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TOPOLOGICAL SOLITARY WAVES WITH ARBITRARY CHARGE AND THE ELECTROMAGNETIC FIELD

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Abstract. This paper deals with a model of solitary waves, in three space dimensions, which are characterized by a topological invariant called charge; these waves behave as relativistic particles. We study the interaction with an electromagnetic field. The Lagrangian density of the system is the sum of three terms: the first is that of the free soliton, the second is the classical Lagrangian density of an electromagnetic field, the third, which is due to the interaction, is chosen so that the electric charge coincides with the topological charge. We prove the existence of a static solution for every fixed value of the charge. The energy functional is strongly unbounded from above, as from below; after a reduction argument, the critical points are found by means of the Principle of Symmetric Criticality.

1. Introduction

Roughly speaking solitary waves are solutions of field equations whose energy travels as a localized packet. They occur in many questions of mathematical physics, such as classical and quantum field theory, non linear optics, fluid mechanics, plasma physics (see e.g. [9], [11], [13], [16]).

In some recent papers ([5], [4], [2], [7]) existence and multiplicity of solitary waves have been proved for a Lorentz invariant equation in 3 space dimensions. Such equation has a long history which starts with the Derrick's celebrated paper [8].

The fields ψ we consider are maps defined in the space-time $\mathbf{R}^4$ and they take values in $\mathcal{M} = \mathbf{R}^4 \setminus \left\{\bar{\xi}\right\}$ with $\bar{\xi} \neq 0$. Since the *internal parameter space* $\mathcal{M}$ is not topologically trivial, a topological invariant can be associated to these fields $\psi = \left(\psi^0, \psi^1, \psi^2, \psi^3\right)$.

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If we interpret this invariant as electric charge, it is natural to analyze the interaction of the soliton and the electromagnetic field. In [3] an equation describing the system soliton-electromagnetic field has been considered and the existence of a non-trivial static solution has been proved. More precisely the existence of the ground state solution, namely the solution having minimal energy, has been proved.

Here, developing some arguments introduced in [2] and [3], we want to analyze the existence of solutions of the system for any fixed value of the charge.

1.1. **The Lagrangian for ψ and the topological charge.** In order to get the Lorentz invariance, the Lagrangian density $\mathcal{L}_1$ for the field ψ has the form

$$\mathcal{L}_1 = \mathcal{L}_1(\psi, \sigma) = -\frac{1}{2}\alpha(\sigma) - V(\psi),$$

where $\alpha : \mathbf{R} \to \mathbf{R}$ and

$$\sigma = |\nabla\psi|^2 - |\psi_t|^2,$$

being $\nabla\psi$ the Jacobian matrix of ψ with respect to the space variable and ψ_t the derivative of ψ with respect to the time variable.

The Euler-Lagrange equations for the action

$$\mathcal{S}_1(\psi) = \int_{t_0}^{t_1}\int_{\mathbf{R}^3}\Big(-\frac{1}{2}\alpha(\sigma) - V(\psi)\Big)dx\,dt$$

are

$$\frac{\partial}{\partial t}\big(\alpha'(\sigma)\,\psi_t\big) - \nabla\big(\alpha'(\sigma)\,\nabla\psi\big) + V'(\psi) = 0, \tag{1.1}$$

where $\nabla(\alpha'(\sigma)\nabla\psi)$ denotes the vector whose j-th component is given by

$$\operatorname{div}\big(\alpha'(\sigma)\,\nabla\psi^j\big)$$

and V' denotes the gradient of V. The static solutions $\psi(x,t) = u(x)$ of (1.1) solve the equation

$$-\nabla\big(\alpha'(\sigma)\,\nabla u\big) + V'(u) = 0. \tag{1.2}$$

When

$$\alpha(\sigma) = \sigma, \tag{1.3}$$

(1.2) reduces to

$$-\Delta u + V'(u) = 0 \quad \text{in } \mathbf{R}^3. \tag{1.4}$$

It can be shown, by a rescaling argument (see [8]), that when $V \geq 0$, then the only finite energy solution of (1.4) is the trivial one.

Here, as in [5], we consider

$$\alpha(\sigma) = \sigma + \frac{\varepsilon}{3}\sigma^3 \tag{1.5}$$

with $\varepsilon > 0$. So the Lagrangian density $\mathcal{L}_1$ becomes

$$\mathcal{L}_1 = -\frac{1}{2}\left(|\nabla\psi|^2 - |\psi_t|^2\right) - \frac{\varepsilon}{6}\left(|\nabla\psi|^2 - |\psi_t|^2\right)^3 - V(\psi).$$

The function $V : \mathbf{R}^4 \setminus \{\bar{\xi}\} \to \mathbf{R}$ is smooth, positive and diverges for $\xi \to \bar{\xi}$.

