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COMPARISON RESULTS FOR A LINEAR ELLIPTIC EQUATION WITH MIXED BOUNDARY CONDITIONS

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Abstract. In this paper we study a linear elliptic equation having mixed boundary conditions, defined in a connected open set Ω of $\mathbb{R}^n$. We prove a comparison result with a suitable "symmetrized" Dirichlet problem which cannot be uniformly elliptic depending on the regularity of $\partial\Omega$. Regularity results for non-uniformly elliptic equations are also given.

1. Introduction

Let Ω be a connected bounded open set of $\mathbb{R}^n$ whose boundary $\partial\Omega$ is made of two manifolds Γ_0 and Γ_1, having Γ_0 positive $(n-1)-$ dimensional Hausdorff measure. We consider the mixed problem

$$\begin{cases} -(a_{ij}(x)u_{x_i})_{x_j} = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_0 \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_1, \end{cases} \tag{1.1}$$

where $a_{ij}(x)$, $i,j = 1,\dots,n$, are bounded, measurable functions on Ω satisfying the ellipticity condition

$$a_{ij}(x)\xi_i\xi_j \geq |\xi|^2 \qquad \text{for a. e. } x \in \Omega, \forall \xi \in \mathbb{R}^n \tag{1.2}$$

and $f \in L^p(\Omega)$, $p > \frac{2n}{n+2}$ if $n > 2$, $p > 1$ if $n = 2$.

It is well known that a way to obtain sharp estimates for solutions of elliptic problems is the comparison with solutions of suitable symmetrized problems using Schwartz symmetrization (see [18], [19], [3], [1]). In this order of ideas we are able to compare the solution of problem (1.1) with the solution of a Dirichlet problem with spherically symmetric data, defined in

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a ball having the same measure as Ω. This problem cannot be uniformly elliptic depending on the shape of Ω.

To be more precise let us define (see also [14]) the function

$$\lambda(t) = \inf_E P_\Omega(E), \qquad t \geq 0, \tag{1.3}$$

where $P_\Omega(E)$ is the perimeter of E relative to Ω and the set E varies in the class of measurable subsets of Ω such that $|E| = t$ and $\partial E \cap \Gamma_0$ does not contain any set of positive $(n-1)-$dimensional Hausdorff measure. Let us suppose that

$$\sup_{t>0} \frac{t^\alpha}{\lambda(t)} < \infty, \qquad 1 - \frac{1}{n} \leq \alpha \leq \frac{3}{2} - \frac{1}{n}. \tag{1.4}$$

We observe that in (1.4), $\alpha < 1 - \frac{1}{n}$ cannot hold for any set Ω, since this condition is denied by the classical isoperimetric inequality. On the other hand (1.4) implies a relative isoperimetric inequality of the kind

$$|E|^\alpha \leq Q P_\Omega(E), \quad \forall E \subset \Omega, \quad E \text{ measurable.} \tag{1.5}$$

If $\alpha = 1 - \frac{1}{n}$, (1.4) means that a relative isoperimetric inequality holds (see e. g. [11], [17], [8]). Depending on the shape of Γ_1, condition (1.4) can fail for $\alpha = 1 - \frac{1}{n}$; this happens, for instance, when Γ_1 has some cuspidal points (see §2 for an example).

We consider the problem

$$\begin{cases} -\left(\nu\left(\omega_n |x|^n\right) w_{x_i}\right)_{xi} = n^2 \omega_n^{2/n} f^\# & \text{in } \Omega^\# \\ w = 0 & \text{on } \partial\Omega^\#, \end{cases} \tag{1.6}$$

where $f^\#$ is the spherically symmetric rearrangement of f (see §2 for the definition), $\Omega^\#$ is the ball centered at the origin, having the same area as Ω, and $\nu(t) = t^{-2+2/n} \lambda^2(t)$. If (1.4) holds true, we prove the following estimates

$$\begin{aligned} & u^\#(x) \leq w(x) \qquad \text{a.e. } x \in \Omega^\#, \\ & \int_\Omega |Du|^2 \, dx \leq n^2 \omega_n^{2/n} \int_{\Omega^\#} \nu \, |Dw|^2 \, dx, \end{aligned} \tag{1.7}$$

where $w \in H_0^1\left(\nu, \Omega^{\#}\right)$ [1] is the solution of problem (1.6). We explicitly observe that problem (1.6) is uniformly elliptic when in (1.4) $\alpha = 1 - \frac{1}{n}$. In this case estimates (1.7) give the same upper bound as in [17] where problem (1.1) is compared with a problem defined in a sector with mixed boundary conditions. When $\alpha > 1 - \frac{1}{n}$ the weak assumption on the regularity of $\partial\Omega$ implies that problem (1.6) is not uniformly elliptic and the estimates of the solution are given in terms of the function $\lambda(t)$. In §2, we give an example of a domain Ω whose boundary is characterized by the behavior of the function $\lambda(t)$ as a power of t with exponent greater than $1-\frac{1}{n}$. Comparison (1.7) allow us to obtain estimates for $u(x)$ in terms of the solution of problem (1.6). Let us observe that under the assumption (1.4), $\frac{1}{\nu}$ belongs to a Marcinkiewitz space M^t. Then §3 is devoted to prove regularity results for a class of degenerate problems, when $\frac{1}{\nu} \in M^t$ (see [14] for Neumann problem). These estimates allow us to obtain existence and regularity results for problem (1.1) under the assumption (1.4). When $\frac{1}{\nu}$ is in Lorentz or Lebesgue spaces a priori estimates for problem (1.6) are given in [4], [20], [7].

