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RICCATI OPERATORS IN NON-REFLEXIVE SPACES

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Abstract. We consider a linear quadratic regulator problem in a general Banach space. The control is by an operator with range in the extrapolated Favard class. The observation operator is unbounded. We prove the existence of a Riccati operator which describes the value function for the optimal control and can be used to synthesize optimal feedback, similarly as in Hilbert spaces.

1. Introduction

We consider the least-square optimal control problem

Problem 1.1.

$$\textit{Minimize } J(\tau, x_0, u) := \int_0^\tau |y(s)|^2 + |u(s)|^2\, ds \quad \textit{subject to}$$
$$x'(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad x(0) = x_0.$$

Here, $A : X \supset D(A) \to X$ is the infinitesimal generator of a C_0-semigroup in a non-reflexive Banach space X, $B : U \to F_{-1}$ maps a Hilbert space U into the extrapolated Favard class $F_{-1} \supset X$ (see below) of the semigroup $T(t)$, and C maps the Favard class $F \subset X$ (see below) into a Hilbert space Y. The initial value x_0 is an element of X.

Although the range of B is only contained in a space larger than X, the properties of the extrapolated Favard class guarantee the existence of a solution $x \in C([0,\tau], X)$, given a function $u \in \mathbf{L}^1([0,\tau], U)$. Since C is not

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defined on all of X, additional hypotheses will be needed to make sense of y. The existence and uniqueness of an optimal control u is then an obvious matter of quadratic optimization in the Hilbert space $\mathbf{L}^2([0,\tau],U)$. We ask whether the optimal control can be described by a state feedback

$$u(t) = F(\tau - t)x(t)$$

with a suitable operator $F(t) : X \to U$.

If the state space X is a Hilbert space, the theory of state feedback pivots around the notion of the Riccati operator, which is a self-adjoint, positive semi-definite operator $P(t) : X \to X$ for $t > 0$ with twofold importance. First, the optimal value of J is given by $\langle x_0, P(\tau)x_0\rangle$. In the language of dynamic programming, $\langle x, P(\tau - t)x\rangle$ is the value function at time t and state vector x. Second, the optimal control is given by the state feedback $u(t) = -B^*P(\tau-t)x(t)$. The Riccati operator solves (at least in a generalized sense) the Riccati differential equation

$$\frac{d}{dt}P(t) = A^*P(t) + P(t)A - P(t)BB^*P(t) + C^*C, \quad P(0) = 0.$$

If $B : U \to X$ is a bounded operator (plus some minor assumptions on C), it is known that such a Riccati operator exists. It is not true, that state feedback exists for general unbounded C and for $B(U) \not\subset X$. For a detailed analysis of the general situation in Hilbert spaces we refer to [11], [12], and [15].

Why should Problem 1.1 be considered in more generality than in Hilbert spaces? Such a setting is necessarily a hybrid, since the control u and the observation y need to live in Hilbert spaces anyhow, to make sense of the cost function. The main type of problems we have in mind are equations with some kind of memory, taking values in Hilbert spaces, frequently just in $\mathbb{R}^n$. Typical equations of this type are delay equations, Volterra integro-differential equations, or models of age-structured populations. Semigroup settings can be applied to such equations by blowing up the state space, working with C_0 semigroups in spaces like $C([-r,0],\mathbb{R}^n)$ or $\mathbf{L}^1([-r,0],\mathbb{R}^n)$. In many examples of this type we encounter the situation that B, though not bounded from U into X, has its range contained in a very small superspace of X called the extrapolated Favard class. This is the case for all examples which can be handled by the settings using Greiner's estimate on the Dirichlet operator [9], the $X^{\odot}$-approach (e.g., [2]), or Hille-Yosida-operators (e.g., [13]).

This paper is a successor of the paper [3], where we have treated the same problem restricted to bounded operators $C : X \to Y$. The unboundedness of C, however, implies some technicalities, so that the method of the proof is entirely different in this paper. We will show that Problem 1.1 admits a Riccati operator $P(t)$ (which now has to map X into a subset of X^*, so that $\langle P(t)x, x\rangle$ makes sense). Again, $\langle P(T)x_0, x_0\rangle$ is the minimal cost to be achieved starting from the initial value x_0, and $u(t) = -B^*P(T-t)x(t)$ is the state feedback synthesis generating the optimal control.

In Section 2, we give a short overview over the notations and basic facts used about the extrapolated Favard class. In Section 3, we set up our assumptions. Starting from the basic hypotheses that the range of B is contained in the extrapolated Favard class and that the observer C is at least defined on the Favard class with measurable output, we find that the contribution of the control function to the cost functional is always finite. Thus, in order to achieve that any initial state has at least one control with finite cost, we are led into the assumption that each initial value has output in $\mathbf{L}^2([0,T], Y)$ if no forcing is applied. Our assumptions resemble therefore closely the assumptions of a Pritchard-Salamon system in Hilbert spaces [11, 15]. In Section 4, we state the main theorem and set up an approximation of C by bounded observers $C_\lambda = \lambda C(\lambda - A)^{-1}$. The proof of the main theorem in Section 5 is by convergence of the Riccati operators of the approximating problems. Probably the key observation in this paper is that $\langle x_0, P_\lambda(t)x_0\rangle$ depends monotonically on λ except for a small error term. As an example we consider a boundary control problem for the heat equation in $\mathbf{L}^1([0,1], \mathbb{R})$ in Section 6.

2. Semigroup matters

In this section we review some notions from semigroup theory. Let X be a Banach space. Throughout the paper, $T(t)$ will be a C_0-semigroup generated by A on X. $D(A)$ will be the domain of A, equipped with the graph norm of A. There exists an estimate $|T(t)x| \le ce^{\omega t}|x|$. Let $\rho > \omega$.

The Favard class F ([1], [8], [14]) is the space of $x \in X$ such that $T(t)x$ is Lipschitz continuous on compact intervals. Either of the following equivalent norms turns F into a Banach space:

$$|x|_F := |x| + \sup_{t>0} \frac{1}{t}|e^{-\omega t}T(t)x - x|, \quad \|x\|_F := |x| + \sup_{\lambda>\rho} \lambda|(\lambda - A)^{-1}x|.$$

$D(A)$ is a closed subspace, isometrically embedded into F. If X is reflexive, $D(A) = F$ (so that the results of our paper are interesting only in non-reflexive state spaces).

