

NON-CLASSICAL BOUNDARY LAYERS FOR FOURTH-ORDER EQUATIONS WITH SINGULAR LIMIT SOLUTION

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Abstract. In this article we study non classical singular perturbation problems involving boundary layers in the interior of the domain. As usual, these problems contain a small parameter which produces, when this parameter approaches zero, classical boundary layers located at the boundary. If we moreover consider a singular source function, we produce also boundary layers inside the domain. Our aim in this article is to study this kind of boundary layers. We consider, here, a model with a fourth-order differential operator, and the open set is a channel to avoid the technicalities due to the curvature of the boundary. Other stationary problems and time dependent problems will be considered elsewhere.

1. INTRODUCTION AND MAIN RESULTS

Many physical problems are characterized by the presence of a small parameter such as a small diffusion coefficient or a small viscosity coefficient, in particular in fluid and solid mechanics and generally this small parameter generates singular perturbations. Many articles are devoted to such singular perturbation in the fluid mechanics, solid mechanics, and mathematical literatures (see for instance Eckhaus [3], Friedrichs [5], Chang and Howes [2], Germain [6], Balian and Peube [1], Lions [10], Vishik and Lyusternik [17] among many other articles). When the small parameter approaches zero, boundary layers appear in the domain. Classically, in the theory of singular perturbations, the boundary layers appear only at the boundary of the domain. This is the case in particular for the famous example of Friedrichs (1941) concerning the structure of these boundary layers, in the long article of Vishik and Lyusternik [17] and in the book of Lions [10]; different aspects of these problems are also considered by Oleinik [12], Frank [4], Lagerström

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[9], Ladyzhenskaya [8]. The singular perturbation phenomenon occurs when the boundary conditions required for the perturbed and limit problems do not match. In studying the behavior of these boundary layers, we usually assume that the source functions and all data are as regular as necessary. However, this is not always the case in physically relevant problems and it is natural to ask and study what will happen in the absence of regularity.

In an earlier article [7], we studied such a problem associated to second-order elliptic problems and we proved that, when the parameter goes to zero and the source function is discontinuous (singular), a discontinuity is produced in the limit solution, which is supplemented by the disparity of one or more boundary conditions. Since the perturbed solution is regular, we note the appearance of boundary layers where singularities exist for the limit solution.

The purpose of this article is to introduce and study similar problems using the terminology of the boundary layer “corrector” introduced by Lions [10] and used in [15] and in many other references. In addition to the properties of this type of boundary layers that had been studied in the second-order elliptic case in [7], we emphasize here on the effect of a fourth-order regularization on the order of magnitude in ε in the estimates of the convergence results which are usually devoted to confirm the choice of the corrector.

For simplicity, we will focus on the following fourth-order linear model:

$$\varepsilon \Delta^2 u^\varepsilon + u^\varepsilon = H, \quad \text{in } \Omega, \quad (1.1)$$

$$u^\varepsilon = \Phi_0, \quad \frac{\partial u^\varepsilon}{\partial \nu} = \Phi_1, \quad \text{on } \Gamma. \quad (1.2)$$

Here Φ_0 and Φ_1 are two given (smooth) functions, Ω is a channel of \mathbb{R}^n and H is the Heaviside function in the direction orthogonal to the walls (z -direction).

For the sake of simplicity, we assume that $n = 3$ and that Ω is of the form $\Omega = (0, L_1) \times (0, L_2) \times (-h, h)$, and we write $\Gamma = (0, L_1) \times (0, L_2) \times \{-h, h\}$. In the x and y directions the boundary condition is periodicity for u^ε :

$$u^\varepsilon \text{ is periodic with periods } L_1 \text{ and } L_2 \text{ in the } x \text{ and } y \text{ directions.} \quad (1.3)$$

Note that the restriction on the dimension of space is not essential since we are treating a linear problem. The choice of dimension three is rather arbitrary. The corresponding inviscid problem is simply

$$u^0 = H(z) = \begin{cases} 1 & \text{if } z \in (0, h), \\ 0 & \text{if } z \in (-h, 0). \end{cases} \quad (1.4)$$

We easily observe that u^0 does not fulfill (1.2) (this produces a classical boundary layer at $z = \pm h$) and that it displays a discontinuity at $z = 0$ producing there a non classical boundary layer which will be our main objective.

On the other hand, it is obvious that $u^\varepsilon \in H^4(\Omega)$ since $H \in L^2(\Omega)$. Hence, u^ε cannot converge to u^0 in $H^1(\Omega)$ as ε approaches to zero since $\partial u^0/\partial z = \delta \notin L^2(\Omega)$, where δ stands for the Dirac distribution concentrated on the surface $z = 0$, and $\partial u^\varepsilon/\partial z \in H^3(\Omega)$. Usually, in the theory of boundary layers, some derivatives of the regularized solution become very large, as ε becomes small, near the boundary only. In our problem this occurs also inside the domain Ω , more precisely where the limit solution is discontinuous. In Section 2, we will illustrate this by just solving the corresponding problem in dimension one, leading to a result useful in the higher dimensional case.

Then in higher dimension, our main results for problem (1.1)–(1.3) are the following ones:

Theorem 1.1. *Let u^ε be the solution of (1.1)–(1.3) and u^0 the solution of (1.4). There exists a corrector θ^ε depending only on x, y and $z/\varepsilon^{1/4}$, and a constant C independent of ε such that*

$$\|u^\varepsilon - u^0 - \theta^\varepsilon\|_{L^2(\Omega)} \leq C \varepsilon^{5/8}, \tag{1.5}$$

$$\|u^\varepsilon - u^0 - \theta^\varepsilon\|_{H^1(\Omega)} \leq C \varepsilon^{3/8}, \tag{1.6}$$

$$\|u^\varepsilon - u^0 - \theta^\varepsilon\|_{H^2(\Omega)} \leq C \varepsilon^{1/8}. \tag{1.7}$$

The above result can be extended to more singular cases, e.g. when we replace, in the right-hand side of (1.1), $H \in L^2(\Omega)$ by $\delta^{(k)} \in H^{-1-k}(\Omega)$, where $\delta^{(k)}$ stands for the k^{th} derivative, in the z -variable, of the Dirac distribution δ ($\delta = dH/dz$). Namely, we study u_k^ε solution of the following equation

$$\varepsilon \Delta^2 u_k^\varepsilon + u_k^\varepsilon = \delta^{(k)}, \quad \text{in } \Omega, \tag{1.8}$$

and verifying the same boundary and periodicity conditions as u^ε , namely (1.2) and (1.3). Here, the limit problem is

$$u_k^0 = \delta^{(k)}, \quad \text{in } \Omega, \tag{1.9}$$

and we have the following result.

Theorem 1.2. *There exists a corrector θ_k^ε , function of $x, y, z/\varepsilon^{1/4}$ and k , and a constant C depending on the data but not on ε and such that*

$$\|u_k^\varepsilon - u_k^0 - \theta_k^\varepsilon\|_{L^2(\Omega)} \leq C \varepsilon^{5/8}, \tag{1.10}$$

$$\|u_k^\varepsilon - u_k^0 - \theta_k^\varepsilon\|_{H^1(\Omega)} \leq C \varepsilon^{3/8}, \quad (1.11)$$

$$\|u_k^\varepsilon - u_k^0 - \theta_k^\varepsilon\|_{H^2(\Omega)} \leq C \varepsilon^{1/8}. \quad (1.12)$$

Remark 1. Throughout this article, C will denote a generic constant depending on all the data except ε . Its value may be different at each occurrence.

Remark 2. The methods used below apply to many other equations as long as the corresponding one-dimensional problem remains explicitly solvable. In particular, we may replace, in (1.1), the term u^ε by $c(x, y)u^\varepsilon$.

Remark 3. Our results can be extended to other source functions with general singularities, in particular a finite linear combination of H and the $\delta^{(k)}$ can be treated.

Remark 4. The existence of u_k^ε solution of (1.2), (1.3), and (1.8), is based on the duality argument, see for instance the book of Lions-Magenes [11].

The rest of this article is organized as follows : In the next section we study the one-dimensional problem corresponding to (1.1) and (1.8) and associated with homogeneous boundary conditions; this will be useful subsequently to describe the corrector. In Section 3, we prove Theorem 1.1 and we derive the first order term in the asymptotic expansion of u^ε with respect to ε . The last section is devoted to the proof of Theorem 1.2 which is concerned with the enhanced singularity for the source function.

2. THE ONE-DIMENSIONAL CASE

It is well known, in the theory of boundary layers, that the most important derivative variations are produced in the direction normal to the boundary. As we will see later on, this is still true for the boundary layer in the interior of the domain. Hence, in view of the geometry of Ω , we will be interested in the one-dimensional problems in the z variable.

We start with a singularity belonging to $L^2(\Omega)$ ($L^\infty(\Omega)$ in fact).

2.1. A discontinuous source function in $L^2(\Omega)$. We intend to solve explicitly the following problem

$$\varepsilon \frac{d^4 u^\varepsilon}{dz^4} + u^\varepsilon = H, \quad \text{in } \Omega = (-h, h), \quad (2.1)$$

$$u^\varepsilon = \frac{du^\varepsilon}{dz} = 0, \quad \text{at } z = \pm h. \quad (2.2)$$

It is obvious that if $u^\varepsilon \in L^2(\Omega)$, then $d^4 u^\varepsilon / dz^4 \in L^2(\Omega)$. Hence, by the Sobolev embedding theorem,

$$u^\varepsilon \in H^4(\Omega) \hookrightarrow C^3(\bar{\Omega}). \tag{2.3}$$

Thus, we write

$$u^\varepsilon(z) = u_1^\varepsilon(z)\chi_{(0,h)} + u_2^\varepsilon(z)\chi_{(-h,0)}, \tag{2.4}$$

and we solve separately (2.1)-(2.2) on $(-h, 0)$ and $(0, h)$, then we match the solutions u_1^ε and u_2^ε and their derivatives of order 1, 2 and 3 at $z = 0$ with the help of the regularity conditions implied by (2.3).