Let us remark that, when $\Gamma_0 = \partial\Omega$, (1.4) holds with $\alpha = 1 - \frac{1}{n}$, since in (1.3) $P_\Omega(E) = P_{\mathbb{R}^n}(E)$ and then (1.4) gives the classical isoperimetric inequality. In this case the comparison result (1.7) is exactly the one obtained by Talenti in [18] for the Dirichlet problem.

When $\mathcal{H}^{n-1}(\Gamma_0) \to 0$, that is problem (1.1) approaches a Neumann problem, (1.4) does not hold since we can take sets $E \subset \Omega$ having measure near to $|\Omega|$ and relative perimeter arbitrarily small. In this case results in this order of ideas have been obtained in [15], where the Neumann problem is compared with two Dirichlet problems defined in two balls having measure $\frac{|\Omega|}{2}$.

2. Comparison results

We begin this section by recalling some definitions that will be useful in the following. Let Ω be an open, bounded set of $\mathbb{R}^n$ and let us consider a measurable function $u : \Omega \longrightarrow \mathbb{R}$. The distribution function of u is defined by

$$\mu(t) = \left|\{x \in \Omega : |u(x)| > t\}\right|, \qquad t \geq 0,$$

[1] Let $\nu \geq 0$, $\nu^{-1} \in L^t(\Omega)$, for some $t \geq 1$; $H_0^1(\nu, \Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H_0^1(\nu,\Omega)} = \left(\int_\Omega \nu\, |Du|^2\, dx\right)^{1/2}.$$

while the decreasing rearrangement of u is defined as the distribution function of μ, i.e.,

$$u^*(s) = \sup\{t \geq 0 : \mu(t) \geq s\}, \qquad s \in [0, |\Omega|].$$

By using the previous notations we also introduce the decreasing spherically symmetric rearrangement of u as follows

$$u^{\#}(x) = u^*(\omega_n |x|^n), \qquad x \in \Omega^{\#},$$

where $\Omega^{\#}$ denotes the ball centered at the origin, having the same measure as Ω, and ω_n is the measure of the unitary ball in $\mathbb{R}^n$.

The theory of rearrangements is well-known and exhaustive treatments of it can be found, for example, in [18], [5], [13].

We are interested in proving a comparison result between the solution of problem (1.1) and the solution of the symmetrized problem (1.6) defined in $\Omega^{\#}$. Our main result is the following:

Theorem 2.1. *Let Ω be a connected bounded open set of $\mathbb{R}^n$ such that its boundary $\partial\Omega$ consists of two manifolds, Γ_0 and Γ_1, with $\mathcal{H}^{n-1}(\Gamma_0) > 0$ and let us suppose that* (1.4) *holds true. Let u and w be solutions of problems* (1.1) *and* (1.6) *with $\nu(\omega_n |x|^n) = \omega_n^{-2+2/n} |x|^{2-2/n} \lambda^2(\omega_n |x|^n)$, respectively, and let f be so regular in order to guarantee the existence of w. Then*

$$(i) \qquad u^{\#}(x) \leq w(x) \qquad \text{a.e. } x \in \Omega^{\#},$$

$$(ii) \qquad \int_\Omega |Du|^2 \, dx \leq n^2 \omega_n^{2/n} \int_{\Omega^{\#}} \nu \, |Dw|^2 \, dx.$$

Proof. For $h > 0$ and $t \geq 0$ we define

$$\varphi_h(x) = \begin{cases} h \operatorname{sign} u & \text{if } |u| > t+h \\ (|u| - t) \operatorname{sign} u & \text{if } t < |u| \leq t+h \\ 0 & \text{otherwise.} \end{cases}$$

We use φ_h as test function in (1.1), then we get

$$\frac{1}{h} \int_{t<|u|\leq t+h} a_{ij} u_{x_i} u_{x_j} dx = \int_{|u|>t+h} f \operatorname{sign} u \, dx + \frac{1}{h} \int_{t<|u|\leq t+h} f(|u|-t) \operatorname{sign} u \, dx.$$

Using (1.2), letting h goes to 0, we have

$$-\frac{d}{dt} \int_{|u|>t} |Du|^2 \, dx \leq \int_{|u|>t} |f| \, dx. \tag{2.1}$$

