varphi_2 - \varphi_1 \le -1$ and

$$N(s,\theta,\tau,\alpha) \le -\{\frac{(p-1-3\alpha)^2(1+3\alpha)}{(p-1-2\alpha)(1+2\alpha)^2} + \frac{p-1-\alpha}{1+\alpha}\}\tau 2 + o(\tau 2) = -\frac{p\alpha^2(2+4\alpha-p)}{(1+\alpha)(1+2\alpha)^2(p-1-3\alpha)}\tau 2 + o(\tau 2).$$

It follows that if $\max\{0, \frac{p-2}{4}\} < \alpha \le \frac{p-1}{4}$, and $\tau > 0$ is small, then there is $\delta > 0$, such that $N(s,\theta,\tau,\alpha) \le -\delta$, for all $\theta \in (0,1)$.

On the other hand, choose $s_0 > 0$ such that $s_0^\alpha = \frac{p-1-3\alpha)}{2(p-1-4\alpha)}$, then we derive from (13) that

$$N(s_0,\theta,\tau,\alpha) \ge \frac{p-1-3\alpha}{2(1+3\alpha)}s_0^{2\alpha} + O(\tau),$$

where we have used the estimates: $\varphi_2 \ge 1$, $\varphi_3/\varphi_2 \le (1+4\alpha)/(1+3\alpha)$ for $\theta \in (0,1)$. As a consequence it follows immediately that $N(s_0,\theta,\tau,\alpha) \ge \delta_1$, for some $\delta_1 > 0$, if τ is small.

Going back to the function $A'(u)$, we can state the following facts: Function $A'(u)$ has the property that $A'(u) > 0$ for either small $u > 0$, or $u = s_0$;

and $A'(u) < 0$ for $u = s$ or large $u > 0$. It follows from the continuities of $A(u)$ and $A'(u)$ that $A(u)$ has at least one local minimum on the interval (s, s_0) and two local maximums in $(0, s)$ and (s_0, ∞) respectively. If $m = \min_{s<u<s_0} A(u)$, $M = \min\{\max_{0<u<s} A(u), \max_{u>s_0} A(u)\}$, then for any $\beta \in (m, M)$ the equation $A(u) = \beta$ has at least four solutions. The existence of four solutions of (11) follows immediately and the proof is complete.

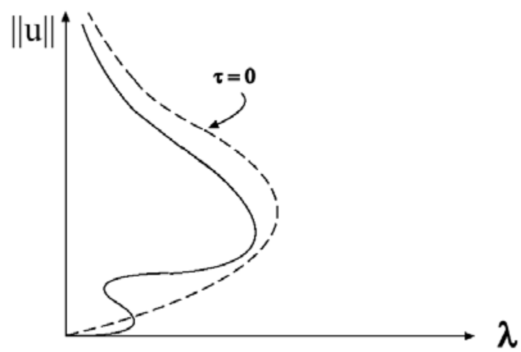

Figure 3.

If $\alpha \in (0, \frac{p-1}{4}]$, i.e., the sublinear case of (11), then in view of the expression in (13) we see clearly that $A'(u) > 0$ for u either small or big. By the proof of Theorem 4 we still have $A'(s_1) < 0$ for small $\tau > 0$, if we assume $2 + 4\alpha - p > 0$, i.e., $\alpha > (p-2)/4$. Summarizing the analysis above, we obtain:

Theorem 5. *If $\alpha \in (\max\{0, \frac{p-2}{4}\}, \frac{p-1}{4}]$ and τ is small, then there exist positive constants $\Lambda_1 < \Lambda_2$ such that problem (11) has at least three ordered solutions if $\lambda \in (\Lambda_1, \Lambda_2)$.*

Remark. If $\tau = 0$, $\alpha \in (0, \frac{p-1}{4})$, then (11) has a unique solution for all $\lambda > 0$; moreover, (11) admits always at least one solution for all $\tau, \lambda > 0$, because $\lim_{u\to 0} A(u) = 0, \lim_{u\to\infty} A(u) = \infty$.