The norm $|x|_{-1} := |(\rho - A)^{-1}x|$ is weaker than $|x|$. The completion of X under $|x|_{-1}$ is denoted by X_{-1} ([8]). In fact, up to equivalence of norms, this space is independent of the choice of $\rho > \omega$. The state space X is densely embedded into X_{-1}. The semigroup $T(t)$ can be continuously extended to a C_0-semigroup $T_{-1}(t)$ on X_{-1}, whose generator $A_{-1} : X \to X_{-1}$ is an extension of A. The resolvent $(\rho - A_{-1})^{-1}$ maps isometrically X_{-1} into X. It is the continuous extension of $(\rho - A)^{-1} : X \to D(A)$.

By the extrapolated Favard class we mean the space $F_{-1} = (\rho - A_{-1})(F)$. It is the same as the Favard class of $T_{-1}(t)$ in X_{-1}. If $f \in \mathbf{L}^1([0,T], F_{-1})$, then $x(t) = \int_0^t T_{-1}(t-s)f(s)\,ds$ is in $C([0,T], X)$ and can be regarded as a mild solution of the problem $x'(t) = Ax(t) + f(t)$. In fact, this property can be used to characterize the extrapolated Favard class. Notice a fine point in the interpretation of the integral: $T_{-1}(t)$ is not a C_0-semigroup on the invariant space F_{-1} equipped with the extrapolated Favard class norm, therefore, $T_{-1}(t)x$ is not even measurable with respect to this norm. However, the integral can be taken in the Banach space X_{-1}, where $T_{-1}(t)$ is strongly continuous. There exists an estimate

$$|\int_0^t T_{-1}(t-s)f(s)\,ds| \le ce^{\omega t}|f|_{L^1([0,t],F_{-1})}. \tag{2.1}$$

If $W(t) : X \to F_{-1}$ is a bounded linear operator such that $W(t)x$ is continuous in t, and $x_0 \in X$, then the equation

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)W(s)x(s)\,ds$$

admits a unique fixed point by contraction, which can be viewed as a mild solution to the problem $x'(t) = Ax(t) + W(t)x(t)$ ([6], though in a slightly different formulation.)

X^* denotes the dual space of X, and $X^{\odot} = \{x \in X^* : \lim_{t\to 0+} T^*(t)x = x\}$ is the largest subspace where $T^*(t)$ forms a strongly continuous semigroup, isometrically embedded in X^*. If X is reflexive, then $X^{\odot} = X^*$. In any case, $X^{\odot}$ is a subspace of F_{-1}^*. In the sun-reflexive ([2]) case, $X^{\odot *} = F_{-1}$ ([14]).

3. HYPOTHESES

Hypothesis 3.1. *Let $T(t)$ be a C_0-semigroup generated by A in a Banach space X. Let U and Y be Hilbert spaces. $B : U \to F_{-1}$ and $C : F \to Y$ are bounded linear operators.*

Remark 3.2. Hypothesis 3.1 implies that for every $x_0 \in X$, $u \in \mathbf{L}^1([0,\tau],U)$ the function $x(.;x_0,u)$ defined by

$$x(t;x_0,u) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)\,ds$$

is contained in $C([0,\tau],X)$.

Since $T(t)$ is not measurable as a semigroup on F, the following is not a consequence of Hypothesis 3.1:

Hypothesis 3.3. *For all $x \in F$, the map $t \to CT(t)x$ is measurable.*

Lemma 3.4. *Assume Hypotheses* 3.1 *and* 3.3. *Let $f \in \mathbf{W}^{1,1}([0,T],\ F_{-1})$. Then the function*

$$x(t) = \int_0^t T_{-1}(t-s)f(s)\,ds$$

takes values in F. The function $Cx(.)$ is measurable with values in Y.

Proof. To show that $x(t) \in F$, we compute

$$(T(h)-1)x(t) = \int_0^t T_{-1}(t+h-s)f(s)\,ds - \int_0^t T_{-1}(t-s)f(s)\,ds$$
$$= \int_{-h}^{t-h} T_{-1}(t-s)f(s+h)\,ds - \int_0^t T_{-1}(t-s)f(s)\,ds$$
$$= \int_{-h}^{0} T_{-1}(t-s)f(s+h)\,ds - \int_{t-h}^t T_{-1}(t-s)f(s)\,ds$$
$$+ \int_0^{t-h} T_{-1}(t-s)(f(s+h)-f(s))\,ds.$$

Using (2.1), we obtain

$$|(T(h)-1)x(t)|$$
$$\leq c\|f\|_{L^1([0,h],F_{-1})} + c\|f\|_{L^1([t-h,t],F_{-1})} + c\|f(.)-f(.-h)\|_{L^1([0,t-h],F_{-1})}$$
$$\leq 2ch \sup_{s\in[0,t]} |f(s)| + ch\|f'(.)\|_{L^1([0,t],F_{-1})} = O(h).$$

To show that $Cx(.)$ is measurable as an F-valued function, we introduce a partition $0 = t_0 < t_1 < \cdots < t_n = T$ and approximate x by

$$\tilde{x}(t) = \int_0^{t_k} T_{-1}(t-s)f(s)\,ds = T(t-t_k)\int_0^{t_k} T_{-1}(t_k-s)f(s)\,ds$$

if $t_k \le t < t_{k+1}$. The last integral is constant on each interval $[t_k, t_{k+1})$, so that Hypothesis 3.3 implies that $C\tilde{x}$ is measurable. Since

$$\|x(t) - \tilde{x}(t)\|_F = \| \int_{t_k}^{t} T_{-1}(t-s) f(s)\, ds \|_F \le c \|f\|_{W^{1,1}([t_k, t_{k+1}], F_{-1})}$$

we infer that $\tilde{x}$ converges to x uniformly on $[0, T]$ with respect to the Favard class norm, when the mesh size $\max(t_{k+1} - t_k)$ goes to zero. Taking limits we infer that Cx is measurable. □

To obtain wellposedness of the optimal control problem we have to assume that both terms in the variation-of-parameters formula give rise to an $\mathbf{L}^2$-observation. We make this assumption in conjunction with a uniform estimate. Notice that by Hypotheses 3.1, 3.3, the expressions on the right-hand-sides in the following hypothesis are well-defined.

