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ON SOME SHARP CONDITIONS FOR LOWER SEMICONTINUITY IN L^1

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Abstract. Let Ω be an open set of $\mathbb{R}^n$ and let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n$ be a nonnegative continuous function, convex with respect to $\xi \in \mathbb{R}^n$. Following the well known theory originated by Serrin [14] in 1961, we deal with the lower semicontinuity of the integral $F(u, \Omega) = \int_\Omega f(x, u(x), Du(x))\, dx$ with respect to the $L^1_{\mathrm{loc}}(\Omega)$ strong convergence. Only recently it has been discovered that dependence of $f(x, s, \xi)$ on the x variable plays a crucial role in the lower semicontinuity. In this paper we propose a mild assumption on x that allows us to consider discontinuous integrands too. More precisely, we assume that $f(x, s, \xi)$ is a nonnegative *Carathéodory* function, convex with respect to ξ, continuous in (s, ξ) and such that $f(\cdot, s, \xi) \in W^{1,1}_{\mathrm{loc}}(\Omega)$ for every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$, with the L^1 norm of $f_x(\cdot, s, \xi)$ locally bounded. We also discuss some other conditions on x; in particular we prove that Hölder continuity of f with respect to x is not sufficient for lower semicontinuity, even in the one dimensional case, thus giving an answer to a problem posed by the authors in [12]. Finally, we investigate the lower semicontinuity of the integral $F(u, \Omega)$, with respect to the strong norm topology of $L^1_{\mathrm{loc}}(\Omega)$, in the vector-valued case, i.e., when $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \to \mathbb{R}$ for some $n \geq 1$ and $m > 1$.

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1. Introduction

Let us consider a functional F of integral type

$$F(u,\Omega) = \int_\Omega f(x, u(x), Du(x))\, dx\,, \tag{1.1}$$

where Ω is an open set of $\mathbb{R}^n$, u varies in the Sobolev class $W^{1,1}_{loc}(\Omega)$, Du denotes the gradient of u and the function $f = f(x,s,\xi)$ is defined for $x \in \Omega$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$.

A classical problem in the calculus of variations is the study of the sequential lower semicontinuity of F on $W^{1,1}_{loc}$ in the norm topology of $L^1_{\mathrm{loc}}(\Omega)$, i.e., to find out sufficient conditions on f to ensure

$$\begin{cases} u_h,\, u \in W^{1,1}_{loc}(\Omega) \\ u_h \to u \quad \text{in } L^1_{loc}(\Omega) \end{cases} \Longrightarrow F(u,\Omega) \le \liminf_{h\to\infty} F(u_h,\Omega)\,. \tag{1.2}$$

A necessary condition for (1.2) is the convexity of $f(x,s,\cdot)$ for every (x,s) fixed. Furthermore, it is possible to give counterexamples to (1.2) if f is not bounded from below. Hence, a reasonable (and classical) set of hypotheses on f, to obtain (1.2), is

$$\begin{cases} f \textit{ is continuous, nonnegative in } \Omega \times \mathbb{R} \times \mathbb{R}^n, \\ f(x,s,\cdot) \textit{ is convex in } \mathbb{R}^n \textit{ for all } (x,s) \in \Omega \times \mathbb{R}. \end{cases} \tag{1.3}$$

However, Aronszajn's counterexample (see [13]) shows that in general (1.3) does not imply (1.2). Thus, more assumption than (1.3) on f is necessary to obtain the lower semicontinuity in (1.2).

The first relevant and well known result in this direction has been given by Serrin [14]. Some extensions of Serrin's Theorem are due to Dal Maso [5] and Ambrosio [1]. De Giorgi, Buttazzo and Dal Maso [7] considered integrands f independent of $x \in \mathbb{R}^n$ and measurable with respect to $s \in \mathbb{R}$. Recently, Fonseca and Leoni [10], [11] studied the vector-valued case too.

Serrin's Theorem states that the lower semicontinuity (1.2) holds if f satisfies (1.3) and *one* of the following (*disjoint*) hypotheses

(a) $f(x,s,\xi) \to +\infty$, as $|\xi| \to +\infty$, for every (x,s) fixed,
(b) $f(x,s,\cdot)$ is strictly convex in $\mathbb{R}^n$, for every $(x,s) \in \Omega \times \mathbb{R}$,
(c) the derivatives f_x, f_ξ and $f_{x\xi}$ exist and are continuous.

Conditions (a) and (b) go in the direction of a strengthening of the convex structure of f in (1.3). In condition (c) the convexity of f is untouched; this is useful, just to make a simple but instructive example, to treat integrands f that for some values of (x,s) are constant, say equal to zero, with respect to ξ. Recently, Gori and Marcellini in [12] were able to improve the case (c) with the following result.

Theorem 1.1. *Let us assume that* $f(x, s, \xi)$ *satisfies* (1.3) *and that, for every compact set* $\Omega' \times H \times K \subset \Omega \times \mathbb{R} \times \mathbb{R}^n$*, there exists a constant* $L = L_{\Omega' \times H \times K}$ *such that*

$$|f(x_1, s, \xi) - f(x_2, s, \xi)| \leq L\, |x_1 - x_2|\,, \tag{1.4}$$

whenever $(x_1, s, \xi)\,, (x_2, s, \xi) \in \Omega' \times H \times K$*. Then the lower semicontinuity inequality in* (1.2) *holds.*

Theorem 1.1 outlines that, what seems to be really relevant in condition (c) of Serrin's Theorem, is a regularity assumption of *local* type and only on the x variable. Thus, the following question arises: is there an optimal assumption of regularity in the x variable to be added to (1.3) in order to get (1.2)? Some answers to this question can be found in this work.

We start discussing the sharpness of Theorem 1.1. In [12] the assumption (1.4) of local Lipschitz continuity of f with respect to x is compared with the weaker assumption of local Hölder continuity of f with respect to x, i.e., it is compared with the following property: for a real number $\alpha \in (0, 1)$ and for every fixed compact set $\Omega' \times H \times K$ contained in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ there exists a constant $L = L_{\Omega' \times H \times K}$ such that

$$|f(x_1, s, \xi) - f(x_2, s, \xi)| \leq L\, |x_1 - x_2|^{\alpha} \tag{1.5}$$

for every $(x_1, s, \xi)\,, (x_2, s, \xi) \in \Omega' \times H \times K$. Through examples, it is shown in [12] that, in general, for every exponent $\alpha \in (0, 1)$, there exists an n-*dimensional* integral $F(u, \Omega)$ (the dimension n depends on α, precisely $n > 4\alpha/(1-\alpha)$) which is not lower semicontinuous in L^1 and whose integrand f satisfies (1.3) and (1.5). This family of counterexamples leaves the following open question: for every $n \in \mathbb{N}$, is there a critical exponent $\alpha(n)$ such that (1.2) is implied by conditions (1.3) and (1.5), when the Hölder condition (1.5) holds for $\alpha \in [\alpha(n), 1)$?

In this paper we give a negative answer to this question showing that, for every $\alpha \in (0, 1)\,$, there exists a *one dimensional* integral F (depending on α) as in (1.1) with the required properties. The counterexample is given in Section 2.

This counterexample seems to suggest that, in a sense, assumption (1.4) in Theorem 1.1 is sharp. However, we have found that it can be significantly improved, once we change our point of view on the kind of regularity in the x variable that we are dealing with. To explain this idea, we first remark that (1.4) can be formulated in the following equivalent way:

$$f(\cdot,s,\xi)\in W^{1,\infty}_{\mathrm{loc}}(\Omega)\,,\quad \forall\,(s,\xi)\in\mathbb{R}\times\mathbb{R}^n,$$

and, for every $\Omega'\times H\times K\subset\subset\Omega\times\mathbb{R}\times\mathbb{R}^n$ there exists a constant $L=L_{\Omega'\times H\times K}$ such that

$$\operatorname*{ess\,sup}_{x\in\Omega'}\left|\frac{\partial f}{\partial x}(x,s,\xi)\right|\le L, \tag{1.6}$$

for every $(s,\xi)\in H\times K$. Thus, we try a comparison of (1.4) with *summability conditions on the weak derivatives* $\partial f/\partial x$, rather than with qualified continuity conditions like Hölder continuity in (1.5). This position reveals very fruitful as the following theorem, which is a direct generalization of Theorem 1.1, shows. It is not more necessary to assume that $f(x,s,\xi)$ is *continuous* with respect to all its arguments, but we can assume that f is a *Carathéodory* function, that is, f is measurable with respect to $x\in\Omega$ and continuous with respect to $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$.

Theorem 1.2. *Let us assume that*

$$\begin{cases} f \text{ is a locally bounded, Carathéodory function in } \Omega\times\mathbb{R}\times\mathbb{R}^n,\\ f \text{ is nonnegative in } \Omega\times\mathbb{R}\times\mathbb{R}^n,\\ f(x,s,\xi) \text{ is convex in } \xi\in\mathbb{R}^n \text{ for all } (x,s)\in\Omega\times\mathbb{R}. \end{cases} \tag{1.7}$$

Moreover, let us suppose that $f(\cdot,s,\xi)\in W^{1,1}_{loc}(\Omega)$ *for every* $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$ *and that, for every compact set* $\Omega'\times H\times K\subset\Omega\times\mathbb{R}\times\mathbb{R}^n$, *there exists a constant* $L=L_{\Omega'\times H\times K}$ *such that*

$$\int_{\Omega'}\left|\frac{\partial f}{\partial x}(x,s,\xi)\right|dx\le L, \tag{1.8}$$

for every $(s,\xi)\in H\times K$. *Then the lower semicontinuity condition in* (1.2) *holds.*

Theorem 1.2 is proved in Sections 3 and 4. Finally, we turn our attention to the *vector-valued* case, i.e., we consider an open set $\Omega\subset\mathbb{R}^n$, with integers $n,m\ge 1$, functions $u:\Omega\to\mathbb{R}^m$ (so that $Du(x)\in\mathbb{R}^{m\times n}$ for $x\in\Omega$), integrals $F(u,\Omega)$ as in (1.1) whose integrands f are defined in $\Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times n}$. This case has been recently exploited by Fonseca and Leoni [10], [11]. In this paper we give some semicontinuity results, presenting new aspects when compared with the theory by Fonseca and Leoni. For a detailed discussion, statements, and proofs we refer to Section 5.

