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PERIODIC SOLUTIONS OF LIÉNARD EQUATIONS AT RESONANCE

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Abstract. In this paper we study the existence of periodic solutions of the second order differential equation

$$x'' + f(x)x' + n^2x + \varphi(x) = p(t), \quad n \in \mathbf{N}.$$

We assume that the limits

$$\lim_{x\to\pm\infty} \varphi(x) = \varphi(\pm\infty), \quad \lim_{x\to\pm\infty} F(x) = F(\pm\infty) \quad \Big(F(x) = \int_0^x f(u)du\Big)$$

exist and are finite. We prove that the given equation has at least one 2π-periodic solution provided that (for $A = \int_0^{2\pi} p(t)\sin nt dt, B = \int_0^{2\pi} p(t)\cos nt dt$) one of the following conditions is satisfied:

$$2(\varphi(+\infty) - \varphi(-\infty)) > \sqrt{A^2 + B^2}$$
$$2n(F(+\infty) - F(-\infty)) > \sqrt{A^2 + B^2}$$
$$2(\varphi(+\infty) - \varphi(-\infty)) = \sqrt{A^2 + B^2}, \quad F(+\infty) \neq F(-\infty)$$
$$2n(F(+\infty) - F(-\infty)) = \sqrt{A^2 + B^2}, \quad \varphi(+\infty) \neq \varphi(-\infty).$$

On the other hand, we prove the non-existence of 2π-periodic solutions provided that the inequality

$$2(\varphi(+\infty) - \varphi(-\infty)) + 2n(F(+\infty) - F(-\infty)) \leq \sqrt{A^2 + B^2}$$

and other conditions hold. We also deal with the existence of 2π-periodic solutions of the equation when φ satisfies a one-sided sublinear condition and F is bounded.

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1. Introduction

We are concerned with the existence of 2π-periodic solutions of the second order differential Liénard equation

$$x'' + f(x)x' + g(x) = p(t), \tag{1.1}$$

where $f, g : \mathbf{R} \to \mathbf{R}$ are continuous, $p : \mathbf{R} \to \mathbf{R}$ is continuous and 2π-periodic.

The existence of 2π-periodic solutions of (1.1) has been widely studied by assuming some dissipativity conditions or by treating sublinear nonlinearities g [11, 15, 16, 18]. When g satisfies the semilinear condition

$$0 < \liminf_{|x|\to+\infty} \frac{g(x)}{x} \le \limsup_{|x|\to+\infty} \frac{g(x)}{x} < +\infty, \tag{1.2}$$

the function $F(x)(=\int_0^x f(u)du)$ is sublinear and the time mapping of the autonomous equation $x''+g(x) = 0$ satisfies some kind of oscillating property, D. Qian [17] proved the existence of 2π-periodic solutions of (1.1).

In the present paper we deal with the existence of periodic solutions of (1.1) when resonance occurs. Consider the equation

$$x'' + f(x)x' + n^2x + \varphi(x) = p(t), \tag{1.3}$$

where n is a positive integer and $\varphi : \mathbf{R} \to \mathbf{R}$ is locally Lipschitz continuous. In this case, a resonant phenomenon may occur. For instance, equation

$$x'' + \sin(x + 1)x' + x + \arctan x = 6\cos t \tag{1.4}$$

has no 2π-periodic solution. In fact, suppose that x is a 2π- periodic solution of (1.4). Multiplying both sides of (1.4) by $\cos t$ and integrating over the interval $[0, 2\pi]$, we have that

$$\begin{aligned} 6\pi = 6\int_0^{2\pi} \cos^2 t dt &= \int_0^{2\pi} \cos t \sin(x + 1)x' dt + \int_0^{2\pi} \cos t \arctan x dt \\ &\le \int_0^{2\pi} |\sin t \cos(x + 1)| dt + \int_0^{2\pi} |\cos t \arctan x| dt \le 2\pi + \pi^2. \end{aligned} \tag{1.5}$$

This is a contradiction. Therefore, (1.4) has no 2π-periodic solution.

When $f \equiv 0$, the existence of 2π-periodic solutions of (1.3) was first studied by A.C. Lazer and D.E. Leach in [10]. When φ satisfies the condition

$$(H_1) \qquad \exists\varphi(\pm\infty) \in \mathbf{R} : \lim_{x\to+\infty} \varphi(x) = \varphi(+\infty), \quad \lim_{x\to-\infty} \varphi(x) = \varphi(-\infty),$$

it was proved in [10] that (1.3) has at least one 2π-periodic solution if

$$2(\varphi(+\infty) - \varphi(-\infty)) > \sqrt{A^2 + B^2} \tag{1.6}$$

holds, where

$$A = \int_0^{2\pi} p(t) \sin nt dt, \quad B = \int_0^{2\pi} p(t) \cos nt dt. \tag{1.7}$$

From then on, many results on equation $x'' + n^2 x + \varphi(x) = p(t)$ have appeared and for the more general one containing a "jumping nonlinearity" [1, 2, 3, 4, 5, 8, 14].

When the damping term appears in (1.3), there are fewer results about the existence of 2π-periodic solutions. Recently, the boundedness of solutions to (1.3) has been proved in [9] by assuming some oddness conditions on f, φ and p. Since the boundedness of solutions is guaranteed by Moser's invariant curves theorem for reversible systems, (1.3) naturally possesses multiple periodic solutions.

The aim of this paper is to study the existence of 2π-periodic solutions of (1.3) when f satisfies some kinds of weak damping conditions. More precisely, we will assume, as for F, that

(H_2) $\quad \exists F(\pm\infty) \in \mathbf{R} : \lim_{x\to+\infty} F(x) = F(+\infty), \quad \lim_{x\to-\infty} F(x) = F(-\infty).$

We are now in a position to state our main results.

Theorem A. *Assume that conditions* (H_1) *and* (H_2) *hold. Then* (1.3) *has at least one* 2π*-periodic solution provided that one of the following conditions is satisfied:*

$$2(\varphi(+\infty) - \varphi(-\infty)) > \sqrt{A^2 + B^2} \tag{1.8}$$

$$2n(F(+\infty) - F(-\infty)) > \sqrt{A^2 + B^2} \tag{1.9}$$

$$2(\varphi(+\infty) - \varphi(-\infty)) = \sqrt{A^2 + B^2}, \quad F(+\infty) \neq F(-\infty) \tag{1.10}$$

$$2n(F(+\infty) - F(-\infty)) = \sqrt{A^2 + B^2}, \quad \varphi(+\infty) \neq \varphi(-\infty). \tag{1.11}$$

Theorem B. *Assume that conditions* (H_1), (H_2) *and*

(H_3) $\quad \forall x \in \mathbf{R} \quad \varphi(-\infty) \leq \varphi(x) \leq \varphi(+\infty), \quad F(-\infty) \leq F(x) \leq F(+\infty)$

hold. Suppose that at least one of the functions φ, F *is not constant. Then* (1.3) *has no* 2π*-periodic solution provided that*

$$2((\varphi(+\infty) - \varphi(-\infty)) + 2n(F(+\infty) - F(-\infty)) \leq \sqrt{A^2 + B^2}. \tag{1.12}$$

In case φ is unbounded, we give the following:

Theorem C. *Assume*

(H_4)

$\exists D > 0 : |\varphi(x)| \leq D,\ x \in (-\infty, 0), \ \lim_{x\to+\infty} \varphi(x) = +\infty, \ \lim_{x\to+\infty} \varphi(x)/x = 0,$

and suppose that F *is bounded. Then* (1.3) *has at least one* 2π*-periodic solution.*