Hypothesis 3.5. *For all $\tau > 0$, there exists $c > 0$ such that for all $x_0 \in F$ and $u \in \mathbf{W}^{1,2}([0, \tau], U)$*

$$\|CT(.)x_0\|_{L^2([0,\tau],Y)} \le c|x_0|,$$
$$\|C(T_{-1} \star Bu)\|_{L^2([0,\tau],Y)} \le c\|u\|_{L^2([0,\tau],U)}.$$

Remark 3.6. Let $\rho > \omega$. By $\mathbf{L}^2_\rho([0, \infty), X)$ we denote the space of X-valued functions such that $e^{-\rho\cdot} f(\cdot) \in \mathbf{L}^2([0, \infty), X)$. Assume that Hypotheses 3.1, 3.3, and 3.5 hold. Then there exists a continuous linear operator

$$\begin{cases} X \times \mathbf{L}^2_\rho([0,\infty), U) & \to \mathbf{L}^2_\rho([0,\infty), Y), \\ (x_0, u) & \mapsto Cx(.; x_0, u) \\ & \text{if } x_0 \in F,\ u \in \mathbf{W}^{1,1}([0,\infty), U) \cap \mathbf{L}^2_\rho([0,\infty), U). \end{cases}$$

In particular, we have the estimate

$$\|Cx(.; x_0, u)\|^2_{L^2_\rho([0,\infty),Y)} \le c(|x_0|^2 + \|u\|^2_{L^2_\rho([0,\infty),U)}). \tag{3.1}$$

By abuse of notation, we denote by $Cx(.; x_0, u)$ the value of this operator, even if $x_0 \in X$ and $u \in \mathbf{L}^2_\rho([0, \infty), U)$ without further regularity.

Proof. This is a straightforward consequence of the hypotheses above with a standard use of the exponential growth of $T(t)$. □

4. APPROXIMATING PROBLEMS AND MAIN RESULT

We will construct the Riccati operator for our problem as the limit case for approximating problems:

Problem 4.1. *With $\lambda > \omega$ and $x \in F_{-1}$ let $C_\lambda x = \lambda C(\lambda - A_{-1})^{-1}x$. Consider the optimal control problem*

$$\text{Minimize } J_\lambda(\tau, x_0, u) := \int_0^\tau |y(s)|^2 + |u(s)|^2\, ds \quad \text{subject to}$$
$$x'(t) = Ax(t) + Bu(t), \quad y(t) = C_\lambda x(t), \quad x(0) = x_0.$$

Since the observer C_λ in the approximating problem above is bounded, we may apply [3] and obtain:

Proposition 4.2. *There exists a unique family $\{P_\lambda(t) : t > 0\}$ of bounded linear operators $X \to X^\odot$ satisfying the following properties:*

(i) *$P_\lambda(t)$ is symmetric, i.e., $\langle P_\lambda(t)x, y\rangle = \langle P_\lambda(t)y, x\rangle$ for all $x, y \in X, t \geq 0$;*
(ii) *$P_\lambda(.)$ is strongly continuous;*
(iii) *P_λ satisfies the integral Riccati equation, i.e., for all $x, y \in X$ and $t \geq 0$*

$$\langle P_\lambda(t)x, y\rangle = \int_0^t \langle C_\lambda T(s)x, C_\lambda T(s)y\rangle\, ds - \int_0^t \langle B^\star P_\lambda(\tau - s)T(s)x, B^\star P_\lambda(\tau - s)T(s)y\rangle\, ds.$$

Moreover, $P_\lambda(.)$ has also the following properties:

(iv) *For all $t > 0, x_0 \in X$ and $u \in \mathbf{L}^2(0, t; U)$,*

$$\langle P_\lambda(t)x_0, x_0\rangle + \int_0^t |u(s) + B^\star P_\lambda(t - s)x(s; x_0, u)|^2\, ds = \int_0^t |C_\lambda x(s; x_0, u)|^2\, ds + \int_0^t |u(s)|^2\, ds.$$

(v) *For all $t > 0$, the functional $J_\lambda(t, x_0, u)$ has a minimum over all admissible controls u and*

$$\langle P_\lambda(t)x_0, x_0\rangle = \min_{u \in \mathbf{L}^2([0,t],U)} J_\lambda(t, x_0, u).$$

The unique minimizer $\hat{u}$ is determined by the feedback law $\hat{u}(s) = -B^\star P_\lambda(t - s)\hat{x}(s)$, where $\hat{x}$ is a solution of the equation $x'(s) = (A - BB^\star P_\lambda(t - s))x(s)$, $x(0) = x_0$.

The following is the main result of our paper:

Theorem 4.3. *Assume Hypotheses* 3.1, 3.3, *and* 3.5. *For $\lambda > \omega$ let $P_\lambda(t)$ denote the Riccati operator of the approximating problem* 4.1. *Then*

(i) *For any* $x_0 \in X$ *and for* $\lambda \to \infty$, $P_\lambda(.)x_0$ *converge to some* $P(.)x_0$ *locally uniformly on* $[0,\infty)$.
(ii) $P(t) \in \mathcal{L}(X, X^\odot)$ *for all* $t \geq 0$, *and* $P(.)x_0$ *is continuous on* $[0,\infty)$.
(iii) *For every* $x_0 \in X$, *the optimal control problem* 1.1 *on any interval* $[0,\tau]$ *has a unique minimizer* $\hat{u}$ *which is given by the feedback law*

$$\hat{u}(t) = -B^* P(\tau - t)\hat{x}(t; x_0), \tag{4.1}$$

where $\hat{x}(t; x_0)$ *is the trajectory of the optimal solution, given by the unique mild solution of the evolution equation*

$$x'(t) = [A_{-1} - BB^* P(\tau - t)]x(t), \quad x(0) = x_0. \tag{4.2}$$

(iv) $P(.)$ *satisfies the Riccati integral equation*

$$\begin{aligned}\langle P(t)y, z\rangle = &\int_0^t \langle CT(s)y, CT(s)z\rangle\, ds \\ &- \int_0^t \langle B^* P(t-s)T(s)y, B^* P(t-s)T(s)z\rangle\, ds\end{aligned} \tag{4.3}$$

for all $y, z \in X$, $t \geq 0$.

5. Proof of the main theorem

Throughout this section we assume that Hypotheses 3.1, 3.3, and 3.5 are satisfied.

5.1. Basic properties of C_λ and $P_\lambda(t)$.

We collect some straightforward consequences of the above hypotheses.