The solutions u_1^ε and u_2^ε have the following form

$$\begin{aligned} u_1^\varepsilon(z) = & 1 + \exp\left(\frac{z-h}{\sqrt{2\varepsilon^{1/4}}}\right) \left\{ A_\varepsilon \cos\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) + B_\varepsilon \sin\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \right\} \\ & + \exp\left(-\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \left\{ C_\varepsilon \cos\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) + D_\varepsilon \sin\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \right\}, \end{aligned} \tag{2.5}$$

for all $z \in (0, h)$, and,

$$\begin{aligned} u_2^\varepsilon(z) = & \exp\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \left\{ E_\varepsilon \cos\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) + F_\varepsilon \sin\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \right\} \\ & + \exp\left(-\frac{z+h}{\sqrt{2\varepsilon^{1/4}}}\right) \left\{ G_\varepsilon \cos\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) + H_\varepsilon \sin\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \right\}, \end{aligned} \tag{2.6}$$

for all $z \in (-h, 0)$. Hence, writing the boundary and regularity conditions ((2.2) and (2.3)) for u_1^ε and u_2^ε leads to a linear system of the form

$$\mathcal{A}_\varepsilon \mathcal{X}_\varepsilon = \mathcal{B}_\varepsilon, \tag{S_\varepsilon}$$

where $\mathcal{B}_\varepsilon = \mathcal{B} = (-1, 0, 0, 0, -1, 0, 0, 0) \in \mathbb{R}^8$ is a vector, $\mathcal{X}_\varepsilon = (A_\varepsilon, B_\varepsilon, \dots, H_\varepsilon)$ is the unknown variable, and $\mathcal{A}_\varepsilon \in \mathfrak{M}_8(\mathbb{R})$ is the real square matrix

$$\mathcal{A}_\varepsilon = \begin{bmatrix} \beta_\varepsilon & \gamma_\varepsilon & \alpha(\varepsilon)\beta_\varepsilon & \alpha(\varepsilon)\gamma_\varepsilon & 0 & 0 & 0 & 0 \\ \beta_\varepsilon - \gamma_\varepsilon & \beta_\varepsilon + \gamma_\varepsilon & -\alpha(\beta_\varepsilon + \gamma_\varepsilon) & \alpha(\beta_\varepsilon - \gamma_\varepsilon) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha(\varepsilon)\beta_\varepsilon & -\alpha(\varepsilon)\gamma_\varepsilon & \beta_\varepsilon & -\gamma_\varepsilon \\ 0 & 0 & 0 & 0 & \alpha(\beta_\varepsilon + \gamma_\varepsilon) & \alpha(\beta_\varepsilon - \gamma_\varepsilon) & \gamma_\varepsilon - \beta_\varepsilon & \beta_\varepsilon + \gamma_\varepsilon \\ \alpha(\varepsilon) & 0 & 1 & 0 & -1 & 0 & -\alpha(\varepsilon) & 0 \\ \alpha(\varepsilon) & \alpha(\varepsilon) & -1 & 1 & -1 & -1 & \alpha(\varepsilon) & -\alpha(\varepsilon) \\ 0 & 2\alpha(\varepsilon) & 0 & -2 & 0 & -2 & 0 & 2\alpha(\varepsilon) \\ -2\alpha(\varepsilon) & 2\alpha(\varepsilon) & 2 & 2 & 2 & -2 & -2\alpha(\varepsilon) & -2\alpha(\varepsilon) \end{bmatrix}.$$

Note that in the matrix \mathcal{A}_ε , the lines 1 and 2 (respectively 3 and 4) express the boundary conditions at $z = h$ (respectively at $z = -h$), the lines 5 to 8

express the matching at $z = 0$ of u_1^ε and u_2^ε and their derivatives up to order three. We have set

$$\alpha = \alpha(\varepsilon) = \exp\left(-\frac{h}{\sqrt{2\varepsilon^{1/4}}}\right), \tag{2.7}$$

and

$$\beta_\varepsilon = \cos\left(\frac{h}{\sqrt{2\varepsilon^{1/4}}}\right), \quad \gamma_\varepsilon = \sin\left(\frac{h}{\sqrt{2\varepsilon^{1/4}}}\right). \tag{2.8}$$

We easily deduce a simple approximate solution of $(\mathcal{S}_\varepsilon)$, denoted by $\bar{\mathcal{X}}_\varepsilon = (\bar{A}_\varepsilon, \dots, \bar{H}_\varepsilon)$. Thanks to the relation $\beta_\varepsilon^3 + \gamma_\varepsilon^3 + \beta_\varepsilon^2\gamma_\varepsilon + \gamma_\varepsilon^2\beta_\varepsilon = \beta_\varepsilon + \gamma_\varepsilon$, which is a simple consequence of the fact that β_ε and γ_ε satisfy $\beta_\varepsilon^2 + \gamma_\varepsilon^2 = 1$, we find $\bar{A}_\varepsilon = -(\beta_\varepsilon + \gamma_\varepsilon)$, $\bar{B}_\varepsilon = \beta_\varepsilon - \gamma_\varepsilon$, $\bar{C}_\varepsilon = -\frac{1}{2}$, $\bar{D}_\varepsilon = -\beta_\varepsilon^2(\alpha(\varepsilon))^2$, $\bar{E}_\varepsilon = \frac{1}{2}$, $\bar{F}_\varepsilon = (\beta_\varepsilon - \gamma_\varepsilon)\alpha(\varepsilon)$, $\bar{G}_\varepsilon = -\frac{1+2\beta_\varepsilon\gamma_\varepsilon}{2}\alpha(\varepsilon)$, $\bar{H}_\varepsilon = -\beta_\varepsilon^2\alpha(\varepsilon)$. When $f_\varepsilon = A_\varepsilon, \dots$, or H_ε , it verifies

$$f_\varepsilon = \bar{f}_\varepsilon + O(\alpha(\varepsilon)), \tag{2.9}$$

so that \bar{f}_ε is an approximation of f_ε at exponential order. This leads to an approximation \bar{u}^ε of u^ε , up to an order subsequently specified:

$$\begin{aligned} \bar{u}^\varepsilon(z) = & 1 + \exp\left(\frac{z-h}{\sqrt{2\varepsilon^{1/4}}}\right) \left\{ \bar{A}_\varepsilon \cos\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) + \bar{B}_\varepsilon \sin\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \right\} \\ & + \exp\left(-\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \left\{ \bar{C}_\varepsilon \cos\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) + \bar{D}_\varepsilon \sin\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \right\}, \end{aligned} \tag{2.10}$$

for all $z \in (0, h)$ and

$$\begin{aligned} \bar{u}^\varepsilon(z) = & \exp\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \left\{ \bar{E}_\varepsilon \cos\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) + \bar{F}_\varepsilon \sin\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \right\} \\ & + \exp\left(-\frac{z+h}{\sqrt{2\varepsilon^{1/4}}}\right) \left\{ \bar{G}_\varepsilon \cos\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) + \bar{H}_\varepsilon \sin\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \right\}, \end{aligned} \tag{2.11}$$

for all $z \in (-h, 0)$. Now, it is easy to see that \bar{u}^ε verifies the following boundary conditions:

$$\begin{aligned} \bar{u}^\varepsilon(-h) &= -(\beta_\varepsilon - \gamma_\varepsilon)(\alpha(\varepsilon))^2, \quad \bar{u}^\varepsilon(h) = -\frac{1}{2}\beta_\varepsilon\alpha(\varepsilon) - 2\beta_\varepsilon^2\gamma_\varepsilon(\alpha(\varepsilon))^3, \\ \frac{d\bar{u}^\varepsilon}{dz}(-h) &= \frac{\varepsilon^{-1/4}}{\sqrt{2}} \left\{ \frac{1}{2}(\beta_\varepsilon + \gamma_\varepsilon) - 2(\beta_\varepsilon - \gamma_\varepsilon)\beta_\varepsilon^2 \right\} \alpha(\varepsilon), \\ \frac{d\bar{u}^\varepsilon}{dz}(h) &= \frac{\varepsilon^{-1/4}}{\sqrt{2}} (\beta_\varepsilon - \gamma_\varepsilon)^2 (\alpha(\varepsilon))^2. \end{aligned}$$

We observe that the boundary values of \bar{u}^ε and $d\bar{u}^\varepsilon/dz$ converge exponentially fast to zero as $\varepsilon \rightarrow 0$. This shows how \bar{u}^ε approximates u^ε near the

boundary. At $z = 0$ where \bar{u}^ε and its derivatives of order 1, 2, and 3 are not continuous, we have:

$$\begin{aligned} \bar{u}^\varepsilon(z = 0^-) &= \frac{1}{2} - \left(\frac{1}{2} + \beta_\varepsilon \gamma_\varepsilon\right) (\alpha(\varepsilon))^2, \\ \bar{u}^\varepsilon(z = 0^+) &= \frac{1}{2} - (\beta_\varepsilon + \gamma_\varepsilon) \alpha(\varepsilon), \\ \frac{d\bar{u}^\varepsilon}{dz}(z = 0^-) &= \frac{\varepsilon^{-1/4}}{2\sqrt{2}} + \frac{\beta_\varepsilon - \gamma_\varepsilon}{\sqrt{2}\varepsilon^{1/4}} \alpha(\varepsilon) + \frac{1}{\sqrt{2}\varepsilon^{1/4}} \left(\frac{1}{2} + \beta_\varepsilon \gamma_\varepsilon - \beta_\varepsilon^2\right) (\alpha(\varepsilon))^2, \\ \frac{d\bar{u}^\varepsilon}{dz}(z = 0^+) &= \frac{\varepsilon^{-1/4}}{2\sqrt{2}} - \frac{\sqrt{2}\gamma_\varepsilon}{\varepsilon^{1/4}} \alpha(\varepsilon) - \frac{\sqrt{2}\beta_\varepsilon^2}{\varepsilon^{1/4}} (\alpha(\varepsilon))^2, \\ \frac{d^2\bar{u}^\varepsilon}{dz^2}(z = 0^-) &= \frac{1}{\varepsilon^{1/2}} \{(\beta_\varepsilon - \gamma_\varepsilon) \alpha(\varepsilon) + \beta_\varepsilon^2\} (\alpha(\varepsilon))^2, \\ \frac{d^2\bar{u}^\varepsilon}{dz^2}(z = 0^+) &= \frac{1}{\varepsilon^{1/2}} \{(\beta_\varepsilon - \gamma_\varepsilon) \alpha(\varepsilon) + 2\beta_\varepsilon^2\} (\alpha(\varepsilon))^2, \\ \frac{d^3\bar{u}^\varepsilon}{dz^3}(z = 0^-) &= -\frac{\varepsilon^{-3/4}}{2\sqrt{2}} + \frac{\beta_\varepsilon - \gamma_\varepsilon}{\sqrt{2}\varepsilon^{3/4}} \alpha(\varepsilon) - \frac{1 + 2\beta_\varepsilon \gamma_\varepsilon + 2\beta_\varepsilon^2}{2\sqrt{2}\varepsilon^{3/4}} (\alpha(\varepsilon))^2, \\ \frac{d^3\bar{u}^\varepsilon}{dz^3}(z = 0^+) &= -\frac{\varepsilon^{-3/4}}{2\sqrt{2}} + \frac{\sqrt{2}\beta_\varepsilon}{\varepsilon^{3/4}} \alpha(\varepsilon) - \frac{2\sqrt{2}\beta_\varepsilon^2}{\varepsilon^{3/4}} (\alpha(\varepsilon))^2. \end{aligned}$$