2. Periodic solutions under bounded perturbations

In what follows we prove the existence of 2π-periodic solutions to

$$x'' + f(x)x' + n^2x + \varphi(x) = p(t), \tag{2.1}$$

when the functions F and φ are bounded. Recall that we have set

$$A = \int_0^{2\pi} p(t)\sin nt dt, \quad B = \int_0^{2\pi} p(t)\cos nt dt. \tag{2.2}$$

The main result of this section is:

Theorem 2.1. *Assume that conditions*

(H_1) $\quad \exists\varphi(\pm\infty) \in \mathbf{R} : \lim_{x\to+\infty} \varphi(x) = \varphi(+\infty), \quad \lim_{x\to-\infty} \varphi(x) = \varphi(-\infty),$

(H_2) $\quad \exists F(\pm\infty) \in \mathbf{R} : \lim_{x\to+\infty} F(x) = F(+\infty), \quad \lim_{x\to-\infty} F(x) = F(-\infty)$

hold. Then (2.1) *has at least one* 2π*-periodic solution provided that one of the following conditions is satisfied:*

$$2(\varphi(+\infty) - \varphi(-\infty)) > \sqrt{A^2 + B^2} \tag{2.3}$$

$$2n(F(+\infty) - F(-\infty)) > \sqrt{A^2 + B^2} \tag{2.4}$$

$$2(\varphi(+\infty) - \varphi(-\infty)) = \sqrt{A^2 + B^2}, \quad F(+\infty) \neq F(-\infty) \tag{2.5}$$

$$2n(F(+\infty) - F(-\infty)) = \sqrt{A^2 + B^2}, \quad \varphi(+\infty) \neq \varphi(-\infty). \tag{2.6}$$

The key idea for the proof of Theorem 2.1 is to expand, after a suitable change of variables, the Poincaré mapping associated to a first order system which is equivalent to (2.1). To this end, consider the following system

$$x' = y - F(x), \quad y' = -(n^2x + \varphi(x)) + p(t). \tag{2.7}$$

It is straightforward to check that (2.7) is equivalent to (2.1). For any given $(x_0, y_0) \in \mathbf{R}^2$, consider the Cauchy problem

$$\begin{cases} x' = y - F(x) \\ y' = -(n^2x + \varphi(x)) + p(t) \end{cases} \quad (x(0), y(0)) = (x_0, y_0). \tag{2.8}$$

Assumptions (H_1), (H_2) and the Lipschitz continuity of φ guarantee existence, uniqueness and global continuability of the solutions of (2.8). We can thus define the Poincaré mapping $P : \mathbf{R}^2 \to \mathbf{R}^2$ as follows:

$$P : (x_0, y_0) \mapsto (x_1, y_1) := (x(2\pi), y(2\pi)).$$

Now we introduce (for $c = \sqrt{2n}$) the following transformation $\Phi : (r,\theta) \in \mathbf{R}^+ \times S^1 \to (x,y) \in \mathbf{R}^2 \backslash \{0\}$:

$$x = \frac{c}{n} r^{\frac{1}{2}} \sin\theta, \quad y = c r^{\frac{1}{2}} \cos\theta. \tag{2.9}$$

Under the transformation Φ, (2.7) becomes

$$\begin{aligned} \frac{d\theta}{dt} &= n + \frac{c}{2n} r^{-\frac{1}{2}} \varphi(\frac{c}{n} r^{\frac{1}{2}} \sin\theta) \sin\theta - \frac{c}{2} r^{-\frac{1}{2}} F(\frac{c}{n} r^{\frac{1}{2}} \sin\theta) \cos\theta \\ &\quad - \frac{c}{2n} r^{-\frac{1}{2}} p(t) \sin\theta \\ \frac{dr}{dt} &= -\frac{c}{n} r^{\frac{1}{2}} \varphi(\frac{c}{n} r^{\frac{1}{2}} \sin\theta) \cos\theta - c r^{\frac{1}{2}} F(\frac{c}{n} r^{\frac{1}{2}} \sin\theta) \sin\theta + \frac{c}{n} r^{\frac{1}{2}} p(t) \cos\theta. \end{aligned} \tag{2.10}$$

Denote by $(r(t),\theta(t))$ the solution of (2.10) satisfying the initial condition $r(0) = r_0$, $\theta(0) = \theta_0$, with $x_0 = \frac{c}{n} r_0^{\frac{1}{2}} \sin\theta_0$, $y_0 = c r_0^{\frac{1}{2}} \cos\theta_0$. Thus, the Poincaré mapping P can be expressed in the form

$$P : (r_0,\theta_0) \to (r_1,\theta_1) := (r(2\pi),\theta(2\pi)).$$

Dividing (2.10) by $r^{\frac{1}{2}}$, we get that

$$\frac{dr^{\frac{1}{2}}}{dt} = -\frac{c}{2n} \varphi(\frac{c}{n} r^{\frac{1}{2}} \sin\theta) \cos\theta - \frac{c}{2} F(\frac{c}{n} r^{\frac{1}{2}} \sin\theta) \sin\theta + \frac{c}{2n} p(t) \cos\theta. \tag{2.11}$$

Now, we integrate (2.11) and apply conditions (H_1), (H_2); we thus have

$$r(t)^{\frac{1}{2}} = r_0^{\frac{1}{2}} + O(1),$$

for all t. Then we get $r(t)^{-\frac{1}{2}} = r_0^{-\frac{1}{2}} + O(r_0^{-1})$, $t \in [0,2\pi]$, for $r_0 \to +\infty$, which, together with the first equality of (2.10), yields

$$\frac{d\theta}{dt} = n + O(r_0^{-\frac{1}{2}}),$$

for $r_0 \to +\infty$. Therefore, we obtain that

$$\theta(t) = \theta_0 + nt + O(r_0^{-\frac{1}{2}}), \quad t \in [0,2\pi], \tag{2.12}$$

for $r_0 \to +\infty$. Combining (2.11) and (2.12), we have

$$\begin{aligned} \frac{dr^{\frac{1}{2}}}{dt} &= -\frac{c}{2n} \varphi[\frac{c}{n} r_0^{\frac{1}{2}} \sin(\theta_0 + nt) + O(1)] \cos(\theta_0 + nt) \\ &\quad - \frac{c}{2} F[\frac{c}{n} r_0^{\frac{1}{2}} \sin(\theta_0 + nt) + O(1)] \sin(\theta_0 + nt) \\ &\quad + \frac{c}{2n} p(t) \cos(\theta_0 + nt) + O(r_0^{-\frac{1}{2}}). \end{aligned} \tag{2.13}$$