Since the target manifold has non-trivial topology ($\pi_3\left(\mathbf{R}^4 \setminus \{\bar{\xi}\}\right) = \mathbf{Z}$), the fields ψ can be classified by a topological invariant. To this end assume that the field ψ is smooth and

$$\lim_{|x|\to\infty} \psi(x,t) = 0. \tag{1.6}$$

As in [5], we can define the *topological charge* $\mathrm{ch}(\psi(\cdot,t)) \in \mathbf{Z}$ by means of the Brower degree (see the Appendix). If (1.6) is uniform with respect to $t \in [t_0, t_1]$, the charge does not depend on t.

There exists also an integral characterization of the charge.

Consider in $\mathbf{R}^4 \setminus \{\bar{\xi}\}$ the unique 3-form η closed but not exact

$$\eta = \sum_{k=0}^{3} \eta_k(\xi)\, d\xi^0 \wedge \ldots \wedge \widehat{d\xi^k} \wedge \ldots \wedge d\xi^3 \tag{1.7}$$

where the hatted symbols are omitted,

$$\eta_k(\xi) = \frac{1}{|S^3|}\frac{(-1)^k\left(\xi^k - \bar{\xi}^k\right)}{\left|\xi - \bar{\xi}\right|^4}$$

and $\left|S^3\right|$ is the measure of the unit 3−sphere.

Fix t and consider the pullback

$$\psi(\cdot,t)^*\eta = \rho(\psi)\, dx^1 \wedge dx^2 \wedge dx^3,$$

where

$$\rho(\psi) = \sum_{k=0}^{3} \eta_k(\psi) \det \frac{\partial\left(\psi^0, \ldots, \widehat{\psi^k}, \ldots, \psi^3\right)}{\partial(x_1, x_2, x_3)}. \tag{1.8}$$

Under suitable assumptions on the asymptotic behavior of $\psi(\cdot,t)$, the topological charge can be written as

$$\mathrm{ch}(\psi(\cdot,t)) = \int_{\mathbf{R}^3} \psi(\cdot,t)^*\eta = \int_{\mathbf{R}^3} \rho(\psi)\, dx. \tag{1.9}$$

In the Appendix we recall some details about the definition of charge given in [5] and we prove the formula (1.9) (see Proposition 20).

1.2. **The interaction with the e.m. field.** Let $(\mathbf{A}, \phi)$ denote the gauge potential associated to the electromagnetic field $(\mathbf{E}, \mathbf{H})$ by the relations

$$\mathbf{E} = -\left(\mathbf{A}_t + \nabla\phi\right), \tag{1.10}$$

$$\mathbf{H} = \nabla \times \mathbf{A}, \tag{1.11}$$

where $\mathbf{A}_t$ is the derivative with respect to t and $\nabla\phi$ is the gradient with respect to x. The Lagrangian density of the electromagnetic field can be defined in the usual way

$$\mathcal{L}_2 = \frac{1}{8\pi}\left(|\mathbf{E}|^2 - |\mathbf{H}|^2\right) = \frac{1}{8\pi}\left(|\mathbf{A}_t + \nabla\phi|^2 - |\nabla \times \mathbf{A}|^2\right).$$

Now we recall the Lagrangian density $\mathcal{L}_3$ describing the interaction between ψ and the electromagnetic field.

First we observe that, by (1.9), the charge density is given by ρ defined in (1.8).

We need to define the current density. To this end consider the 3-form on $\mathbf{R}^4$ given by the pullback of η by ψ

$$\psi^*\eta = \sum_{\substack{0\le k\le 3\\ 1\le h\le 4}} \eta_k(\psi) \det \frac{\partial(\psi^0, \ldots, \widehat{\psi^k}, \ldots, \psi^3)}{\partial(x_1, \ldots, \widehat{x_h}, \ldots, x_4)} dx^1 \wedge \ldots \wedge \widehat{dx^h} \wedge \ldots \wedge dx^4, \tag{1.12}$$

where $x_4 = t$. The Hodge $*$ operator applied to (1.12) gives a 1-form on $\mathbf{R}^4$

$$*\left(\psi^*\eta\right). \tag{1.13}$$

The vector field associated to (1.13) can be written $(\mathbf{J}, \rho)$; indeed the 4-th component of this vector field is just the charge density ρ defined in (1.8).

So it is natural to interpret $\mathbf{J} = (J_1, J_2, J_3)$ as the current density. Direct calculations show that, for $i = 1, 2, 3$,

$$J_i\left(\psi\right) = (-1)^i \sum_{k=0}^{3} \eta_k\left(\psi\right) \det \frac{\partial\left(\psi^0, \ldots, \widehat{\psi^k}, \ldots, \psi^3\right)}{\partial\left(x_1, \ldots, \widehat{x_i}, \ldots, x_3, t\right)}. \tag{1.14}$$

Now we can write the Lagrangian density $\mathcal{L}_3$ of the interaction

$$\mathcal{L}_3 = (\mathbf{J}(\psi, \nabla\psi, \psi_t)|\mathbf{A}) - \rho(\psi, \nabla\psi)\phi.$$