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2. Hölder continuity versus Lipschitz continuity

In this section, we propose an example that originates from one in [12]. This kind of construction is inspired by an old example given by Aronszajn in 1941 (see Pauc [13], page 54 and following), and later exploited by Dal Maso (see Section 4 in [5]) in 1980.

As explained in the introduction, for every fixed $\alpha \in (0,1)$, we seek for a one dimensional integrand $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to [0,+\infty)$, that satisfies (1.3) and hypothesis (1.5) for the chosen α, but such that lower semicontinuity inequality (1.2) does not hold for some sequences u_h.

Theorem 2.1. *For every $\alpha \in (0,1)$ there exists a sequence $(u_h)_{h\in\mathbb{N}} \in W^{1,\infty}(0,1)$, which uniformly converges to $u \equiv 0$, and there exists a function $a : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ with the following properties:*

(i) *$a(x,s)$ is bounded and uniformly continuous for $(x,s) \in \overline{\Omega} \times \mathbb{R}$;*

(ii) *for every $s \in \mathbb{R}$, $a(x,s)$ is Hölder continuous (of exponent α) with respect to $x \in \overline{\Omega}$; more precisely, there exists a constant L such that*

$$|a(x_1,s) - a(x_2,s)| \le L\,|x_1 - x_2|^{\alpha}, \quad \forall\, x_1, x_2 \in \overline{\Omega},\ \forall\, s \in \mathbb{R};$$

(iii) *if we define*

$$f(x,s,\xi) = |a(x,s)\xi - 1|, \qquad (x,s,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R},$$

then $f(x,s,\xi)$ satisfies (1.3), the Hölder continuity property (1.5) and

$$\lim_{h\to+\infty} \int_{-1}^{1} f\left(x, u_h, u_h'\right)\,dx = 0\,; \qquad \int_{-1}^{1} f\left(x,0,0\right)\,dx = 2\,. \tag{2.1}$$

The proof is carried over into several steps.

Step 1 (Definition of u_h): Let $\Omega = (-1,1)$. For $h \in \mathbb{N}$, let $v_h = v_h(x)$, $x \in [0,1]$, be defined by

$$v_h(x) = \begin{cases} 0 & \text{for } 0 \le x \le \lambda_h \\ x - \lambda_h & \text{for } \lambda_h < x < 1 - \lambda_h \\ 1 - 2\lambda_h & \text{for } 1 - \lambda_h \le x \le 1, \end{cases} \tag{2.2}$$

where λ_h is a sequence of real numbers converging to zero, such that $\lambda_h \in (0,1/2)$ for every $h \in \mathbb{N}$. We extend v_h as an even function in the interval

$[-1,0)$. We have

$$\left\{x\in\Omega:\ v_h'(x)=0\right\}=(-1,-(1-\lambda_h))\cup(-\lambda_h,\lambda_h)\cup(1-\lambda_h,1)$$

and thus

$$\operatorname{meas}\left\{x\in\Omega:\ v_h'(x)=0\right\}=4\lambda_h\,,\quad\forall\,h\in\mathbb{N}. \tag{2.3}$$

Since $v_h(-1)=v_h(1)$, we again extend $v_h(x)$ by periodicity from $[-1,1]$ to all $\mathbb{R}$. For every $h\in\mathbb{N}$ let $u_h(x)$ be defined by

$$u_h(x)=c_h+\frac{1}{a_h}\,v_h\left(b_hx\right), \tag{2.4}$$

with $\{a_h\}$, $\{b_h\}$ sequences of integer numbers diverging to $+\infty$ as $h\to+\infty$ and c_h converging to zero. Therefore, u_h converges to $u\equiv 0$ in $L^\infty(\Omega)$.

Step 2 (Definition of a): Passing to a subsequence if necessary, we can assume that the graphs of the functions u_h are disjoint. Let us define

$$E_h=\left\{x\in(-1,1):\ u_h'(x)=0\right\}=\left\{x\in(-1,1):\ v_h'(b_hx)=0\right\}.$$

We have

$$u_h'(x)=\frac{b_h}{a_h}\,v_h'(b_hx)=\begin{cases}\pm\frac{b_h}{a_h} & \text{if } x\in[-1,1]-E_h\\ 0 & \text{if } x\in E_h\end{cases}$$

and thus by periodicity and by (2.3), we have

$$\operatorname{meas}\left(E_h\right)=\operatorname{meas}\left\{x\in\Omega:\ u_h'(x)=0\right\}=4\lambda_h\,,\quad\forall\,h\in\mathbb{N}. \tag{2.5}$$

For every $h\in\mathbb{N}$ and for $x\in[-1,1]-E_h$ and $s=u_h(x)$, we define

$$a(x,s)=\frac{1}{u_h'(x)}=\pm\frac{a_h}{b_h}\,,\qquad x\in[-1,1]-E_h\,. \tag{2.6}$$

We also define $a(x,0)$ by continuity:

$$a(x,s)=0,\quad \text{if } x\in[-1,1] \text{ and } s=0;$$

to this aim, since $u_h(x)\to 0$ as $h\to+\infty$, we impose the condition

$$\{|a(x,s)|:x\in[-1,1]-E_h\,,\ \ s=u_h(x)\}=\frac{a_h}{b_h}\to 0\,,\ \text{ as }\ h\to+\infty. \tag{2.7}$$

At this stage $a(x,s)$ has been defined as a continuous function on a closed subset of $\overline{\Omega}\times\mathbb{R}$. In the next step, we will extend it to the full $\overline{\Omega}\times\mathbb{R}$.

Step 3 (Extension of a to $\overline{\Omega}\times\mathbb{R}$): For every fixed $h\in\mathbb{N}$, $a(x,s)$ has been defined in Step 2 at the points $(x,s)\in S_h$, where S_h is the subset of the graph of u_h given by $S_h = \{(x,s) : x \in [-1,1] - E_h, \; ; s = u_h(x)\}$. By the analytic expression of v_h and u_h in (2.2), (2.4), we have

$$0 \le v_h \le 1 - 2\lambda_h\,, \qquad c_h \le u_h(x) = c_h + \frac{1}{a_h}\, v_h\,(b_h x) \le c_h + \frac{1-2\lambda_h}{a_h}\,;$$

therefore, the set S_h is contained in the rectangle

$$R_h = \Big\{(x,s) : \;\; x \in [-1,1]\,, \quad c_h \le s < c_h + \frac{1}{a_h}\Big\}\,.$$

We will extend $a(x,s)$ to the larger rectangle $R'_h \subset [-1,1] \times \mathbb{R}$, given by

$$R'_h = \Big\{(x,s) : \;\; x \in [-1,1]\,, \quad \frac{1}{2}\, c_h \le s \le \frac{3}{2} c_h + \frac{1}{a_h}\Big\}\,.$$

Passing possibly to a subsequence, we can assume that $R'_h \cap R'_k = \emptyset$ if $h \neq k$. Then we first extend a equal to zero out of the union $\bigcup_h R'_h$ (and in particular, by continuity, $a(x,s) = 0$ when $s = c_h/2$ and $s = 3c_h/2 + 1/a_h$).

In order to estimate the oscillation $|a(x_1,s_1) - a(x_2,s_2)|$ when (x_1,s_1), (x_2,s_2) vary in R'_h, we first consider (x_1,s_1), $(x_2,s_2) \in S_h$ and we prove the following Lipschitz estimate (with constant depending on h)

$$|a(x_1,s_1) - a(x_2,s_2)| \le \frac{a_h}{\lambda_h} \cdot |x_1 - x_2|\,. \tag{2.8}$$

In fact, under the conditions $x_1, x_2 \in [-1,1] - E_h$, $s_1 = u_h(x_1)$, $s_2 = u_h(x_2)$, when (x_1,s_1), (x_2,s_2) are in the same connected component of S_h, then $a(x_1,s_1) = a(x_2,s_2)$; otherwise $b_h \cdot |x_1 - x_2| \ge 2\lambda_h$. By the definition of $a(x,s)$ in (2.6), we have

$$|a(x_1,s_1) - a(x_2,s_2)| \le 2\,\frac{a_h}{b_h} \; \le a_h \cdot \frac{|x_1 - x_2|}{\lambda_h}\,.$$

Of course this estimate also gives

$$|a(x_1,s_1) - a(x_2,s_2)| \le \frac{a_h}{\lambda_h} \cdot (|x_1 - x_2| + |s_1 - s_2|)\,, \tag{2.9}$$

for every $h \in \mathbb{N}$ and for every (x_1,s_1), $(x_2,s_2) \in S_h$ with $s_1 = u_h(x_1)$, $s_2 = u_h(x_2)$.