Consequently, \bar{u}^ε satisfies the equation

$$\begin{aligned} \varepsilon \frac{d^4\bar{u}^\varepsilon}{dz^4} + \bar{u}^\varepsilon &= H - \varepsilon \left[\left(\frac{1}{2} + \beta_\varepsilon \gamma_\varepsilon\right) - (\beta_\varepsilon + \gamma_\varepsilon) \alpha(\varepsilon) \right] \alpha(\varepsilon) \delta^{(3)} \\ &\quad - \frac{\varepsilon^{3/4}}{\sqrt{2}} \left[\beta_\varepsilon + \gamma_\varepsilon + \left(\frac{1}{2} + \beta_\varepsilon \gamma_\varepsilon + \beta_\varepsilon^2\right) \alpha(\varepsilon) \right] \alpha(\varepsilon) \delta'' + \varepsilon^{1/2} \beta_\varepsilon^2 (\alpha(\varepsilon))^2 \delta' \\ &\quad + \frac{\varepsilon^{1/4}}{\sqrt{2}} \left[\beta_\varepsilon + \gamma_\varepsilon + \left(\frac{1}{2} + \beta_\varepsilon \gamma_\varepsilon - \beta_\varepsilon^2\right) \alpha(\varepsilon) \right] \alpha(\varepsilon) \delta. \end{aligned} \tag{2.12}$$

Because of the appearance of $\delta, \delta', \delta''$ and $\delta^{(3)}$ in the right-hand side of (2.12), we cannot apply the standard energy method to prove the convergence of $u^\varepsilon - \bar{u}^\varepsilon$ to 0 in suitable Sobolev spaces. But, if we replace in (2.5) and in (2.6), $A_\varepsilon, \dots, H_\varepsilon$ by their asymptotic expansions, as $\varepsilon \rightarrow 0$, using (2.9), we conclude from the expression of \bar{u}^ε (2.10)-(2.11) that

$$\begin{aligned} u^\varepsilon(z) &= \bar{u}^\varepsilon(z) + O(\alpha(\varepsilon)) \exp\left(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \left\{ \cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right\} \chi_{(0,h)} \\ &\quad + O(\alpha(\varepsilon)) \exp\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \left\{ \cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right\} \chi_{(-h,0)} + O(\alpha(\varepsilon)) \end{aligned}$$

$$\left\{ \exp\left(\frac{z-h}{\sqrt{2}\varepsilon^{1/4}}\right) + \exp\left(-\frac{z+h}{\sqrt{2}\varepsilon^{1/4}}\right) \right\} \left\{ \cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right\}, \tag{2.13}$$

for all $z \in (-h, h)$.

By simple computations, we see that any of the three functions $g = \exp(-z/\varepsilon^{1/4})\{\cos(z/\varepsilon^{1/4}) + \sin(z/\varepsilon^{1/4})\}\chi_{(0,h)}$, or $\exp(z/\varepsilon^{1/4})\{\cos(z/\varepsilon^{1/4}) + \sin(z/\varepsilon^{1/4})\}\chi_{(-h,0)}$ or $\{\exp((z-h)/\varepsilon^{1/4}) + \exp(-(z+h)/\varepsilon^{1/4})\}\{\cos(z/\varepsilon^{1/4}) + \sin(z/\varepsilon^{1/4})\}\chi_{(-h,h)}$ verifies the following estimate

$$\|g\|_{H^m(\Omega)} \leq C \varepsilon^{(1-2m)/8}, \quad \forall m \geq 0.$$

Using this bound for $m = 0$ together with the triangle inequality applied to (2.13), we obtain

$$\|u^\varepsilon - \bar{u}^\varepsilon\|_{L^2(\Omega)} \leq C \varepsilon^{1/8} \alpha(\varepsilon). \tag{2.14}$$

Hence, by differentiating (2.13) m times with respect to z and knowing that the $O(\alpha(\varepsilon))$ terms are independent of z , we deduce that \bar{u}^ε is an approximation of u^ε at exponential order in all the Sobolev spaces $H^m(\Omega)$.

Proposition 2.1. *There exists a constant C independent of ε such that*

$$\|u^\varepsilon - \bar{u}^\varepsilon\|_{H^m(\Omega)} \leq C \varepsilon^{(1-2m)/8} \alpha(\varepsilon), \quad \forall m \in \mathbb{N}. \tag{2.15}$$

Remark 5. Notice that the singularities $\delta, \delta', \delta''$ and $\delta^{(3)}$ appearing in the right-hand side of (2.12) can be eliminated, as we will see later, by introducing an appropriate corrector function; we may then apply the standard energy estimates and obtain an analogous result to (2.15). However, the estimate (2.15) will be sufficient to derive the corrector in the multidimensional case subsequently.

2.2. A singular source function in $H^s(\Omega), s < 0$. In this section, we are interested in the one-dimensional problem corresponding to (1.8). As before, this problem is considered in the normal direction (z):

$$\varepsilon \frac{d^4 v^\varepsilon}{dz^4} + v^\varepsilon = \delta^{(k)}, \quad \text{in } \Omega = (-h, h), \tag{2.16}$$

$$v^\varepsilon = \frac{dv^\varepsilon}{dz} = 0, \quad \text{at } z = \pm h. \tag{2.17}$$

To derive the explicit solution of (2.16)-(2.17), we have to consider four cases depending on the value of k .

Case 1. If $k = 4q, q \in \mathbb{N}$, then we define recursively $v_0^\varepsilon = v^\varepsilon$ and

$$v_4^\varepsilon = v_0^\varepsilon - \frac{\delta^{(4(q-1))}}{\varepsilon}, \quad v_8^\varepsilon = v_4^\varepsilon + \frac{\delta^{(4(q-2))}}{\varepsilon^2}, \dots, \quad v_{4q+4}^\varepsilon = v_{4q}^\varepsilon + (-1)^{q+1} \frac{\delta^{(-4)}}{\varepsilon^{q+1}}.$$

By summing these equations, we deduce that

$$v^\varepsilon = - \sum_{j=1}^{q+1} (-1)^j \frac{\delta^{(4(q-j))}}{\varepsilon^j} + v_{4q+4}^\varepsilon, \tag{2.18}$$

where we have used this convention

$$\delta^{(-i)} = H_i, \quad \forall i \in \{1, 2, 3, 4\}, \tag{2.19}$$

H_i denoting the integrand of H_{i-1} for $i \in \{2, 3, 4\}$ (by setting $H_1 = H$), i.e.,

$$H_i(z) = \frac{z^{i-1}}{(i-1)!} \chi_{(0,h)}, \quad \forall i \in \{1, 2, 3, 4\}.$$

Hence, v_{4q+4}^ε verifies the equations

$$\varepsilon \frac{d^4 v_{4q+4}^\varepsilon}{dz^4} + v_{4q+4}^\varepsilon = (-1)^{q+1} \frac{H_4}{\varepsilon^{q+1}}, \quad \text{in } (-h, h), \tag{2.20}$$

$$v_{4q+4}^\varepsilon(z = h) = (-1)^{q+1} \frac{h^3}{6\varepsilon^{q+1}}, \quad v_{4q+4}^\varepsilon(z = -h) = 0, \tag{2.21}$$

$$\frac{d v_{4q+4}^\varepsilon}{dz}(h) = (-1)^{q+1} \frac{h^2}{2\varepsilon^{q+1}}, \quad \frac{d v_{4q+4}^\varepsilon}{dz}(-h) = 0, \tag{2.22}$$

$$v_{4q+4}^\varepsilon \text{ is of class } C^3 \text{ at } z = 0. \tag{2.23}$$

Thus, we see that we have reduced the singularity of the right-hand side. Solving again (2.20) separately on $(-h, 0)$ and $(0, h)$, matching the solutions at $z = 0$ with the help of (2.23) and applying the boundary conditions (2.21) and (2.22), we obtain a linear system of type $(\mathcal{S}_\varepsilon)$ with the same matrix \mathcal{A}_ε and with $\mathcal{B}_\varepsilon = (0, 0, \dots, 0, 2\sqrt{2}(-1)^q/\varepsilon^{q+1/4}) \in \mathbb{R}^8$.

The resolution of this system leads to an approximation $\bar{v}_{4q+4}^\varepsilon$ for v_{4q+4}^ε and consequently (thanks to (2.18)) to an approximation for v^ε called \bar{v}^ε which is given by

$$\begin{aligned} \bar{v}^\varepsilon(z) = & - \sum_{j=1}^q (-1)^j \frac{\delta^{(4(q-j))}}{\varepsilon^j} + \frac{\sqrt{2} (-1)^q}{4\varepsilon^{q+1/4}} \left\{ \exp\left(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right. \\ & \times \left[\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right] \chi_{(0,h)} \\ & \left. + \exp\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \left[\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) - \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right] \chi_{(-h,0)} \right\}. \end{aligned} \tag{2.24}$$

Note that the approximation \bar{v}^ε of v^ε is defined in the following sense

$$v^\varepsilon = \bar{v}^\varepsilon + e.s.t., \tag{2.25}$$

where *e.s.t.* stands for exponentially small(er) terms on $(-h, h)$ (exponentially small in all usual norms such as $\mathcal{C}^k, H^m, \dots$).