Lemma 5.1. *For all* $\lambda > \omega$, $\mu > \omega$, $\mu \neq \lambda$,

$$C_\lambda = \frac{\lambda}{\mu} C_\mu + \lambda\big(1 - \frac{\lambda}{\mu}\big) C_\mu (\lambda - A)^{-1}. \tag{5.1}$$

Proof. We apply λC to the resolvent identity

$$(\lambda - A)^{-1} = (\mu - A)^{-1} + (\mu - \lambda)(\mu - A)^{-1}(\lambda - A)^{-1}.$$

Lemma 5.2. *Fix* $x_0 \in X$ *and* $\tau > 0$.

(i)

$$\lim_{\lambda \to \infty} C_\lambda T(.)x_0 = Cx(.; x_0, 0) \text{ in } \mathbf{L}^2([0,\tau], Y).$$

In particular, the net $(C_\lambda T(.)x_0)_{\lambda > \rho}$ *is compact in* $\mathbf{L}^2([0,\tau], Y)$.

(ii)

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \|C_\lambda\|_{\mathcal{L}(X,Y)} = 0.$$

Proof. (i) follows from the fact that $C_\lambda T(t)x_0 = Cx(t;\lambda(\lambda - A)^{-1}x_0, 0)$ and Hypothesis 3.5. To prove (ii), observe that

$$|C_\lambda x_0|^2 = |C\int_0^\infty \lambda e^{-\lambda t} T(t)x_0\,dt|^2 \le [\int_0^\infty \lambda^2 e^{-2(\lambda-\rho)t}\,dt]$$
$$[\int_0^\infty |e^{-\rho t} CT(t)x_0|^2\,dt] = \frac{\lambda^2}{2(\lambda-\rho)} \|Cx(.;x_0,0)\|^2_{L^2_\rho([0,\infty),Y)}.$$

The desired result follows from Remark 3.6. □

The following lemmas establish the boundedness and equicontinuity of P_λ. We start with a variant of the Cauchy-Schwarz inequality:

Lemma 5.3. *Let $A \in \mathcal{L}(X, X^\star)$ define a real symmetric bilinear form $\langle Ax, y\rangle$ on $X \times X$. Suppose there exist non-negative constants a, b such that the inequalities*

$$-a|x|^2 \le \langle Ax, x\rangle \le b|x|^2$$

hold for all $x \in X$. Then the estimate

$$|Ax| \le \sqrt{a+b}\,\left[\langle Ax, x\rangle + a|x|^2\right]^{1/2} + a|x|$$

is true for all $x \in X$.

Proof. Choose some $x, y \in X$. From the quadratic inequality

$$\langle A(x+ty), x+ty\rangle + a|x+ty|^2 \ge 0$$

which holds for all $t \in \mathbb{R}$, we infer

$$[\langle Ax, y\rangle + a|x|\,|y|]^2 \le [\langle Ax, x\rangle + a|x|^2]\,[\langle Ay, y\rangle + a|y|^2].$$

The statement now follows as $|Ax| = \sup_{|y|=1} |\langle Ax, y\rangle|$.

Lemma 5.4. *Fix $\rho > \omega$. The Riccati operators of the approximating problem satisfy*

(i) $\|P_\lambda(t)\|_{\mathcal{L}(X,X^\odot)}$ *is uniformly bounded for t in compact intervals and $\lambda \in [\rho, \infty)$.*
(ii) *If $0 \le s \le t$, then $\langle P_\lambda(s)x_0, x_0\rangle \le \langle P_\lambda(t)x_0, x_0\rangle$.*
(iii) *For any $x_0 \in X$, $\tau > 0$, the family $\{q_\lambda : \lambda \in [\rho, \infty)\}$ of functions*

$$q_\lambda : \begin{cases} [0,\tau] & \to \mathbb{R}, \\ t & \mapsto \langle P_\lambda(t)x_0, x_0\rangle \end{cases}$$

is equicontinuous.

Proof. To prove (i), notice that $\langle P_\lambda(t)x_0, x_0\rangle$ is the optimal value for $J_\lambda(t, x_0, u)$. Therefore, using the control $u = 0$ and utilizing (3.1), we obtain

$$\langle P_\lambda(t)x_0, x_0\rangle \leq \|C_\lambda x(., x_0, 0)\|^2_{L^2([0,t],Y)}$$
$$= \|Cx(., \lambda(\lambda - A)^{-1}x_0, 0)\|^2_{L^2([0,t],Y)} \leq c|x_0|^2.$$

Lemma 5.3 above implies that $P_\lambda(t)$ is uniformly bounded.

To prove (ii), let u be a control minimizing $J_\lambda(t, x_0, u)$. Then

$$\langle P_\lambda(s)x_0, x_0\rangle \leq J_\lambda(s, x_0, u) \leq J_\lambda(t, x_0, u) = \langle P_\lambda(t)x_0, x_0\rangle.$$

To prove (iii), choose $0 \leq s \leq t \leq \tau$ and put $t - s = h$. Let $\tilde{u}$ be a λ-optimal control on the interval $[0, s]$ for the initial condition $T(h)x_0$. Then

$$\langle P_\lambda(s)T(h)x_0, T(h)x_0\rangle = \int_0^s |C_\lambda x(\sigma; T(h)x_0, \tilde{u})|^2\, d\sigma + \int_0^s |\tilde{u}(\sigma)|^2\, d\sigma.$$

Let $u(\sigma) = 0$ for $\sigma \in [0, h]$ and $u(\sigma) = \tilde{u}(\sigma - h)$ for $\sigma \in (h, t]$. Then

$$\langle P_\lambda(t)x_0, x_0\rangle \leq \int_0^t |C_\lambda x(\sigma; x_0, u)|^2\, d\sigma + \int_0^t |u(\sigma)|^2\, d\sigma$$
$$= \int_0^h |C_\lambda T(\sigma)x_0|^2\, d\sigma + \int_0^s |C_\lambda x(\sigma; T(h)x_0, \tilde{u})|^2\, d\sigma + \int_0^s |\tilde{u}(\sigma)|^2\, d\sigma$$
$$= \int_0^h |C_\lambda T(\sigma)x_0|^2\, d\sigma + \langle P_\lambda(s)T(h)x_0, T(h)x_0\rangle.$$