Again, by the definition of $a(x,s)$ in (2.6), we have

$$\max\{|a(x,s)| : \; (x,s) \in S_h\} = \frac{a_h}{b_h}\,;$$

if $(x_1,s_1) \in S_h$, $(x_2,s_2) \in R'_h$ with either $s_2 = c_h/2$ and $s_2 = 3c_h/2 + 1/a_h$, then $a(x_2,s_2) = 0$ and $|s_1 - s_2| \ge c_h/2$. Therefore, we obtain

$$|a(x_1,s_1) - a(x_2,s_2)| = |a(x_1,s_1)| \le \frac{a_h}{b_h}$$

$$\leq \frac{a_h}{b_h} \cdot \frac{|s_1 - s_2|}{c_h/2} \leq \frac{a_h}{\lambda_h} \cdot (|x_1 - x_2| + |s_1 - s_2|)$$

if we pose the condition

$$c_h \geq 2\frac{\lambda_h}{b_h}\,. \tag{2.10}$$

This proves that the Lipschitz estimate (2.9) holds at every (x_1, s_1), $(x_2, s_2) \in R'_h$, where $a(x, s)$ has been already defined. By using Mac Shane lemma, we can extend it to the rectangle R'_h with the same Lipschitz constant as in (2.9). That is, we have

$$|a(x_1, s_1) - a(x_2, s_2)| \leq \frac{a_h}{\lambda_h} \cdot (|x_1 - x_2| + |s_1 - s_2|)\ , \tag{2.11}$$

for every $h \in \mathbb{N}$ and for every (x_1, s_1), $(x_2, s_2) \in R'_h$. Moreover, we can assume that a is bounded in R'_h and that the following bound holds

$$\max\left\{|a(x, s)| :\ (x, s) \in R'_h\right\} = \frac{a_h}{b_h}\,, \tag{2.12}$$

for every $h \in \mathbb{N}$ and for every $(x, s) \in R'_h$.

Step 4 (Hölder continuity of a): To test Hölder continuity of $a(x, s)$ with respect to x, we fix h and s and we estimate the quantity

$$\sup\left\{\frac{|a(x_1, s) - a(x_2, s)|}{|x_1 - x_2|^\alpha} :\ \ (x_1, s), (x_2, s) \in R'_h\right\}. \tag{2.13}$$

Let $t > 0$ be a new real parameter that we will choose later. We estimate the supremum in (2.13) separately for $|x_1 - x_2| \geq t$ and for $|x_1 - x_2| \leq t$.

Under the further condition $|x_1 - x_2| \geq t$, the supremum in (2.13) can be estimate by computing separately the maximum value of the numerator and the minimum value of the denominator. By (2.12) we have

$$\max\left\{|a(x_1, s) - a(x_2, s)| :\ \ (x_1, s), (x_2, s) \in R'_h\right\}$$

$$\leq 2\max\left\{|a(x, s)| :\ (x, s) \in R'_h\right\} = \frac{2a_h}{b_h}\,.$$

For the same s-values, since $|x_1 - x_2| \geq t$, we obtain

$$\sup\left\{\frac{|a(x_1, s) - a(x_2, s)|}{|x_1 - x_2|^\alpha} : (x_1, s), (x_2, s) \in R'_h |x_1 - x_2| \geq t\right\} \leq \frac{2a_h}{b_h\, t^\alpha}. \tag{2.14}$$

While, if $|x_1 - x_2| \leq t$, we use the Lipschitz estimate (2.11) (with constant depending on h) with $s_1 = s_2 \equiv s$

$$|a(x_1, s) - a(x_2, s)| \leq \frac{a_h}{\lambda_h} \cdot |x_1 - x_2|$$

and we obtain

$$\sup\left\{\frac{|a(x_1,s)-a(x_2,s)|}{|x_1-x_2|^{\alpha}}:\ (x_1,s),(x_2,s)\in R'_h\ \ |x_1-x_2|\le t\right\}\le \frac{a_h}{\lambda_h}\cdot t^{1-\alpha}\,. \tag{2.15}$$

From (2.14) and (2.15) we deduce that

$$\sup\left\{\frac{|a(x_1,s)-a(x_2,s)|}{|x_1-x_2|^{\alpha}}:\ \ (x_1,s),(x_2,s)\in R'_h\right\}$$

$$\le \max\left\{\frac{2a_h}{b_h}\,t^{-\alpha};\frac{a_h}{\lambda_h}\cdot t^{1-\alpha}\right\}\le 2a_h\cdot\max\left\{\frac{t^{-\alpha}}{b_h};\frac{t^{1-\alpha}}{\lambda_h}\right\}. \tag{2.16}$$

The above inequality is valid for every $t>0$. We consider the minimum of the right hand side with respect to $t>0$, which is assumed when $\frac{t^{-\alpha}}{b_h}=\frac{t^{1-\alpha}}{\lambda_h}$, i.e., when $t=\frac{\lambda_h}{b_h}$. We obtain that the Hölder quotient in the left hand side of (2.16) is less than or equal to

$$2\,\frac{a_h}{(\lambda_h)^{\alpha}\,(b_h)^{1-\alpha}}\ . \tag{2.17}$$

Previously, we estimated the Hölder continuity with respect to x of $a(x,s)$ in R'_h, for every fixed $h\in\mathbb{N}$. Thus, to obtain the Hölder continuity of $a(x,s)$ with respect to x, with $(x,s)\in\overline{\Omega}\times\mathbb{R}$, we impose the further condition that the sequence in (2.17) remains bounded, i.e., there exists $L>0$ such that

$$\frac{a_h}{(\lambda_h)^{\alpha}\,(b_h)^{1-\alpha}}\le L\,,\qquad\forall\,h\in\mathbb{N}. \tag{2.18}$$

Step 5 (Lower semicontinuity test): Let us prove that

$$\lim_{h\to+\infty}\int_{-1}^{1} f\left(x,u_h,u'_h\right)\,dx=0\ . \tag{2.19}$$

By the definition (2.6), for every $h\in\mathbb{N}$, we obtain

$$f\left(x,u_h,Du_h\right)=\left|a(x,u_h)\cdot u'_h-1\right|=0\,,\qquad\forall\,x\in[-1,1]-E_h\,.$$

Thus, since $u'_h=0$ on E_h, we have

$$\int_{-1}^{1} f\left(x,u_h,u'_h\right)\,dx=\int_{E_h} f\left(x,u_h,u'_h\right)\,dx$$

$$\le\int_{E_h}\left\{\max_{\Omega\times\mathbb{R}}|a(x,s)|\cdot\left|u'_h\right|+1\right\}dx=4\,\lambda_h\,, \tag{2.20}$$

which converges to zero as $h\to+\infty$, since $\lambda_h\to 0$ as $h\to+\infty$.

Step 6 (Sufficient conditions): Looking above, we required the following limit relations (see in particular (2.7), (2.18), (2.20) and (2.10))

$$\begin{cases} a_h \to +\infty\,, \quad b_h \to +\infty\,, \quad c_h \to 0\,, \quad \lambda_h \to 0\,, \\ \frac{a_h}{b_h} \to 0\,, \\ \frac{a_h}{(\lambda_h)^\alpha (b_h)^{1-\alpha}} \le L\,, \ \ c_h \ge 2\,\frac{\lambda_h}{b_h}\,, \quad \forall\, h \in \mathbb{N}\,. \end{cases} \tag{2.21}$$

It remains to exhibit sequences of real parameters which satisfy the limit relations (2.21). To this aim, we consider for example, a parameter $q > 1/(1-\alpha)$, $q \in \mathbb{N}$, $q > 1$ and

$$a_h = h, \quad b_h = h^q, \quad \frac{a_h}{(\lambda_h)^\alpha (b_h)^{1-\alpha}} = 1.$$

Since $q > 1$, then $\frac{a_h}{b_h} \to 0$. Moreover, the last condition gives

$$(\lambda_h)^\alpha = \frac{a_h}{(b_h)^{1-\alpha}} = h^{1-q(1-\alpha)}$$

and we obtain that $\lambda_h \to 0$, since $q > 1/(1-\alpha)$. Finally, we pose

$$\frac{c_h}{2} = \frac{\lambda_h}{b_h} = h^{\frac{1-q(1-\alpha)}{\alpha} - q} = h^{\frac{1-q(1-\alpha)-q\alpha}{\alpha}} = h^{\frac{1-q}{\alpha}}$$

which converges to zero, since $q > 1$.

3. Preliminaries

In this section we give some preliminary results (standard in the context of lower semicontinuity problems) that will be used in the proof of the Theorem 1.2.

Definition 3.1. Let Ω be a bounded open set in $\mathbb{R}^n$. We say that a function $v \in W^{1,\infty}(\Omega)$ is *piecewise affine* in Ω if there exists a finite partition of Ω into open sets $\{\Omega_j\}_{j=1}^N$, $N \in \mathbb{N}$, and a set of measure zero, i.e.,

$$\Omega_j \cap \Omega_h = \emptyset, \quad \forall j,\, h \in \{1, ..., N\},\, j \neq h, \quad \left|\Omega \setminus \bigcup_{j=1}^N \Omega_j\right| = 0,$$

such that v is affine on each Ω_j, i.e., there exists a $\xi_j \in \mathbb{R}^n$ and a $q_j \in \mathbb{R}$ such that

$$v(x) = (\xi_j, x) + q_j, \quad \forall x \in \Omega_j,\, j \in \{1, ..., N\}.$$

We shall denote the space of such functions with $Aff(\Omega)$.

The following theorem holds (see Theorem 1.8, Chapter 2 in [4]).

Theorem 3.2. *Let Ω be a bounded open set with Lipschitz boundary and let $u \in W_0^{1,p}(\Omega)$, $1 \leq p < \infty$. Then for every $\varepsilon > 0$ there exists $v_\varepsilon \in Aff_0(\Omega)$ such that*

$$||u - v_\varepsilon||_{W^{1,p}(\Omega)} < \varepsilon.$$

Here, the classical generalization of Lusin's Theorem by Scorza Dragoni, that has revealed to be very useful when dealing with Carathéodory integrands (see [9]).

Theorem 3.3 (Scorza Dragoni)**.** *Let Ω be an open set of $\mathbb{R}^n$ of finite measure, let $g : \Omega \times \mathbb{R} \to \mathbb{R}^n$ be a Carathéodory function. Then, for every $\tau > 0$, there exists an open set Ω_τ with $|\Omega_\tau| < \tau$ such that g is continuous on $(\Omega \backslash \Omega_\tau) \times \mathbb{R}$.*

The following approximation result has been given by De Giorgi (see [6]).