Integrating both sides of the above equality we get that

$$
\begin{aligned}
r_1^{\frac{1}{2}} &= r_0^{\frac{1}{2}} - \frac{c}{2n}\int_0^{2\pi} \varphi[\frac{c}{n} r_0^{\frac{1}{2}} \sin(\theta_0+nt) + O(1)]\cos(\theta_0+nt)dt \\
&- \frac{c}{2}\int_0^{2\pi} F[\frac{c}{n} r_0^{\frac{1}{2}} \sin(\theta_0+nt) + O(1)]\sin(\theta_0+nt)dt \qquad (2.14)\\
&+ \frac{c}{2n}\int_0^{2\pi} p(t)\cos(\theta_0+nt)dt + O(r_0^{-\frac{1}{2}}),
\end{aligned}
$$

for $r_0 \to +\infty$. Similarly, we have

$$
\begin{aligned}
\theta_1 &= \theta_0 + 2n\pi + \frac{c}{2n} r_0^{-\frac{1}{2}} \int_0^{2\pi} \varphi[\frac{c}{n} r_0^{\frac{1}{2}} \sin(\theta_0+nt) + O(1)]\sin(\theta_0+nt)dt \\
&- \frac{c}{2} r_0^{-\frac{1}{2}} \int_0^{2\pi} F[\frac{c}{n} r_0^{\frac{1}{2}} \sin(\theta_0+nt) + O(1)]\cos(\theta_0+nt)dt \qquad (2.15)\\
&- \frac{c}{2n} r_0^{-\frac{1}{2}} \int_0^{2\pi} p(t)\sin(\theta_0+nt)dt + O(r_0^{-1}),
\end{aligned}
$$

for $r_0 \to +\infty$. Set $\rho = r^{\frac{1}{2}}$. From (2.14) and (2.15) we have that, for $\rho_0 \to +\infty$,

$$
\begin{aligned}
\rho_1 &= \rho_0 - \frac{c}{2n}\int_0^{2\pi} \varphi[\frac{c}{n}\rho_0 \sin(\theta_0+nt) + O(1)]\cos(\theta_0+nt)dt \\
&- \frac{c}{2}\int_0^{2\pi} F[\frac{c}{n}\rho_0 \sin(\theta_0+nt) + O(1)]\sin(\theta_0+nt)dt \\
&+ \frac{c}{2n}\int_0^{2\pi} p(t)\cos(\theta_0+nt)dt + O(\rho_0^{-1}) \\
\theta_1 &= \theta_0 + 2n\pi + \frac{c}{2n}\rho_0^{-1}\int_0^{2\pi} \varphi[\frac{c}{n}\rho_0 \sin(\theta_0+nt) + O(1)]\sin(\theta_0+nt)dt \\
&- \frac{c}{2}\rho_0^{-1}\int_0^{2\pi} F[\frac{c}{n}\rho_0 \sin(\theta_0+nt) + O(1)]\cos(\theta_0+nt)dt \\
&- \frac{c}{2n}\rho_0^{-1}\int_0^{2\pi} p(t)\sin(\theta_0+nt)dt + O(\rho_0^{-2}).
\end{aligned}
$$

An estimate of the above integrals is provided by the following:

Lemma 2.2. *Assume that conditions* (H_1) *and* (H_2) *hold. Then we have*

$$
\lim_{\rho_0\to+\infty}\int_0^{2\pi} \varphi[\frac{c}{n}\rho_0 \sin(\theta_0+nt) + O(1)]\cos(\theta_0+nt)dt = 0,
$$

$$
\lim_{\rho_0\to+\infty}\int_0^{2\pi} F[\frac{c}{n}\rho_0 \sin(\theta_0+nt) + O(1)]\sin(\theta_0+nt)dt = 2(F(+\infty) - F(-\infty)),
$$

$$\lim_{\rho_0\to+\infty}\int_0^{2\pi}\varphi[\frac{c}{n}\rho_0\sin(\theta_0+nt)+O(1)]\sin(\theta_0+nt)dt=2(\varphi(+\infty)-\varphi(-\infty)),$$
$$\lim_{\rho_0\to+\infty}\int_0^{2\pi}F[\frac{c}{n}\rho_0\sin(\theta_0+nt)+O(1)]\cos(\theta_0+nt)dt=0.$$

The proof of this lemma is easy. We omit it here. We are now in position to give the

Proof of Theorem 2.1. From Lemma 2.2, we know that (ρ_1, θ_1) can be expressed in the form

$$\rho_1=\rho_0-\frac{c}{2n}[2n(F(+\infty)-F(-\infty))+\sqrt{A^2+B^2}\sin(\theta_0-\alpha)]+o(1)$$
$$\theta_1=\theta_0+2n\pi+\frac{c}{2n}\rho_0^{-1}[2(\varphi(+\infty)-\varphi(-\infty))-\sqrt{A^2+B^2}\cos(\theta_0-\alpha)]+o(\rho_0^{-1}),$$

$(\rho_0\to+\infty)$, where A and B are given in (2.2) and $\alpha=\arctan B/A$ for $A\geq 0$ or $\alpha=\pi+\arctan B/A$ for $A<0$. Define the mapping $P_0:\mathbf{R}^+\times S^1\to\mathbf{R}^+\times S^1$ by $P_0(\rho_0,\theta_0)=(\rho_1,\theta_1)$. In what follows, we examine separately the different situations when at least one of the conditions in Theorem 2.1 is satisfied.

(1) Assume that $2(\varphi(+\infty)-\varphi(-\infty))>\sqrt{A^2+B^2}$. In this case, it follows from the asymptotic expression of the mapping P_0 that, for ρ_0 large enough, the image (ρ_1,θ_1) of (ρ_0,θ_0) under P_0 does not belong to the line $\theta=\theta_0$. Therefore, the Poincaré-Bohl theorem [12] ensures the existence of at least one fixed point (ρ_0,θ_0) of the mapping P_0. Thus, the Poincaré mapping P possesses at least one fixed point (r_0,θ_0). Hence, (2.1) has at least one 2π-periodic solution.

(2) Assume that $2n(F(+\infty)-F(-\infty))>\sqrt{A^2+B^2}$. In this case, from the asymptotic expression of the mapping P_0, we know that for ρ_0 large enough it follows that $\rho_1<\rho_0$. This means that the image (ρ_1,θ_1) of the point (ρ_0,θ_0) under the mapping P_0 lies inside the circle $\{(\rho,\theta):\rho=\rho_0,\theta\in\mathbf{R}\}$. Thus, the Brouwer fixed point theorem ensures the existence of at least one fixed point of the mapping P_0. Therefore, the Poincaré mapping P has at least one fixed point. Thus, (2.1) has at least one 2π-periodic solution.

(3) Assume that $2(\varphi(+\infty)-\varphi(-\infty))=\sqrt{A^2+B^2}$ and $F(+\infty)\neq F(-\infty)$. First, we assume $\sqrt{A^2+B^2}>0$; we can thus introduce the transformation

$$\gamma=\frac{2n}{c\sqrt{A^2+B^2}}\rho,\quad \phi=\theta-\alpha.$$

Then, for γ_0 large enough, the mapping P_0 becomes

$$\mathcal{P}_0:\begin{cases}\gamma_1= & \gamma_0-a-\sin\phi_0+o(1)\\ \phi_1= & \phi_0+2n\pi+\gamma_0^{-1}(1-\cos\phi_0)+o(\gamma_0^{-1}),\end{cases}$$

where $a = 2n(F(+\infty) - F(-\infty))/\sqrt{A^2+B^2}$. Since $F(+\infty) \neq F(-\infty)$, we have $a \neq 0$. From the expression of (γ_1, ϕ_1) we have that, for γ_0 large enough,

$$\sin\phi_1 = \sin\phi_0 + \gamma_0^{-1}\cos\phi_0(1-\cos\phi_0) + o(\gamma_0^{-1}).$$

Therefore,

$$\gamma_1 \sin\phi_1 = \gamma_0\sin\phi_0 + \cos\phi_0 - a\sin\phi_0 - 1 + o(1).$$