This implies that

$$|\langle P_\lambda(t)x_0, x_0\rangle - \langle P_\lambda(s)x_0, x_0\rangle| = \langle P_\lambda(t)x_0, x_0\rangle - \langle P_\lambda(s)x_0, x_0\rangle$$
$$\leq \int_0^h |C_\lambda T(\sigma)x_0|^2\, d\sigma + \langle P_\lambda(s)T(h)x_0, T_h(x_0)\rangle - \langle P_\lambda(s)x_0, x_0\rangle$$
$$\leq \int_0^h |C_\lambda T(\sigma)x_0|^2 d\sigma + |\langle P_\lambda(s)[T(h)x_0 - x_0], T(h)x_0\rangle|$$
$$+ |\langle P_\lambda(s)x_0, T(h)x_0 - x_0\rangle|.$$

The net $(C_\lambda T(.)x_0)_{\lambda \geq \lambda_0}$ is compact in $\mathbf{L}^2(0, \tau; Y)$ (Lemma 5.2) and $||P_\lambda(.)||_{\mathcal{L}(X, X^\odot)}$ is uniformly bounded. Therefore, for any $\epsilon > 0$ there is $\delta > 0$ such that

$$|\langle P_\lambda(t)x_0, x_0\rangle - \langle P_\lambda(s)x_0, x_0\rangle| < \epsilon$$

for all $\lambda \geq \rho$, $|t - s| < \delta$.

5.2. **Asymptotic monotonicity of P_λ with respect to λ.**

The following two lemmas are the key observation in this paper. They imply that $\langle P_\lambda x_0, x_0\rangle$ is monotonically increasing in λ except for a small error term.

Lemma 5.5. *For any $\tau > 0$, $\epsilon > 0$, $0 < \delta < \tau$, there exists λ_1, such that for all $\mu \geq \lambda \geq \lambda_1$, $x_0 \in X$, $u \in \mathbf{L}^2([0,\tau],U)$, $t \in [0,\tau-\delta]$ the following estimate holds:*

$$\int_0^t |C_\lambda x(s;x_0,u)|^2 ds \leq \int_0^{t+\delta} |C_\mu x(s;x_0,u)|^2 ds + \epsilon(|x_0|^2 + \|u\|_{L^2}^2). \quad (5.2)$$

Proof. We extend $u \in \mathbf{L}^2([0,\tau],U)$ by 0 to the interval $[\tau,+\infty)$. For the sake of simplicity we will use the notation $x(.)$ instead of $x(.;x_0,u)$. We fix $\rho > \omega$. For $\mu > \lambda > \lambda_1 > \rho$ we have the following estimates:

$$(\lambda - A)^{-1}x(t) = \int_0^\infty e^{-\lambda s}T(s)\big[T(t)x_0 + \int_0^t T_{-1}(t-\sigma)Bu(\sigma)\,d\sigma\big]\,ds$$

$$= \int_0^\infty e^{-\lambda s}T(t+s)x_0\,ds$$

$$+ \int_0^\infty e^{-\lambda s}\Big[\Big(\int_0^{t+s} - \int_t^{t+s}\Big)T_{-1}(t+s-\sigma)Bu(\sigma)\,d\sigma\Big]ds$$

$$= \int_0^\infty e^{-\lambda s}x(t+s)\,ds - \int_t^\infty e^{-\lambda(\sigma-t)}\int_0^\infty e^{-\lambda\xi}T_{-1}(\xi)Bu(\sigma)\,d\xi\,d\sigma$$

$$= \int_0^\delta e^{-\lambda s}x(t+s)\,ds + \int_\delta^\infty e^{-\lambda s}x(t+s)\,ds$$

$$- \int_t^\infty e^{-\lambda(\sigma-t)}(\lambda - A_{-1})^{-1}Bu(\sigma)\,d\sigma.$$

Substituting this expression into the formula (5.1) we obtain

$$\begin{aligned} C_\lambda x(t) &= \frac{\lambda}{\mu}C_\mu x(t) + \Big(1 - \frac{\lambda}{\mu}\Big)\int_0^\delta \lambda e^{-\lambda s}C_\mu x(t+s)\,ds \\ &+ \Big(1 - \frac{\lambda}{\mu}\Big)\int_\delta^\infty \lambda e^{-\lambda s}C_\mu x(t+s)\,ds \\ &- \lambda(\mu-\lambda)C\int_t^\infty e^{-\lambda(\sigma-t)}(\mu - A)^{-1}(\lambda - A_{-1})^{-1}Bu(\sigma)\,d\sigma \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

These terms can be estimated as follows:

$$\|I_1\|_{L^2([0,t],Y)} \leq \frac{\lambda}{\mu}\|C_\mu x(.)\|_{L^2([0,t+\delta],Y)};$$

$$|I_2|^2_{L^2([0,t];Y)} = \Big(1-\frac{\lambda}{\mu}\Big)^2 \int_0^t \Big|\int_0^\delta \lambda e^{-\lambda s} C_\mu x(\xi+s)\,ds\Big|^2 d\xi$$
$$\le \Big(1-\frac{\lambda}{\mu}\Big)^2 \int_0^\delta \lambda e^{-\lambda s} \|C_\mu x(.)\|^2_{L^2([0,t+\delta],Y)}\,ds \quad \text{(Jensen inequality)}$$
$$\le \Big(1-\frac{\lambda}{\mu}\Big)^2 \|C_\mu x(.)\|^2_{L^2([0,t+\delta],Y)}\,.$$

The next integral is estimated using (3.1):

$$\|I_3\|^2_{L^2([0,t],Y)} \le \int_0^t \big[\int_\delta^\infty \lambda e^{-\lambda s}|C_\mu x(\xi+s)|\,ds\big]^2\,d\xi$$
$$\le \int_0^t e^{2\rho\xi}\big[\int_\delta^\infty \lambda^2 e^{-2(\lambda-\rho)s}\,ds\big]\big[\int_{\delta+\xi}^\infty |e^{-\rho\sigma}C_\mu x(\sigma)|^2 d\sigma\big]\,d\xi$$
$$\text{(Hölder inequality)} \le c\frac{\lambda^2}{\lambda-\rho}e^{-2\lambda\delta}\big[|x_0|^2 + \|u\|^2_{L^2([0,\tau],U)}\big].$$

For the last integral we have

$$|I_4(s)|_Y \le ||C_\lambda||_{\mathcal{L}(X,Y)} \int_s^\infty e^{-\lambda(\sigma-s)}|\mu(\mu-A_{-1})^{-1}Bu(\sigma)|_X\,d\sigma$$
$$\le c||C_\lambda|| \int_s^\infty e^{-\lambda(\sigma-s)}|(\rho-A_{-1})\mu(\mu-A_{-1})^{-1}Bu(\sigma)|_{X_{-1}}\,d\sigma$$
(since $\rho - A_{-1}$ is an isomorphism of X onto X_{-1})
$$\le c||C_\lambda||\,||B||_{\mathcal{L}(U,F_{-1})}e^{\rho s}\int_0^\infty e^{-(\lambda-\rho)\xi}|e^{-\rho(\xi+s)}u(\xi+s)|\,d\xi$$
(since $\|A_{-1}\mu(\mu-A_{-1})^{-1}\|_{\mathcal{L}(F_{-1},X_{-1})}$ is uniformly bounded.)