Lemma 3.4. *Assume that $f = f(x, s, \xi)$ satisfies* (1.7) *and has compact support in $\Omega \times \mathbb{R}$. Then there exists an increasing sequence of functions $\{f_j(x, s, \xi)\}_{j \in \mathbb{N}}$ such that:*

(i) *$f_j(x, s, \xi)$ converges to $f(x, s, \xi)$ on $\Omega \times \mathbb{R} \times \mathbb{R}^n$;*

(ii) *for every $j \in \mathbb{N}$, $f_j(x, s, \xi)$ satisfies* (1.7)*, and has compact support in $\Omega \times \mathbb{R}$;*

(iii) *for every $j \in \mathbb{N}$ there exist constants M_j with the following properties:*

$$\begin{cases} |f_j(x,s,\xi)| \leq M_j\,(1 + |\xi|)\,, \ \forall\,(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \\ |f_j(x,s,\xi_1) - f_j(x,s,\xi_2)| \leq M_j\,|\xi_1 - \xi_2|\,, \ \forall\,(x,s) \in \Omega \times \mathbb{R},\ \xi_1, \xi_2 \in \mathbb{R}^n. \end{cases} \tag{3.1}$$

Proof. Here follows a description of the f_j. The complete proof can be found in [6]. For a suitable sequence of mollifiers $\{\alpha^{(t)}\}_{t=1}^\infty$, i.e., $\alpha^{(t)} \in C_c^\infty(\mathbb{R}^n)$, with $\alpha^{(t)} \geq 0$ and $\int_{\mathbb{R}^n} \alpha^{(t)}(\eta)\,d\eta = 1$, the functions

$$\begin{aligned} a_h^{(t)}(x,s) &= -\int_{\mathbb{R}^n} f(x,s,\eta)\alpha_{\xi_h}^{(t)}(\eta)\,d\eta\,, \quad \forall\,h = 1,2,\ldots,n, \qquad (3.2) \\ a_0^{(t)}(x,s) &= \int_{\mathbb{R}^n} f(x,s,\eta)\Big\{(n+1)\alpha^{(t)}(\eta) + \sum_{h=1}^n \eta_h \alpha_{\xi_h}^{(t)}(\eta)\Big\}\,d\eta\,, \end{aligned}$$

are considered, and then put together in

$$f_j(x,s,\xi) = \max_{1 \leq j \leq t} \Big\{0, a_0^{(t)}(x,s) + \sum_{h=1}^n a_h^{(t)}(x,s)\xi_h\Big\}.$$

4. Proof of the lower semicontinuity theorem

The proof is structured in the following way. We first prove a lower semicontinuity result under technical hypotheses (Lemma 4.1); this is the core of the proof. Then we show how to apply Lemma 3.4 in order to get Theorem 1.2 from the previous results.

Lemma 4.1. *Let us assume that f satisfies* (1.7) *and that:*

(i) *the derivative $f_\xi(x,s,\xi)$ exists and is a Carathéodory function in $\Omega\times\mathbb{R}\times\mathbb{R}^n$ such that $f_\xi(\cdot,s,\xi)\in W^{1,1}_{loc}(\Omega,\mathbb{R}^n)$, for every $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$; moreover, for each $\Omega'\times H\times K\subset\subset\Omega\times\mathbb{R}\times\mathbb{R}^n$, there exists a constant $L_{\Omega'\times H\times K}$ such that*

$$\int_{\Omega'}\Big|\frac{\partial f_\xi}{\partial x}(x,s,\xi)\Big|dx\le L_{\Omega'\times H\times K},\quad\forall\ (s,\xi)\in H\times K;$$

(ii) *there exists a constant M such that,*

$$\begin{cases}|f_\xi(x,s,\xi)|\le M,\ \forall\ (x,s,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}^n,\\ |f_\xi(x,s,\xi_1)-f_\xi(x,s,\xi_2)|\le M\,|\xi_1-\xi_2|\,,\ \forall\ (x,s)\in\Omega\times\mathbb{R},\ \xi_1,\xi_2\in\mathbb{R}^n.\end{cases}\tag{4.1}$$

Then (1.2) *holds.*

Proof. Let $\{\gamma_i\}_{i\in\mathbb{N}}$ be an increasing sequence of smooth functions with compact support in $\Omega\times\mathbb{R}$, converging pointwise to 1 in $\Omega\times\mathbb{R}$. Since $\gamma_i(x,s)f(x,s,\xi)$ is an increasing sequence of functions which pointwise converges to $f(x,s,\xi)$, by a standard argument, it is sufficient to prove the stated lower semicontinuity assuming directly that there exists $\Omega'\times H\subset\subset\Omega\times\mathbb{R}$ such that $f(x,s,\xi)=0$, for every $(x,s,\xi)\in(\Omega\backslash\Omega')\times(\mathbb{R}\backslash H)\times\mathbb{R}^n$, (which implies $F(u,\Omega)=F(u,\Omega')$, $\forall u\in W^{1,1}_{loc}(\Omega)$). In this position hypotheses (i) of Lemma 4.1 become $f_\xi(\cdot,s,\xi)\in W^{1,1}(\Omega,\mathbb{R}^n)$, for every $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$, and

$$\int_{\Omega}\Big|\frac{\partial f_\xi}{\partial x}(x,s,\xi)\Big|dx\le L_K\,,\quad\forall\,(s,\xi)\in\mathbb{R}\times K\,,\tag{4.2}$$

for a suitable constant L_K depending on $K\subset\subset\mathbb{R}^n$.

Let $u_h,u\in W^{1,1}_{\mathrm{loc}}(\Omega)$ such that $u_h\to u$ in $L^1_{\mathrm{loc}}(\Omega)$. We will prove that

$$\liminf_{h\to+\infty}F(u_h,\Omega)\ge F(u,\Omega)\,.\tag{4.3}$$

Without loss of generality, we can assume that

$$\liminf_{h\to+\infty}F(u_h,\Omega)=\lim_{h\to+\infty}F(u_h,\Omega)<\infty,$$

and that u_h converges almost everywhere to u in Ω. By the local boundedness of f and by (4.1), it results that there exists a constant M' such that

$$|f(x,s,\xi)| \le M'\,(1+|\xi|)\,, \quad \forall (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n.$$

In particular we have $F(u,\Omega) < \infty$.

Since $u \in W^{1,1}(\Omega')$, and since $\partial\Omega'$ can be supposed Lipschitz, we can extend u outside Ω' to a function of $W_0^{1,1}(\Omega'')$, where $\Omega'' \subset\subset \Omega$ has Lipschitz boundary and $\Omega' \subset\subset \Omega''$.

Let us fix $\varepsilon > 0$. By Theorem 3.2, there exists $v_\varepsilon \in Aff_0(\Omega'')$ such that, in particular,

$$\int_{\Omega'} |Du - Dv_\varepsilon|\, dx \le \varepsilon\,; \tag{4.4}$$

by Fatou Lemma and the finiteness of $F(u,\Omega)$, we can also choose v_ε such that

$$\int_{\Omega'} f(x,u,Dv_\varepsilon)\,dx \ge \int_{\Omega'} f(x,u,Du)\,dx\, - \varepsilon\,. \tag{4.5}$$

Since $v_\varepsilon \in Aff(\Omega')$, referring to Definition 3.1 we shall consider $\{\Omega'_j\}_{j=1}^N$ such that $Dv_\varepsilon = \xi_j \in \mathbb{R}^n$ in Ω'_j.

Now let us take a sequence of functions $\{\beta_k^\varepsilon\}_{k=1}^\infty \subset C^\infty(\Omega')$ such that,

$$\begin{cases} 0 \le \beta_k^\varepsilon(x) \le \beta_{k+1}^\varepsilon(x) \le 1, \quad \forall x \in \Omega', \\ \beta_k^\varepsilon|_{\Omega'_j} \in C_c^\infty(\Omega'_j), \quad \lim_{k\to\infty} \beta_k^\varepsilon(x) = 1, \quad a.e. \quad x \in \Omega'. \end{cases}$$

By Beppo Levi's Theorem, there exists an index k_ε such that, for every $k \ge k_\varepsilon$, we have

$$\int_{\Omega'} \beta_k^\varepsilon(x) f(x,u,Dv_\varepsilon)\,dx \ge \int_{\Omega'} \beta_k^\varepsilon(x) f(x,u,Du)\,dx\, - 2\varepsilon\,. \tag{4.6}$$

We estimate the difference of the integrands in (4.3)

$$\begin{aligned} f(x,u_h,Du_h) - f(x,u,Du) &= f(x,u_h,Du_h) - f(x,u_h,Dv_\varepsilon) \\ &+ f(x,u_h,Dv_\varepsilon) - f(x,u,Dv_\varepsilon) + f(x,u,Dv_\varepsilon) - f(x,u,Du)\,. \end{aligned} \tag{4.7}$$

By the convexity of $f(x,s,\xi)$ with respect to ξ, for the first difference in the right side of (4.7) we have:

$$\begin{aligned} f(x,u_h,Du_h) - f(x,u_h,Dv_\varepsilon) &\ge (f_\xi(x,u_h,Dv_\varepsilon), Du_h - Dv_\varepsilon) \\ &= (f_\xi(x,u_h,Dv_\varepsilon), Du_h) - (f_\xi(x,u,Dv_\varepsilon), Du) \\ &+ (f_\xi(x,u,Dv_\varepsilon), Du - Dv_\varepsilon) + (f_\xi(x,u,Dv_\varepsilon) - f_\xi(x,u_h,Dv_\varepsilon), Dv_\varepsilon)\,. \end{aligned}$$