Remark 1. Since η is closed, the pullback $\psi^*\eta$ is closed, then

$$d\left(\psi^*\eta\right) = 0, \tag{1.15}$$

which can be written as the *continuity equation*

$$\nabla \mathbf{J} + \frac{\partial \rho}{\partial t} = 0.$$

1.3. **Statement of the result.** The total action is

$$\mathcal{S} = \mathcal{S}\left(\psi, \mathbf{A}, \phi\right) = \mathcal{S}_1\left(\psi\right) + \mathcal{S}_2\left(\mathbf{A}, \phi\right) + \mathcal{S}_3\left(\psi, \mathbf{A}, \phi\right),$$

where

$$\mathcal{S}_i = \int_{t_0}^{t_1} \int_{\mathbf{R}^3} \mathcal{L}_i dx dt.$$

With the above choices, the Euler-Lagrange equations

$$d\mathcal{S} = 0, \tag{1.16}$$

become respectively

$$\Box\psi + \varepsilon\Box_6\psi + V'\left(\psi\right) = F, \tag{1.17}$$

$$\nabla \times \left(\nabla \times \mathbf{A}\right) = 4\pi \mathbf{J}\left(\psi\right) - \frac{\partial}{\partial t}\left(\mathbf{A}_t + \nabla\phi\right), \tag{1.18}$$

$$-\nabla\left(\mathbf{A}_t + \nabla\phi\right) = 4\pi\rho\left(\psi\right), \tag{1.19}$$

where $\Box_6\psi$ is the vector whose j-th component is given by

$$\frac{\partial}{\partial t}\left[\left(\left|\nabla\psi\right|^2 - \left(\psi_t\right)^2\right)^2 \psi_t^j\right] - \operatorname{div}\left[\left(\left|\nabla\psi\right|^2 - \left(\psi_t\right)^2\right)^2 \nabla\psi^j\right].$$

The right hand side F of (1.17), which derives from the interaction term $\mathcal{S}_3$, depends on ψ (and its first and second derivatives) and on $\mathbf{A}$ and ϕ (and their first derivatives). Observe that, when $F = 0$, (1.17) reduces to the equation studied in [5].

Remark 2. From (1.10), (1.11), (1.18) and (1.19) we obtain the second couple of the Maxwell equations

$$\nabla \times \mathbf{H} = 4\pi \mathbf{J}\left(\psi\right) + \mathbf{E}_t \tag{1.20}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho\left(\psi\right). \tag{1.21}$$

In this paper we confine ourselves to study static solutions, i.e., solutions which do not depend on t; so, from now on, instead of ψ we shall use the symbol u.

Remark 3. Since the equations (1.17), (1.18) and (1.19) are Lorentz invariant, the static solutions give rise to travelling solutions by a Lorentz transformation (see [3]).

If we look for static solutions, the equations (1.17), (1.18) and (1.19) become

$$-\Delta u - \varepsilon \Delta_6 u + V'(u) = G, \tag{1.22}$$

$$\nabla \times (\nabla \times \mathbf{A}) = 0, \tag{1.23}$$

$$-\Delta \phi = 4\pi \rho (u), \tag{1.24}$$

where G depends on ψ (and its first and second derivatives) and ϕ (and their first derivatives).

Remark 4. Since $\frac{\partial \psi^a}{\partial t}$ appears as a factor in J_i, for $i = 1, 2, 3$ (see (1.14)), in the static case we have

$$\mathbf{J} = 0,$$

namely, as it is quite natural, there is no electric current.

Of course $\mathbf{A} = 0$ is solution of (1.23). Then we shall look for fields u and ϕ satisfying (1.22) and (1.24).

Assume now that

V1) $V \in C^2 \left(\mathbf{R}^4 \setminus \{ \bar{\xi} \}, \mathbf{R} \right)$;

V2) $V(\xi) \geq V(0) = 0$;

V3) the Hessian matrix $V''(0)$ is non degenerate;

V4) there exist c, $r > 0$ such that, if $|\xi| < r$, then

$$V\left(\bar{\xi} + \xi \right) \geq c \left| \xi \right|^{-6}.$$

In order to get the existence of infinitely many solutions, we assume, as in [2], that V satisfies a suitable symmetry property. Take a reference frame in the target space $\mathbf{R}^4$ so that

$$\bar{\xi} = (1, 0, 0, 0).$$

For any $\xi \in \mathbf{R}^4$, we set

$$\xi = (\xi^0, \widetilde{\xi}), \tag{1.25}$$

with $\xi^0 \in \mathbf{R}$ and $\widetilde{\xi} \in \mathbf{R}^3$. With this notation assume:

V5) for every ξ and for every g in the orthogonal group $O(3)$,

$$V(\xi^0, g\widetilde{\xi}) = V(\xi^0, \widetilde{\xi}).$$

The main result of the paper is the following Theorem.