We now proceed to evaluate the left hand side of (2.1) by the following inequalities:

$$-\frac{d}{dt}\int_{|u|>t}|Du|\,dx \le \left(-\mu'(t)\right)^{\frac{1}{2}}\left(-\frac{d}{dt}\int_{|u|>t}|Du|^2\,dx\right)^{\frac{1}{2}}$$

and

$$-\frac{d}{dt}\int_{|u|>t}|Du|\,dx = P_\Omega\left(\{x\in\Omega : |u|>t\}\right) \ge \lambda\left(\mu(t)\right), \qquad (2.2)$$

where $\mu(t)$ denotes the distribution function of $u(x)$. We gather

$$\frac{\lambda^2\left(\mu(t)\right)}{\left(-\mu'(t)\right)} \le -\frac{d}{dt}\int_{|u|>t}|Du|^2\,dx \le \int_{|u|>t}|f|\,dx \le \int_0^{\mu(t)} f^*(r)dr; \qquad (2.3)$$

the last inequality in (2.3) is the Hardy-Littlewood theorem. We rewrite (2.3) in the form

$$1 \le \lambda^{-2}\left(\mu(t)\right)\left(-\mu'(t)\right)\int_0^{\mu(t)} f^*(r)dr,$$

and, by integrating between 0 and t, we obtain

$$t \le \int_0^t \lambda^{-2}\left(\mu(\tau)\right)\left(-\mu'(\tau)\right)\int_0^{\mu(\tau)} f^*(r)dr = \int_{\mu(t)}^{\Omega}\lambda^{-2}(\tau)d\tau\int_0^{\tau} f^*(r)dr.$$

By definition of decreasing rearrangement, we then get

$$u^*(s) \le \int_s^{|\Omega|}\lambda^{-2}(\tau)d\tau\int_0^{\tau} f^*(r)dr = w^*(s), \qquad s\in[0,|\Omega|], \qquad (2.4)$$

where w is the solution of problem (1.6), i.e., (i).

In order to prove (ii), using Hölder inequality, (2.2) and (2.3), we obtain

$$\begin{aligned}\int_\Omega |Du|^2\,dx &\le \int_0^{|\Omega|}\left\{\frac{1}{\left(-\mu'(t)\right)}\left(-\frac{d}{dt}\int_{|u|>t}|Du|^2\,dx\right)\right\}\left(-d\mu(t)\right)\\ &\le \int_0^{|\Omega|}\left\{\frac{1}{\lambda\left(\mu(t)\right)}\left(-\frac{d}{dt}\int_{|u|>t}|Du|^2\,dx\right)\right\}^2\left(-d\mu(t)\right)\\ &\le \int_0^{|\Omega|}\left\{\frac{1}{\lambda(s)}\int_0^s f^*(r)dr\right\}^2 ds = n^2\omega_n^{2/n}\int_{\Omega^\#}\nu\,|Dw|^2\,dx,\end{aligned}$$

where $\nu = \nu\left(\omega_n|x|^n\right)$. □

Remark 1. From (i) we can deduce the following estimates

$$\text{ess sup}\,|u| \le \text{ess sup}\,|w|; \qquad (2.5)$$

$$\int_\Omega |u|^q\, dx \le \int_{\Omega^\#} |w|^q\, dx, \qquad q>0.$$

Remark 2. If $\alpha = 1-\frac{1}{n}$, a relative isoperimetric inequality holds true

$$|E|^{1-\frac{1}{n}} \le QP_\Omega(E), \qquad \forall E\subset\Omega,\ E \text{ measurable},$$

where $Q = Q\left(\Gamma_1, E\right)$. Then we can write explicitly $\lambda(t)$ in (2.4) if we are able to evaluate the constant Q. In [17] Pacella and Tricarico proved that, if $\mathcal{H}^{n-1}\left(\Gamma_1\right)>0$, then

$$Q = \left(n\beta_n^{1/n}\right)^{-1},$$

where β_n is the Lebesgue measure of the unitary sector[2] in $\mathbb{R}^n$, with amplitude $\beta\in(0,\frac{\pi}{2}]$. In this case we can write (2.4) as follows

$$u^*(s) \le \frac{1}{n^2\beta_n^{2/n}}\int_s^{|\Omega|} dr\, r^{-2+\frac{2}{n}}\int_0^r f^*(\tau)d\tau, \qquad s\in(0,|\Omega|)$$

and our comparison results reduce to those ones contained in [17], between a solution u of (1.1) and the solution of a symmetrized mixed problem in a sector with amplitude β.

In what follows we give an example of a domain Ω where hypotheses (1.4) is satisfied for $\alpha > 1-\frac{1}{n}$ computing $\lambda(t)$ and evaluating $\sup_{t>0}\frac{t^\alpha}{\lambda(t)}$. We consider the set $\Omega\subset\mathbb{R}^2$ delimited by the x-axis, the parabola $y=x^2$ and the circle centered on the x-axis and orthogonal to the parabola in the point $\left(\frac{1}{2},\frac{1}{4}\right)$ (see figure), that is the "cone"

$$\Omega = \left\{(x,y)\in\mathbb{R}^2 : 0<x<\frac{\sqrt{2}}{4}+\sqrt{2},\ 0<y<x^2,\ 0<\left(x-\frac{1}{4}\right)^2+y^2<\frac{1}{8}\right\}.$$

Let $\Gamma_0 = \{(x,y)\in\partial\Omega : (x-\frac{1}{4}t)^2+y^2=\frac{1}{8}\}$ and $\Gamma_1=\partial\Omega-\Gamma_0$. We will prove that

$$\sup_{t>0}\frac{t^\alpha}{\lambda(t)}<\infty \quad \text{ if } \alpha=\frac{2}{3}.$$