Now by multiplying for β_k^ε and integrating over Ω', we obtain

$$\begin{aligned}
&\int_{\Omega'} \beta_k^\varepsilon(x) \{f(x,u_h,Du_h) - f(x,u,Du)\}\, dx \\
&\ge \int_{\Omega'} \beta_k^\varepsilon(x) \{(f_\xi(x,u_h,Dv_\varepsilon), Du_h) - (f_\xi(x,u,Dv_\varepsilon), Du)\}\, dx \\
&+ \int_{\Omega'} \beta_k^\varepsilon(x) \left(f_\xi(x,u,Dv_\varepsilon), Du - Dv_\varepsilon\right) dx \\
&+ \int_{\Omega'} \beta_k^\varepsilon(x) \left(f_\xi(x,u,Dv_\varepsilon) - f_\xi(x,u_h,Dv_\varepsilon), Dv_\varepsilon\right) dx \\
&+ \int_{\Omega'} \beta_k^\varepsilon(x) \{f(x,u_h,Dv_\varepsilon) - f(x,u,Dv_\varepsilon)\}\, dx \\
&+ \int_{\Omega'} \beta_k^\varepsilon(x) \{f(x,u,Dv_\varepsilon) - f(x,u,Du)\}\, dx.
\end{aligned}$$

We note that, by (4.1), $|f_\xi(x,s,Dv_\varepsilon(x))| \le M$ for every $(x,s) \in \Omega \times \mathbb{R}$ and for every v_ε; so, by (4.4), we have

$$\int_{\Omega'} \beta_k^\varepsilon(x) \left(f_\xi(x,u,Dv_\varepsilon), Du - Dv_\varepsilon\right) dx \ge -M \int_{\Omega'} |Du - Dv_\varepsilon|\; dx \ge -M\varepsilon\,;$$

moreover, since $(x,s) \to f(x,s,Dv_\varepsilon(x))$ and $(x,s) \to f_\xi(x,s,Dv_\varepsilon(x))$ are bounded functions, by the Lebesgue's dominated convergence theorem we have,

$$\lim_{h\to\infty} \int_{\Omega'} \beta_k^\varepsilon(x) \{f(x,u_h,Dv_\varepsilon) - f(x,u,Dv_\varepsilon)\}\, dx = 0,$$

$$\lim_{h\to\infty} \int_{\Omega'} \beta_k^\varepsilon(x) \left(f_\xi(x,u,Dv_\varepsilon) - f_\xi(x,u_h,Dv_\varepsilon), Dv_\varepsilon\right) dx = 0;$$

eventually we recall (4.6), to get that, for every $\varepsilon > 0$ and for every v_ε, $k \ge k_\varepsilon$, we have:

$$\begin{aligned}
&\liminf_{h\to\infty} \int_{\Omega'} \beta_k^\varepsilon(x) \{f(x,u_h,Du_h) - f(x,u,Du)\}\, dx \\
&\ge \liminf_{h\to\infty} \int_{\Omega'} \beta_k^\varepsilon(x)\{(f_\xi(x,u_h,Dv_\varepsilon), Du_h) - (f_\xi(x,u,Dv_\varepsilon), Du)\}dx - M\varepsilon - 2\varepsilon.
\end{aligned}$$

Then, to complete the proof, it remains to show that for every fixed v_ε and $k \ge k_\varepsilon$, we have

$$\lim_{h\to+\infty} \int_{\Omega'} \beta_k^\varepsilon(x)\{(f_\xi(x,u_h,Dv_\varepsilon), Du_h) - (f_\xi(x,u,Dv_\varepsilon), Du)\}\, dx = 0. \quad (4.8)$$

Indeed, since $0 \leq \beta_k^\varepsilon(x) \leq 1$, we have

$$\liminf_{h\to\infty} \int_{\Omega'} f(x,u_h,Du_h)dx \geq \int_{\Omega'} \beta_k^\varepsilon(x) f(x,u,Du)dx - M\varepsilon - 2\varepsilon,$$

for every $k \geq k_\varepsilon$; letting $k \to \infty$, by Beppo Levi's theorem, we obtain

$$\liminf_{h\to\infty} \int_{\Omega'} f(x,u_h,Du_h)dx \geq \int_{\Omega'} f(x,u,Du)dx - M\varepsilon - 2\varepsilon.$$

Now the dependence from v_ε is vanished and so we can let $\varepsilon \to 0$ to gain the conclusion (4.3).

So it remains to prove (4.8); we stress that it suffices to prove this relation for every v_ε and k *fixed.* In order to achieve this we prove that

$$\Big| \int_{\Omega'} \beta_k^\varepsilon(x) \left\{ (f_\xi(x,u_h,Dv_\varepsilon),Du_h) - (f_\xi(x,u,Dv_\varepsilon),Du) \right\} \, dx \Big| \\ \leq \int_{\Omega'} \sum_{i=1}^n \Big| \int_{u(x)}^{u_h(x)} \frac{\partial}{\partial x_i} \left(\beta_k^\varepsilon(x) f_{\xi_i}(x,s,\xi_j)\right) ds \Big| dx. \tag{4.9}$$

Indeed let us consider the function $g(x,s) = \beta_k^\varepsilon(x) f_\xi(x,s,Dv_\varepsilon(x))$. For σ small enough

$$g_\sigma(x,s) = \int_{B_\sigma(x)} k_\sigma(x-y) g(y,s) dy,$$

is defined on $\Omega' \times \mathbb{R}$, with $g_\sigma(\cdot,s) \in C^\infty(\Omega';\mathbb{R}^n)$ for every s fixed (here k_σ is the usual mollifier). In particular, for every small σ, and u, $v \in W^{1,1}(\Omega)$,

$$\int_{\Omega'} (g_\sigma(x,v(x)),Dv(x)) - (g_\sigma(x,u(x)),Du(x)) \, dx \\ = -\int_{\Omega'} \Big\{ \sum_{i=1}^n \int_{u(x)}^{v(x)} \frac{\partial g_\sigma^{(i)}}{\partial x_i}(x,s) ds \Big\} dx\,;$$

now we prove that we can pass to the limit as $\sigma \to 0$ in this formula to gain (4.9). If we look, for example, at

$$\lim_{\sigma\to 0} \int_{\Omega'} (g_\sigma(x,v(x)),Dv(x)) \, dx = \int_{\Omega'} (g(x,v(x)),Dv(x)) \, dx\,,$$

because of the boundedness of g and the summability of Dv, we only need to prove that

$$\lim_{\sigma\to 0} g_\sigma(x,v(x)) = g(x,v(x)), \tag{4.10}$$

for a.e. $x \in \Omega'$. The following proof of (4.10) has been suggested to us by the referee, shortening the original argument. Let us note that, from standard properties of convolutions, we only know that for every fixed $s \in \mathbb{R}$ there exists a set K_s of full measure in Ω' such that $\lim_{\sigma\to 0} g_\sigma(x,s) = g(x,s)$ for every $x \in K_s$. Yet to prove (4.10) we need to show that a set of full measure K_s with this property can be chosen independently from s. To show this, we apply Scorza-Dragoni's Theorem in order to find a numerable disjoint family of compact sets $(K_i)_{i=1}^\infty$ such that $\mathcal{L}^n\left(\Omega' \setminus \bigcup_{i=1}^\infty K_i\right) = 0$ and g is (uniformly) continuous on $K_i \times \mathbb{R}$ for every $i \in \mathbb{R}$. Then we consider the set $C = \bigcup_{i=1}^\infty C_i$ where we put $C_i = \{x \in K_i : \theta^n(K_i, x) = 1\}$, (here $\theta^n(K_i,x) = \lim_{\rho\to 0}\left(\mathcal{L}^n(B_\rho(x)\cap K_i)/\omega_n\rho^n\right)$ is as usual the n-dimensional density of the set K_i at the point x). By Lebesgue Density Theorem $\mathcal{L}^n(K_i \setminus C_i) = 0$, and hence C is of full measure in Ω'. By proving that

$$\lim_{\sigma\to 0} g_\sigma(x,s) = g(x,s), \quad \forall (x,s) \in C \times \mathbb{R}, \tag{4.11}$$

we conclude the proof of (4.10). To see the validity of (4.11) it is sufficient to show that every $x \in C$ is a Lebesgue point of $g(\cdot, s)$ for every $s \in \mathbb{R}$. Let us fix $x \in C_i$ for some i and $\varepsilon > 0$. By the choice of K_i there exists $\rho_\varepsilon > 0$ such that

$$|g(x,s) - g(y,s)| \le \varepsilon, \quad \forall (y,s) \in (B_{\rho_\varepsilon}(x) \cap K_i) \times \mathbb{R},$$

and hence, for every $\rho < \rho_\varepsilon$, we have

$$\frac{1}{\omega_n \rho^n}\int_{B_\rho(x)} |g(x,s) - g(y,s)|dy \le \varepsilon \frac{\mathcal{L}^n(B_\rho(x)\cap K_i)}{\omega_n\rho^n} + 2M\frac{\mathcal{L}^n(B_\rho(x)\setminus K_i)}{\omega_n\rho^n},$$

where we have used the boundedness of g. Letting $\rho \to 0$, by the definition of n-dimensional density and by $x \in C_i$, we find

$$\lim_{\rho\to 0}\frac{1}{\omega_n\rho^n}\int_{B_\rho(x)} |g(x,s) - g(y,s)|dy \le \varepsilon\theta^n(K_i,x) + 2M\theta^n(\mathbb{R}^n \setminus K_i, x) = \varepsilon,$$

for every $\varepsilon > 0$. This proves (4.11), and then, as explained, (4.10).