Theorem 5. *Under the previous assumptions V1)-V5), for every $N \in \mathbf{N}$, there exist two fields*

$$u_N \in W^{1,2}\left(\mathbf{R}^3, \mathbf{R}^4\right) \cap W^{1,6}\left(\mathbf{R}^3, \mathbf{R}^4\right)$$
$$\phi_N : \mathbf{R}^3 \to \mathbf{R} \text{ with } \int_{\mathbf{R}^3} |\nabla \phi_N|^2 \, dx < +\infty$$

such that $\mathrm{ch}(u_N) = N$ *and* u_N, $\mathbf{A}$ =0, ϕ_N *are static solutions of the Euler-Lagrange equation* (1.16). *Moreover,* $\phi_{N_1} \neq \phi_{N_2}$ *if* $N_1 \neq N_2$.

Remark 6. The solutions $u = (u^0, \tilde{u})$ and ϕ are symmetric in the following sense: for every $x \in \mathbf{R}^3$ and for every $g \in O(3)$

$$\left(u^0(gx), \tilde{u}(gx)\right) = \left(u^0(x), g\tilde{u}(x)\right), \quad \phi(gx) = \phi(x).$$

We point out that the field $u = (u^0, \tilde{u})$ exhibits the same symmetry introduced by Skyrme in [14].

2. The functional setting

We have to find the fields u, ϕ which solve (1.22) and (1.24). Such solutions can be characterized (see also Lemma 2.1 of [3]) as critical points of the functional

$$f(u, \phi) = \int_{\mathbf{R}^3} \left(\frac{1}{2}|\nabla u|^2 + \frac{\varepsilon}{6}|\nabla u|^6 + V(u)\right)dx - \frac{1}{2}\int_{\mathbf{R}^3} |\nabla \phi|^2 dx + \int_{\mathbf{R}^3} \phi\rho(u)dx. \tag{2.1}$$

Of course, we need to choose the right function spaces for the functional f.

Let E denote the completion of $C_0^\infty(\mathbf{R}^3, \mathbf{R}^4)$ with respect to the norm

$$\|u\|_E = \|u\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla u\|_{L^6}.$$

By Sobolev embedding theorems

$$E = W^{1,2}\left(\mathbf{R}^3, \mathbf{R}^4\right) \cap W^{1,6}\left(\mathbf{R}^3, \mathbf{R}^4\right),$$

so the functions of E are continuous and they decay to zero at infinity.

Consider the open set $\Lambda = \{u \in E : u(x) \neq \bar{\xi}\}$ in E. Let $D = D^{1,2}(\mathbf{R}^3, \mathbf{R})$ denote the completion of $C_0^\infty\left(\mathbf{R}^3, \mathbf{R}\right)$ with respect to the norm

$$\|\phi\|_D = \|\nabla \phi\|_{L^2}.$$

Clearly, D is continuously embedded in $L^6\left(\mathbf{R}^3, \mathbf{R}\right)$.

The functional f is C^2 in $\Lambda \times D$ (for the details see Section 2 in [3]).

On $\Lambda \times D$ we can define the following $O(3)$-action: for every $(u,\phi) \in \Lambda \times D$, for every $g \in O(3)$

$$\begin{cases} (T_g u)(x) = \left(u^0(gx), g^{-1}\widetilde{u}(gx)\right), \\ (T_g \phi)(x) = \phi(gx). \end{cases} \tag{2.2}$$

Proposition 7. *The functional f is invariant under the action* (2.2), *i.e., for every $g \in O(3)$, $u \in \Lambda$ and $\phi \in D$*

$$f(T_g u, T_g \phi) = f(u, \phi).$$

The first part of the functional f, namely

$$\int_{\mathbf{R}^3} \left(\frac{1}{2}|\nabla u|^2 + \frac{\varepsilon}{6}|\nabla u|^6 + V(u)\right) dx - \frac{1}{2}\int_{\mathbf{R}^3} |\nabla \phi|^2 \, dx,$$

is obviously invariant. The invariance of the functional

$$(u,\phi) \mapsto \int_{\mathbf{R}^3} \phi \rho(u) \, dx$$

will be a consequence of the following Lemma.

Lemma 8. *For every $g \in O(3)$ and $u \in \Lambda$ and for almost every $x \in \mathbf{R}^3$*

$$\rho(T_g u)(x) = \rho(u)(gx). \tag{2.3}$$

In the proof of Lemma 8 we shall use the following property of the orthogonal matrices.

Lemma 9. *Let $G \in O(4)$ and A a 4×3 matrix. Let G_k^- be the matrix obtained from G taking away the k-th row and A_k^+ the 4×4 matrix obtained by adding to A, as k-th column, the transpose of the k-th row of G. Then*

$$\det\left(G_k^- \cdot A\right) = \det(G \cdot A_k^+) = \det G \det(A_k^+). \tag{2.4}$$

Proof of Lemma 8. It is sufficient to prove (2.3) in $\Lambda \cap C_0^\infty\left(\mathbf{R}^3, \mathbf{R}^4\right)$, which is dense in Λ. So we fix $g \in O(3)$, $u \in \Lambda \cap C_0^\infty\left(\mathbf{R}^3, \mathbf{R}^4\right)$ and $x \in \mathbf{R}^3$.