To construct an example with arbitrarily many solutions, we consider

$$\begin{cases} -u'' = \lambda(u + \sin\sqrt{u}), \quad u(t) > 0 \quad t \in (a, b) \\ u(a) = u(b) = 0. \end{cases} \tag{14}$$

Let $\lambda_1 = \frac{\pi^2}{(b-a)^2}$, which is the first eigenvalue of Laplacian, then we have:

Theorem 6. *For any integer $r \geq 1$, there is $\delta > 0$ such that if $\lambda \in (\lambda_1 - \delta, \lambda_1 + \delta)$, then problem (14) has at least r distinct solutions.*

Proof. First note that $u+\sin\sqrt{u}>0$, for all $u>0$, therefore we can apply the previous analysis. Let

$$s(\theta,u_0)=\tfrac{1}{2}(1-\theta^2)u^2+2\{\sin\sqrt{u}-\sin\sqrt{\theta u}-\sqrt{u}(\cos\sqrt{u}-\sqrt{\theta}\cos\sqrt{\theta u})\}$$

$$S(u_0)=\int_0^1\frac{u_0d\theta}{\sqrt{s(\theta,u_0)}},$$

then $S(u_0)$ is continuous on $(0,+\infty)$ and $S(u_0)\to 0$, as $u_0\to 0^+$. Further, we get from (8) that the solvability of (14) is equivalent to the solvability of

$$S(u_0)=(b-a)\sqrt{\tfrac{\lambda}{2}}. \tag{15}$$

For a given integer $n\geq 1$, we have that for all $\theta\in[0,1)$

$$\begin{aligned} s(\theta,(2n\pi)^2)&=8(n\pi)^4(1-\theta^2)-2\{\sin(2n\pi\sqrt{\theta})+2n\pi(1-\sqrt{\theta}\cos(2n\pi\sqrt{\theta}))\}\\ &\leq 8(n\pi)^4(1-\theta^2)-2\{\sin(2n\pi\sqrt{\theta})+2n\pi(1-\sqrt{\theta})\}\\ &=8(n\pi)^4(1-\theta^2)-2\{2n\pi(1-\sqrt{\theta})-\sin(2n\pi(1-\sqrt{\theta}))\}\\ &<8(n\pi)^4(1-\theta^2), \end{aligned}$$

consequently,

$$S((2n\pi)^2)>\int_0^1\frac{(2n\pi)^2d\theta}{\sqrt{8(n\pi)^4(1-\theta^2)}}=\frac{\pi}{\sqrt{2}}.$$

Similarly, we obtain for $\theta\in[0,1)$

$$s(\theta,(2n-1)^2\pi^2)>(2n-1)^4\pi^4(1-\theta^2)/2$$

and thus $S((2n-1)^2\pi^2)<\frac{\pi}{\sqrt{2}}$.

Whence given an integer $r\geq 1$, choose $\delta=\min_{1\leq n\leq r}|\frac{2S((n\pi)^2)}{(b-a)^2}-\lambda_1|$, then we see that equation (15) has at least r distinct solutions, if $\lambda\in(\lambda_1-\delta,\lambda_1+\delta)$, and thus problem (14) has also at least r solutions. The proof is complete.

Before we proceed to the discussion of nodal solutions, we consider a nonpositone problem [2, 4],

$$\begin{cases}\Delta_p u=\lambda(u^q-u^r), & u(t)>0 \quad t\in(a,b)\\ u(a)=u(b)=0, & \lambda>0, q+1>p>r+1>0.\end{cases} \tag{16}$$

By the identity (7), we get $|u'(t)|^p=p'[F(u_0)-F(u(t))]$, where $u_0=\max_{a<t<b}u(t)>0$. Since $F(u)=\lambda(\frac{u^{q+1}}{1+q}-\frac{u^{1+r}}{1+r})$, $u(a)=0$ we see that $F(0)=0$, and $F(u_0)>0$, which implies $u_0\geq\gamma$, where γ satisfies $\gamma^{q-r}=\frac{1+q}{1+r}$.

In view of equation (8), we have that u_0 satisfies

$$\int_0^1 \frac{u_0 d\varphi}{\sqrt[p]{\varphi_1 u_0^{1+q} - \varphi_2 u_0^{1+r}}} = \frac{c_p(b-a)\lambda^{1/p}}{2}, \tag{17}$$

where $\varphi_1 = (1-\varphi^{1+q})/(1+q)$, $\varphi_2 = (1-\varphi^{1+r})/(1+r)$. Define

$$\mathcal{F}(u_0) = \int_0^1 \frac{u_0 d\varphi}{\sqrt[p]{\varphi_1 u_0^{1+q} - \varphi_2 u_0^{1+r}}},$$