Similarly, we can obtain that

$$\gamma_1\cos\phi_1 = \gamma_0\cos\phi_0 - \sin\phi_0 - a\cos\phi_0 + o(1).$$

Furthermore, we get

$$\begin{aligned}\gamma_1\sin\phi_1 - \gamma_0\sin\phi_0 &= \cos\phi_0 - a\sin\phi_0 - 1 + o(1)\\ \gamma_1\cos\phi_1 - \gamma_0\cos\phi_0 &= -\sin\phi_0 - a\cos\phi_0 + o(1).\end{aligned}$$

Since $a \neq 0$, we know that $(\gamma_1\sin\phi_1 - \gamma_0\sin\phi_0, \gamma_1\cos\phi_1 - \gamma_0\cos\phi_0) \neq (0,0)$ for γ_0 large enough. Define the vectors

$$\begin{aligned}&V_1(\phi_0) = (-\cos\phi_0, \sin\phi_0),\\ &V_2(\gamma_0,\phi_0) = (\gamma_1\sin\phi_1 - \gamma_0\sin\phi_0, \gamma_1\cos\phi_1 - \gamma_0\cos\phi_0),\\ &V_3(\phi_0) = (\cos\phi_0 - a\sin\phi_0 - 1, -\sin\phi_0 - a\cos\phi_0).\end{aligned}$$

Next, we shall estimate the rotation number of the vector $V_2(\gamma_0,\phi_0)$ around the origin when ϕ_0 increases from 0 to 2π. To this end, it is sufficient to estimate the rotation number of the vector $V_3(\phi_0)$ around the origin when ϕ_0 increases from 0 to 2π.

The inner product of the vectors $V_1(\phi_0), V_3(\phi_0)$ is

$$< V_1(\phi_0), V_3(\phi_0) >= \cos\phi_0 - 1 \leq 0. \tag{2.16}$$

Since the vector $(-\cos\phi_0, \sin\phi_0)$ rotates clockwise exactly once with respect to the origin when ϕ_0 increases from 0 to 2π, we know from (2.16) that the vector $V_3(\phi_0)$ also rotates clockwise exactly once with respect to the origin when ϕ_0 increases from 0 to 2π. Therefore, the Brouwer degree $deg(V_3(\phi_0), B(\Gamma_0), 0) = 1$, for Γ_0 large enough, where $B(\Gamma_0) = \{(\gamma,\phi) : \gamma \leq \Gamma_0\}$. Furthermore, the Brouwer degree $deg(V_2(\gamma_0,\phi_0), B(\Gamma_0), 0) = 1$. This shows that $V_2(\gamma_0,\phi_0)$ as a mapping has at least one zero point. Then the mapping $\mathcal{P}_0$ has at least one fixed point. Hence, (2.1) has at least one 2π-periodic solution.

If $2(\varphi(+\infty) - \varphi(-\infty)) = \sqrt{A^2+B^2} = 0$ and $F(+\infty) \neq F(-\infty)$, then the mapping P_0 can be expressed in the form:

$$\begin{cases}\rho_1 = \rho_0 - c(F(+\infty) - F(-\infty)) + o(1)\\ \theta_1 = \theta_0 + 2n\pi + o(\rho_0^{-1}),\end{cases}$$

for $\rho_0 \to +\infty$.

If $F(+\infty) - F(-\infty) > 0$, then $\rho_1 < \rho_0$ for ρ_0 large enough. Therefore, the mapping P_0 has at least one fixed point. Thus, (2.1) has at least one 2π-periodic solution.

If $F(+\infty) - F(-\infty) < 0$, then we can consider the inverse mapping P_0^{-1} of P_0. The inverse mapping $P_0^{-1} : (\rho_0, \theta_0) \to (\rho_{-1}, \theta_{-1})$ has the form

$$\begin{cases} \rho_{-1} = \rho_0 + c(F + \infty) - F(-\infty)) + o(1) \\ \theta_{-1} = \theta_0 - 2n\pi + o(\rho_0^{-1}), \end{cases}$$

for $\rho_0 \to +\infty$. Obviously, P_0^{-1} has at least one fixed point. Furthermore, P_0 also has at least one fixed point. Thus, (2.1) has at least one 2π- periodic solution.

(4) Assume that $2(F(+\infty) - F(-\infty)) = \sqrt{A^2 + B^2}$ and $\varphi(+\infty) \neq \varphi(-\infty)$. If $\sqrt{A^2 + B^2} > 0$, then we take again the transformation

$$\gamma = \frac{2n}{c\sqrt{A^2 + B^2}}\rho, \quad \phi = \theta - \alpha.$$

Then the mapping P_0 becomes

$$\mathcal{P}_0 : \begin{cases} \gamma_1 = \gamma_0 - 1 - \sin\phi_0 + o(1) \\ \phi_1 = \phi_0 + 2n\pi + \gamma_0^{-1}(b - \cos\phi_0) + o(\gamma_0^{-1}), \end{cases}$$

(for $\gamma_0 \to +\infty$), where $b = 2n(\varphi(+\infty) - \varphi(-\infty))/\sqrt{A^2 + B^2}$. Since $\varphi(+\infty) \neq \varphi(-\infty)$, we have $b \neq 0$. Furthermore, arguing as in step (3), we get

$$\begin{aligned} \gamma_1 \sin\phi_1 - \gamma_0 \sin\phi_0 &= b\cos\phi_0 - \sin\phi_0 - 1 + o(1) \\ \gamma_1 \cos\phi_1 - \gamma_0 \cos\phi_0 &= -b\sin\phi_0 - \cos\phi_0 + o(1). \end{aligned}$$

Since $b \neq 0$, we know that $(\gamma_1 \sin\phi_1 - \gamma_0 \sin\phi_0, \gamma_1 \cos\phi_1 - \gamma_0 \cos\phi_0) \neq (0, 0)$, for γ_0 large enough. Define

$$\begin{aligned} \tilde{V}_1(\phi_0) &= (-\cos\phi_0, \sin\phi_0), \\ \tilde{V}_2(\gamma_0, \phi_0) &= (\gamma_1 \sin\phi_1 - \gamma_0 \sin\phi_0, \gamma_1 \cos\phi_1 - \gamma_0 \cos\phi_0), \\ \tilde{V}_3(\phi_0) &= (b\cos\phi_0 - \sin\phi_0 - 1, -b\sin\phi_0 - \cos\phi_0). \end{aligned}$$

The inner product of $\tilde{V}_1(\phi_0)$ and $\tilde{V}_3(\phi_0)$ is

$$< \tilde{V}_1(\phi_0), \tilde{V}_3(\phi_0) >= \cos\phi_0 - b.$$

If $|b| \geq 1$, then we can put ourselves into the situation considered in step (3) and conclude that $\tilde{V}_3(\phi_0)$ makes exactly one rotation around the origin when ϕ_0 increases from 0 to 2π.

If $|b| < 1$, then there exist $\phi_1 < \phi_2 < \phi_3 = \phi_1 + 2\pi$ such that

$$\cos\phi_i = b, \quad i = 1, 2, 3,$$

and

$$\cos\phi_0 - b \leq 0, \phi_0 \in [\phi_1, \phi_2]; \quad \cos\phi_0 - b \geq 0, \phi_0 \in [\phi_2, \phi_3]. \tag{2.17}$$

However, at the points ϕ_i, $i = 1, 2, 3$, we have that

$$b\cos\phi_i - \sin\phi_i - 1 = cos^2\phi_i - \sin\phi_i - 1 = -(1 + \sin\phi_i)\sin\phi_i$$

and

$$-b\sin\phi_i - \cos\phi_i = -cos\phi_i \sin\phi_i - \cos\phi_i = -(1 + \sin\phi_i)\cos\phi_i.$$

Therefore, at the points ϕ_i, $i = 1, 2, 3$, the vector $\tilde{V}_3(\phi_0)$ is in the opposite direction of the vector $(\sin\phi_i, \cos\phi_i)$. From this fact and (2.16) we know that $\tilde{V}_3(\phi_0)$ also makes exactly one rotation around the origin when ϕ_0 increases from 0 to 2π. The remainder is the same as in step (3).