Case 2. If $k = 4q + 1, q \in \mathbb{N}$, we define recursively

$$v_1^\varepsilon = v^\varepsilon, \quad \text{and} \quad v_{4j+1}^\varepsilon = v_{4j-3}^\varepsilon + (-1)^j \frac{\delta^{(4(q-j)+1)}}{\varepsilon^j}, \quad \forall j \geq 1. \quad (2.26)$$

Writing (2.26) for $j = 1, 2, \dots, q + 1$, using the convention (2.19), and summing these equations, we find

$$v^\varepsilon = - \sum_{j=1}^{q+1} (-1)^j \frac{\delta^{(4(q-j)+1)}}{\varepsilon^j} + v_{4q+5}^\varepsilon, \quad (2.27)$$

where v_{4q+5}^ε satisfies

$$\varepsilon \frac{d^4 v_{4q+5}^\varepsilon}{dz^4} + v_{4q+5}^\varepsilon = (-1)^{q+1} \frac{H_3}{\varepsilon^{q+1}}, \quad \text{in } (-h, h), \quad (2.28)$$

$$v_{4q+5}^\varepsilon(z = h) = (-1)^{q+1} \frac{h^2}{2\varepsilon^{q+1}}, \quad v_{4q+5}^\varepsilon(z = -h) = 0, \quad (2.29)$$

$$\frac{d v_{4q+5}^\varepsilon}{dz}(h) = (-1)^{q+1} \frac{h}{\varepsilon^{q+1}}, \quad \frac{d v_{4q+5}^\varepsilon}{dz}(-h) = 0, \quad (2.30)$$

$$v_{4q+5}^\varepsilon \text{ is of class } C^3 \text{ at } z = 0. \quad (2.31)$$

Thus, solving (2.28)–(2.31) is equivalent to solving a linear system of type $(\mathcal{S}_\varepsilon)$ with $\mathcal{B}_\varepsilon = (0, 0, \dots, 0, -2(-1)^{q+1}/\varepsilon^{q+1/2}, 0) \in \mathbb{R}^8$. We then obtain the explicit expression of v_{4q+5}^ε of which we give an approximation to within an *e.l.t.* Hence, we deduce, thanks to (2.27) an approximation \bar{v}^ε of v^ε in this case ($k = 4q + 1$)

$$\begin{aligned} \bar{v}^\varepsilon(z) &= - \sum_{j=1}^q (-1)^j \frac{\delta^{(4(q-j)+1)}}{\varepsilon^j} + \frac{(-1)^{q+1}}{2 \varepsilon^{q+1/2}} \sin\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \\ &\quad \times \left\{ \exp\left(-\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \chi_{(0,h)} + \exp\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \chi_{(-h,0)} \right\}. \end{aligned} \quad (2.32)$$

Case 3. For $k = 4q + 2, q \in \mathbb{N}$, we define similarly

$$v_2^\varepsilon = v^\varepsilon, \quad \text{and} \quad v_{4j+2}^\varepsilon = v_{4j-2}^\varepsilon + (-1)^j \frac{\delta^{(4(q-j)+2)}}{\varepsilon^j}, \quad \forall j \geq 1. \quad (2.33)$$

Using again the convention (2.19), we write (2.33) for $j = 1, 2, \dots, q + 1$ and summing these $(q + 1)$ equations, we obtain

$$v^\varepsilon = - \sum_{j=1}^{q+1} (-1)^j \frac{\delta^{(4(q-j)+2)}}{\varepsilon^j} + v_{4q+6}^\varepsilon, \tag{2.34}$$

where v_{4q+6}^ε satisfies

$$\varepsilon \frac{d^4 v_{4q+6}^\varepsilon}{dz^4} + v_{4q+6}^\varepsilon = (-1)^{q+1} \frac{H_2}{\varepsilon^{q+1}}, \quad \text{in } (-h, h), \tag{2.35}$$

$$v_{4q+6}^\varepsilon(z = h) = (-1)^{q+1} \frac{h}{\varepsilon^{q+1}}, \quad v_{4q+6}^\varepsilon(z = -h) = 0, \tag{2.36}$$

$$\frac{d v_{4q+6}^\varepsilon}{dz}(h) = \frac{(-1)^{q+1}}{\varepsilon^{q+1}}, \quad \frac{d v_{4q+6}^\varepsilon}{dz}(-h) = 0, \tag{2.37}$$

$$v_{4q+6}^\varepsilon \text{ is of class } C^3 \text{ at } z = 0. \tag{2.38}$$

As before, this leads to a linear system of type $(\mathcal{S}_\varepsilon)$ with

$$\mathcal{B}_\varepsilon = (0, \dots, 0, (-1)^q \sqrt{2}/\varepsilon^{q+1/2}, 0, 0),$$

a vector of \mathbb{R}^8 . Furthermore, we deduce an approximation of v^ε called \bar{v}^ε and given by

$$\begin{aligned} \bar{v}^\varepsilon(z) = & - \sum_{j=1}^q (-1)^j \frac{\delta^{(4(q-j)+2)}}{\varepsilon^j} + \frac{\sqrt{2}(-1)^{q+1}}{4\varepsilon^{q+3/4}} \left\{ \exp\left(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right. \\ & \times \left[\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) - \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right] \chi_{(0,h)} \\ & \left. + \exp\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \left[\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right] \chi_{(-h,0)} \right\}. \end{aligned} \tag{2.39}$$

Case 4. Similarly for $k = 4q + 3$, $q \in \mathbb{N}$, we write $v_3^\varepsilon = v^\varepsilon$ and we define recursively

$$v_{4j+3}^\varepsilon = v_{4j-1}^\varepsilon + (-1)^j \frac{\delta^{(4(q-j)+3)}}{\varepsilon^j}, \quad \forall j \geq 1. \tag{2.40}$$

As in the above cases, by summing the equations (2.40), we deduce that

$$v^\varepsilon = - \sum_{j=1}^{q+1} (-1)^j \frac{\delta^{(4(q-j)+3)}}{\varepsilon^j} + v_{4q+7}^\varepsilon, \tag{2.41}$$

where v_{4q+7}^ε satisfies the following equations

$$\varepsilon \frac{d^4 v_{4q+7}^\varepsilon}{dz^4} + v_{4q+7}^\varepsilon = (-1)^{q+1} \frac{H}{\varepsilon^{q+1}}, \quad \text{in } (-h, h), \tag{2.42}$$

$$v_{4q+7}^\varepsilon(z = h) = \frac{(-1)^{q+1}}{\varepsilon^{q+1}}, \quad v_{4q+7}^\varepsilon(z = -h) = 0, \tag{2.43}$$

$$\frac{d v_{4q+7}^\varepsilon}{dz}(h) = \frac{d v_{4q+7}^\varepsilon}{dz}(-h) = 0, \tag{2.44}$$

$$v_{4q+7}^\varepsilon \text{ is of class } C^3 \text{ at } z = 0. \tag{2.45}$$

We apply the same techniques to resolve the system of equations (2.42) to (2.45) (system of type $(\mathcal{S}_\varepsilon)$ with $\mathcal{B}_\varepsilon = (0, \dots, 0, -1/\varepsilon^{q+1}, 0, 0, 0) \in \mathbb{R}^8$) and we derive an approximation for v_{4q+7}^ε and then for v^ε

$$\begin{aligned} \bar{v}^\varepsilon(z) = & - \sum_{j=1}^q (-1)^j \frac{\delta^{(4(q-j)+3)}}{\varepsilon^j} + \frac{(-1)^q}{2 \varepsilon^{q+1}} \cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \\ & \times \left\{ \exp\left(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \chi_{(0,h)} - \exp\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \chi_{(-h,0)} \right\}. \end{aligned} \tag{2.46}$$

Finally, we gather all these results in the following proposition:

Proposition 2.2. *The solution v^ε of (2.16)-(2.17) possesses an approximation to within an e.s.t. denoted by \bar{v}^ε , i.e.,*

$$v^\varepsilon = \bar{v}^\varepsilon + e.s.t., \tag{2.47}$$

where \bar{v}^ε is given by (2.24), (2.32), (2.39) and (2.46) according to the value of k , the e.s.t. term being exponentially small(er) in all classical spaces (such as C^k, H^m, \dots).

Remark 6. We can obviously derive the full explicit expressions of u^ε or v^ε , but the results of Proposition 2.1 and Proposition 2.2, which make explicit their (simpler) approximations \bar{u}^ε and \bar{v}^ε , will be sufficient to derive the corrector and prove the convergence results in higher dimensions.

3. PROOF OF THEOREM 1.1

This section is devoted to the proof of Theorem 1.1. First, we start by defining the corrector and then we prove the estimates (1.5), (1.6) and (1.7).