then $\mathcal{F}(u_0)$ is well defined on the interval $[\gamma, \infty)$, due to the fact $\gamma^{q-r}\varphi_1 - \varphi_2 > 0, \varphi \in (0,1)$. By a direct calculation and the assumption $1+q > p > 1+r$, we get

$$\mathcal{F}'(u_0) = \int_0^1 \frac{(1-\frac{1+q}{p})\varphi_1 u_0^{1+q} - (1-\frac{1+r}{p})\varphi_2 u_0^{1+r}}{\sqrt[p]{(\varphi_1 u_0^{1+q} - \varphi_2 u_0^{1+r})^{1+p}}} d\varphi < 0.$$

Thus, $\mathcal{F}(u_0)$ is strictly decreasing on $[\gamma, \infty)$. Back to the equation (17), and choose $K > 0$, such that $K^p = 2\mathcal{F}(\gamma)/(c_p(b-a))$, we get:

Theorem 7. *Problem* (16) *has a unique solution when* $\lambda \in (0, K]$, *and has no solution for* $\lambda > K$.

Remark. If the function $f(u)$ satisfies $f(u) \geq 0$, $u > 0$, then any positive solution of (6) is obviously concave; but for the solution of (16) it is concave only on $(a+\delta, b-\delta)$; and is convex on $(a, a+\delta) \cup (b-\delta, b)$.

Finally, we sketch the discussion of nodal solutions of (6). Let us define two functions $f_+(u) = f(u)$, $f_-(u) = -f(-u)$, $u \geq 0$. Then $f_+ = f_-$ if f is odd. Throughout this part we assume that f_+, f_- are non-negative and $F_+(u), F_-(u)$ are strictly increasing on $(0, \infty)$.

Let $u(t)$ be a nodal solution of (6) with $r \geq 1$ interior nodal points, say, $a_1, \cdots a_r \in (a, b)$. Then $u(t), v(t) = -u(t)$ solve the following problems respectively

$$\begin{cases} \Delta_p u = f_+(u), \quad u(t) > 0, \quad t \in (a_{2i}, a_{2i+1}) \\ u(a_{2i}) = u(a_{2i+1}) = 0, \end{cases}$$

$$\begin{cases} \Delta_p v = f_-(v), \quad v(t) > 0, \quad t \in (a_{2i+1}, a_{2i+2}) \\ u(a_{2i+1}) = u(a_{2i+2}) = 0, \end{cases}$$

where $i = 0, 1, 2, \cdots$, $a_0 = a$, $a_{r+1} = b$. Via the previous discussion about positive solutions of (6), we obtain

$$\int_0^1 \frac{\alpha_i d\theta}{\sqrt[p]{F_+(\alpha_i) - F_+(\theta\alpha_i)}} = \frac{c_p(a_{2i+1} - a_{2i})}{2},$$

$$\int_0^1 \frac{\beta_i d\theta}{\sqrt[p]{F_-(\beta_i) - F_-(\theta\beta_i)}} = \frac{c_p(a_{2i+2} - a_{2i+1})}{2},$$

where $\alpha_i = \max_{a_{2i}<t<a_{2i+1}} u(t)$, $\beta_i = \max_{a_{2i+1}<t<a_{2i+2}} v(t)$.

On the other hand, it follows from identity (7) that $F_+(\alpha_0) = F_-(\beta_1) = F_+(\alpha_2) = F_-(\beta_3) = \cdots$. Here we have used the relations $F_+(u) = F(u), F_-(u) = F(-u)$, $u \geq 0$. Then it follows from the monotonicity of F_+, F_- that $\alpha_0 = \alpha_2 = \cdots$; $\beta_1 = \beta_3 = \cdots$; $a_1 - a_0 = a_3 - a_1 = \cdots, a_2 - a_1 = a_4 - a_3 = \cdots$. Thus, r must be finite and $u(t)$ is just a periodic extension of a ground wave. If u has $m \geq 1$ upper waves and $n \geq 1$ lower waves ($|m - n| \leq 1$), then we can have similar representations like in Theorem 3 for the upper and lower waves,