To have the thesis of the lemma it remains to prove that, for each $i = 1, ..., n$,

$$\lim_{\sigma\to 0}\int_{\Omega'}\Big\{\int_{u(x)}^{v(x)} \frac{\partial g_\sigma^{(i)}}{\partial x_i}(x,s)ds\Big\}dx = \int_{\Omega'}\Big\{\int_{u(x)}^{v(x)} \frac{\partial g^{(i)}}{\partial x_i}(x,s)ds\Big\}dx.$$

Indeed

$$\begin{aligned}&\int_{\Omega'}\Big|\int_{u(x)}^{v(x)} \frac{\partial g_\sigma^{(i)}}{\partial x_i}(x,s)ds - \int_{u(x)}^{v(x)} \frac{\partial g^{(i)}}{\partial x_i}(x,s)ds\Big|dx \\ &\le \int_{\Omega'}\Big\{\int_H \Big|\frac{\partial g_\sigma^{(i)}}{\partial x_i}(x,s) - \frac{\partial g^{(i)}}{\partial x_i}(x,s)\Big|ds\Big\}dx\end{aligned}$$

$$= \int_H \Big\{ \int_{\Omega'} \Big| \frac{\partial g_\sigma^{(i)}}{\partial x_i}(x,s) - \frac{\partial g^{(i)}}{\partial x_i}(x,s) \Big| dx \Big\} ds.$$

Once again, by standard properties of convolution, we have that, for each $s \in \mathbb{R}$,

$$t_\sigma(s) = \int_{\Omega'} \Big| \frac{\partial g_\sigma^{(i)}}{\partial x_i}(x,s) - \frac{\partial g^{(i)}}{\partial x_i}(x,s) \Big| dx \to 0$$

when $\sigma \to 0$; moreover it results

$$|t_\sigma(s)| \le \int_{\Omega'} \Big| \frac{\partial g_\sigma^{(i)}}{\partial x_i}(x,s) \Big| dx + \int_{\Omega'} \Big| \frac{\partial g^{(i)}}{\partial x_i}(x,s) \Big| dx \le 2L\,.$$

Since $t_\sigma(s)$ is bounded on H uniformly on σ, by Lebesgue's dominated convergence Theorem we conclude. Hence, (4.9) is established.

By (4.9), we have

$$\Big| \int_{\Omega'} \beta_k^\varepsilon(x) \left\{ (f_\xi(x,u_h,Dv_\varepsilon), Du_h) - (f_\xi(x,u,Dv_\varepsilon), Du) \right\} \, dx \Big|$$

$$\le \sum_{i=1}^n \sum_{j=1}^N \int_{\Omega'_j} \Big| \int_{u(x)}^{u_h(x)} \Big| \frac{\partial \beta_k^\varepsilon}{\partial x_i}(x) \Big| \Big| f_{\xi_i}(x,s,\xi_j) \Big| ds \Big| dx$$

$$+ \sum_{i=1}^n \sum_{j=1}^N \int_{\Omega'_j} \Big| \int_{u(x)}^{u_h(x)} \Big| \frac{\partial f_{\xi_i}}{\partial x_i}(x,s,\xi_j) \Big| ds \Big| dx. \tag{4.12}$$

Using the first inequality in (4.1), we have

$$\sum_{i=1}^n \sum_{j=1}^N \int_{\Omega'_j} \Big| \int_{u(x)}^{u_h(x)} \Big| \frac{\partial \beta_k^\varepsilon}{\partial x_i}(x) \Big| \Big| f_{\xi_i}(x,s,\xi_j) \Big| ds \Big| dx \le nM \| D\beta_k^\varepsilon \|_\infty \int_{\Omega'} |u_h - u| dx,$$

which goes to zero for $h \to +\infty$. On the other hand,

$$\sum_{i=1}^n \sum_{j=1}^N \int_{\Omega'_j} \Big| \int_{u(x)}^{u_h(x)} \Big| \frac{\partial f_{\xi_i}}{\partial x_i}(x,s,\xi_j) \Big| ds \Big| dx \le n \sum_{j=1}^N \int_{D_{j,h}} \Big| \frac{\partial f_\xi}{\partial x}(x,s,\xi_j) \Big| dx ds,$$

for

$$D_{j,h} = \{ (x,s) \in \Omega'_j \times H : \min\{u_h(x), u(x)\} \le s \le \max\{u_h(x), u(x)\} \}.$$

Now

$$|D_{j,h}| \le \int_{\Omega'} |u_h - u| \, dx \to 0, \quad \text{as} \quad h \to \infty,$$

while, if we define $K=\{\xi_j\}_{j=1}^N$ and consider L_K as in hypothesis (4.2), we have that, for every j,

$$\int_{\Omega\times\mathbb{R}}\Big|\frac{\partial f_\xi}{\partial x}(x,s,\xi_j)\Big|dxds=\int_H ds\int_{\Omega'}\Big|\frac{\partial f_\xi}{\partial x}(x,s,\xi_j)\Big|dx\le |H|L_K.$$

Hence,

$$\lim_{h\to\infty}\int_{D_{j,h}}\Big|\frac{\partial f_\xi}{\partial x}(x,s,\xi_j)\Big|dxds=0,\quad \forall j=1,...,N,$$

from which we conclude that, for ε fixed,

$$\lim_{h\to\infty}\Big|\int_{\Omega'}\beta_k^\varepsilon(x)\left\{(f_\xi(x,u_h,Dv_\varepsilon),Du_h)-(f_\xi(x,u,Dv_\varepsilon),Du)\right\}\,dx\Big|$$

$$\le\lim_{h\to\infty}nM\,||D\beta_k^\varepsilon||_\infty\int_{\Omega'}|u_h-u|\,dx$$

$$+\lim_{h\to\infty}nN\max_{1\le j\le N}\int_{D_{j,h}}\Big|\frac{\partial f_\xi}{\partial x}(x,s,\xi_j)\Big|dxds=0.$$

This ends the proof.

We are ready for the proof of Theorem 1.2. We start preparing the integrand f to satisfy the hypotheses of Lemma 3.4, then we manipulate De Giorgi's approximating functions in order to apply on them Lemma 4.1. As usual, the increasing convergence of the approximating integrands give the result for the initial functional.

Proof of Theorem 1.2. Arguing as in the beginning of Lemma 4.1, we can assume that there exists $\Omega'\times H\subset\subset\Omega\times\mathbb{R}$ such that

$$f(x,s,\xi)=0,\quad \forall(x,s,\xi)\in\big(\Omega\backslash\Omega'\big)\times(\mathbb{R}\backslash H)\times\mathbb{R}^n.$$

In this position the hypotheses of Theorem 1.2 become

$$f(\cdot,s,\xi)\in W^{1,1}(\Omega)\,,\quad \forall\ (s,\xi)\in\mathbb{R}\times\mathbb{R}^n, \tag{4.13}$$

and

$$\int_\Omega\Big|\frac{\partial f}{\partial x}(x,s,\xi)\Big|dx\le L_K\,,\qquad \forall\,(s,\xi)\in\mathbb{R}\times K\,, \tag{4.14}$$

for a suitable constant L_K depending on $K\subset\subset\mathbb{R}^n$.

Since f has compact support in (x,s), we can approximate it with an increasing sequence $\{f_j(x,s,\xi)\}_{j\in\mathbb{N}}$ as in Lemma 3.4. We would like to apply Lemma 4.1 to such functions, but this is not possible. So we denote by φ_ρ a mollifier in $\mathbb{R}^n$ ($\varphi_\rho \in C_c^\infty(\mathbb{R}^n)$, $\varphi_\rho \geq 0$, $\varphi_\rho(\eta) = 0$ if $|\eta| \geq \rho$, $\int_{\mathbb{R}^n} \varphi_\rho(\eta)\, d\eta = 1$) and we consider the functions

$$g_j(x,s,\xi) = \int_{\mathbb{R}^n} f_j(x,s,\xi-\eta)\varphi_{\rho_j}(\eta)\, d\eta,$$

where we define, for M_j as in the statement of Lemma 3.4, $\rho_j = (jM_j)^{-1}$. By the Lipschitz continuity (3.1) of f_j with respect to $\xi \in \mathbb{R}^n$, we have

$$\begin{aligned} |g_j(x,s,\xi) - f_j(x,s,\xi)| &\leq \int_{\mathbb{R}^n} |f_j(x,s,\xi-\eta) - f_j(x,s,\xi)|\, \varphi_{\rho_j}(\eta)\, d\eta \\ &\leq M_j \int_{B_{\rho_j}(0)} |\eta|\, \varphi_{\rho_j}(\eta)\, d\eta \leq \frac{1}{j}\,, \end{aligned}$$

so that, for every $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$,

$$f_j(x,s,\xi) - \frac{2}{j} \leq g_j(x,s,\xi) - \frac{1}{j} \leq f_j(x,s,\xi) \leq f(x,s,\xi)\,. \tag{4.15}$$

By the monotone convergence theorem we have

$$\lim_{j\to+\infty} \int_{\Omega'} f_j\left(x,u(x),Du(x)\right)\, dx \;=\; \int_{\Omega'} f\left(x,u(x),Du(x)\right)\, dx\,.$$

Thus, if we consider the sequence of integrals

$$F_j(u,\Omega') = \int_{\Omega'} \left\{ g_j\left(x,u(x),Du(x)\right) - \frac{1}{j} \right\} dx\,, \tag{4.16}$$

by (4.15) we obtain that $F_j(u,\Omega')$ converges, as $j \to +\infty$, to the main integral $F(u,\Omega') = F(u,\Omega)$, and that, at the same time, $F_j(u,\Omega') \leq F(u,\Omega')$ for every $j \in \mathbb{N}$. Therefore, it suffices to show that every $F_j(u,\Omega')$ is lower semicontinuous. To achieve this we shall invoke Lemma 4.1, and so all that we have to do is to verify that the g_j are in the hypotheses of such Lemma.

Clearly every g_j is Carathéodory, non negative, convex in the ξ variable, locally bounded, and has compact support in (x,s), being a convolution in ξ of $f_j \leq f$. Furthermore, $(g_j)_\xi$ exists and is a Carathéodory function.