3.1. Construction of the corrector θ^ε . Based on the resolution of an analogous one-dimensional problem to (2.1)-(2.2) but with nontrivial boundary conditions, (for which we may prove a result analogous to Proposition 2.1), we propose the following corrector the validity of which will be rigorously justified later:

$$\bar{\theta}^{\varepsilon,0} = \bar{U}^\varepsilon - u^0. \tag{3.1}$$

Here \bar{U}^ε is an approximation of U^ε solution of the following system

$$(\mathcal{C}_\varepsilon) \quad \begin{cases} \varepsilon \frac{\partial^4 U^\varepsilon}{\partial z^4} + U^\varepsilon = H, & \text{in } (-h, h), \\ U^\varepsilon = \Phi_0(x, y, \cdot), & \text{at } z = \pm h, \\ \frac{\partial U^\varepsilon}{\partial z} = -\Phi_1(x, y, \cdot), & \text{at } z = \pm h. \end{cases}$$

The resolution of $(\mathcal{C}_\varepsilon)$ is based on the same techniques as those used to solve (2.1)-(2.2) in Section 2.1. Thus, the expression of \bar{U}^ε can be made explicit. After lengthy but straight forward calculations, the expression of $\bar{\theta}^{\varepsilon,0}$ is given by

$$\begin{aligned} \bar{\theta}^{\varepsilon,0}(x, y, z) = \exp\left(\frac{z-h}{\sqrt{2\varepsilon^{1/4}}}\right) & \left\{ \left[-(\beta_\varepsilon + \gamma_\varepsilon) + (\beta_\varepsilon + \gamma_\varepsilon)\Phi_0(h) \right. \right. \\ & \left. \left. + \sqrt{2}\varepsilon^{1/4}\gamma_\varepsilon\Phi_1(h) \right] \cos\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) + \left[\beta_\varepsilon - \gamma_\varepsilon \right. \right. \\ & \left. \left. - (\beta_\varepsilon - \gamma_\varepsilon)\Phi_0(h) - \sqrt{2}\varepsilon^{1/4}\beta_\varepsilon\Phi_1(h) \right] \sin\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \right\} \\ & - \frac{1}{2} \exp\left(-\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \cos\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right), \end{aligned} \tag{3.2}$$

for all $(x, y, z) \in \Omega^+$ and

$$\begin{aligned} \bar{\theta}^{\varepsilon,0}(x, y, z) = \exp\left(-\frac{z+h}{\sqrt{2\varepsilon^{1/4}}}\right) & \left\{ \left[(\beta_\varepsilon + \gamma_\varepsilon)\Phi_0(-h) \right. \right. \\ & \left. \left. - \sqrt{2}\varepsilon^{1/4}\gamma_\varepsilon\Phi_1(-h) \right] \cos\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) + \left[(\beta_\varepsilon - \gamma_\varepsilon) \right. \right. \\ & \left. \left. \times \Phi_0(-h) - \sqrt{2}\varepsilon^{1/4}\beta_\varepsilon\Phi_1(-h) \right] \sin\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \right\} \\ & + \frac{1}{2} \exp\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right) \cos\left(\frac{z}{\sqrt{2\varepsilon^{1/4}}}\right), \end{aligned} \tag{3.3}$$

for all $(x, y, z) \in \Omega^-$. Here we have set

$$\Omega^+ = \{(x, y, z) \in \Omega / z > 0\}, \quad \Omega^- = \{(x, y, z) \in \Omega / z < 0\}.$$

Note that in (3.2), (3.3) and hereafter, some terms such as $\Phi_i(\pm h)$ may depend on x or y , but for simplicity this dependence is not mentioned in our writing. Secondly, observe that the functions $\bar{\theta}^{\varepsilon,0}$ and their derivatives are not continuous at $z = 0$. Hence, after some computations, the equation and

boundary conditions for $\bar{\theta}^{\varepsilon,0}$ read

$$\begin{aligned} \varepsilon \frac{\partial^4 \bar{\theta}^{\varepsilon,0}}{\partial z^4} + \bar{\theta}^{\varepsilon,0} &= -\varepsilon \delta^{(3)} + \varepsilon \alpha(\varepsilon) \lambda_3(x, y) \delta^{(3)} + \frac{\varepsilon^{3/4}}{\sqrt{2}} \alpha(\varepsilon) \lambda_2(x, y) \delta'' \\ &+ \frac{\varepsilon^{1/2}}{2} \alpha(\varepsilon) \lambda_1(x, y) \delta' + \frac{\varepsilon^{1/4}}{2\sqrt{2}} \alpha(\varepsilon) \lambda_0(x, y) \delta, \quad \text{in } \Omega, \end{aligned} \tag{3.4}$$

$$\bar{\theta}^{\varepsilon,0} = -u^0 + \Phi_0 - \frac{\text{sign}(z)}{4} \alpha(\varepsilon), \quad \text{on } \Gamma, \tag{3.5}$$

$$\frac{\partial \bar{\theta}^{\varepsilon,0}}{\partial \nu} = \Phi_1 - \frac{\varepsilon^{-1/4}}{4} \alpha(\varepsilon), \quad \text{on } \Gamma, \tag{3.6}$$

where

$$\lambda_i(x, y) = \lambda_i^+(x, y) - \lambda_i^-(x, y), \quad i \in \{0, 1, 2, 3\}, \tag{3.7}$$

and

$$\begin{aligned} \lambda_3^+(x, y) &= -(\beta_\varepsilon + \gamma_\varepsilon) + (\beta_\varepsilon + \gamma_\varepsilon) \Phi_0(h) + \sqrt{2} \gamma_\varepsilon \varepsilon^{1/4} \Phi_1(h), \\ \lambda_3^-(x, y) &= (\beta_\varepsilon + \gamma_\varepsilon) \Phi_0(-h) - \sqrt{2} \gamma_\varepsilon \varepsilon^{1/4} \gamma_\varepsilon \Phi_1(-h), \\ \lambda_2^+(x, y) &= \frac{1}{2} - 2\gamma_\varepsilon + 2\gamma_\varepsilon \Phi_0(h) - \sqrt{2} (\beta_\varepsilon - \gamma_\varepsilon) \varepsilon^{1/4} \Phi_1(h), \\ \lambda_2^-(x, y) &= \frac{1}{2} - 2\gamma_\varepsilon \Phi_0(-h) + \sqrt{2} (\gamma_\varepsilon - \beta_\varepsilon) \varepsilon^{1/4} \Phi_1(-h), \\ \lambda_1^+(x, y) &= 2 [(\beta_\varepsilon - \gamma_\varepsilon) + (\gamma_\varepsilon - \beta_\varepsilon) \Phi_0(h) - \sqrt{2} \beta_\varepsilon \varepsilon^{1/4} \Phi_1(h)], \\ \lambda_1^-(x, y) &= 2 [-(\beta_\varepsilon - \gamma_\varepsilon) \Phi_1(h) + \sqrt{2} \beta_\varepsilon \varepsilon^{1/4} \Phi_1(-h)], \\ \lambda_0^+(x, y) &= -1 + 4\beta_\varepsilon - 4\beta_\varepsilon \Phi_0(h) - 2\sqrt{2} (\beta_\varepsilon + \gamma_\varepsilon) \varepsilon^{1/4} \Phi_1(h), \\ \lambda_0^-(x, y) &= -1 + 4\beta_\varepsilon \Phi_0(-h) - 2\sqrt{2} (\gamma_\varepsilon + \beta_\varepsilon) \varepsilon^{1/4} \Phi_1(-h). \end{aligned}$$

More precisely, the coefficients λ_p^\pm are given by

$$\lambda_p^+(x, y) = \frac{\partial^p \bar{\theta}^{\varepsilon,0}}{\partial z^p} \Big|_{z=0^+}, \quad \lambda_p^-(x, y) = \frac{\partial^p \bar{\theta}^{\varepsilon,0}}{\partial z^p} \Big|_{z=0^-}, \quad \forall p \in \{0, 1, 2\},$$

and λ_3^\pm satisfy

$$\bar{\theta}^{\varepsilon,0}(x, y, 0^+) = -\frac{1}{2} + \lambda_3^+(x, y) \alpha(\varepsilon), \quad \bar{\theta}^{\varepsilon,0}(x, y, 0^-) = \frac{1}{2} + \lambda_3^-(x, y) \alpha(\varepsilon).$$

We note the presence of the term $\varepsilon \delta^{(3)}$ in the right-hand side of the equation (3.4) of $\bar{\theta}^{\varepsilon,0}$. This term will absorb the main singularity produced by replacing, formally, u^ε by u^0 in (1.1). The other singularities ($-\varepsilon \alpha(\varepsilon) \lambda_3(x, y) \delta^{(3)}$, $-\varepsilon^{3/4} \alpha(\varepsilon) \lambda_2(x, y) \delta''$, ...) are of lower order since they are multiplied by $\alpha(\varepsilon)$ which is an *e.s.t.* in $H^m(\Omega)$ for all $m \in \mathbb{N}$. However, we cannot apply, for

the moment, the standard energy method for $u^\varepsilon - u^0 - \bar{\theta}^{\varepsilon,0}$ since the right-hand side of its equation does not belong to $L^2(\Omega)$. In order to resolve this difficulty, we define the main form of the corrector θ^ε as follows:

$$\theta^\varepsilon = \bar{\theta}^{\varepsilon,0} - \varphi^\varepsilon, \tag{3.8}$$

where φ^ε is designed to absorb the above mentioned singularities:

$$\varphi^\varepsilon(x, y, z) = \sum_{p=0}^3 \left[\lambda_{3-p}^+(x, y) \frac{z^p}{p! \varepsilon^{p/4}} \chi_{(0,h)} - \lambda_{3-p}^-(x, y) \frac{(-z)^p}{p! \varepsilon^{p/4}} \chi_{(-h,0)} \right] \alpha(\varepsilon). \tag{3.9}$$

Thanks to (3.4)–(3.6) and (3.8), we deduce that θ^ε verifies the following equations

$$\varepsilon \frac{\partial^4 \theta^\varepsilon}{\partial z^4} + \theta^\varepsilon = -\varepsilon \delta^{(3)} - \varphi^\varepsilon, \quad \text{in } \Omega, \tag{3.10}$$

$$\theta^\varepsilon = -u^0 + \Phi_0 - \frac{\text{sign}(z)}{4} \alpha(\varepsilon) - \varphi^\varepsilon, \quad \text{on } \Gamma, \tag{3.11}$$

$$\frac{\partial \theta^\varepsilon}{\partial \nu} = \Phi_1 - \frac{\varepsilon^{-1/4}}{4} \alpha(\varepsilon) + \frac{\partial \varphi^\varepsilon}{\partial z}, \quad \text{on } \Gamma, \tag{3.12}$$

The corrector θ^ε is now completely defined.

3.2. The L^2 , H^1 , and H^2 estimates. Let $w^\varepsilon = u^\varepsilon - u^0 - \theta^\varepsilon$ which satisfies

$$\varepsilon \Delta^2 w^\varepsilon + w^\varepsilon = \varphi^\varepsilon - \varepsilon \sum_{\substack{i,j=1 \\ i+j \neq 6}}^3 \frac{\partial^4 \theta^\varepsilon}{\partial x_i^2 \partial x_j^2}, \quad \text{in } \Omega, \tag{3.13}$$

$$w^\varepsilon = \frac{\text{sign}(z)}{4} \alpha(\varepsilon) + \varphi^\varepsilon, \quad \text{on } \Gamma, \tag{3.14}$$

$$\frac{\partial w^\varepsilon}{\partial \nu} = \frac{\varepsilon^{-1/4}}{4} \alpha(\varepsilon) - \frac{\partial \varphi^\varepsilon}{\partial z}, \quad \text{on } \Gamma; \tag{3.15}$$

here we have denoted x, y , and z by x_1, x_2 , and x_3 .