Therefore,

$$\|I_4\|^2_{L^2([0,t],Y)} \le c||C_\lambda||^2 \int_0^t e^{2\rho s}\Big[\int_0^\infty e^{-(\lambda-\rho)\xi}|e^{-\rho(s+\xi)}u(s+\xi)|d\xi\Big]^2 ds$$
$$\le c||C_\lambda||^2 \int_0^t \frac{1}{(\lambda-\rho)^2}e^{2\rho s}\int_0^\infty (\lambda-\rho)e^{-(\lambda-\rho)\xi}|e^{-\rho(s+\xi)}u(s+\xi)|^2 d\xi\,ds$$
(Jensen inequality)
$$\le c\frac{||C_\lambda||^2}{\lambda^2}\frac{\lambda^2 e^{2\rho t}}{(\lambda-\rho)^2}\int_0^\infty (\lambda-\rho)e^{-(\lambda-\rho)\xi}\int_0^t |e^{-\rho(s+\xi)}u(s+\xi)|^2\,ds d\xi$$
$$\le c\frac{||C_\lambda||^2}{\lambda^2}\|u\|^2_{L^2([0,\tau],U)}.$$

The assertion of the lemma follows now from these estimates. □

Lemma 5.6. *For any $\epsilon > 0$, $\tau > 0$, $\delta \in (0, \tau)$ there exists some $\lambda_1 > \omega$ such that the inequality*

$$\langle P_\lambda(t)x_0, x_0\rangle \leq \langle P_\mu(t+\delta)x_0, x_0\rangle + \epsilon|x_0|^2 \tag{5.3}$$

holds for all $\mu \geq \lambda \geq \lambda_1$, $t \in [0, \tau - \delta]$, and $x_0 \in X$.

Proof. Given $\tau > 0$, $\epsilon > 0$, and $\delta > 0$, choose λ_1 according to Lemma 5.5. Let $\lambda_1 < \lambda < \mu$, and let $\tilde{u}$ be a μ-optimal control on the interval $[0, t+\delta]$ for the initial condition x_0. Notice that by optimality and (3.1)

$$\begin{aligned}\|\tilde{u}\|^2_{L^2([0,t+\delta],U)} &\leq J(t+\delta, x_0, \tilde{u}) \leq J(t+\delta, x_0, 0) \\ &= \|Cx(.; x_0, 0)\|^2_{L^2([0,t+\delta],Y)} \leq c|x_0|^2.\end{aligned}$$

Define $\tilde{u}(s) = 0$ for $s \in (t+\delta, \tau]$. Then, by (5.2),

$$\begin{aligned}\langle P_\lambda(t)x_0, x_0\rangle &\leq \int_0^t |C_\lambda x(s; x_0, \tilde{u})|^2 ds + \int_0^t |\tilde{u}(s)|^2\, ds \\ \leq &\int_0^{t+\delta} |C_\mu x(s; x_0, \tilde{u})|^2\, ds + \epsilon|x_0|^2 + (1+\epsilon)\int_0^{t+\delta} |\tilde{u}(s)|^2\, ds \\ = &\langle P_\mu(t+\delta)x_0, x_0\rangle + \epsilon|x_0|^2 + \epsilon\|\tilde{u}\|^2_{L^2([0,t+\delta],U)}.\end{aligned}$$

5.3. **Completion of the proof of Theorem 4.3.**

We are now in the position to finish up the proof of the main theorem:

Proof of Theorem 4.3, (i) and (ii). Let $\tau > 0$. We fix $x_0 \in X$ and $\rho > \omega$. We write $\gamma_\lambda(t) = \langle P_\lambda(t)x_0, x_0\rangle$. By Lemma 5.4[(i)], $\gamma_\lambda(t)$ is uniformly bounded for $\lambda \geq \rho$ and $t \in [0, \tau]$. Therefore, $\gamma(t) = \limsup_{\lambda\to\infty} \gamma_\lambda(t)$ is uniformly bounded for $t \in [0, \tau]$. From Lemmas 5.6 and the equicontinuity of γ_λ (Lemma 5.4[(iii)]) we obtain that for any $\eta > 0$ there exist $\delta > 0$ and λ_1, such that for $\lambda_1 < \lambda < \mu$, $t \in [0, \tau]$,

$$\gamma_\mu(t) - \gamma_\lambda(t) = (\gamma_\mu(t) - \gamma_\mu(t+\delta)) + (\gamma_\mu(t+\delta) - \gamma_\lambda(t)) \geq -\eta.$$

Consequently, the limit superior of γ_λ equals the limit inferior, i.e.,

$$\gamma(t) = \lim_{\lambda\to\infty} \gamma_\lambda(t).$$

By equicontinuity of the family γ_λ, the limit is uniform in $t \in [0, \tau]$ and γ is continuous.