Next we verify hypothesis (4.1). By the Lipschitz continuity (1.6) of f_j, we have

$$\begin{aligned} |g_j(x,s,\xi_1) - g_j(x,s,\xi_2)| &\leq \int_{\mathbb{R}^n} |f_j(x,s,\xi_1-\eta) - f_j(x,s,\xi_2-\eta)|\, \varphi_{\rho_j}(\eta)\, d\eta \\ &\leq M_j\, |\xi_1 - \xi_2|\,, \end{aligned}$$

and so $|(g_j)_\xi(x,s,\xi)| \leq M_j$. By the definition of convolution

$$|(g_j)_\xi(x,s,\xi_1) - (g_j)_\xi(x,s,\xi_2)|$$

$$\leq \int_{\mathbb{R}^n} |f_j(x,s,\xi_1-\eta) - f_j(x,s,\xi_2-\eta)| \left| (\varphi_{\rho_j})_\xi (\eta) \right| \, d\eta \leq M_j T_j \, |\xi_1 - \xi_2| \, ,$$

where

$$T_j = \int_{\mathbb{R}^n} \left| (\varphi_{\rho_j})_\xi (\eta) \right| \, d\eta.$$

Therefore, the assumption (4.1) is satisfied with $M = \max\{M_j,\, M_j T_j\}$.

It remains to study the properties of weak derivability and summability of $(g_j)_\xi$ in the x variable. We shall start examining such properties for the coefficients $a_h^{(t)}$ in Lemma 3.4. From (3.2) we have that $a_h^{(t)}$ is Carathéodory. For every $\psi \in C_c^\infty(\Omega)$, by (4.13) we have that

$$\begin{aligned}
\int_\Omega a_h^{(t)}(x,s) \frac{\partial \psi}{\partial x_i}(x) dx &= \int_\Omega \Big(- \int_{\mathbb{R}^n} f(x,s,\eta) \alpha_{\xi_h}^{(t)}(\eta) \, d\eta \Big) \frac{\partial \psi}{\partial x_i}(x) dx \\
&= - \int_{\mathbb{R}^n} \Big(\int_\Omega f(x,s,\eta) \frac{\partial \psi}{\partial x_i}(x) dx \Big) \alpha_{\xi_h}^{(t)}(\eta) \, d\eta \\
&= \int_{\mathbb{R}^n} \Big(\int_\Omega \frac{\partial f}{\partial x_i}(x,s,\eta) \psi(x) dx \Big) \alpha_{\xi_h}^{(t)}(\eta) \, d\eta \\
&= \int_\Omega \Big(\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i}(x,s,\eta) \alpha_{\xi_h}^{(t)}(\eta) d\eta \Big) \psi(x) dx \, ;
\end{aligned}$$

furthermore, thanks to the main assumption (4.14), we obtain

$$\begin{aligned}
\int_\Omega \Big| \frac{\partial a_h^{(t)}}{\partial x_i}(x,s) \Big| dx &= \int_\Omega \Big| \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i}(x,s,\eta) \alpha_{\xi_h}^{(t)}(\eta) d\eta \Big| dx \\
&\leq \left\| \alpha_{\xi_h}^{(t)} \right\|_\infty \int_{spt(\alpha)} \Big(\int_\Omega \Big| \frac{\partial f}{\partial x_i}(x,s,\eta) \Big| dx \Big) d\eta \\
&\leq \left\| \alpha_{\xi_h}^{(t)} \right\|_\infty \left| spt(\alpha^{(t)}) \right| L_{spt(\alpha^{(t)})} \, .
\end{aligned}$$

Hence, $a_h^{(t)}(\cdot, s) \in W^{1,1}(\Omega)$, and there exists a constant $R_h^{(t)}$ such that

$$\int_\Omega \Big| \frac{\partial a_h^{(t)}}{\partial x_i}(x,s) \Big| dx \leq R_h^{(t)}, \quad \forall s \in \mathbb{R}.$$

The same analysis carried over for $a_h^{(t)}$ applies unchanged to $a_0^{(t)}$, so that $a_0^{(t)}(x,s)$ is Carathéodory, with $a_0^{(t)}(\cdot,s) \in W^{1,1}(\Omega)$, and there exists a constant $R_0^{(t)}$ such that

$$\int_\Omega \Big| \frac{\partial a_0^{(t)}}{\partial x_i}(x,s) \Big| dx \leq R_0^{(t)}, \quad \forall s \in \mathbb{R}.$$

In particular, $a_0^{(t)}(\cdot,s)+\sum_{i=1}^n a_h^{(t)}(\cdot,s)\xi_h \in W^{1,1}(\Omega)$, from which,

$$f_j(\cdot,s,\xi)=\max_{t=1,\ldots,j}\Big\{0,a_0^{(t)}(\cdot,s)+\sum_{h=1}^n a_h^{(t)}(\cdot,s)\xi_i\Big\}\in W^{1,1}(\Omega),$$

and finally we have $(g_j)_\xi(\cdot,s,\xi)\in W^{1,1}(\Omega)$, with

$$\frac{\partial (g_j)_\xi}{\partial x_i}(x,s,\xi)=\int_{\mathbb{R}^n}\frac{\partial f_j}{\partial x_i}(x,s,\xi-\eta)\left(\varphi_{\rho_j}\right)_\xi(\eta)\,d\eta.$$

Since,

$$\int_\Omega\Big|\frac{\partial f_j}{\partial x_i}(x,s,\xi)\Big|dx\le\sum_{t=1}^j\Big\{\int_\Omega\Big|\frac{\partial}{\partial x_i}\Big(a_0^{(t)}(x,s)+\sum_{h=1}^n a_h^{(t)}(x,s)\xi_h\Big)\Big|dx\Big\}$$

$$\le\sum_{t=1}^j\int_\Omega\Big|\frac{\partial a_0^{(t)}}{\partial x_i}(x,s)\Big|dx+|\xi|\sum_{t=1}^j\Big\{\max_{h=1,\ldots,n}\int_\Omega\Big|\frac{\partial a_h^{(t)}}{\partial x_i}(x,s)\Big|dx\Big\}\le S_j\big(1+|\xi|\big),$$

for a suitable constant S_j (depending on $R_0^{(t)}$, $R_h^{(t)}$, j), we deduce that

$$\int_\Omega\Big|\frac{\partial (g_j)_\xi}{\partial x_i}(x,s,\xi)\Big|dx\le\int_\Omega dx\int_{\mathbb{R}^n}\Big|\frac{\partial f_j}{\partial x_i}(x,s,\xi-\eta)\Big|\,\Big|\left(\varphi_{\rho_j}\right)_\xi(\eta)\Big|\,d\eta$$

$$\le\int_{B_{\rho_j}(0)}\Big|\left(\varphi_{\rho_j}\right)_\xi(\eta)\Big|\,d\eta\int_\Omega\Big|\frac{\partial f_j}{\partial x_i}(x,s,\xi-\eta)\Big|dx$$

$$\le\int_{B_{\rho_j}(0)}S_j\left(1+|\xi-\eta|\right)\Big|\left(\varphi_{\rho_j}\right)_\xi(\eta)\Big|\,d\eta\le S_jT_j(1+|\xi|+\rho_j).$$

Hence, if $K\subset\subset\mathbb{R}^n$ for $(L_j)_K=S_jT_j(1+T+\rho_j)$, (where T is equal to the radius of a ball centered at the origin and containing K), we have

$$\int_\Omega\Big|\frac{\partial (g_j)_{\xi_i}}{\partial x_i}(x,s,\xi)\Big|dx\le(L_j)_K,\quad\forall(s,\xi)\in\mathbb{R}\times K.$$

So Lemma 4.1 applies to every g_j, providing the lower semicontinuity of each F_j. By the preceding arguments this end the proof of Theorem 1.2.

5. Some results in the vector-valued case

In this section we deal with a function $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to [0,\infty)$, where Ω is an open set of $\mathbb{R}^n$, n, $m \geq 1$, and for every $(x,s) \in \Omega \times \mathbb{R}^m$ the function $f(x,s,\cdot)$ is convex in $\mathbb{R}^{m\times n}$. We seek conditions on f sufficient to prove

$$\begin{cases} u_h,\, u \in W^{1,1}_{loc}(\Omega;\mathbb{R}^m) \\ u_h \to u \quad \text{in } L^1_{loc}(\Omega;\mathbb{R}^m) \end{cases} \Longrightarrow F(u,\Omega) \leq \liminf_{h\to\infty} F(u_h,\Omega)\,. \tag{5.1}$$

The lower semicontinuity results by Serrin [14] were proved for the scalar case $m = 1$. Some effort has been spent to understand the question of their validity in the vector-valued case $m > 1$. See in particular Eisen's [8] and Černý-Malý [2], [3] counterexamples; see also the results by Fonseca and Leoni [10].

The main difference with respect to the scalar case is the need of an assumption of coercivity of the integrand, at least when we assume the dependence on the s variable, as Eisen's counterexample shows [8]. Under the investigation of the scalar case made in the previous sections, we became aware of the fact that, in particular, two results, without coercivity assumptions, hold in the vector-valued case too, provided the dependence on the s variable is dropped. The first one is a vector-valued version of Theorem 1.1.

Proposition 5.1. *Let us assume that* $f(x,\xi)$ *satisfies* (1.3) *and, for every* $\xi \in \mathbb{R}^{m\times n}$, $f(\cdot,\xi) \in W^{1,\infty}_{loc}(\Omega)$. *Let us also assume that, for every* $\Omega' \subset\subset \Omega$, $K \subset\subset \mathbb{R}^{m\times n}$, *there exists a constant* $L = L_{\Omega'\times K}$ *such that*

$$\operatorname*{ess\,sup}_{x\in\Omega'} \left|\frac{\partial}{\partial x} f(x,\xi)\right| \leq L,$$

for every $\xi \in K$. *Then the lower semicontinuity* (5.1) *holds.*

The second result we give is an extension to the vector-valued case of the Serrin's result related to strictly convex integrands. Since the counterexample by Černý and Malý [3], is natural here to drop the dependence on s.

Proposition 5.2. *Let us assume that* $f(x,\xi)$ *satisfies* (1.3) *and that* $f(x,\cdot)$ *is strictly convex for every* $x \in \Omega$. *Then* (5.1) *holds.*

For comparison with Proposition 5.4 below, we quote a result proved in [10].