Now, we intend to apply the standard energy estimates to (3.13)–(3.13). For this purpose, we transform the boundary conditions of w^ε into the homogeneous ones by setting

$$\tilde{w}^\varepsilon = w^\varepsilon - \psi^\varepsilon, \tag{3.16}$$

where ψ^ε is a polynomial function in z “small” in all $H^m(\Omega)$ and given by

$$\psi^\varepsilon(x, y, z) = \left[-\frac{\alpha(\varepsilon)}{2} + \varphi^\varepsilon(-h) - \varphi^\varepsilon(h) + h(\varphi_z^\varepsilon(h) + \varphi_z^\varepsilon(-h) - \frac{\varepsilon^{-1/4}}{2} \alpha(\varepsilon)) \right] \frac{z^3}{4h^3}$$

$$\begin{aligned}
 & - \left[\varphi_z^\varepsilon(-h) - \varphi_z^\varepsilon(h) \right] \frac{z^2}{4h} - \left[\frac{3}{h} \left(-\frac{\alpha(\varepsilon)}{2} + \varphi^\varepsilon(-h) - \varphi^\varepsilon(h) \right) \right. \\
 & + \varphi_z^\varepsilon(h) + \varphi_z^\varepsilon(-h) - \left. \frac{\varepsilon^{-1/4}}{2} \alpha(\varepsilon) \right] \frac{z}{4} + \frac{\varphi^\varepsilon(-h) + \varphi^\varepsilon(h)}{2} \\
 & + \frac{h}{4} \left(\varphi_z^\varepsilon(-h) - \varphi_z^\varepsilon(h) \right), \quad \forall (x, y, z) \in \Omega. \tag{3.17}
 \end{aligned}$$

Precise estimates for ψ^ε are given below, see (3.24). Hence, \tilde{w}^ε satisfies

$$\varepsilon \Delta^2 \tilde{w}^\varepsilon + \tilde{w}^\varepsilon = \varphi^\varepsilon - \varepsilon \sum_{\substack{i,j=1 \\ i+j \neq 6}}^3 \frac{\partial^4 \theta^\varepsilon}{\partial x_i^2 \partial x_j^2} - \psi^\varepsilon - \varepsilon \sum_{\substack{i,j=1 \\ i+j \neq 6}}^3 \frac{\partial^4 \psi^\varepsilon}{\partial x_i^2 \partial x_j^2}, \quad \text{in } \Omega, \tag{3.18}$$

$$\tilde{w}^\varepsilon = \frac{\partial \tilde{w}^\varepsilon}{\partial \nu} = 0, \quad \text{on } \Gamma, \tag{3.19}$$

and

$$\tilde{w}^\varepsilon \text{ is periodic with periods } L_1 \text{ and } L_2 \text{ in the } x \text{ and } y \text{ directions.} \tag{3.20}$$

We multiply (3.18) by \tilde{w}^ε , integrate over Ω and we apply the Green formula to the fourth order term. Thanks to (3.19) and (3.20), all the boundary terms canceled and we obtain

$$\begin{aligned}
 \varepsilon \|\Delta \tilde{w}^\varepsilon\|_{L^2(\Omega)}^2 + \|\tilde{w}^\varepsilon\|_{L^2(\Omega)}^2 & = \left(\varphi^\varepsilon + \varepsilon \sum_{\substack{i,j=1 \\ i+j \neq 6}}^3 \frac{\partial^4 \theta^\varepsilon}{\partial x_i^2 \partial x_j^2}, \tilde{w}^\varepsilon \right)_{L^2(\Omega)} \\
 & + \left(-\psi^\varepsilon + \varepsilon \sum_{\substack{i,j=1 \\ i+j \neq 6}}^3 \frac{\partial^4 \psi^\varepsilon}{\partial x_i^2 \partial x_j^2}, \tilde{w}^\varepsilon \right)_{L^2(\Omega)}. \tag{3.21}
 \end{aligned}$$

Notice that the differentiation in the tangential direction (x) or (y) cancels the singularities δ and δ' , produced by the discontinuity of θ^ε at $z = 0$, in the term $\partial^2 \theta^\varepsilon / \partial z^2$. Consequently, the right-hand side of (3.21) is regular since the term $\sum_{\substack{i,j=1 \\ i+j \neq 6}}^3 (\partial^4 \theta^\varepsilon / \partial x_i^2 \partial x_j^2)$ is in $L^2(\Omega)$. By applying the Cauchy-Schwarz and Young inequalities for the right-hand side of (3.21), we deduce that

$$\begin{aligned}
 \varepsilon \|\Delta \tilde{w}^\varepsilon\|_{L^2(\Omega)}^2 + \|\tilde{w}^\varepsilon\|_{L^2(\Omega)}^2 & \leq c \|\psi^\varepsilon\|_{L^2(\Omega)}^2 + c \|\varphi^\varepsilon\|_{L^2(\Omega)}^2 \\
 + c\varepsilon^2 \sum_{\substack{i,j=1 \\ i+j \neq 6}}^3 \left\| \frac{\partial^4 \theta^\varepsilon}{\partial x_i^2 \partial x_j^2} \right\|_{L^2(\Omega)}^2 & + c\varepsilon^2 \sum_{\substack{i,j=1 \\ i+j \neq 6}}^3 \left\| \frac{\partial^4 \psi^\varepsilon}{\partial x_i^2 \partial x_j^2} \right\|_{L^2(\Omega)}^2. \tag{3.22}
 \end{aligned}$$

On the other hand, we infer from the explicit expression of φ^ε given by (3.9) that

$$\|\varphi^\varepsilon\|_{L^2(\Omega)} \leq c \varepsilon^{-3/4} \alpha(\varepsilon), \tag{3.23}$$

and from (3.17) follows the following estimate

$$\|\psi^\varepsilon\|_{H^m(\Omega)} \leq c \varepsilon^{-3/4} \alpha(\varepsilon), \quad \forall m \in \mathbb{N}. \tag{3.24}$$

Hence, taking into account (3.24) and differentiating the expression of θ^ε ((3.2), (3.3) and (3.8)) with respect to x_i and x_j two times, we deduce that

$$\sum_{\substack{i,j=1 \\ i+j \neq 6}}^3 \left\| \frac{\partial^4 \theta^\varepsilon}{\partial x_i^2 \partial x_j^2} \right\|_{L^2(\Omega)} \leq C \left\{ \left\| \frac{\partial^2 \theta^\varepsilon}{\partial z^2} \right\|_{L^2(\Omega^+)} + \left\| \frac{\partial^2 \theta^\varepsilon}{\partial z^2} \right\|_{L^2(\Omega^-)} \right\} \leq c\varepsilon^{-3/8}. \tag{3.25}$$

Combining (3.23), (3.24) and (3.25) into (3.22), we obtain the estimates (1.5) and (1.7). Finally, the estimate (1.6) is obtained by interpolation. This ends the proof of Theorem 1.1.

4. PROOF OF THEOREM 1.2

In this section, we will show how Theorem 1.1 can be extended to the solution u_k^ε of an equation with a stronger singularity for the source function as in (1.8). We will derive the correctors corresponding to the problem (1.8) with the boundary conditions (1.2) and then prove the convergence results as in Section 3. For this purpose and based on the well known technique in boundary layers theory which consists in neglecting the tangential derivatives if compared to the normal ones, we start in a first step by considering the space dimension one. In a second step, we reduce the singularity of the right-hand side of the equation (1.8). Thus, we propose to look for v_k^ε solution of this system

$$(C_\varepsilon^k) \quad \begin{cases} \varepsilon \frac{\partial^4 v_k^\varepsilon}{\partial z^4} + v_k^\varepsilon = \delta^{(k)}, & \text{in } (-h, h), \\ v_k^\varepsilon = \Phi_0(x, y, \cdot), & \text{at } z = \pm h, \\ \frac{\partial v_k^\varepsilon}{\partial z} = -\Phi_1(x, y, \cdot), & \text{at } z = \pm h. \end{cases}$$

At this stage, we intend to use the techniques applied in Section 2.2. Similarly, we have to consider four cases according to the values of k , and we will give the form of the corrector for the four cases. However, we will prove the convergence results of Theorem 1.2 just for one case and the others can be treated in a similar way.

Case 1. For $k = 4q + 3, q \in \mathbb{N}$, as in Subsection 2.2, we may reduce the singularity of the source function. By applying a simple recursive formula and by taking into account the convention (2.19), we deduce that

$$v_{4q+3}^\varepsilon = - \sum_{j=1}^{q+1} (-1)^j \frac{\delta^{(4(q-j)+3)}}{\varepsilon^j} + \tilde{v}_{4q+3}^\varepsilon, \tag{4.1}$$

where $\tilde{v}_{4q+3}^\varepsilon$ verifies the following system

$$\begin{cases} \varepsilon \frac{\partial^4 \tilde{v}_{4q+3}^\varepsilon}{\partial z^4} + \tilde{v}_{4q+3}^\varepsilon = \frac{(-1)^{q+1}}{\varepsilon^{q+1}} H, & \text{in } (-h, h), \\ \tilde{v}_{4q+3}^\varepsilon = \Phi_0(x, y, h) + \frac{(-1)^{q+1}}{\varepsilon^{q+1}}, & \text{at } z = h, \\ \tilde{v}_{4q+3}^\varepsilon = \Phi_0(x, y, -h), & \text{at } z = -h, \\ \frac{\partial \tilde{v}_{4q+3}^\varepsilon}{\partial z} = -\Phi_1(x, y, \cdot), & \text{at } z = \pm h. \end{cases}$$

Let us recall that the resolution of the above system leads to a linear system of type $(\mathcal{S}_\varepsilon)$ with $\mathcal{B}_\varepsilon = (\Phi_0(x, y, h), -\sqrt{2}\varepsilon^{1/4}\Phi_1(x, y, h), \Phi_0(x, y, -h), -\sqrt{2}\varepsilon^{1/4}\Phi_1(x, y, -h), (-1)^q/\varepsilon^{q+1}, 0, 0, 0)$. The explicit expression of v_{4q+3}^ε is then deduced and as usual we only need to define an approximation of v_{4q+3}^ε , in the sense of Proposition 2.1 and Proposition 2.2, denoted by $\bar{v}_{4q+3}^\varepsilon$.