Let $\delta(u)(x)$ be the vector whose k-th component is

$$\delta_k(u)(x) = (-1)^k \det(\nabla u(x))_k^-, \tag{2.5}$$

where $(\nabla u(x))_k^-$ is obtained from $\nabla u(x)$ taking away the k-th row. With this notation we can write

$$\rho(u)(x) = \frac{1}{|S^3|}\frac{1}{\left|u(x) - \bar{\xi}\right|^4}\left(u(x) - \bar{\xi} \mid \delta(u)(x)\right).$$

We set $v = T_g u$. It is easy to see that

$$v(x) = Gu(gx),$$

where G is the following matrix, written in block form

$$G = \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \in O(4).$$

We have

$$\rho(v)(x) = \frac{1}{|S^3|} \frac{1}{\left|v(x) - \bar{\xi}\right|^4} \left(v(x) - \bar{\xi} \mid \delta(v)(x)\right).$$

Since $G\bar{\xi} = \bar{\xi}$, we have

$$v(x) - \bar{\xi} = G\left(u(gx) - \bar{\xi}\right), \tag{2.6}$$

then

$$\left|v(x) - \bar{\xi}\right|^2 = \left|u(gx) - \bar{\xi}\right|^2. \tag{2.7}$$

Now we have to study $\delta(v)(x)$, where, by (2.5),

$$\delta_k(v)(x) = (-1)^k \det(\nabla v(x))_k^-.$$

We know that

$$\nabla v(x) = G \cdot \nabla u(gx) \cdot g,$$

moreover, it is easy to see that

$$(\nabla v(x))_k^- = G_k^- \cdot \nabla u(gx) \cdot g;$$

hence,

$$\delta_k(v)(x) = (-1)^k \det\left(G_k^- \cdot \nabla u(gx)\right) \det g. \tag{2.8}$$

By Lemma 9

$$\begin{aligned} \det\left(G_k^- \cdot \nabla u(gx)\right) &= \det G \cdot \det\left((\nabla u(gx))_k^+\right) \\ &= \det G \cdot \sum_{i=0}^{3} (-1)^{k+i} G_{ki} \det(\nabla u(gx))_i^- \\ &= (-1)^k \det G \cdot \sum_{i=0}^{3} G_{ki} \cdot (\delta_i(u)(gx)). \end{aligned}$$

Substituting in (2.8), we get

$$\delta_k(v)(x) = \sum_{i=0}^{3} G_{ki} \cdot \delta_i(u)(gx)$$

which means

$$\delta(v)(x) = G(\delta(u)(gx)). \tag{2.9}$$

So, by (2.7)-(2.6)-(2.9), we conclude

$$\begin{aligned}\rho\left(T_g u\right)(x) &= \frac{1}{\left|S^3\right|}\frac{1}{\left|v\left(x\right)-\bar{\xi}\right|^4}\left(v\left(x\right)-\bar{\xi}\mid\delta\left(v\right)\left(x\right)\right)\\ &= \frac{1}{\left|S^3\right|}\frac{1}{\left|u\left(gx\right)-\bar{\xi}\right|^4}\left(G\left(u\left(gx\right)-\bar{\xi}\right)\mid G\cdot\delta\left(u\right)\left(gx\right)\right)\\ &= \frac{1}{\left|S^3\right|}\frac{1}{\left|u\left(gx\right)-\bar{\xi}\right|^4}\left(u\left(gx\right)-\bar{\xi}\mid\delta\left(u\right)\left(gx\right)\right)=\rho\left(u\right)\left(gx\right).\end{aligned}$$

□

The functional f is strongly indefinite, i.e., it is unbounded both from above and from below, even modulo compact perturbations; then it is not convenient to work directly with it. In order to find weak solutions of our problem, we deal as in [3] with a "reduced" functional bounded from below.

The partial derivative of f with respect to ϕ, evaluated at $\varphi\in C_0^\infty\left(\mathbf{R}^3,\mathbf{R}\right)$ is

$$f_\phi\left(u,\phi\right)\left[\varphi\right]=-\int_{\mathbf{R}^3}\nabla\phi\cdot\nabla\varphi dx+\int_{\mathbf{R}^3}\rho\left(u\right)\varphi dx.$$

It is easy to see that

$$f_\phi\left(u,\phi\right)=0$$

if and only if

$$-\Delta\phi=\rho\left(u\right). \tag{2.10}$$

For any fixed $u\in\Lambda$,

$$\rho(u)\in L^1\left(\mathbf{R}^3,\mathbf{R}\right)\cap L^2\left(\mathbf{R}^3,\mathbf{R}\right);$$

so, by Lemma 2.4 of [3], there exists a unique $\phi\in D$ which solves (2.10). Then we can define the C^∞ map

$$u\in\Lambda\mapsto\phi\left[u\right]=\left(-\Delta\right)^{-1}\left(\rho\left(u\right)\right)\in D.$$

Now consider the functional

$$J:\Lambda\to\mathbf{R},\quad J\left(u\right)=f\left(u,\phi\left[u\right]\right).$$

A direct calculus shows that, if u is a critical point of J, then $\left(u,\phi\left[u\right]\right)$ is a critical point of f.