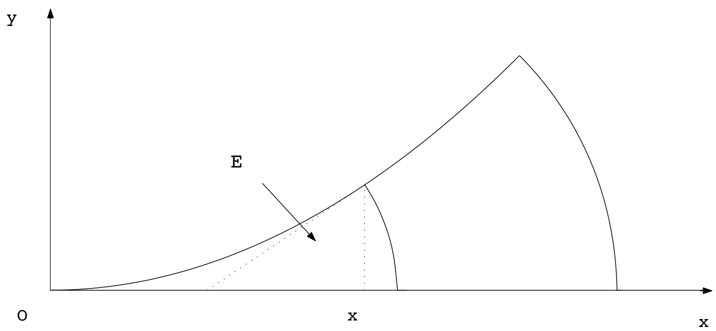
First of all we prove the existence of a set $E\subseteq\Omega$ having measure t and minimum perimeter between all the measurable subsets of Ω such that $\partial E\cap\Gamma_0$ does not contain any subset of positive 1-dimensional Hausdorff measure. The existence of such a set is proved in [11] if the boundary of Ω is locally

[2]The sector $A(\beta,R)$ in $\mathbb{R}^n$, $\beta\in[0,\pi[$ and $R>0$ is defined as the set

$$A(\beta,R)=\{x\in\mathbb{R}^n : 0<|x|<R,(x,\xi)>\cos\beta\cdot|x|\}$$

where ξ is the vector $(1,0,...,0)$ and $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^n$.

We call unitary sector with amplitude β the set $A(\beta,1)$.



Lipschitz. Since $\partial\Omega$ has only one point where it is not Lipschitz, i.e., $(0,0)$, and $(0,0)$ does not belong to $\partial E \cap \Omega$, we can adapt the proof contained in [11] to our case and by means of a simple extension argument (see [10]) we can prove that such a set exists. It is easy to realize that $\partial E \cap \Omega$ must be concave towards E and $\partial E \cap \Omega$ must be a curve having one terminal point on $y = 0$ and the other one on $y = x^2$. The regular case is studied in [8].

Then we can suppose that $\partial E \cap \Omega$ is the curve of polar equation $\rho = \rho(\vartheta), \vartheta \in [0, \gamma_t]$. If we impose that $|E| = t$ and $\partial E \cap \Omega$ has minimum length, we find that $\rho(\vartheta)$ satisfies the following Euler equation

$$\frac{\rho}{\sqrt{\rho^2 + \rho'^2}} - \frac{d}{d\vartheta}\frac{\rho'}{\sqrt{\rho^2 + \rho'^2}} + 2\mu\rho = 0, \quad (\mu = \text{Lagrange multiplier}) \tag{2.6}$$

the boundary condition

$$\rho'(0) = 0 \tag{2.7}$$

and the trasversality condition

$$\rho'(\gamma_t) = -\frac{\sin^2 \gamma_t}{\cos \gamma_t \, \left(1 + \sin^2 \gamma_t\right)}. \tag{2.8}$$

Since solutions of equation (2.6) are a family of circles, the unique solution of (2.6) that satisfies (2.7) and (2.8) is the circular arc orthogonal to $\partial\Omega$ for $\vartheta = 0$ and $\vartheta = \gamma_t$ (see figure).

Let $x_0 = x_0(t)$ be the first cartesian coordinate of the point $(\rho(\gamma_t), \gamma_t)$; it is easy to find that $\partial E \cap \Omega$ is the circular arc centered in $\left(\frac{x_0}{2}, 0\right)$ and x_0 is the unique solution of

$$|E| = \frac{x_0^2}{8}\arctan(2x_0)\left(1+4x_0^2\right) + \frac{x_0^3}{12} = t.$$

The length of $\partial E \cap \Omega$ is $\frac{x_0}{2}\arctan(2x_0)\sqrt{1+4x_0^2}$ and the quotient

$$\frac{t^{2/3}}{\lambda(t)} = \frac{\left[\frac{x_0^2}{8}\arctan(2x_0)\left(1+4x_0^2\right) + \frac{x_0^3}{12}\right]^{2/3}}{\frac{x_0}{2}\arctan(2x_0)\sqrt{1+4x_0^2}}$$

is a strictly increasing function of x_0 and then of t. Then sup $\frac{t^{2/3}}{\lambda(t)}$ is achieved when $t = |\Omega|$ and it is $\frac{1}{\sqrt[3]{9}}\frac{(3\pi+2)^{2/3}}{\sqrt{2}\pi}$.

3. Regularity results for solutions of degenerate elliptic equations

In this section we consider the problem

$$\begin{cases} -\left(a_{ij}(x)z_{x_i}\right)_{x_j} = f & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is an open, bounded subset of $\mathbb{R}^n$, $a_{ij}(x)$ are such that

$$a_{ij}(x)\xi_i\xi_j \geq \nu(x)\left|\xi\right|^2, \qquad \text{a. e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^n \tag{3.2}$$

and $\nu \in L^1(\Omega)$ is a non-negative function.

We investigate the regularity of solutions of (3.1) when $\frac{1}{\nu}$ belongs to the Marcinkiewitz space M^t and f is in a Lorentz space. Let us first recall the definition of these spaces.