If $A = B = 0$ and $\varphi(+\infty) \neq \varphi(-\infty)$, then the mapping P_0 has the form:

$$\begin{cases} \rho_1 = \rho_0 + o(1) \\ \theta_1 = \theta_0 + 2n\pi + \frac{c}{n}(\varphi(+\infty) - \varphi(-\infty))\rho_0^{-1} + o(\rho_0^{-1}), \end{cases}$$

for $\rho_0 \to +\infty$. As in step (1) we conclude that (2.1) also has at least one 2π- periodic solution. □

Remark 2.3. From the proof of Theorem 2.1, we know that the same conclusion of Theorem 2.1 still holds if conditions (2.3)-(2.4)-(2.5)-(2.6) are replaced by

$$2(\varphi(-\infty) - \varphi(+\infty) > \sqrt{A^2 + B^2},$$

$$2n(F(-\infty) - F(+\infty)) > \sqrt{A^2 + B^2},$$

$$2(\varphi(-\infty) - \varphi(+\infty)) = \sqrt{A^2 + B^2}, \quad F(+\infty) \neq F(-\infty),$$

$$2n(F(-\infty) - F(+\infty)) = \sqrt{A^2 + B^2}, \quad \varphi(+\infty) \neq \varphi(-\infty),$$

respectively.

Remark 2.4. If $A = 0 = B$, then it follows from Theorem 2.1 that equation

$$x'' + n^2 x + \varphi(x) = p(t)$$

has at least one 2π-periodic solution provided that $\varphi(+\infty) \neq \varphi(-\infty)$ holds. In this way, we partly generalize the result of A.C. Lazer and D.E. Leach [10].

3. No periodic solutions under bounded perturbations

In what follows we give sufficient conditions for the non-existence of 2π-periodic solutions to

$$x'' + f(x)x' + n^2x + \varphi(x) = p(t), \tag{3.1}$$

where the functions F and φ are bounded. We will use the well-known fact that equation $x'' + n^2x = p(t)$ has a 2π-periodic solution if and only if

$$\int_0^{2\pi} p(t)\sin nt dt = 0, \quad \int_0^{2\pi} p(t)\cos nt dt = 0.$$

The main result of this Section is:

Theorem 3.1. *Assume that conditions*

(H_1) $\quad \exists\varphi(\pm\infty) \in \mathbf{R} : \lim_{x\to+\infty} \varphi(x) = \varphi(+\infty), \quad \lim_{x\to-\infty} \varphi(x) = \varphi(-\infty),$

(H_2) $\quad \exists F(\pm\infty) \in \mathbf{R} : \lim_{x\to+\infty} F(x) = F(+\infty), \quad \lim_{x\to-\infty} F(x) = F(-\infty)$

(H_3) $\quad \forall\, x \in \mathbf{R} \quad \varphi(-\infty) \le \varphi(x) \le \varphi(+\infty), \quad F(-\infty) \le F(x) \le F(+\infty)$

hold. Suppose that at least one of the functions φ, F is not constant. Then (3.1) *has no 2π-periodic solution provided that*

$$2((\varphi(+\infty) - \varphi(-\infty)) + 2n(F(+\infty) - F(-\infty)) \le \sqrt{A^2 + B^2} \tag{3.2}$$

holds.

Proof. We follow the arguments in [10]. Assume by contradiction that equation (3.1) has a 2π-periodic solution $y(\cdot)$. Without loss of generality, we may assume that F is not a constant function (the other case can be treated similarly). Let α be a constant such that

$$A\sin n\alpha + B\cos n\alpha = \sqrt{A^2 + B^2},$$

with A, B given as above (cf. (1.7)). It follows from the fact that $y(\cdot)$ is a 2π-periodic solution of (3.1) that $y(\cdot + \alpha)$ is a 2π-periodic solution of equation

$$x'' + f(x)x' + n^2x + \varphi(x) = p(t + \alpha).$$

Set $q(t) = p(t + \alpha) - f(y(t))y'(t) - \varphi(y(t))$. Then $y(\cdot + \alpha)$ is a 2π-periodic solution of

$$x'' + n^2x = q(t).$$

Therefore, we have that

$$\int_0^{2\pi} q(t)\sin nt dt = 0, \quad \int_0^{2\pi} q(t)\cos nt dt = 0. \tag{3.3}$$

By a simple calculation, we know that

$$\int_0^{2\pi} p(t+\alpha)\cos nt dt = \sqrt{A^2+B^2}. \tag{3.4}$$

From the second equality of (3.3), we get

$$\int_0^{2\pi} p(t+\alpha)\cos nt dt = \int_0^{2\pi} f(y(t))y'(t)\cos nt dt + \int_0^{2\pi} \varphi(y(t))\cos nt dt. \tag{3.5}$$

Integration by parts yields

$$\int_0^{2\pi} f(y(t))y'(t)\cos nt dt = n\int_0^{2\pi} F(y(t))\sin nt dt.$$

Set

$$J_+ = \{t : \sin nt \geq 0, t \in [0, 2\pi]\}, \quad J_- = \{t : \sin nt \leq 0, t \in [0, 2\pi]\}.$$

Obviously, both J_+ and J_- consist of n connected intervals. If $t \in J_+$, from condition (H_3) we know that

$$F(y(t))\sin nt \leq F(+\infty)\sin nt.$$

Therefore,

$$\int_{J_+} F(y(t))\sin nt dt \leq \int_{J_+} F(+\infty)\sin nt dt.$$

Furthermore,

$$\int_{J_+} F(y(t))\sin nt dt \leq 2F(+\infty). \tag{3.6}$$

Similarly, we get

$$\int_{J_-} F(y(t))\sin nt dt \leq -2F(-\infty). \tag{3.7}$$

Since the function F is not constant, only one of the inequalities (3.6), (3.7) can be an equality. Then we have that

$$\int_0^{2\pi} f(y(t))y'(t)\cos nt dt < 2n(F(+\infty) - F(-\infty)). \tag{3.8}$$

On the other hand, by the same method, we obtain

$$\int_0^{2\pi} \varphi(y(t))\cos nt dt \leq 2(\varphi(+\infty) - \varphi(-\infty)). \tag{3.9}$$

It follows from (3.4), (3.5), (3.8) and (3.9) that

$$\begin{aligned}\sqrt{A^2+B^2} &= \int_0^{2\pi} p(t+\alpha)\cos nt dt \\ &< 2(\varphi(+\infty) - \varphi(-\infty)) + 2n(F(+\infty) - F(-\infty)).\end{aligned}$$

This is a contradiction with assumption (3.2). Hence, (2.1) has no 2π-periodic solution. $\square$

From Theorem 3.1 and Massera's theorem [13] we have the following result.