Theorem 5.3 (Fonseca - Leoni)**.** *Let us assume that* f *is lower semicontinuous on* $\Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n}$ *and it is convex with respect to* $\xi \in \mathbb{R}^{m\times n}$. *Moreover,* *for every* $(x_0, s_0) \in \Omega \times \mathbb{R}^m$ *there exists* $\delta_0 > 0$ *and a continuous function* $g : B_{\delta_0}(x_0, s_0) \to \mathbb{R}^{m\times n}$ *such that*

$$(x, s) \longmapsto f(x, s, g(x, s)) \in L^\infty(B_{\delta_0}(x_0, s_0)). \tag{5.2}$$

Finally, let us assume that either $f(x_0, s_0, \xi) = 0$ *for every* $\xi \in \mathbb{R}^{m\times n}$*, or there exists* $\delta_0 > 0$ *and* $c_0 > 0$ *such that*

$$f(x, s, \xi) \geq c_0 |\xi| - \frac{1}{c_0}, \quad \forall (x, s) \in B_{\delta_0}(x_0, s_0).$$

Then the lower semicontinuity (5.1) *holds.*

A question posed in [10] is about the necessity of assumption (5.2). This hypothesis comes out by the use of the approximation theorem by Ambrosio (Lemma 1.5, Statement A, in [1]). A counterexample by Černý and Malý [2] shows that such assumption in general cannot be dropped. However, it may be interesting to note that, if we assume a local coerciveness with superlinear growth, then Theorem 5.3 holds without (5.2), as stated in the next proposition.

Proposition 5.4. *Let us assume that* f *is lower semicontinuous on* $\Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n}$ *and convex in the* ξ *variable. Then* (5.1) *holds if we also assume that, for every* $(x_0, s_0) \in \Omega \times \mathbb{R}^m$*, either* $f(x_0, s_0, \xi) = 0$ *for every* $\xi \in \mathbb{R}^{m\times n}$*, or there exists* $\delta_0 > 0$*,* $c_0 > 0$ *and* $p_0 > 1$ *such that*

$$f(x, s, \xi) \geq c_0 |\xi|^{p_0} - \frac{1}{c_0}, \quad \forall (x, s) \in B_{\delta_0}(x_0, s_0). \tag{5.3}$$

We notice that, under assumption (5.3), if the coerciveness constant does not depend on s as in Proposition 5.4, we can consider more generally that f is only measurable with respect to x (see Proposition 5.6 in [12]).

Below we give some details of the proofs. We start with a lemma useful in the proof of Proposition 5.1.

Lemma 5.5. *Let us assume that* f *satisfies* (1.3) *and that:* (i) *there exists an open set* $\Omega' \subset\subset \Omega$*, such that* $f(x, \xi) = 0$*, for every* $x \in \Omega \backslash \Omega'$ *and for every* $\xi \in \mathbb{R}^{m\times n}$*;* ($ii$) *the derivative* $f_\xi(x, \xi)$ *exists and is a continuous function in* $\Omega \times \mathbb{R}^{m\times n}$ *such that* $f_\xi(\cdot, \xi) \in W^{1,\infty}(\Omega; \mathbb{R}^{m\times n})$*, for every* $\xi \in \mathbb{R}^{m\times n}$*;* ($iii$) *there exists a constant* M *such that*

$$\begin{cases} |f_\xi(x, \xi)| \leq M, \ \forall \ (x, \xi) \in \Omega \times \mathbb{R}^{m\times n}, \\ |f_\xi(x, \xi_1) - f_\xi(x, \xi_2)| \leq M |\xi_1 - \xi_2|, \ \forall x \in \Omega, \ \xi_1, \xi_2 \in \mathbb{R}^{m\times n}. \end{cases} \tag{5.4}$$

Then the lower semicontinuity inequality (5.1) *holds.*

Proof. Let $u_h, u \in W^{1,1}_{loc}(\Omega;\mathbb{R}^m)$ such that $u_h \to u$ in $L^1_{loc}(\Omega;\mathbb{R}^m)$. Without loss of generality, we can assume that

$$\liminf_{h\to+\infty} F(u_h,\Omega) = \lim_{h\to+\infty} F(u_h,\Omega) < \infty,$$

and that u_h converges almost everywhere to u in Ω. We also remark that, from the first assumption in (5.4), we have $F(u,\Omega)<\infty$, so that $F(u_h,\Omega) = F(u_h,\Omega')$ and $F(u,\Omega) = F(u,\Omega')$. As in the proof of Lemma 4.1, for every $\varepsilon>0$, there exists $v_\varepsilon \in Aff(\Omega'';\mathbb{R}^m)$ such that,

$$\int_{\Omega'} |Du - Dv_\varepsilon|\, dx \le \varepsilon\,, \quad \int_{\Omega'} f(x, Dv_\varepsilon)\, dx \ge \int_{\Omega'} f(x,Du)\, dx - \varepsilon\,. \tag{5.5}$$

Since $v_\varepsilon \in Aff(\Omega';\mathbb{R}^m)$, referring to Definition 3.1, we consider $\{\Omega'_j\}_{j=1}^N$ such that $Dv_\varepsilon = \xi_j \in \mathbb{R}^{m\times n}$ in Ω'_j. Now let us consider the sequence of functions $\{\beta^\varepsilon_k\}_{k=1}^\infty \subset C^\infty(\Omega')$ as in Lemma 4.1 and using the same argument we have semicontinuity provided that, for every v_ε and k fixed,

$$\lim_{h\to+\infty} \int_{\Omega'} \beta^\varepsilon_k(x) \left\{ (f_\xi(x,Dv_\varepsilon), Du_h) - (f_\xi(x,Dv_\varepsilon), Du) \right\} dx = 0\,. \tag{5.6}$$

Decomposing Ω' on the sets where Dv_ε is constant and observing that $\beta^\varepsilon_k(x) f(x,\xi_j) \in W^{1,\infty}_0(\Omega'_j;\mathbb{R}^{m\times n})$, we have that

$$\Big| \int_{\Omega'} \beta^\varepsilon_k(x) \left\{ (f_\xi(x,Dv_\varepsilon), Du_h) - (f_\xi(x,Dv_\varepsilon), Du) \right\} dx \Big|$$

$$\le \sum_{j=1}^N \Big| \int_{\Omega'_j} (\beta^\varepsilon_k(x) f_\xi(x,\xi_j), Du_h - Du) \Big| dx$$

$$= \sum_{j=1}^N \Big| \int_{\Omega'_j} \sum_{i=1}^n \sum_{t=1}^m \beta^\varepsilon_k(x) f_{\xi_{t,i}}(x,\xi_j) \cdot \frac{\partial}{\partial x_i} \left(u_h^{(t)} - u^{(t)} \right) \Big| dx$$

$$= \sum_{j=1}^N \Big| \int_{\Omega'_j} \sum_{i=1}^n \sum_{t=1}^m \frac{\partial}{\partial x_i} \left(\beta^\varepsilon_k(x) f_{\xi_{t,i}}(x,\xi_j) \right) \cdot \left(u_h^{(t)} - u^{(t)} \right) \Big| dx$$

$$\le \sum_{j=1}^N \sum_{t=1}^m \int_{\Omega'_j} C_j \left| u_h^{(t)} - u^{(t)} \right| dx \to 0,$$

for $h\to+\infty$, where $C_j = \sup\{ |\frac{\partial}{\partial x_i}(\beta^\varepsilon_k(x) f_\xi(x,\xi_j))| : x \in \Omega'_j \}$.

Proof of Proposition 5.1. We only have to use the same (in fact simpler) argument used by the authors in [12] for the proof of Theorem 1.6.

Proof of Proposition 5.2. We refer here to theorems and lemmas in the paper [14] by Serrin: the proof of Theorem 12, valid in the scalar case, is based on Lemmas 4, 5, 7, 8. However, Lemmas 4, 5, 8 are also true in the vectorial setting; moreover the presence of the variable s is not determinant for their validity. So Lemma 7 is the only one which makes Serrin's Theorem 12 work only in the scalar setting; on the contrary, if f is independent of s, we can easily extend Lemma 7 to the vectorial setting and complete the proof.

Proof of Proposition 5.4. The usual blow up method reduce the problem of lower semicontinuity to that of proving the inequality

$$\liminf_{k\to\infty}\int_Q f(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k w_k(y), Dw_k(y))dy \geq f(x_0,u(x_0),Du(x_0)),$$

where $x_0 \in \Omega$, Q is the unit cube of $\mathbb{R}^n$, ε_k tends decreasing to zero, and $w_k \in W^{1,1}(Q;\mathbb{R}^m)$ converges strongly $L^1(Q;\mathbb{R}^m)$ to the affine limit $w_0(y) := (Du(x_0), y)$, $y \in Q$. This is trivial if, for the fixed value $(x_0, u(x_0))$, $f(x_0, u(x_0), \cdot) = 0$. Otherwise, there exist δ_0, c_0 positive and $p_0 > 1$ such that

$$f(x,s,\xi) \geq c_0\,|\xi|^{p_0} - \frac{1}{c_0},$$

holds for every $(x,s,\xi) \in \overline{B_{\delta_0}(x_0,u(x_0))}\times\mathbb{R}^{m\times n}$. If we set $M = \overline{B_{\delta_0}(x_0,u(x_0))}$ we are in the hypotheses of Lemma 1.5, statement B by Ambrosio [1]; thus there exist continuous functions $a_h : M \to \mathbb{R}$ and $b_h : M \to \mathbb{R}^{m\times n}$ such that

$$f(x,s,\xi) = \sup_{h\in\mathbb{N}}\left\{a_h(x,s) + (b_h(x,s),\xi)\right\},$$

for every $(x,s,\xi) \in \overline{B_{\delta_0}(x_0,u(x_0))} \times \mathbb{R}^{m\times n}$. Now we can conclude the proof as in Fonseca-Leoni [10], Theorem 1.1.

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