If we put

$$\overline{f}(s) = \frac{1}{s}\int_0^s f^*(r)dr$$

we say that a function $f \in L(p,q)$, $1 < p < \infty$, $1 \leq q \leq \infty$, if the quantity

$$||f||_{p,q} = \begin{cases} \left(\int_0^\infty \left(\overline{f}(s)s^{1/p}\right)^q \frac{ds}{s}\right)^{1/q} & 1 \leq q < \infty \\ \sup_{s>0} \overline{f}(s)s^{1/p} & q = \infty \end{cases}$$

is finite. We remark that $L(p,p) = L^p(\Omega)$ and for $1 < q < p < r < \infty$ the following inclusions hold

$$L^r(\Omega) \subset L(p,1) \subset L(p,q) \subset L^p(\Omega) \subset L(p,r) \subset L(p,\infty) \subset L^q(\Omega).$$

Now let z be a measurable function on Ω, $1 < t < \infty$ and $\frac{1}{t} + \frac{1}{t'} = 1$. Let

$$\|z\|_{M^t} = \min\left\{C \geq 0 : \int_K |z(x)|dx \leq C|K|^{1/t'} \text{ for all measurable } K \subset \Omega\right\};$$

M^t is the set of measurable functions z on Ω satisfying $||z||_{M^t} < \infty$. It is easy to verify that M^t is a Banach space under the norm $||\cdot||_{M^t}$ and $M^t = L(t, \infty)$.

Before entering into details, let us recall some other preliminary results. We first introduce a relation between nonnegative functions in $L^1(\Omega)$. We say that f is dominated by g, and we write $f \prec g$ if

$$\int_0^s f^*(r)dr \leq \int_0^s g^*(r)dr \quad \forall s \in [0, |\Omega|) \text{ and } \int_0^{|\Omega|} f^*(r)dr = \int_0^{|\Omega|} g^*(r)dr. \tag{3.3}$$

We explicitly observe that the definition given in (3.3) makes sense also if f and g are defined in different sets with the only restriction that these sets should have the same measure. Various properties and characterizations of such a relation are given, for example, in [9] and [2]. We only recall the following:

Theorem 3.1. *The following statements are equivalent:*

i) $f \prec g$;

ii) $\int_\Omega f(x)\eta(x)dx \leq \int_0^{|\Omega|} g^*(s)\eta^*(s)ds, \qquad \int_\Omega f(x)dx = \int_\Omega g(x)dx$ *for all non negative* $\eta \in L^\infty(\Omega)$;

iii) $\int_0^{|\Omega|} f^*(s)\eta^*(s)ds \leq \int_0^{|\Omega|} g^*(s)\eta^*(s)ds, \qquad \int_\Omega f(x)dx = \int_\Omega g(x)dx$ *for all non negative* $\eta \in L^\infty(\Omega)$.

Obviously the above theorem implies that

$$||f|| \leq ||g|| \tag{3.4}$$

in any Lorentz space.

Now, let z be a measurable function in Ω. We consider for any $s \in [0, |\Omega|]$ a subset $E(s) \subset \Omega$ such that

1) $|E(s)| = s$;

2) $s_1 < s_2 \implies E(s_1) \subset E(s_2)$;

3) $E(s) = \{x \in \Omega : |z(x)| > t\}$ if $s = \mu(t)$.

For any $f \in L^1(\Omega)$, $f \geq 0$, there exists a function F such that

$$\int_{E(s)} f\, dx = \int_0^s F(t)dt.$$

In general, F is not a rearrangement of f, but $F \prec f$ (see [4], [3]). Roughly speaking, we say that F is built from f on the level sets of z.

The main tool to obtain our regularity result for solutions of problem (3.1) is the following theorem due to Alvino and Trombetti [4].

Theorem 3.2. *Let* $\nu \geq 0$, $\nu \in L^1(\Omega)$, $\frac{1}{\nu} \in L^t(\Omega)$ *for some* $t \geq 1$ *and* $f \in L^p(\Omega)$, $\frac{1}{p} = \frac{1}{2} - \frac{1}{2t} + \frac{1}{n}$. *If* $z \in H_0^1(\nu, \Omega)$ *is a solution of* (3.1), *then*

$$z^{\#}(x) \leq \frac{1}{n^2 \omega_n^{2/n}} \int_{\omega_n |x|^n}^{|\Omega|} \frac{r^{-2+2/n}}{\underline{\nu}(r)} \, dr \int_0^r f^*(s) ds \qquad x \in \Omega^{\#}; \tag{3.5}$$

moreover, the following estimate holds

$$\int_\Omega \nu |Dz|^2 \, dx \leq \int_0^{|\Omega|} \frac{1}{\underline{\nu}(r)} \Big(\frac{r^{-1+1/n}}{n \omega_n^{1/n}} \int_0^r f^*(s) ds \Big)^2 dr, \tag{3.6}$$

where $\frac{1}{\underline{\nu}}$ *is built from* $\frac{1}{\nu}$ *on the level sets of* z.

By (3.6), if we assume that (3.2) holds with

$$\begin{aligned} \nu &\in L^1(\Omega), \ \nu(x) \geq 0 \text{ for a. e. } x \in \Omega \\ \frac{1}{\nu} &\in M^t \ \text{ for some } t > 1 \end{aligned} \tag{3.7}$$

and we suppose that

$$f \in L(p,2) \quad \text{with } \frac{1}{p} = \frac{1}{2} + \frac{1}{n} - \frac{1}{2t} \tag{3.8}$$

we obtain an estimate of the norm $||z||_{H_0^1(\nu,\Omega)}$ in terms of the norm of f and $\frac{1}{\nu}$.