Corollary 3.2. *Assume that the assumptions of Theorem* 3.1 *are satisfied. Then, all the solutions of* (3.1) *are unbounded provided*

$$2(\varphi(+\infty) - \varphi(-\infty)) + 2n(F(+\infty) - F(-\infty)) \le \sqrt{A^2 + B^2}.$$

If condition (H_3) is not satisfied, then we have

Corollary 3.3. *Assume that conditions* (H_1), (H_2) *and*

$$2((\varphi(+\infty) - \varphi(-\infty)) + 2n(F(+\infty) - F(-\infty)) < \sqrt{A^2 + B^2}$$

hold. Then there exists a positive constant r_0 *such that every solution* $x(\cdot)$ *of* (2.1) *such that* $x(0) = x_0, x'(0) = y_0$ *satisfying the inequality* $x_0^2 + y_0^2 \ge r_0^2$ *is not a* 2π*-periodic solution of* (2.1).

The proof of Corollary 3.3 can be performed by means of a slight modification of the proof of Theorem 3.1.

4. Periodic solutions under unbounded perturbations

If the function φ is unbounded, the method used in Section 2 is not applicable. We will use a phase-plane analysis method to prove the existence of 2π-periodic solutions of (2.1). For convenience, we recall that (2.1) is equivalent to

$$x' = y - F(x), \quad y' = -(n^2 x + \varphi(x)) + p(t). \tag{4.1}$$

In what follows, we use the transformation $\Phi : (r, \theta) \in \mathbf{R}^+ \times S^1 \to (x, y) \in \mathbf{R}^2 \setminus \{0\}$ introduced in (2.9) and develop some useful estimates.

Lemma 4.1. *Assume that*

(H_4)

$$\exists D > 0 : |\varphi(x)| \le D,\ x \in (-\infty, 0), \quad \lim_{x\to+\infty} \varphi(x) = +\infty,\ \lim_{x\to+\infty} \varphi(x)/x = 0,$$

holds and suppose that F *is bounded. Then there exist two positive constants* $R_0 > \delta > 1$ *such that*

(1) $\delta^{-1} r_0 \le r(t) \le \delta r_0$, *for* $r_0 \ge R_0$, $t \in [0, 2\pi]$,
(2) $\theta'(t) > 0$, *for* $r_0 \ge R_0$, $t \in [0, 2\pi]$.

Lemma 4.1 can be easily proved by using (4.1).

Now, denote by $\tau(r_0, \theta_0)$ the time required for the motion $(r(t), \theta(t))$ to complete one turn around the origin. Then we have

Lemma 4.2. *Assume that the assumptions of Lemma* 4.1 *hold. Then, for* $r_0 \to +\infty$*, we have*

$$\tau(r_0, \theta_0) = \frac{2\pi}{n} + o(1),$$

uniformly in $\theta_0 \in [0, 2\pi]$.

Proof. From condition (H_4) and Lemma 4.1, we know that

$$\lim_{r_0 \to +\infty} r^{-\frac{1}{2}} \varphi(\frac{c}{n} r^{\frac{1}{2}} \sin\theta) \sin\theta = 0. \tag{4.2}$$

According to the first equality of (4.1), we have

$$\tau(r_0, \theta_0) = \int_0^{2\pi} \frac{d\theta}{n + \frac{c}{2n} r^{-\frac{1}{2}} \varphi(\frac{c}{n} r^{\frac{1}{2}} \sin\theta) \sin\theta - \frac{c}{2} r^{-\frac{1}{2}} F(\frac{c}{n} r^{\frac{1}{2}} \sin\theta) \cos\theta - \frac{c}{2n} r^{-\frac{1}{2}} p(t) \sin\theta}.$$

Since F is bounded, from (4.2) we obtain that

$$\tau(r_0, \theta_0) = \int_0^{2\pi} \frac{d\theta}{n + o(1)} = \frac{2\pi}{n} + o(1),$$

for $r_0 \to +\infty$.

Lemma 4.3. *Assume that the assumptions of Lemma* 4.1 *are satisfied. Then* $\tau(r_0, \theta_0) > \frac{2\pi}{n+1}$, *for* $r_0 \to +\infty$.

The proof of Lemma 4.3 follows directly from Lemma 4.2.

Lemma 4.4. *Assume that the assumptions of Lemma* 4.1 *are satisfied. Then* $\tau(r_0, \theta_0) < \frac{2\pi}{n}$ *for* $r_0 \to +\infty$.

Proof. We follow the ideas in [3]. From $\lim_{x \to +\infty} \varphi(x) = +\infty$, we know that, for any sufficiently large constant $H > 0$, there is a constant $a > 0$ such that

$$\varphi(x) \geq H, \text{ for } x \geq a. \tag{4.3}$$

Moreover, since φ is bounded in the interval $(-\infty, 0)$, there is a constant $K > 0$ such that $|\varphi(x)| \leq K$, for $x \in (-\infty, 0)$. The boundedness of F also ensures that there is a constant $C > 0$ such that $|F(x)| \leq C$, for $x \in (-\infty, +\infty)$. For r_0 large enough, let $\sigma \in (0, \pi/10)$ be defined by

$$\sigma = \frac{n\pi\sqrt{\delta}a}{2c} r_0^{-\frac{1}{2}},$$

where δ is given in Lemma 4.1. From now on, without loss of generality, we assume $\theta_0 \in [0, \sigma] \subset [0, \pi/10]$. The other cases can be treated similarly.

From (4.1) we have

$$\tau(\theta_0, I_0) = \sum_{i=1}^{6} \tau_i(I_0, \theta_0),$$

with

$$\tau_1(I_0,\theta_0) = \int_{\theta_0}^{\sigma} \frac{d\theta}{\Psi}, \qquad \tau_2(I_0,\theta_0) = \int_{\sigma}^{\pi-\sigma} \frac{d\theta}{\Psi}, \quad \tau_3(I_0,\theta_0) = \int_{\pi-\sigma}^{\pi+\sigma} \frac{d\theta}{\Psi},$$
$$\tau_4(I_0,\theta_0) = \int_{\pi+\sigma}^{2\pi-\sigma} \frac{d\theta}{\Psi}, \quad \tau_5(I_0,\theta_0) = \int_{2\pi-\sigma}^{2\pi} \frac{d\theta}{\Psi}, \quad \tau_6(I_0,\theta_0) = \int_{2\pi}^{2\pi+\theta_0} \frac{d\theta}{\Psi},$$

where $\Psi = n + \frac{c}{2n} r^{-\frac{1}{2}}[\varphi(\frac{c}{n} r^{\frac{1}{2}} \sin\theta)\sin\theta - nF(\frac{c}{n} r^{\frac{1}{2}} \sin\theta)\cos\theta - p(t)\sin\theta]$. Next, we shall estimate the above integrals. If $\theta \in [\theta_0, \sigma]$, then

$$0 \le \frac{c}{n} r^{\frac{1}{2}} \sin\theta \le \frac{c\sqrt{\delta}}{n} r_0^{\frac{1}{2}} \sin\sigma \le \frac{c\sqrt{\delta}}{n} r_0^{\frac{1}{2}} \sigma = \frac{1}{2}\pi a\delta.$$