$$u(t) = \begin{cases} P_+^{-1}(c_p(t - a_{2i})), & t \in [a_{2i}, \frac{a_{2i}+a_{2i+1}}{2}]; \\ P_+^{-1}(c_p(a_{2i+1} - t)), & t \in [\frac{a_{2i}+a_{2i+1}}{2}, a_{2i+1}]; \\ -P_-^{-1}(c_p(t - a_{2i+1})), & t \in [a_{2i+1}, \frac{a_{2i+1}+a_{2i+2}}{2}]; \\ -P_-^{-1}(c_p(a_{2i+2} - t)), & t \in [\frac{a_{2i+1}+a_{2i+2}}{2}, a_{2i+2}]; \quad i = 0, 1, 2, \cdots \end{cases}$$

where

$$P_+(s) = \int_0^s (F_+(u_0) - F_+(u))^{-1/p}\, du,$$
$$P_-(s) = \int_0^s (F_-(v_0) - F_-(u))^{-1/p}\, du,$$

and $u_0 = \max u(t)$, $v_0 = -\min u(t) > 0$ are determined by the following nonlinear system,

$$m\int_0^1 \frac{u_0 d\theta}{\sqrt[p]{F_+(u_0) - F_+(\theta u_0)}} + n\int_0^1 \frac{v_0 d\theta}{\sqrt[p]{F_-(v_0) - F_-(\theta v_0)}} = \frac{c_p(b-a)}{2}$$
$$F_+(u_0) = F_-(v_0). \tag{18}$$

If $f(u)$ is an odd function, then the problem is much easier, and the above system reduces to a single equation

$$\int_0^1 \frac{\alpha d\theta}{\sqrt[p]{F(\alpha) - F(\theta\alpha)}} = \frac{c_p(b-a)}{2(m+n)}$$

and solutions of (6) are periodic and are also symmetric with respect to the middle point of interval $[a, b]$.

To illustrate a little about the non-symmetric case, let us take, for simplicity, $f_+ = \lambda u^q$, $f_- = \mu u^r$, $\lambda, \mu > 0$, $q, r > -1$. Then by solving the second equation in (18) and substituting in the first equation, we see that system (18) is equivalent to

$$c_+ m\lambda^{-\frac{1}{1+q}} x^{\frac{1}{1+q}-\frac{1}{p}} + c_- n\mu^{-\frac{1}{1+r}} x^{\frac{1}{1+r}-\frac{1}{p}} = \frac{c_p(b-a)}{2}, \quad x > 0 \tag{19}$$

where

$$c_+ = (1+q)^{\frac{1}{1+q}} \int_0^1 \frac{d\theta}{\sqrt[p]{1-\theta^{1+q}}},$$
$$c_- = (1+r)^{\frac{1}{1+r}} \int_0^1 \frac{d\theta}{\sqrt[p]{1-\theta^{1+r}}}, \frac{\lambda}{1+q} u_0^{1+q} = x = \frac{\mu}{1+r} v_0^{1+r}.$$

If $q = r = p - 1$, this leads to the eigenvalue problems and equation (19) reduces to

$$m\lambda^{-\frac{1}{p}} + n\mu^{-\frac{1}{p}} = \frac{b-a}{2c}, \tag{20}$$

$c = \int_0^1 \sqrt[p]{\frac{p}{(1-\theta^p)p'}} d\theta$. Equation (20) determines the so called Fucik spectrum curve [7] when $p = 2$, for which the problem (6) has a nontrivial solution.

If $q, r > p - 1$ (or $q, r < p - 1$), then the left hand side function in (19), denoted by $B_{mn}(x)$, is monotone and has a property $B_{mn}(0) = 0(\infty)$, $B_{mn}(\infty) = \infty(0)$. Thus (19) has a unique solution for any interval $[a, b]$. On the contrary, if $q < p - 1 < r$ or $r < p - 1 < q$, then $B_{mn}(0) = B_{mn}(\infty) = \infty$, $B_{mn}(x)$ is bounded from below by a positive constant, which implies that (19) is solvable only for large $b - a$. In other words, problem (6) in this case has nodal solutions only if the interval $[a, b]$ is sufficiently long [6] and thus has only finitely many nodal solutions.

Whence for the following boundary value problem, we have

$$\begin{cases} \Delta_p u = \lambda u_+^q - \mu u_-^r, & t \in (a, b), u_+ = \max\{u, 0\}, u_- = u_+ - u \\ u(a) = u(b) = 0. \end{cases} \tag{21}$$

Theorem 8. *Given constants $\lambda, \mu > 0, q, r > -1$, the following alternative is true*

1) *If either $q, r > p - 1$ or $q, r < p - 1$, then problem* (21) *has a "unique"* (*up to a translation*) *nodal solution with m upper waves and n lower waves for any integer $m, n \geq 0$.*

2) *If $q = r = p - 1$, then problem* (21) *has a nodal solution with m upper waves and n lower waves if and only if λ, μ satisfy the equation* (20).