More precisely we have the following:

Theorem 3.3. *If* $z \in H_0^1(\nu, \Omega)$ *is solution of* (3.1) *under the assumptions* (3.2), (3.7), (3.8), *then*

$$\int_\Omega \nu |Dz|^2 \, dx \leq \frac{1}{n^2 \omega_n^{2/n}} \left[\frac{p}{2(p-1)} + 2 - \frac{2}{n} \right] \left|\left| \frac{1}{\nu} \right|\right|_{M^t} ||f||_{p,2}^2 . \tag{3.9}$$

Remark. We explicitly observe that under the assumption (1.4) the function $\frac{1}{\nu(x)} = \omega_n^{2-2/n} |x|^{-2+2/n} \lambda^{-2} (\omega_n |x|^n) \in M^t(\Omega^{\#})$ and hence by *iii)* of Theorem 2.1 and (3.9) we obtain the following estimate for the solution u of problem (1.1) with $f \in L(p,2)$

$$\int_\Omega |Du|^2 \, dx \leq \left[\frac{p}{2(p-1)} + 2 - \frac{2}{n} \right] \left|\left| \frac{1}{\nu} \right|\right|_{M^t} ||f||_{p,2}^2 .$$

The a priori estimate implies the solvability of problem (1.1) under assumption (1.4).

Proof. By (3.6) we have

$$\int_\Omega \nu\,|Dz|^2\,dx \le \int_0^{|\Omega|} \Big(\frac{d}{dr}\int_0^r \frac{1}{\underline{\nu}(\sigma)}d\sigma\Big)\frac{r^{-2+2/n}}{n^2\omega_n^{2/n}}\Big(\int_0^r f^*(s)ds\Big)^2 dr.$$

Integrating by parts we get

$$\begin{aligned}||z||^2_{H_0^1(\nu,\Omega)} \le \frac{1}{n^2\omega_n^{2/n}}\Big\{ &|\Omega|^{-2+2/n}\Big(\int_0^{|\Omega|}\frac{1}{\underline{\nu}(s)}ds\Big)\Big(\int_0^{|\Omega|} f^*(s)ds\Big)^2 \\ &- \lim_{r\to 0}\Big(\int_0^r \frac{1}{\underline{\nu}(s)}ds\Big)r^{-2+2/n}\Big(\int_0^r f^*(s)ds\Big)^2 \\ &+ \Big(2-\frac{2}{n}\Big)\int_0^{|\Omega|} dr\Big(\int_0^r \frac{1}{\underline{\nu}(s)}ds\Big)r^{-3+2/n}\Big(\int_0^r f^*(\sigma)d\sigma\Big)^2 \\ &- 2\int_0^{|\Omega|} dr\Big(\int_0^r \frac{1}{\underline{\nu}(s)}ds\Big)r^{-2+2/n}\Big(\int_0^r f^*(s)ds\Big)f^*(r)\Big\} \\ &= I_1 - I_2 + I_3 - I_4.\end{aligned} \tag{3.10}$$

By recalling the definition of norm in M^t and (3.4) we get

$$\begin{aligned}&\Big(\int_0^r \frac{1}{\underline{\nu}(s)}ds\Big)r^{-2+2/n}\Big(\int_0^r f^*(s)ds\Big)^2 \\ &\le \Big\|\frac{1}{\nu}\Big\|_{M^t} r^{-1-1/t+2/n}\Big(\int_0^r f^*(s)ds\Big)^2 \le \frac{p}{2(p-1)}\Big\|\frac{1}{\nu}\Big\|_{M^t}||f||^2_{p,2}\,;\end{aligned} \tag{3.11}$$

thus, $0 \le I_2 < \infty$. By the same arguments we get

$$I_1 \le \frac{1}{n^2\omega_n^{2/n}}\frac{p}{2(p-1)}\Big\|\frac{1}{\nu}\Big\|_{M^t}\|f\|^2_{p,2}. \tag{3.12}$$

On the other hand

$$\begin{aligned}I_3 &\le \frac{1}{n^2\omega_n^{2/n}}\Big(2-\frac{2}{n}\Big)\Big\|\frac{1}{\nu}\Big\|_{M^t}\int_0^{|\Omega|} r^{2/n-1/t}\overline{f}(r)^2 dr \\ &= \frac{1}{n^2\omega_n^{2/n}}\Big(2-\frac{2}{n}\Big)\Big\|\frac{1}{\nu}\Big\|_{M^t}||f||^2_{p,2}\,,\end{aligned} \tag{3.13}$$

and, reasoning in the same way, $0 \le I_4 < \infty$. Thus, by (3.12) and (3.13), disregarding non-positive terms in the right-hand side of (3.10), we get

$$||z||^2_{H_0^1(\nu,\Omega)} \le \frac{1}{n^2\omega_n^{2/n}}\left[\frac{p}{2(p-1)} + 2 - \frac{2}{n}\right]\left\|\frac{1}{\nu}\right\|_{M^t}||f||^2_{p,2}\,,$$