Hence,

$$\begin{aligned}
&\tau_1(I_0,\theta_0) \\
&= \int_{\theta_0}^{\sigma} \frac{d\theta}{n + \frac{c}{2n} r^{-\frac{1}{2}}[\varphi(\frac{c}{n} r^{\frac{1}{2}} \sin\theta)\sin\theta - nF(\frac{c}{n} r^{\frac{1}{2}} \sin\theta)\cos\theta - p(t)\sin\theta]} \\
&= \int_{\theta_0}^{\sigma} \frac{d\theta}{n + O(r_0^{-\frac{1}{2}})} = (\frac{\sigma}{n} - \frac{\theta_0}{n}) + O(r_0^{-1}),
\end{aligned} \tag{4.4}$$

for $r_0 \to +\infty$. If $\theta \in [\sigma, \pi - \sigma]$, then

$$\frac{c}{n} r^{\frac{1}{2}} \sin\theta \ge \frac{c}{n\sqrt{\delta}} r_0^{\frac{1}{2}} \sin\sigma \ge \frac{2c}{n\sqrt{\delta}\pi} r_0^{\frac{1}{2}} \sigma = a,$$

which, together with (4.3), implies that

$$\varphi(\frac{c}{n} r^{\frac{1}{2}} \sin\theta) \ge H, \quad \forall\theta \in [\sigma, \pi - \sigma]. \tag{4.5}$$

Therefore, from Lemma 4.1 we have that

$$\begin{aligned}
\Psi &= n + \frac{c}{2n} r^{-\frac{1}{2}}[\varphi(\frac{c}{n} r^{\frac{1}{2}} \sin\theta)\sin\theta - nF(\frac{c}{n} r^{\frac{1}{2}} \sin\theta)\cos\theta - p(t)\sin\theta] \\
&\ge n + \frac{c}{2n} r^{-\frac{1}{2}}[H\sin\theta - ||p||_\infty \sin\theta - nC] \\
&\ge n + \frac{c}{2n\sqrt{\delta}} r_0^{-\frac{1}{2}}(H - ||p||_\infty)\sin\theta - \frac{1}{2} cC\sqrt{\delta} r_0^{-\frac{1}{2}} \\
&= n + \frac{c}{2n\sqrt{\delta}} r_0^{-\frac{1}{2}}[(H - ||p||_\infty)\sin\theta - n\delta C].
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
&\tau_2(I_0,\theta_0)\\
&=\int_\sigma^{\pi-\sigma}\frac{d\theta}{n+\frac{c}{2n}r^{-\frac12}[\varphi(\frac{c}{n}r^{\frac12}\sin\theta)\sin\theta-nF(\frac{c}{n}r^{\frac12}\sin\theta)\cos\theta-p(t)\sin\theta]}\\
&\le\int_\sigma^{\pi-\sigma}\frac{d\theta}{n+\frac{c}{2n\sqrt{\delta}}r_0^{-\frac12}[(H-||p||_\infty)\sin\theta-n\delta C]}\\
&=\frac1n\int_\sigma^{\pi-\sigma}\frac{d\theta}{1+\frac{c}{2n^2\sqrt{\delta}}r_0^{-\frac12}[(H-||p||_\infty)\sin\theta-n\delta C]} \qquad (4.6)\\
&=\frac1n\int_\sigma^{\pi-\sigma}\{1-\frac{c}{2n^2\sqrt{\delta}}r_0^{-\frac12}[(H-||p||_\infty)\sin\theta-n\delta C]\}d\theta+O(r_0^{-1})\\
&=(\frac{\pi}{n}-\frac{2\sigma}{n})-\frac{c}{2n^3\sqrt{\delta}}r_0^{-\frac12}[2(H-||p||_\infty)\cos\sigma-n\delta C(\pi-2\sigma)]+O(r_0^{-1})\\
&=(\frac{\pi}{n}-\frac{2\sigma}{n})-\frac{c}{2n^3\sqrt{\delta}}r_0^{-\frac12}[2(H-||p||_\infty)\cos\sigma-n\delta C\pi]+O(r_0^{-1}),
\end{aligned}
$$

for $r_0\to+\infty$. If $\theta\in[\pi-\sigma,\pi]$, then

$$0\le\frac{c}{n}r^{\frac12}\sin\theta\le\frac{c\sqrt{\delta}}{n}r_0^{\frac12}\sin\sigma\le\frac{c\sqrt{\delta}}{n}r_0^{\frac12}\sigma=\frac12\pi a\delta.$$

Thus,

$$
\begin{aligned}
&\int_{\pi-\sigma}^{\pi}\frac{d\theta}{n+\frac{c}{2n}r^{-\frac12}[\varphi(\frac{c}{n}r^{\frac12}\sin\theta)\sin\theta-nF(\frac{c}{n}r^{\frac12}\sin\theta)\cos\theta-p(t)\sin\theta]}\\
&=\int_{\pi-\sigma}^{\pi}\frac{d\theta}{n+O(r_0^{-\frac12})}=\frac{\sigma}{n}+O(r_0^{-1}),
\end{aligned}
$$

for $r_0\to+\infty$. Using the same method, we can get

$$
\begin{aligned}
&\int_{\pi}^{\pi+\sigma}\frac{d\theta}{n+\frac{c}{2n}r^{-\frac12}[\varphi(\frac{c}{n}r^{\frac12}\sin\theta)\sin\theta-nF(\frac{c}{n}r^{\frac12}\sin\theta)\cos\theta-p(t)\sin\theta]}\\
&=\int_{\pi}^{\pi+\sigma}\frac{d\theta}{n+O(r_0^{-\frac12})}=\frac{\sigma}{n}+O(r_0^{-1}),
\end{aligned}
$$

for $r_0\to+\infty$. Hence,

$$\tau_3(I_0,\theta_0)$$

$$= \int_{\pi-\sigma}^{\pi+\sigma} \frac{d\theta}{n + \frac{c}{2n} r^{-\frac{1}{2}} [\varphi(\frac{c}{n} r^{\frac{1}{2}} \sin\theta) \sin\theta - nF(\frac{c}{n} r^{\frac{1}{2}} \sin\theta) \cos\theta - p(t) \sin\theta]}$$

$$= \int_{\pi-\sigma}^{\pi+\sigma} \frac{d\theta}{n + O(r_0^{-\frac{1}{2}})} = \frac{2\sigma}{n} + O(r_0^{-1}), \tag{4.7}$$

for $r_0 \to +\infty$. Hence, from (4.4)-(4.6)-(4.7) we obtain that

$$\tau_1(r_0, \theta_0) + \tau_2(r_0, \theta_0) + \tau_3(r_0, \theta_0)$$
$$\leq (\frac{\pi}{n} + \frac{\sigma}{n} - \frac{\theta_0}{n}) - \frac{c}{2n^3\sqrt{\delta}} r_0^{-\frac{1}{2}} [2(H - ||p||_\infty) \cos\sigma - n\delta C\pi] + O(r_0^{-1}),$$

for $r_0 \to +\infty$. Similarly, applying condition $\varphi(x) \geq -K$, $x \in (-\infty, 0)$, we can prove that

$$\tau_4(r_0, \theta_0) + \tau_5(r_0, \theta_0) + \tau_6(r_0, \theta_0)$$
$$\leq (\frac{\pi}{n} + \frac{\theta_0}{n} - \frac{\sigma}{n}) + \frac{c}{2n^3\sqrt{\delta}} r_0^{-\frac{1}{2}} [2(K + ||p||_\infty) \cos\sigma + n\delta C\pi] + O(r_0^{-1}),$$

for $r_0 \to +\infty$. Thus, we have

$$\tau(r_0, \theta_0) \leq \frac{2\pi}{n} - \frac{c}{2n^3\sqrt{\delta}} r_0^{-\frac{1}{2}} [2(H - K - 2||p||_\infty) \cos\sigma - 2n\delta C\pi] + O(r_0^{-1}),$$

for $r_0 \to +\infty$. For the conclusion it is now sufficient to take H large enough such that $H - K - 2||p||_\infty - 2n\delta C\pi > 0$. □

We are now in position to give the main result of this section.