The functional J is C^2 and it does not exhibit the strong indefiniteness of f. Indeed, by definition of $\phi\left[u\right]$, we have

$$-\Delta\phi\left[u\right]=\rho\left(u\right). \tag{2.11}$$

If we multiply by $\phi[u]$ and we integrate by parts, we obtain

$$\int_{\mathbf{R}^3} |\nabla \phi[u]|^2 \, dx = \int_{\mathbf{R}^3} \rho(u) \, \phi[u] \, dx$$

and so we can write

$$J(u) = \int_{\mathbf{R}^3} \left(\frac{1}{2} |\nabla u|^2 + \frac{\varepsilon}{6} |\nabla u|^6 + V(u) \right) dx + \frac{1}{2} \int_{\mathbf{R}^3} |\nabla \phi[u]|^2 \, dx.$$

Proposition 10. *The functional J is invariant with respect to the $O(3)$-action on Λ, i.e., for every $u \in \Lambda$ and $g \in O(3)$*

$$J(T_g u) = J(u).$$

This Proposition is an obvious consequence of the invariance of f and of the following Lemma.

Lemma 11. *For every $u \in \Lambda$ and $g \in O(3)$*

$$\phi[T_g u] = T_g \phi[u]. \tag{2.12}$$

Proof. Let $u \in \Lambda$, $g \in O(3)$ be fixed. We set, for short

$$\phi_0 = \phi[u], \quad \phi_1 = \phi[T_g u] \quad \phi_2 = T_g \phi_0$$

Then, by (2.11) and Lemma 8, for almost every $x \in \mathbf{R}^3$, we have

$$-\Delta \phi_1(x) = \rho(T_g u)(x) = \rho(u)(gx) = -\Delta \phi_0(gx) = -\Delta \phi_2(x).$$

So we deduce

$$\phi_1 = \phi_2.$$

□

Now we consider the subspace of E of the fixed points

$$F = \{u \in E : \forall g \in O(3) : T_g u = u\}.$$

The subset $\Lambda_F = \Lambda \cap F$ is a natural constraint to find the critical points of J.

Proposition 12. *If $u \in \Lambda_F$ is a critical point of $J|_{\Lambda_F}$, then u is a critical point of J and so the pair $(u, \phi[u])$ is a critical point of f in Λ.*

Proof. The first part of the statement is a consequence of Proposition 10 and the well known Principle of Symmetric Criticality (see [12] and Section 4 of [2]). The second part follows from the previous considerations. □

Remark 13. From (2.12) we deduce that if $u \in \Lambda_F$, then $\phi[u]$ is a fixed point for the $O(3)$-action on D. Therefore, if u is a critical point of $J|_{\Lambda_F}$, then the solution $(u, \phi[u])$ is symmetric in the sense of Remark 6.

3. Proof of Theorem 5

For every $N \geq 1$, consider the subset

$$\Lambda_F^N = \{u \in \Lambda_F : \mathrm{ch}(u) = N\}.$$

First we recall that the subsets Λ_F^N are not empty (see Subsection 5.1 of [2]); notice that the proof of this fact uses an alternative definition of the topological charge (see Proposition 22).

We are going to prove that the functional J attains its minimum in Λ_F^N. So we need some properties of functional J proved in [3].

Lemma 14. *The functional J is coercive in Λ, i.e., for every sequence $\{u_n\} \subset \Lambda$, with $\|u_n\|_E \to \infty$, we have*

$$\lim_n J(u_n) = +\infty.$$

See Lemma 3.1 in [3].

Lemma 15. *If $\{u_n\} \subset \Lambda$ converges weakly to $u \in \partial\Lambda$, then*

$$\int_{\mathbf{R}^3} V(u_n)\, dx \to +\infty.$$

See Lemma 3.2 in [3].

From the previous lemma it follows obviously that for every $M > 0$ there exists $d > 0$ such that for every $u \in \Lambda$

$$\int_{\mathbf{R}^3} V(u)\, dx < M \Rightarrow \min_{x \in \mathbf{R}^3} \left| u(x) - \bar{\xi} \right| \geq d. \tag{3.1}$$

Lemma 16. *For every sequence $\{u_n\} \subset \Lambda$, if $u_n \rightharpoonup u \in \Lambda$ and $J(u_n)$ is bounded from above, then*

$$J(u) \leq \liminf_n J(u_n).$$

See Proposition 3.5 in [3].

The following Lemma takes the role of the Splitting Lemma in [5] (Lemma 4.1) and [3] (Proposition 3.8).

Lemma 17. *Let $\{u_n\} \subset \Lambda_F$ such that $J(u_n)$ is bounded. Then there exists $u \in \Lambda_F$ such that, up to subsequence,*

$$u_n \rightharpoonup u \tag{3.2}$$

$$J(u) \leq \liminf_n J(u_n) \tag{3.3}$$

$$\mathrm{ch}(u) = \lim_n \mathrm{ch}(u_n). \tag{3.4}$$

Before to prove this Lemma, we recall the following result proved by Strauss in [15] and by Berestycki and Lions in [6].