that is, (3.9). □

Using the pointwise estimate (3.9) it is possible to prove regularity results for solutions of problem (3.1) in Lorentz spaces. We know that, if $f \in L(p,2)$, with $\frac{1}{p} = \frac{1}{2} + \frac{1}{n} - \frac{1}{2t}$, then the solution $z \in H_0^1(\Omega)$ belongs, by Sobolev embeddings, to the Lorentz space $L(\beta,2)$, with $\frac{1}{\beta} = \frac{1}{2} - \frac{1}{n} + \frac{1}{2t}$. The theorem that follows shows in Lorentz spaces how the summability of z improves by improving the summability of f until we have $z \in L^\infty(\Omega)$ when $f \in L(q,1)$ with $\frac{1}{q} = \frac{2}{n} - \frac{1}{t}$. Such results extend, in the linear case, the results contained in [7] where the case $\frac{1}{\nu} \in L(t,h)$, $1 \le h \le t$, is considered. The following theorem holds.

Theorem 3.4. *Let $z \in H_0^1(\nu,\Omega)$ be a solution of (3.1) under the assumptions (3.2), (3.7) and let $f \in L(q,k)$, with $\frac{1}{q} < \frac{1}{2} + \frac{1}{n} - \frac{1}{2t}$. Then the following results hold:*

i) if $\frac{1}{q} = \frac{2}{n} - \frac{1}{t}$, $k = 1$, then $z \in L^\infty(\Omega)$. Besides

$$||z||_\infty \le \frac{1}{n^2\omega_n^{2/n}} \left\| \frac{1}{\nu} \right\|_{M^t} ||f||_{q,1};$$

ii) if $\frac{1}{q} = \frac{2}{n} - \frac{1}{t}$, $k > 1$, then $z \in L_{\Phi_k}$ where L_{Φ_k} denotes the Orlicz space generated by the function $\Phi_k(s) = \exp(|s|^{k'}) - 1$. Besides

$$\int_\Omega e^{\alpha\, z^{k'}} dx \le C, \tag{3.14}$$

where α is a constant depending on n, k, q, $\left|\left|\frac{1}{\nu}\right|\right|$, $||f||$ and C is a constant depending on n, k, q, $\left|\left|\frac{1}{\nu}\right|\right|$, $||f||$ and $|\Omega|$;

iii) if $\frac{1}{q} > \frac{2}{n} - \frac{1}{t}$, $k \ge 1$, then $z \in L(\beta,k)$ with $\frac{1}{\beta} = \frac{1}{q} + \frac{1}{t} - \frac{2}{n}$. Besides

$$||z||_{\beta,k} \le \left(\frac{\beta^2}{\beta-1}\right) \left\| \frac{1}{\nu} \right\|_{M^t} ||f||^*_{q,k}.$$

Proof. i) From (3.5) we get

$$||z||_\infty \le \frac{1}{n^2\omega_n^{2/n}} \int_0^{|\Omega|} \frac{r^{-2+2/n}}{\underline{\nu}(r)} dr \int_0^r f^*(s)ds;$$

since the function $\frac{1}{\underline{\nu}}$ is dominated by $\frac{1}{\nu}$ and the function

$$g(r) = r^{-2+2/n} \int_0^r f^*(s)ds$$

is decreasing, from Theorem 3.1 we deduce

$$\begin{aligned}||z||_\infty &\le \frac{1}{n^2\omega_n^{2/n}}\int_0^{|\Omega|}\Big(\frac{1}{\nu(r)}\Big)^* r^{-2+2/n}\Big(\int_0^r f^*(s)ds\Big)dr\\ &\le \frac{1}{n^2\omega_n^{2/n}}\Big\|\frac{1}{\nu}\Big\|_{M^t}\int_0^{|\Omega|} r^{2/n-1/t}\overline{f}(r)\frac{dr}{r} = \frac{1}{n^2\omega_n^{2/n}}\Big\|\frac{1}{\nu}\Big\|_{M^t}||f||_{q,1}.\end{aligned}$$

ii) From (3.5), integrating by parts we get

$$\begin{aligned}z^*(s) \le{}& \frac{1}{n^2\omega_n^{2/n}}\Big(\int_0^{|\Omega|}\frac{1}{\underline{\nu}(r)}dr\Big)|\Omega|^{-2+2/n}\Big(\int_0^{|\Omega|} f^*(r)dr\Big) \qquad (3.15)\\ &-\Big(\int_0^s \frac{1}{\underline{\nu}(r)}dr\Big)s^{-2+2/n}\Big(\int_0^s f^*(r)dr\Big)\\ &+\Big(2-\frac{2}{n}\Big)\int_s^{|\Omega|}\Big(\int_0^r\frac{1}{\underline{\nu}(\sigma)}d\sigma\Big)r^{-3+2/n}\Big(\int_0^r f^*(\sigma)d\sigma\Big)dr\\ &-\int_s^{|\Omega|}\Big(\int_0^r\frac{1}{\underline{\nu}(\sigma)}d\sigma\Big)r^{-2+2/n}f^*(r)dr\\ &= I_1 - I_2 + I_3 - I_4.\end{aligned}$$