Theorem 4.5. *Assume* (H_4) *and suppose that* F *is bounded. Then equation*

$$x'' + f(x)x' + n^2 x + \varphi(x) = p(t) \tag{4.8}$$

has at least one 2π*-periodic solution.*

Proof. From Lemma 4.3 and Lemma 4.4, we know that

$$2n\pi < \theta(2\pi) - \theta_0 < 2(n+1)\pi,$$

for r_0 large enough. This means that (when r_0 is large enough) the image (r_1, θ_1) of (r_0, θ_0) under the Poincaré mapping P does not belong to the line $\theta = \theta_0$. Therefore, the Poincaré-Bohl theorem [12] ensures the existence of at least one fixed point of the mapping P. Thus, we conclude that (4.8) has at least one 2π-periodic solution. □

Remark 4.6. If condition (H_4) is replaced by

(H_4') $$\lim_{|x|\to+\infty} sgn(x)\varphi(x) = +\infty \text{ (or } -\infty), \qquad \lim_{|x|\to+\infty} \varphi(x)/x = 0,$$

then the same conclusion of Theorem 4.5 holds.

5. Remarks

We can use the methods developed in Section 2 and Section 3 to deal with the existence of 2π-periodic solutions of equation

$$x'' + f(x') + n^2 x + \varphi(x) = p(t),\ n \in \mathbf{N}, \tag{5.1}$$

where $f : \mathbf{R} \to \mathbf{R}$ is locally Lipschitz continuous. Assume that

(f_1) $\qquad \exists f(\pm\infty) \in \mathbf{R}\ :\ \lim_{x\to+\infty} f(x) = f(+\infty), \quad \lim_{x\to-\infty} f(x) = f(-\infty),$

and consider the following system (equivalent to (5.1))

$$x' = y, \quad y' = -(n^2 x + f(y) + \varphi(x)) + p(t). \tag{5.2}$$

By taking the transformation Φ as in Section 2, we know that (5.2) becomes

$$\begin{cases} \dfrac{d\theta}{dt} = n + \dfrac{c}{2n} r^{-\frac{1}{2}} [f(cr^{\frac{1}{2}} \cos\theta) \sin\theta + \varphi(\dfrac{c}{n} r^{\frac{1}{2}} \sin\theta) \sin\theta] - \dfrac{c}{2n} r^{-\frac{1}{2}} p(t) \sin\theta \\ \dfrac{dr}{dt} = -\dfrac{c}{n} r^{\frac{1}{2}} [f(cr^{\frac{1}{2}} \cos\theta) \cos\theta + \varphi(\dfrac{c}{n} r^{\frac{1}{2}} \sin\theta) \cos\theta] + \dfrac{c}{n} r^{\frac{1}{2}} p(t) \cos\theta. \end{cases} \tag{5.3}$$

Using the same notation of Section 2, we can show that the map $P_0 : (\rho_0, \theta_0) \to (\rho_1, \theta_1)$ related to (5.3) has the form

$$\begin{cases} \rho_1 = \rho_0 - \dfrac{c}{2n} [2(f(+\infty) - f(-\infty)) + \sqrt{A^2 + B^2} \sin(\theta_0 - \alpha)] + o(1) \\ \theta_1 = \theta_0 + 2n\pi + \dfrac{c}{2} \rho_0^{-1} [2(\varphi(+\infty) - \varphi(-\infty)) - \sqrt{A^2 + B^2} \cos(\theta_0 - \alpha)] + o(\rho_0^{-1}), \end{cases}$$

for $\rho_0 \to +\infty$ and with $\alpha = \arctan B/A$ for $A \geq 0$ or $\alpha = \pi + \arctan B/A$ for $A < 0$. From this asymptotic expression of the mapping P_0 we obtain the following conclusion.

Theorem 5.1. *Assume that conditions (H_1) and (f_1) are satisfied. Then (5.1) has at least one 2π-periodic solution provided that one of the following conditions is satisfied:*

$$2(\varphi(+\infty) - \varphi(-\infty)) > \sqrt{A^2 + B^2} \tag{5.4}$$

$$2(f(+\infty) - f(-\infty)) > \sqrt{A^2 + B^2} \tag{5.5}$$

$$2(\varphi(+\infty) - \varphi(-\infty)) = \sqrt{A^2 + B^2}, \quad f(+\infty) \neq f(-\infty) \tag{5.6}$$

$$2(f(+\infty) - f(-\infty)) = \sqrt{A^2 + B^2}, \quad \varphi(+\infty) \neq \varphi(-\infty). \tag{5.7}$$

Theorem 5.1 generalizes a result in [7]. Indeed, in [7] it was proved that the equation $x'' + f(x') + x = p(t)$ has at least one 2π-periodic solution provided that condition (f_1) and the inequality $2(f(+\infty)-f(-\infty)) > \sqrt{a^2+b^2}$ hold, with $a=\int_0^{2\pi} p(t)\sin t dt$, $b=\int_0^{2\pi} p(t)\cos t dt$. If

$$(f_2) \qquad f(-\infty) \le f(x) \le f(+\infty)$$

holds, then by the same method used in the proof of Theorem 3.1 we can prove

Theorem 5.2. *Assume that* (H_1)*, the first inequality of* (H_3)*,* (f_1)*,* (f_2) *hold and that at least one of the functions* f, φ *is not constant. Then* (5.1) *has no* 2π*-periodic solutions provided that*

$$2(\varphi(+\infty)-\varphi(-\infty)) + 2(f(+\infty)-f(-\infty)) \le \sqrt{A^2+B^2}$$

holds.

6. Examples

In this section we give, as applications of the above theorems, three examples.

Example 6.1. Consider equation

$$x'' + n^2 x + \frac{2x'}{1+x^2} + \arctan x = 2\cos nt, \quad n \in \mathbf{N}.$$

Since $\displaystyle\int_0^x \frac{2}{1+u^2}du = 2\arctan x$, we have $2n(F(+\infty)-F(-\infty)) = 4n\pi$. On the other hand, $\sqrt{A^2+B^2} = 2\pi$. Theorem 2.1 ensures that this equation possesses at least one 2π-periodic solution.

Example 6.2. Let $f(x) = 2x\cos x^2$. Obviously, f is unbounded. It is easy to show that $F(x) = \sin x^2$. Therefore, F is bounded. Define $\varphi(x) = \frac{x}{1+\sqrt{x}}$ for $x \ge 0$ and $\varphi(x) = \sin x$ for $x \le 0$. Then φ is bounded in the interval $(-\infty, 0)$ and $\lim_{x\to+\infty}\varphi(x) = +\infty$, $\lim_{x\to+\infty}\varphi(x)/x = 0$. By Theorem 4.5 we know that the equation

$$x'' + f(x)x' + n^2 x + \varphi(x) = p(t)$$

has at least one 2π-periodic solution.

Example 6.3. Consider equation

$$x'' + n^2 x + 2\arctan x' + \frac{1}{1+x^2} = \cos nt, \quad n \in \mathbf{N}.$$

Let $f(x) = 2\arctan x$. Then $f(+\infty) = \pi$, $f(-\infty) = -\pi$. Therefore, $2(f(+\infty)-f(-\infty)) = 4\pi$. Moreover, it is easy to show that $\sqrt{A^2+B^2} = \pi$. According to Theorem 5.1, this equation has at least one 2π-periodic solution.

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