Lemma 18. *Let $n \geq 2$. Every radial function u in $W^{1,2}(\mathbf{R}^n)$ is almost everywhere equal to a function $U(x)$, continuous for $x \neq 0$, such that*

$$|U(x)| \leq C_n |x|^{\frac{1-n}{2}} \|u\|_{W^{1,2}}, \qquad |x| \geq 1, \tag{3.5}$$

where C_n depends only on n.

Proof of Lemma 17. From Lemma 14 and the boundedness of $J(u_n)$, we deduce that the sequence $\{u_n\}$ is bounded in E.

Then there exists $u \in E$ such that, up to subsequence, $u_n \rightharpoonup u$ in E.

By Lemma 15, the weak limit u belongs to Λ; moreover, since F is weakly closed (indeed it is closed and convex), $u \in F$.

Lemma 16 gives immediately (3.3).

Lastly, since for every vector field $u \in F$, the scalar field $|u|$ is radial (see the proof of Proposition 4 in [2]), we can apply Lemma 18 to $u_n - u$, which is a bounded sequence, and we obtain that, fixed $\varepsilon > 0$, there exists $R > 0$ such that for $|x| \geq R$

$$|u_n(x) - u(x)| \leq C_3 \frac{\|u_n - u\|_E}{R} \leq \varepsilon. \tag{3.6}$$

On the other hand, in the ball of center 0 and radius R, the sequence u_n is uniformly convergent to u, therefore, for $|x| \leq R$ and k sufficiently large

$$|u_n(x) - u(x)| \leq \varepsilon. \tag{3.7}$$

So, by (3.6) and (3.7), $u_n \to u$ in L^∞.

Then from the invariance of Brower degree with respect to the uniform convergence we obtain (3.4). □

Proof of Theorem 5. Fix $N \geq 1$. If we apply Lemma 17 to a minimizing sequence in Λ_F^N, we deduce that J attains its minimum.

Let $u_N \in \Lambda_F^N$ be a minimizer; by Proposition 12, $(u_N, \phi[u_N])$ is a critical point of f.

Moreover, if $N_1 \neq N_2$, then $\phi[u_{N_1}] \neq \phi[u_{N_2}]$. Indeed, by (2.11), for every $N \in \mathbf{N}$

$$-\int_{\mathbf{R}^3} \Delta\phi[u_N]dx = \int_{\mathbf{R}^3} \rho(u_N)\, dx = N.$$

Thereby the theorem is completely proved. □

Remark 19. Using the above notations, up to a subsequence,

$$\lim_{N} f\left(u_N, \phi[u_N]\right) = \lim_{N} J\left(u_N\right) = +\infty.$$

Indeed, from Proposition 3.8 (Splitting Lemma) of [3], we can deduce that the functional J is coercive with respect to the charge.

APPENDIX A. REMARKS ON THE TOPOLOGICAL CHARGE

As in the previous sections, we consider fields

$$u \in \Lambda = \{u \in W^{1,2}\left(\mathbf{R}^3, \mathbf{R}^4\right) \cap W^{1,6}\left(\mathbf{R}^3, \mathbf{R}^4\right) : u(x) \neq \bar{\xi}\}$$

with $\bar{\xi} = (1, 0, 0, 0)$.

First of all we recall the definition of topological charge given in [5].

Consider the unit sphere Σ having center at $\bar{\xi}$; on the sphere Σ we fix the north pole $N = 2\bar{\xi}$. Then we consider the projection

$$P : \xi \in \mathbf{R}^4 \backslash \{\bar{\xi}\} \mapsto \frac{\xi - \bar{\xi}}{\left|\xi - \bar{\xi}\right|} + \bar{\xi} \in \Sigma.$$

For every $u \in \Lambda$, we set

$$\mathrm{ch}(u) = \deg(P \circ u, S(u), N), \tag{A.1}$$

where

$$S(u) = \{x \in \mathbf{R}^3 : |u(x)| > 1\}.$$

Indeed, if $P \circ u(x) = N$, then $|u(x)| > 1$ and the degree in (A.1) is well defined.

From the topological point of view, the choices of the $S(u)$ and of the value N on Σ have no intrinsic meaning. Indeed we point out that

$$\deg(P \circ u, S(u), N) = \lim_{R \to +\infty} \deg(P \circ u, B_R, N),$$

where $B_R = \{x \in \mathbf{R}^3 : |x| < R\}$. On the other hand, for every $Q \in \Sigma \backslash \{0\}$ we have

$$\lim_{R \to +\infty} \deg(P \circ u, B_R, N) = \lim_{R \to +\infty} \deg(P \circ u, B_R, Q).$$

Indeed, since $|u\left(x\right)| \to 0$ for $|x| \to +\infty$, when R is sufficiently large, $P\left(u\left(\partial B_R\right)\right)$ is sufficiently near to 0 so that N and Q belongs to the same connected component of $\Sigma \backslash P(u(\partial B_R))$. So we could write

$$\mathrm{ch}(u) = \deg(P \circ u, \mathbf{R}^3). \tag{A.2}$$

In this paper and in [3] the authors use an integral characterization of the topological charge.