Let us consider the first and second integrals in the right-hand side of (3.15), i.e., I_1, I_2. By assumptions we have

$$I_1, I_2 \le \Big\|\frac{1}{\underline{\nu}}\Big\|_{M^t}|\Omega|^{2/n-1/t-1}\Big(\int_0^{|\Omega|} f^*(r)dr\Big) \le C_1\Big\|\frac{1}{\nu}\Big\|_{M^t}||f||_{q,k}.$$

On the other hand $-\infty < I_2 \le 0$. Moreover, by Hölder inequality we get

$$\begin{aligned}I_3 &\le \Big(2-\frac{2}{n}\Big)\Big\|\frac{1}{\underline{\nu}}\Big\|_{M^t}\int_s^{|\Omega|} r^{-2-1/t+2/n}\Big(\int_0^r f^*(\sigma)d\sigma\Big)dr\\ &\le \Big(2-\frac{2}{n}\Big)\Big|\frac{1}{\nu}\Big\|_{M^t}\Big(\log\frac{|\Omega|}{s}\Big)^{1-1/k}\Big(\int_s^{|\Omega|} r^{2k/n-k/t}\overline{f}(r)^k\frac{dr}{r}\Big)^{1/k}\\ &= \Big(\log\frac{|\Omega|}{s}\Big)^{1-1/k}\Big(2-\frac{2}{n}\Big)\Big\|\frac{1}{\nu}\Big\|_{M^t}||f||_{q,k}\end{aligned}$$

and, reasoning in the same way, we prove that $-\infty < I_4 \le 0$. Thus, disregarding non-positive terms in (3.15) we gather

$$z^*(s) \le C_1 + C_2\Big(\log\frac{|\Omega|}{s}\Big)^{1-1/k}$$

and then there exists $\alpha > 0$ such that

$$\int_0^{|\Omega|} e^{[\alpha z^*(s)]^{k'}} ds < +\infty.$$

iii) Reasoning as in the previous case, disregarding non-positive terms in (3.15), we can write

$$\begin{aligned} z^*(s) &\leq \frac{1}{n^2\omega_n^{2/n}}\Big(\int_0^{|\Omega|} \frac{1}{\underline{\nu}(r)}dr\Big)\,|\Omega|^{-2+2/n}\Big(\int_0^{|\Omega|} f^*(r)dr\Big) \\ &\quad + \frac{1}{n^2\omega_n^{2/n}}\Big(2-\frac{2}{n}\Big)\int_s^{|\Omega|}\Big(\int_0^r \frac{1}{\underline{\nu}(\sigma)}d\sigma\Big)r^{-3+2/n}\Big(\int_0^r f^*(\sigma)d\sigma\Big)dr \\ &\leq C_1\Big\|\frac{1}{\underline{\nu}}\Big\|_{M^t}||f||_{q,k} + C_2\Big\|\frac{1}{\underline{\nu}}\Big\|_{M^t}\int_s^{|\Omega|} r^{-2-1/t+2/n}\Big(\int_0^r f^*(\sigma)d\sigma\Big)dr. \end{aligned}$$

Thus,

$$\begin{aligned} \overline{z}(s) &= \frac{1}{s}\int_0^s z^*(r)dr \\ &\leq C_1\Big\|\frac{1}{\underline{\nu}}\Big\|_{M^t}||f||_{q,k} + C_2\frac{1}{s}\int_0^s ds\int_r^{|\Omega|}\sigma^{-2-1/t+2/n}\int_0^\sigma f^*(\rho)d\rho \\ &\leq C_1\Big\|\frac{1}{\nu}\Big\|_{M^t}||f||_{q,k} + C_2\Big\|\frac{1}{\nu}\Big\|_{M^t}\int_0^{|\Omega|}\frac{r^{2/n-1/t}}{\max\{s,r\}}\overline{f}(r)dr. \end{aligned}$$

By definition of norm in Lorentz space $L(\beta,k)$, using Theorem 319 in [12] (the estimate is obvious if $k=1$), we have

$$\begin{aligned} &\Big(||z||^*_{\beta,k}\Big)^k \leq C_1\Big\|\frac{1}{\nu}\Big\|^k_{M^t}||f||^k_{q,k} \\ &+ C_2\Big\|\frac{1}{\nu}\Big\|^k_{M^t}\int_0^\infty\Big(\int_0^{|\Omega|}\frac{s^{1/\beta-1/k}}{\max\{s,r\}r^{1/\beta-1/k}}r^{2/n+1/\beta-1/t-1/k}\overline{f}(r)dr\Big)^k ds \\ &\leq C_1\Big\|\frac{1}{\underline{\nu}}\Big\|^k_{M^t}||f||^k_{q,k} + C_2\Big\|\frac{1}{\nu}\Big\|^k_{M^t}\Big(\frac{\beta^2}{\beta-1}\Big)^k\int_0^\infty\overline{f}(r)^k r^{k(2/n+1/\beta-1/t)-1}dr \\ &= C_1\Big\|\frac{1}{\underline{\nu}}\Big\|^k_{M^t}||f||^k_{q,k} + C_2\Big(\frac{\beta^2}{\beta-1}\Big)^k\Big\|\frac{1}{\nu}\Big\|^k_{M^t}\Big(||f||_{q,k}\Big)^k, \end{aligned}$$

where $\frac{1}{q} = \frac{1}{\beta} + \frac{2}{n} - \frac{1}{t}$. □

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