We are now ready to define the primary form of the corrector, $\bar{\theta}_{4q+3}^{\varepsilon,0} = \bar{v}_{4q+3}^\varepsilon - u_{4q+3}^0$, as follows

$$\begin{aligned} \bar{\theta}_{4q+3}^{\varepsilon,0}(x, y, z) &= - \sum_{j=0}^q (-1)^j \frac{\delta^{(4(q-j)+3)}}{\varepsilon^j} + \exp\left(\frac{z-h}{\sqrt{2}\varepsilon^{1/4}}\right) \\ &\quad \times \left\{ \nu_\varepsilon(x, y) \cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \xi_\varepsilon(x, y) \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right\} \\ &\quad - \frac{(-1)^{q+1}}{2\varepsilon^{q+1}} \exp\left(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right), \end{aligned} \tag{4.2}$$

for all $(x, y, z) \in \Omega^+$ and

$$\begin{aligned} \bar{\theta}_{4q+3}^{\varepsilon,0}(x, y, z) &= - \sum_{j=0}^q (-1)^j \frac{\delta^{(4(q-j)+3)}}{\varepsilon^j} + \exp\left(-\frac{z+h}{\sqrt{2}\varepsilon^{1/4}}\right) \\ &\quad \times \left\{ \eta_\varepsilon(x, y) \cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \sigma_\varepsilon(x, y) \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right\} \\ &\quad + \frac{(-1)^{q+1}}{2\varepsilon^{q+1}} \exp\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right), \end{aligned} \tag{4.3}$$

for all $(x, y, z) \in \Omega^-$, where we have set

$$\begin{aligned} \nu_\varepsilon(x, y) &= \beta_\varepsilon + \gamma_\varepsilon + \Phi_0(h) + \sqrt{2} \varepsilon^{1/4} \gamma_\varepsilon \Phi_1(h), \\ \xi_\varepsilon(x, y) &= -(\beta_\varepsilon - \gamma_\varepsilon) \Phi_0(h) - \sqrt{2} \varepsilon^{1/4} \beta_\varepsilon \Phi_1(h), \\ \eta_\varepsilon(x, y) &= (\beta_\varepsilon + \gamma_\varepsilon) \Phi_0(-h) - \sqrt{2} \varepsilon^{1/4} \gamma_\varepsilon \Phi_1(-h), \\ \sigma_\varepsilon(x, y) &= (\beta_\varepsilon - \gamma_\varepsilon) \Phi_0(-h) - \sqrt{2} \varepsilon^{1/4} \beta_\varepsilon \Phi_1(-h). \end{aligned}$$

Recall that the above function is not continuous at $z = 0$. More precisely, $\bar{\theta}_{4q+3}^{\varepsilon,0}$ contains two kind of singularities; the first one is aimed at absorbing the main singularity produced by u_k^0 at $z = 0$ and the second one is a consequence of the approximations used previously, where $\tilde{v}_{4q+3}^\varepsilon$ was approximated by $\bar{v}_{4q+3}^\varepsilon$. As in Section 3, we define the final form of the corrector by setting

$$\theta_{4q+3}^\varepsilon = \bar{\theta}_{4q+3}^{\varepsilon,0} - \varphi_k^\varepsilon, \tag{4.4}$$

where φ_k^ε is a correcting function which is “small” in all the spaces $H^m(\Omega)$, $\forall m \in \mathbb{N}$ and given by

$$\begin{aligned} \varphi_k^\varepsilon(x, y, z) &= \sum_{p=0}^3 \left[\lambda_k^{p,+}(x, y) \frac{z^{3-p}}{(3-p)! \varepsilon^{p/4}} \chi_{(0,h)} \right] \alpha(\varepsilon) \\ &\quad - \sum_{p=0}^3 \left[\lambda_k^{p,-}(x, y) \frac{(-z)^{3-p}}{(3-p)! \varepsilon^{p/4}} \chi_{(-h,0)} \right] \alpha(\varepsilon). \end{aligned} \tag{4.5}$$

Here we have set, for all $k \in \mathbb{N}$:

$$\begin{aligned} \lambda_k^{3,+}(x, y) &= \nu_\varepsilon(x, y), & \lambda_k^{3,-}(x, y) &= \eta_\varepsilon(x, y), \\ \lambda_k^{2,+}(x, y) &= \nu_\varepsilon(x, y) + \xi_\varepsilon(x, y), & \lambda_k^{2,-}(x, y) &= -\eta_\varepsilon(x, y) + \sigma_\varepsilon(x, y), \\ \lambda_k^{1,+}(x, y) &= \xi_\varepsilon(x, y), & \lambda_k^{1,-}(x, y) &= -\sigma_\varepsilon(x, y), \\ \lambda_k^{0,+}(x, y) &= 2[\xi_\varepsilon(x, y) - \nu_\varepsilon(x, y)], & \lambda_k^{0,-}(x, y) &= 2[\eta_\varepsilon(x, y) + \sigma_\varepsilon(x, y)]. \end{aligned}$$

After some computations, we see that $\theta_{4q+3}^\varepsilon$ verifies the following equations

$$\varepsilon \frac{\partial^4 \theta_{4q+3}^\varepsilon}{\partial z^4} + \theta_{4q+3}^\varepsilon = -\varepsilon \delta^{(k+4)} - \varphi^\varepsilon, \quad \text{in } \Omega, \tag{4.6}$$

$$\theta^\varepsilon = \Phi_0 - \frac{\text{sign}(z)}{4\varepsilon^{q+1}} \alpha(\varepsilon) - \varphi^\varepsilon, \quad \text{on } \Gamma, \tag{4.7}$$

$$\frac{\partial \theta^\varepsilon}{\partial \nu} = \Phi_1 - \frac{1}{4\varepsilon^{q+5/4}} \alpha(\varepsilon) + \frac{\partial \varphi^\varepsilon}{\partial z}, \quad \text{on } \Gamma. \tag{4.8}$$

Now we define $w_{4q+3}^\varepsilon = u_{4q+3}^\varepsilon - u_{4q+3}^0 - \theta_{4q+3}^\varepsilon$. Thanks to (1.2), (1.9), (4.12) and (4.13), we deduce the boundary values of w_{4q+3}^ε which read

$$\begin{aligned} w_{4q+3}^\varepsilon &= \frac{\text{sign}(z)}{4\varepsilon^{q+1}}\alpha(\varepsilon) + \varphi^\varepsilon, & \text{on } \Gamma, \\ \frac{\partial w_{4q+3}^\varepsilon}{\partial \nu} &= \frac{1}{4\varepsilon^{q+5/4}}\alpha(\varepsilon) - \frac{\partial \varphi^\varepsilon}{\partial z}, & \text{on } \Gamma. \end{aligned}$$

Afterwards, in order to apply the standard energy methods, we transform the boundary conditions of w_{4q+3}^ε into the homogeneous ones by setting

$$\tilde{w}_{4q+3}^\varepsilon = w_{4q+3}^\varepsilon - \psi_{4q+3}^\varepsilon, \tag{4.9}$$

where ψ_{4q+3}^ε is a “small” polynomial function in z , given by

$$\begin{aligned} &\psi_{4q+3}^\varepsilon(x, y, z) \\ &= \left[-\frac{\alpha(\varepsilon)}{2\varepsilon^{q+1}} + \varphi^\varepsilon(-h) - \varphi^\varepsilon(h) + h(\varphi_z^\varepsilon(h) + \varphi_z^\varepsilon(-h) - \frac{\alpha(\varepsilon)}{2\varepsilon^{q+5/4}}) \right] \frac{z^3}{4h^3} \\ &- \left[\varphi_z^\varepsilon(-h) - \varphi_z^\varepsilon(h) \right] \frac{z^2}{4h} - \left[\frac{3}{h} \left(-\frac{\alpha(\varepsilon)}{2\varepsilon^{q+5/4}} + \varphi^\varepsilon(-h) - \varphi^\varepsilon(h) \right) \right. \\ &\left. + \varphi_z^\varepsilon(h) + \varphi_z^\varepsilon(-h) - \frac{\alpha(\varepsilon)}{2\varepsilon^{q+5/4}} \right] \frac{z}{4} + \frac{\varphi^\varepsilon(-h) + \varphi^\varepsilon(h)}{2} + \frac{h}{4} (\varphi_z^\varepsilon(-h) - \varphi_z^\varepsilon(h)), \end{aligned} \tag{4.10}$$

$\forall (x, y, z) \in \Omega$. Hence, $\tilde{w}_{4q+3}^\varepsilon$ verifies the following equations

$$\begin{aligned} \varepsilon \Delta^2 \tilde{w}_{4q+3}^\varepsilon + \tilde{w}_{4q+3}^\varepsilon &= \varphi_{4q+3}^\varepsilon - \varepsilon \sum_{\substack{i,j=1 \\ i+j \neq 6}}^3 \frac{\partial^4 \theta_{4q+3}^\varepsilon}{\partial x_i^2 \partial x_j^2} \\ &- \psi_{4q+3}^\varepsilon - \varepsilon \sum_{\substack{i,j=1 \\ i+j \neq 6}}^3 \frac{\partial^4 \psi_{4q+3}^\varepsilon}{\partial x_i^2 \partial x_j^2}, \quad \text{in } \Omega, \end{aligned} \tag{4.11}$$

$$\tilde{w}_{4q+3}^\varepsilon = \frac{\partial \tilde{w}_{4q+3}^\varepsilon}{\partial \nu} = 0, \quad \text{on } \Gamma, \tag{4.12}$$

and

$$\tilde{w}_{4q+3}^\varepsilon \text{ is periodic with periods } L_1 \text{ and } L_2 \text{ in the } x \text{ and } y \text{ directions.} \tag{4.13}$$