3) *If either* $q < p-1 < r$ *or* $r < p-1 < q$, *then problem* (21) *has a nodal solution with* m *upper waves and* n *lower waves if and only if* $b-a \geq \Omega(m,n)$; *moreover, the solution is "unique" if* $b-a = \Omega(m,n)$, *and* (21) *has two solutions if* $b-a > \Omega(m,n)$, *where* $\Omega(m,n) = \min_{\tau>0} 2B_{mn}(\tau)/c_p$.

Finally, we consider one more example in the non-symmetrical case. Given positive constants $q, r, s, \varepsilon, q > p-1 > r, s$, we study the following nonlinear eigenvalue problem

$$\begin{cases} \Delta_p u = \lambda(u_+^q + \varepsilon u_+^s - u_-^r), & t \in (a,b), \\ u(a) = u(b) = 0. \end{cases} \tag{22}$$

Theorem 9. *For any* $\lambda > 0$ *problem* (22) *has at least one nodal solution with* m *upper waves and* n *lower waves. Moreover, there exist* $\varepsilon_0, \Upsilon > 0$, *which depend on* p, q, r, s *such that*

1) *The nodal solution is "unique", if* $\varepsilon > \Upsilon$

2) *If* $\varepsilon < \varepsilon_0$, *then for any integer* $m, n \geq 1$, *there exist positive constants* $\lambda_- < \lambda_+$, *which depend only on* $p, q, r, s, m, n, b-a$ *so that problem* (22) *has at least three nodal solutions with* m *upper waves and* n *lower waves, when* $\lambda \in (\lambda_-, \lambda_+)$.

Proof. Since $F_+(u) = \lambda(\frac{u^{q+1}}{1+q} + \varepsilon\frac{u^{s+1}}{1+s})$, $F_-(v) = \frac{\lambda}{1+r}v^{r+1}$, we see that system (18) takes the following form

$$\begin{aligned} & m\int_0^1 \frac{u_0^{1-\frac{1+s}{p}}\,d\theta}{\sqrt[p]{\tilde{\varepsilon}\theta_s + \theta_q u_0^{q-r}}} + n\tilde{c}v_0^{1-\frac{1+r}{p}} = \frac{b-a}{2}\sqrt[p]{\frac{\lambda p}{1+q}} \\ & u_0^{q+1} + \tilde{\varepsilon}u_0^{s+1} = \frac{1+q}{1+r}v_0^{r+1}, \end{aligned} \tag{23}$$

where $\tilde{\varepsilon} = \frac{1+q}{1+s}\varepsilon$, $\tilde{c} = \sqrt[p]{\frac{1+r}{1+q}}\int_0^1 \frac{d\theta}{\sqrt[p]{1-\theta^{1+r}}}$, $\theta_s = 1-\theta^{1+s}$, $\theta_q = 1-\theta^{1+q}$.

By the second equation in (23), we obtain

$$v_0 = \sqrt[1+r]{\frac{1+r}{1+q}}(u_0^{1+q} + \tilde{\varepsilon}u_0^{1+s})^{\frac{1}{1+r}},$$

and thus system (23) is equivalent to a single equation

$$m\int_0^1 \frac{u_0^{s_1}\,d\theta}{\sqrt[p]{\tilde{\varepsilon}\theta_s + \theta_q u_0^{q-r}}} + nC(u_0^{1+q} + \tilde{\varepsilon}u_0^{1+s})^{r_1} = \frac{b-a}{2}\sqrt[p]{\frac{\lambda p}{1+q}}, \tag{24}$$

where $s_1 = 1 - \frac{1+s}{p}$, $r_1 = \frac{1}{1+r} - \frac{1}{p} > 0$, $C = (\frac{1+r}{1+q})^{\frac{1}{1+r}}\int_0^1 \frac{d\theta}{\sqrt[p]{1-\theta^{1+r}}}$.