For each fixed $\delta > 0$, $t \in [0, \tau - \delta]$, we use Lemma 5.4[ii] and Lemma 5.3 with $A = P_\lambda(t+\delta) - P_\lambda(t)$ to obtain

$$|P_\lambda(t+\delta)x_0 - P_\lambda(t)x_0| \leq \sqrt{\|P_\lambda(t+\delta)\|_{\mathcal{L}(X,X^\odot)}}\,(\gamma_\lambda(t+\delta) - \gamma_\lambda(t))^{1/2}.$$

Lemma 5.6 and an application of Lemma 5.3 to $A = P_\mu(t+\delta) - P_\lambda(t)$ yield that for each $\delta > 0$ and $\epsilon > 0$ there exists λ_1, such that for $\lambda_1 < \lambda < \mu$

$$|P_\mu(t+\delta)x_0 - P_\lambda(t)x_0| \le \epsilon + c\,(\gamma_\mu(t+\delta) - \gamma_\lambda(t))^{1/2}.$$

Given $\epsilon > 0$, choose $\delta > 0$ such that $\gamma_\lambda(t+\delta) - \gamma_\lambda(t) \le \epsilon$ for all $\lambda > \rho$ and $t \in [0,\tau]$. For these ϵ, δ pick λ_1 as above. Finally, choose $\lambda_2 > \lambda_1$ such that for $\lambda > \lambda_2$ and $t \in [0,\tau]$ we have $|\gamma(t) - \gamma_\lambda(t)| \le \epsilon$. If $\lambda_2 < \lambda < \mu$, then

$$\begin{aligned}
&|P_\mu(t)x_0 - P_\lambda(t)x_0| \le |P_\mu(t)x_0 - P_\mu(t+\delta)x_0| + |P_\mu(t+\delta)x_0 - P_\lambda(t)x_0| \\
\le& c\sqrt{\gamma_\mu(t+\delta) - \gamma_\mu(t)} + \epsilon + c\sqrt{\gamma_\mu(t+\delta) - \gamma_\lambda(t)} \le c\sqrt{\epsilon} + \epsilon \\
+& c\sqrt{[\gamma_\mu(t+\delta) - \gamma(t+\delta)] + [\gamma(t+\delta) - \gamma(t)] + [\gamma(t) - \gamma_\lambda(t)]} \le \epsilon + c\sqrt{\epsilon}.
\end{aligned}$$

We infer therefore that $P_\lambda(t)x_0$ is a Cauchy net in $X^\odot$, converging uniformly to some $P(t)x_0 \in X^\odot$. From the uniform convergence we obtain that $P(t)$ is strongly continuous. □

Proof of Theorem 4.3 (iii), (iv). To prove (iii) we take limits in Remark 4.2(iv). Thus, for any $\in \mathbf{L}^2([0,\tau], U)$, we have

$$J(\tau, x_0, u) = \langle P(\tau)x_0, x_0\rangle + \int_0^\tau |u(t) + B^*P(\tau - t)x(t; x_0, u)|^2\, dt.$$

Therefore, the minimal value for J is $\langle P(\tau)x_0, x_0\rangle$, obtained by the feedback law $u(t) = B^*P(\tau - t)x(t)$. Notice that $BB^*P(\tau - t) : X \to F_{-1}$ is a strongly continuous family of bounded operators, so that the evolution equation determined by the feedback law, i.e.,

$$x(t) = T(t)x_0 + \int_0^t T(t-s)BB^*P(\tau - s))x(s)\, ds, \quad x(0) = x_0$$

admits a unique solution.

To prove (iv), we take limits in Remark 4.2(iii). This is possible since $P_\lambda(t)x_0 \to P(t)x_0$, B^* is continuous on $X^\odot$, and $C_\lambda T(.)x \to Cx(.; x_0, 0)$ in $\mathbf{L}^2([0,\tau], Y)$.

6. Example

Consider the following boundary control problem for the heat equation on an interval:

Problem 6.1.

$$\textit{Minimize } J(T; x_0, u) := \int_0^T \Big[\Big(\int_0^1 g(\xi)x(t,\xi)\, d\xi\Big)^2 + u(t)^2\Big]\, dt \quad \textit{subject to}$$

$$\frac{\partial}{\partial t}x(t,\xi) = \frac{\partial^2}{\partial \xi^2}x(t,\xi), \ \textit{for } t \geq 0, \ \xi \in [0,1],$$
$$x(t,0) = 0, \ \frac{\partial}{\partial \xi}x(t,1) = u(t), \quad x(0,\xi) = x_0(\xi), \quad u \in \mathbf{L}^2([0,T],\mathbb{R}).$$

Here $x_0 \in \mathbf{L}^1([0,1],\mathbb{R})$ *and* $g \in \mathbf{L}^p([0,1],\mathbb{R})$ *with* $p > 1$.

To rewrite this problem in the setting of Problem 1.1, we define

$$X = \mathbf{L}^1([0,1],\mathbb{R}), \quad U = Y = \mathbb{R},$$
$$Ax = \frac{d^2}{d\xi^2}x, \quad D(A) = \{x \in \mathbf{W}^{2,1}([0,1],\mathbb{R}) : x(0) = 0, \ \frac{d}{d\xi}x(1) = 0\},$$
$$Cx = \int_0^1 g(\xi)x(\xi)\,d\xi, \quad Bu = -uA_{-1}b \ \text{ with } b(\xi) = \xi.$$

It is known that A generates an analytic semigroup $T(t)$ on X. Since $\frac{d^2}{d\xi^2}b = 0$ and the boundary conditions $b(0) = 0$, $\frac{d}{d\xi}b(1) = 1$ hold, the boundary value problem can be rewritten

$$\frac{d}{dt}x(t) = A(x(t) - u(t)b) = A_{-1}x(t) + Bu(t).$$

Notice that C is an unbounded operator on X unless $g \in \mathbf{L}^\infty([0,1],\mathbb{R})$.

The following lemmas show that this problem fits into the setting exposed in Section 3:

Lemma 6.2. *The vector* b *is contained in the Favard class of* $T(t)$. *Consequently,* B *maps* $\mathbb{R}$ *into the extrapolated Favard class.*

Proof. Notice that b solves

$$\frac{d^2}{d\xi^2}b = 0, \quad b(0) = 0, \quad \frac{d}{d\xi}b(1) = 1.$$

By [5, Theorem 16] it is sufficient to check Greiner's condition ([9]): For $\lambda > 0$, the solution b_λ of

$$(\lambda - \frac{d^2}{d\xi^2})b_\lambda = 0, \quad b_\lambda(0) = 0, \quad \frac{d}{d\xi}b_\lambda(1) = 1$$

satisfies an estimate

$$\|b_\lambda\|_{L^1([0,1],\mathbb{R})} \leq \frac{c}{\lambda}.$$

This estimate is obtained by straightforward integration from

$$b_\lambda(\xi) = \frac{\sinh(\sqrt{\lambda}\xi)}{\sqrt{\lambda}\cosh(\sqrt{\lambda})}.$$