We multiply (4.11) by $\tilde{w}_{4q+3}^\varepsilon$, integrate over Ω , apply the boundary and periodicity conditions ((4.12) and (4.13)) and obtain

$$\varepsilon \|\Delta \tilde{w}_{4q+3}^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\tilde{w}_{4q+3}^\varepsilon\|_{L^2(\Omega)}^2 \leq \|\psi_{4q+3}^\varepsilon\|_{L^2(\Omega)}^2 + \|\varphi_{4q+3}^\varepsilon\|_{L^2(\Omega)}^2$$

$$+ \varepsilon^2 \sum_{\substack{i,j=1 \\ i+j \neq 6}}^3 \left\| \frac{\partial^4 \theta_{4q+3}^\varepsilon}{\partial x_i^2 \partial x_j^2} \right\|_{L^2(\Omega)}^2 + \varepsilon^2 \sum_{\substack{i,j=1 \\ i+j \neq 6}}^3 \left\| \frac{\partial^4 \psi_{4q+3}^\varepsilon}{\partial x_i^2 \partial x_j^2} \right\|_{L^2(\Omega)}^2. \tag{4.14}$$

Furthermore, observe that

$$\left\| \varphi_{4q+3}^\varepsilon \right\|_{L^2(\Omega)} \leq c \varepsilon^{-q-7/4} \alpha(\varepsilon), \tag{4.15}$$

and, that (4.10), ψ_{4q+3}^ε verifies

$$\left\| \psi_{4q+3}^\varepsilon \right\|_{H^m(\Omega)} \leq c \varepsilon^{-q-7/4} \alpha(\varepsilon), \quad \forall m \in \mathbb{N}. \tag{4.16}$$

Using the expression of $\theta_{4q+3}^\varepsilon$ given by (4.2), (4.3) and (4.4), we deduce that

$$\sum_{\substack{i,j=1 \\ i+j \neq 6}}^3 \left\| \frac{\partial^4 \theta_{4q+3}^\varepsilon}{\partial x_i^2 \partial x_j^2} \right\|_{L^2(\Omega)} \leq C \left\{ \left\| \frac{\partial^3 \theta_{4q+3}^\varepsilon}{\partial x \partial z^2} \right\|_{L^2(\Omega^+)} + \left\| \frac{\partial^3 \theta_{4q+3}^\varepsilon}{\partial x \partial z^2} \right\|_{L^2(\Omega^-)} \right\} \leq c \varepsilon^{-3/8}. \tag{4.17}$$

Combining (4.15), (4.16) and (4.17) into (4.14) and since $\alpha(\varepsilon)$ is given by (2.7), we infer the estimates (1.10) and (1.12); the estimate (1.11) is then obtained by a simple interpolation.

Case 2. Similarly, for $k = 4q + 2$, $q \in \mathbb{N}$, we start by reducing the singularity of the right-hand side of the first equation in $(\mathcal{C}_\varepsilon^k)$. By induction, the right-hand side $\delta^{(k)}$ will be transformed to $(-1/\varepsilon)^{q+1} H_2$ and we get a linear system of type $(\mathcal{S}_\varepsilon)$ with $\mathcal{B}_\varepsilon = (\Phi_0(x, y, h), -\sqrt{2}\varepsilon^{1/4}\Phi_1(x, y, h), \Phi_0(x, y, -h), -\sqrt{2}\varepsilon^{1/4}\Phi_1(x, y, -h), 0, (-1)^q\sqrt{2}/\varepsilon^{q+3/4}, 0, 0)$. The resolution of this system leads to the definition of an approximate solution from which we remove the limit solution u_{4q+3}^0 . Finally, we obtain the primary form of the corrector as follows

$$\begin{aligned} \bar{\theta}_{4q+2}^{\varepsilon,0}(x, y, z) &= - \sum_{j=0}^q (-1)^j \frac{\delta^{(4(q-j)+2)}}{\varepsilon^j} + \exp\left(\frac{z-h}{\sqrt{2}\varepsilon^{1/4}}\right) \\ &\times \left\{ \nu_\varepsilon(x, y) \cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \xi_\varepsilon(x, y) \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right\} \\ &+ \frac{(-1)^{q+1}\sqrt{2}}{4\varepsilon^{q+3/4}} \exp\left(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \left[\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) - \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right], \end{aligned} \tag{4.18}$$

for all $(x, y, z) \in \Omega^+$ and,

$$\bar{\theta}_{4q+2}^{\varepsilon,0}(x, y, z) = - \sum_{j=0}^q (-1)^j \frac{\delta^{(4(q-j)+2)}}{\varepsilon^j} + \exp\left(-\frac{z+h}{\sqrt{2}\varepsilon^{1/4}}\right) \tag{4.19}$$

$$\begin{aligned} & \times \left\{ \eta_\varepsilon(x, y) \cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \sigma_\varepsilon(x, y) \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right\} \\ & + \frac{(-1)^{q+1}\sqrt{2}}{4\varepsilon^{q+3/4}} \exp\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \left[\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right], \end{aligned}$$

for all $(x, y, z) \in \Omega^-$. We note that $\bar{\theta}_{4q+2}^{\varepsilon,0}$ contains two kinds of discontinuities at $z = 0$; the first one is aimed at absorbing the main singularity produced by the discontinuity of u_k^0 at $z = 0$, namely $\varepsilon\delta^{(k+4)}$, that we will keep here; the second one results from the approximations used above. In order to eliminate the second singularity, we introduce another corrector function and obtain the final form of the corrector $\theta_{4q+2}^\varepsilon = \bar{\theta}_{4q+2}^{\varepsilon,0} - \varphi_k^\varepsilon$, where φ_k^ε is given by (3.9). Hence, the equation for $\theta_{4q+2}^\varepsilon$ reads

$$\varepsilon\Delta^2\theta_{4q+2}^\varepsilon + \theta_{4q+2}^\varepsilon = -\varepsilon\delta^{(k+4)} - \varphi_k^\varepsilon + \varepsilon \sum_{\substack{i,j=1 \\ i+j \neq 6}}^3 \frac{\partial^4 \theta_{4q+2}^\varepsilon}{\partial x_i^2 \partial x_j^2}. \tag{4.20}$$

Finally, since the right-hand side of (4.20) is regular ($\in L^2(\Omega)$), we may apply the standard energy estimates for $\tilde{w}_{4q+2}^\varepsilon = w_{4q+2}^\varepsilon - \psi_{4q+2}^\varepsilon$, where $w_{4q+2}^\varepsilon = u_{4q+3}^\varepsilon - u_{4q+3}^0 - \theta_{4q+3}^\varepsilon$ and, ψ_{4q+2}^ε is a “small” polynomial function in z .

Case 3. For $k = 4q + 1$, $q \in \mathbb{N}$, we deduce from the resolution of the corresponding one-dimensional problem the primary form of the corrector

$$\begin{aligned} \bar{\theta}_{4q+1}^{\varepsilon,0}(x, y, z) &= - \sum_{j=0}^q (-1)^j \frac{\delta^{(4(q-j)+1)}}{\varepsilon^j} + \exp\left(\frac{z-h}{\sqrt{2}\varepsilon^{1/4}}\right) \\ & \times \left\{ \nu_\varepsilon(x, y) \cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \xi_\varepsilon(x, y) \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right\} \\ & + \frac{(-1)^{q+1}}{2\varepsilon^{q+1/2}} \exp\left(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right), \end{aligned} \tag{4.21}$$

for all $(x, y, z) \in \Omega^+$ and

$$\begin{aligned} \bar{\theta}_{4q+1}^{\varepsilon,0}(x, y, z) &= - \sum_{j=0}^q (-1)^j \frac{\delta^{(4(q-j)+1)}}{\varepsilon^j} + \exp\left(-\frac{z+h}{\sqrt{2}\varepsilon^{1/4}}\right) \\ & \times \left\{ \eta_\varepsilon(x, y) \cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \sigma_\varepsilon(x, y) \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right\} \\ & + \frac{(-1)^{q+1}}{2\varepsilon^{q+1/2}} \exp\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right), \end{aligned} \tag{4.22}$$

for all $(x, y, z) \in \Omega^-$. Now, we define as follows the final form of the corrector by removing the singularities produced by the approximations used above

$$\theta_{4q+1}^\varepsilon = \bar{\theta}_{4q+1}^{\varepsilon,0} - \varphi_k^\varepsilon.$$

Finally, the corrector $\theta_{4q+1}^\varepsilon$ verifies an equation similar to (4.20) and then we may apply the same techniques, as in the first case, to prove the estimates stated in Theorem 1.2.

Case 4. For $k = 4q$, $q \in \mathbb{N}$, we follow the same steps as in the above cases and we obtain first the primary form of the corrector which is given by

$$\begin{aligned} \bar{\theta}_{4q}^{\varepsilon,0}(x, y, z) &= - \sum_{j=0}^q (-1)^j \frac{\delta^{(4(q-j))}}{\varepsilon^j} + \exp\left(\frac{z-h}{\sqrt{2}\varepsilon^{1/4}}\right) \\ &\quad \times \left\{ \nu_\varepsilon(x, y) \cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \xi_\varepsilon(x, y) \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right\} \\ &\quad - \frac{(-1)^{q+1}\sqrt{2}}{4\varepsilon^{q+1/4}} \exp\left(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \left[\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right], \end{aligned} \tag{4.23}$$

for all $(x, y, z) \in \Omega^+$ and

$$\begin{aligned} \bar{\theta}_{4q}^{\varepsilon,0}(x, y, z) &= - \sum_{j=0}^q (-1)^j \frac{\delta^{(4(q-j)+2)}}{\varepsilon^j} + \exp\left(-\frac{z+h}{\sqrt{2}\varepsilon^{1/4}}\right) \\ &\quad \times \left\{ \eta_\varepsilon(x, y) \cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \sigma_\varepsilon(x, y) \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right\} \\ &\quad - \frac{(-1)^{q+1}\sqrt{2}}{4\varepsilon^{q+1/4}} \exp\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \left[\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) - \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \right], \end{aligned} \tag{4.24}$$

for all $(x, y, z) \in \Omega^-$.

Second, we define the final form of the corrector which reads in this case

$$\theta_{4q}^\varepsilon = \bar{\theta}_{4q}^{\varepsilon,0} - \varphi_k^\varepsilon,$$

where φ_k^ε is still given by (3.9) small in all $L^2(\Omega)$. Hence, the corrector is now well defined and it verifies an equation similar to (4.20). This completes the study of all the cases and ends the proof of Theorem 1.2. \square

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