Now let us consider the function

$$\begin{aligned}E(u_0) &= m\int_0^1 \frac{u_0^{s_1}d\theta}{\sqrt[p]{\tilde{\varepsilon}\theta_s+\theta_q u_0^{q-s}}} + nC(u_0^{1+q}+\tilde{\varepsilon}u_0^{1+s})^{r_1}\\ &= m\int_0^1 \frac{u_0^{s_1}}{\sqrt[p]{\tilde{\varepsilon}\Theta+u_0^{q-s}}}d\mu(\theta) + nC(u_0^{1+q}+\tilde{\varepsilon}u_0^{1+s})^{r_1},\end{aligned}$$

where $\Theta = \frac{\theta_s}{\theta_q} = \frac{1-\theta^{1+s}}{1-\theta^{1+q}}$, $d\mu(\theta) = \frac{d\theta}{\sqrt[p]{\theta_q}}$.

First of all, $E(u_0)$ is differentiable on $(0,\infty)$ and has by the assumption, properties: $E(u_0)\to 0$, $u_0\to 0$; $E(u_0)\to+\infty$, $u_0\to+\infty$, which gives the existence of at least one solution for (24) and hence also for (21).

Further by a direct differentiation, we obtain

$$\begin{aligned}E'(u_0) &= m\int_0^1 \frac{u_0^{s_1-1}[s_1\tilde{\varepsilon}\Theta+(s_1-\frac{q-s}{p})u_0^{q-s}]}{(\tilde{\varepsilon}\Theta+u_0^{q-s})^{\frac{1+p}{p}}}d\mu(\theta)\\ &\quad + nCr_1(u_0^{1+q}+\tilde{\varepsilon}u_0^{1+s})^{r_1-1}[(1+q)u_0^q+\tilde{\varepsilon}(1+s)u_0^s].\end{aligned}$$

In view of $s_1-\frac{q-s}{p} = 1-\frac{q+1}{p} < 0$, $\Theta\in[\frac{1+s}{1+q},1]$, we see therefore that $E'(u_0)\geq 0$ on $(0,1]$, if $\tilde{\varepsilon}\geq\frac{(1+q)(q+1-p)}{(1+s)(p-1-s)}$.

On the other hand, if $u_0>1$, then we have the following estimate

$$\begin{aligned}E'(u_0) &\geq m\int_0^1 \frac{u_0^{s_1-1}(s_1-\frac{q-s}{p})u_0^{q-s}}{(\tilde{\varepsilon}\Theta+u_0^{q-s})^{\frac{1+p}{p}}}d\mu(\theta) + nC(1+s)r_1(u_0^{1+q}+\tilde{\varepsilon}u_0^{1+s})^{r_1}u_0^{-1}\\ &\geq m\int_0^1 \frac{(s_1-\frac{q-s}{p})u_0^{q+s_1-s-1}}{u_0^{\frac{(q-s)(1+p)}{p}}}d\mu(\theta) + nC(1+s)r_1(u_0^{1+q}+\tilde{\varepsilon}u_0^{1+s})^{r_1}u_0^{-1}\\ &= mu_0^{-\frac{q+1}{p}}(1-\frac{q+1}{p})\int_0^1 d\mu(\theta) + nC(1+s)r_1(u_0^{1+q}+\tilde{\varepsilon}u_0^{1+s})^{r_1}u_0^{-1}\\ &\geq \{m(1-\frac{q+1}{p})\int_0^1 d\mu(\theta) + nC(1+s)r_1\tilde{\varepsilon}^{r_1}\}u_0^{-1} > 0\end{aligned}$$

if $\tilde{\varepsilon}$ is large, say, $\tilde{\varepsilon}\geq(\frac{2(q+1-p)}{pCr_1(1+s)}\int_0^1 d\mu(\theta))^{\frac{1}{r_1}}$, where we have used the fact $|n-m|\leq 1$ and thus $m\leq 2n$. Thus, if we choose

$$\Upsilon = \max\Big\{\frac{(1+q)(q+1-p)}{(1+s)(p-1-s)},(\frac{2(q+1-p)}{pr_1C(1+s)}\int_0^1 d\mu(\theta))^{\frac{1}{r_1}}\Big\},$$

then we have $E'(u_0) > 0$ for all $u_0 > 0$. Thus equation (24) has only one solution for any $\lambda > 0$ and the proof of 1) is complete.