Lemma 6.3. *In the setting above, $X^\odot = \{y \in \mathcal{C}([0,T],\mathbb{R}) : y(0) = 0\}$. For $y \in X^\odot$, we have $B^*(y) = y(1)$.*

Proof. In $X^* = \mathbf{L}^\infty([0,1],\mathbb{R})$, the adjoint of the generator A^* is given by

$$A^* y = \frac{d^2}{d\xi^2} y, \quad \text{defined on}$$

$$D(A^*) = \{y \in \mathbf{W}^{2,\infty}([0,1],\mathbb{R}) : y(0) = 0,\ \frac{d}{d\xi} y(1) = 0\}.$$

Now, $X^\odot = \overline{D(A^*)} = \{y \in \mathcal{C}([0,1],\mathbb{R}) : y(0) = 0\}$. For $y \in D(A^*)$, we have (using the boundary conditions)

$$B^* y = \langle B^* y, 1\rangle = -\langle y, A_{-1} b\rangle = -\langle A^* y, b\rangle = -\int_0^1 \frac{d^2}{d\xi^2} y(\xi)\xi \, d\xi = y(1).$$

This result extends to $y \in X^\odot$ by continuity of B^*. $\square$

Lemma 6.4. *For $\Re(\lambda) \geq 0$, the resolvent of A is given by*

$$[(\lambda - A)^{-1} x](\xi) = \int_0^1 K(\lambda, \xi, \eta) x(\eta)\, d\eta$$

with

$$K(\lambda,\xi,\eta) = \begin{cases} \frac{1}{\sqrt{\lambda}} \frac{\sinh(\sqrt{\lambda}\eta)\cosh(\sqrt{\lambda}(1-\xi))}{\cosh(\sqrt{\lambda})} & \text{if } \eta \leq \xi, \\ \frac{1}{\sqrt{\lambda}} \frac{\sinh(\sqrt{\lambda}\xi)\cosh(\sqrt{\lambda}(1-\eta))}{\cosh(\sqrt{\lambda})} & \text{if } \eta \geq \xi. \end{cases}$$

In particular, there exists a constant M independent of $\lambda > 0$, $\xi, \eta \in [0,1]$ such that

$$|K(i\lambda, \xi, \eta)| \leq \frac{M}{\sqrt{\lambda}} e^{-\sqrt{\lambda/2}|\eta - \xi|}. \tag{6.1}$$

Proof. A direct computation shows that

$$(\lambda - \frac{\partial^2}{\partial \xi^2}) K(\lambda, \xi, \eta) = \delta_\eta$$

in the sense of distributions, where δ_η denotes the Dirac measure at the point η. Moreover, K satisfies the boundary conditions

$$K(\lambda, 0, \eta) = 0, \quad \frac{\partial}{\partial \xi} K(\lambda, 1, \eta) = 0.$$

Therefore, K is the Green's function for the boundary value problem $(\lambda - A)y = x$. Estimate (6.1) is straightforward, using

$$|\sinh(\sqrt{i\lambda})| \leq e^{\sqrt{\lambda/2}}, \quad c e^{\sqrt{\lambda/2}} \leq |\cosh(\sqrt{i\lambda})| \leq e^{\sqrt{\lambda/2}}$$

with some $c > 0$. $\square$

Lemma 6.5. *There exists a constant c such that for $x \in \mathbf{L}^1([0,1],\mathbb{R})$,*

$$\int_0^\infty |CT(t)x|^2\,dt \le c\|x\|^2_{L^1([0,1],\mathbb{R})}.$$

Consequently, Hypothesis 3.5 *is satisfied.*

Proof. Let $1/q + 1/p = 1$. The function $t \mapsto T(t)x$ is continuous on $(0,\infty)$ with values in $\mathbf{L}^q([0,1],\mathbb{R})$ for all $x \in \mathbf{L}^1([0,1],\mathbb{R})$. Using Laplace transforms and Plancherel's Theorem, we estimate the integral

$$\int_{-\infty}^\infty |C(i\lambda - A)^{-1}x|^2\,d\lambda.$$

Since the spectrum of A considered on $\mathbf{L}^q([0,1],\mathbb{R})$ is bounded away from the imaginary axis, we need only to check the convergence of the integral at infinity. By definition of C and Lemma 6.4, we have

$$\begin{aligned}
&\int_1^\infty |C(i\lambda - A)^{-1}x|^2\,d\lambda \\
&= \int_1^\infty \Big|\int_0^1 g(\xi)\int_0^1 K(i\lambda,\xi,\eta)x(\eta)\,d\eta\,d\xi\Big|^2 d\lambda \\
&\le \int_1^\infty \Big(\frac{M}{\sqrt{\lambda}}\int_0^1\int_0^1 |g(\xi)|e^{-\sqrt{\lambda/2}|\eta-\xi|}|x(\eta)|\,d\eta\,d\xi\Big)^2\,d\lambda \\
&\le M^2\int_1^\infty \lambda^{-1}\Big(\int_0^1 \|g\|_{L^p([0,1],\mathbb{R})}\ \|e^{-\sqrt{\lambda/2}|\eta-\cdot|}\|_{L^q([0,1],\mathbb{R})}\ |x(\eta)|\,d\eta\Big)^2\,d\lambda \\
&\le M^2\int_1^\infty \lambda^{-1}\left(2^{3/2q}q^{-1/q}\lambda^{-1/2q}\|x\|_{L^1([0,1],\mathbb{R})}\right)^2\,d\lambda\|g\|^2_{L^p([0,1],\mathbb{R})} \\
&= M^2 2^{3/q}q^{-1-2/q}\|x\|^2_{L^1([0,1],\mathbb{R})}\|g\|^2_{L^p([0,1],\mathbb{R})}.
\end{aligned}$$

As a consequence of these Lemmas, we can apply Theorem 4.3:

Proposition 6.6. *There exists a family of operators $\{P(t) : t \ge 0\}$ mapping $\mathbf{L}^1([0,1],\mathbb{R})$ into $\mathcal{C}([0,T],\mathbb{R})$ such that for any initial state $x_0 \in \mathbf{L}^1([0,T],\mathbb{R})$ the value*

$$\langle P(t)x_0, x_0\rangle = \int_0^1 (P(t)x_0)(\xi)x_0(\xi)\,d\xi$$

is the optimal value of $J(t;x_0,u)$ to be achieved. The optimal control is realized by the feedback synthesis $u(t) = -[P(T-t)x(t)](1)$.

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