To show 2), we observe that

$$\int_0^1 \frac{u_0^{s_1}}{\sqrt[p]{\tilde{\varepsilon}\Theta + u_0^{q-s}}} d\mu(\theta) \to u_0^{1-\frac{1+q}{p}} \int_0^1 d\mu(\theta), \quad (u_0^{1+q} + \tilde{\varepsilon} u_0^{1+s})^{r_1} \to u_0^{(1+q)r_1}$$

on compacta, as $\tilde{\varepsilon} \to 0$. Therefore,

$$E(u_0) \to m u_0^{1-\frac{1+q}{p}} \int_0^1 d\mu(\theta) + nC u_0^{(1+q)r_1} \stackrel{\text{def}}{=} \tilde{E}(u_0), \text{ as } \tilde{\varepsilon} \to 0.$$

As we have already noted the function $\tilde{E}(u_0)$ has the property

$$\tilde{E}(u_0) \to +\infty, u_0 \to 0; \quad \tilde{E}(u_0) \to +\infty, \ u_0 \to +\infty,$$

we may assume that E_0, the minimum of $\tilde{E}(u_0)$, is achieved at the point $u_0 = \tilde{u}$, then we choose $\hat{u} \in (0, \tilde{u})$ such that $\tilde{E}(\hat{u}) \geq 10E_0$. Hence, we can choose $\varepsilon_0 > 0$ so that if $\tilde{\varepsilon} < \varepsilon_0$, $|E(u_0) - \tilde{E}(u_0)| \leq E_0$ for all $u_0 \in [\hat{u}, \tilde{u}]$. Consequently, $E(\hat{u}) \geq 9E_0$, $E(\tilde{u}) \leq 2E_0$. Finally, let $\lambda_- = \frac{1+q}{p'}(\frac{4E_0}{b-a})^p$, $\lambda_+ = \frac{1+q}{p'}(\frac{18E_0}{b-a})^p$, then we get by the continuity of $E(u_0)$ and the facts that $E(0+) = 0$, $E(+\infty) = +\infty$ if $\lambda \in (\lambda_-, \lambda_+)$, then equation (24) has one root on each interval $(0, \hat{u}), (\hat{u}, \tilde{u})$, and $(\tilde{u}, +\infty)$ respectively. The proof of 2) is also complete.

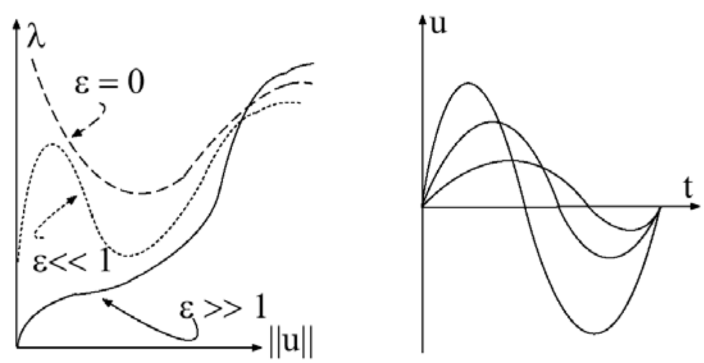

Figure 4.

Remark. The equation in (6) is invariant under transformation $s = b-a-t$, thus $u(b-a-t)$ solves (6) too, if $u(t)$ is a solution of (6). For an odd function $f(u)$, (6) is also invariant under change of variable $v = -u$, which yields another solution for (6). As we already noted, for the odd case the first invariance of the problem does not provide us more solutions, and the second invariance does. Solutions with same number of nodes must share nodes also and they are different from each other only in the amplitudes. In the non-symmetrical case like problems (21) and (23), solutions with same number of nodes have different nodes and different amplitudes. Thus the first invariance does indeed produce more solutions (see Figure 4) and our statements about "uniqueness" and "multiplicity" in Theorems 6 and 7 are up to this invariance, for instance in 2) of Theorem 7, problem (23) actually has at least six solutions.

In conclusion, we would like to remark that our problem is of 1D character. A natural question is: what is the structure of solutions of (1) and (2) in higher dimensional domains. The structures of solutions depend heavily on the geometry of the underlying domain. It would be hopeless to classify the structure of solutions for any given domain. But, we may still hope that a structure result like Theorem 1 remains valid in higher dimensional domains with certain kind of symmetry, like balls or convex domains. Finally, we should point out that many properties of the solutions of (6) are known in the literature, we have included them in our discussions just for the safety of self-contained of the paper.

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