In $\mathbf{R}^4\backslash\{\bar{\xi}\}$ they consider the 3-form which is closed but not exact

$$\eta = \sum_{0\leq k\leq 3} \frac{(-1)^k}{|S^3|} \frac{(\xi^k - \bar{\xi}^k)}{|\xi - \bar{\xi}|^4} d\xi^0 \wedge \cdots \wedge \widehat{d\xi^k} \wedge \cdots \wedge d\xi^3.$$

For every $u \in \Lambda$, the pull-back of η by u is

$$u^*\eta = \sum_{0\leq k\leq 3} \eta_k(u) \det \frac{\partial(u^0, \dots, \widehat{u^k}, \dots, u^3)}{\partial(x_1, x_2, x_3)} dx^1 \wedge dx^2 \wedge dx^3. \tag{A.3}$$

Proposition 20. *For every $u \in \Lambda$,*

$$\mathrm{ch}(u) = \int_{\mathbf{R}^3} \rho(u) dx. \tag{A.4}$$

We just give an outline of the proof of (A.4).

It is well known that the topological degree can be characterized by means of the integrals of differential forms (see e.g. [10], Theorem 14.1.1).

Theorem 21. *If M and N are connected, compact, n-dimensional manifolds (without boundary), $f : M \to N$ is a smooth map and Ω is a differential form of rank n; then*

$$\int_M f^*\Omega = \deg(f) \int_N \Omega. \tag{A.5}$$

Indeed for this kind of maps, the degree of f is well defined independently on the value $y \in N$.

Theorem 21 can be generalized for several situations (manifolds with boundary, non-compact manifolds and proper maps, ...).

Proof of Proposition 20. Consider the following 3-form

$$\Omega = \frac{1}{|S^3|} \sum_{0\leq k\leq 3} (-1)^k (\xi^k - \bar{\xi}^k) d\xi^0 \wedge \cdots \wedge \widehat{d\xi^k} \wedge \cdots \wedge d\xi^3$$

such that

$$\int_\Sigma \Omega = 1.$$

It is easy to see that

$$P^*\Omega = \eta,$$

where η is the 3-form introduced in (1.7). Therefore, for every $u \in \Lambda$

$$(P \circ u)^*\Omega = u^*(P^*\Omega) = u^*\eta. \tag{A.6}$$

Then (A.6) and (A.5) give

$$\int_{\mathbf{R}^3} u^*\eta = \int_{\mathbf{R}^3} (P\circ u)^*\Omega = \deg(P\circ u, \mathbf{R}^3)\int_\Sigma \Omega = \deg(P\circ u, \mathbf{R}^3).$$

Observe that $u^*\eta$ has finite integral on $\mathbf{R}^3$, in fact

$$u^*\eta = \rho(u)dx^1\wedge dx^2\wedge dx^3$$

and $\rho(u)\in L^1\left(\mathbf{R}^3,\mathbf{R}\right)\cap L^2\left(\mathbf{R}^3,\mathbf{R}\right)$. □

In [2], in order to show that the subsets Λ_F^N are not empty, the authors gave a simpler evaluation of the topological charge. The following proposition shows that these different definitions give the same value.

We shall write

$$\xi = \left(\xi^0, \widetilde{\xi}\right), \tag{A.7}$$

with $\xi^0\in\mathbf{R}$ and $\tilde{\xi}\in\mathbf{R}^3$. Analogously we write the field u as follows

$$u = \left(u^0, \tilde{u}\right).$$

Proposition 22. *For every* $u\in\Lambda$

$$\mathrm{ch}\,(u) = \deg\left(\widetilde{u}, S_0\,(u)\,, 0\right),$$

where $S_0\,(u) = \left\{x\in\mathbf{R}^3 : u^0\,(x) > 1\right\}.$

Proof. First of all, we prove that

$$\mathrm{ch}(u) = \deg\left(P\circ u, S\,(u)\,, N\right) = \deg\left(P\circ u, S_0\,(u)\,, N\right). \tag{A.8}$$

It is easy to see that $S_0\,(u)\subset S\,(u)$; let $K = \overline{S\,(u)\setminus S_0\,(u)}$. Since

$$(P\circ u)\,(x) = N \Leftrightarrow \begin{cases} u^0\,(x) > 1 \\ \widetilde{u}\,(x) = 0 \end{cases},$$

we have that $N\notin (P\circ u)\,(K)$. So, for the excision property of the degree, we get (A.8). Since $(P\circ u)\,(S_0\,(u))\subset \Sigma\setminus\{0\}$, in order to evaluate $\deg\left(P\circ u, S_0\,(u)\,, N\right)$ we need a local chart on $\Sigma\setminus\{0\}$. Let π be the stereographic projection of $\Sigma\setminus\{0\}$ on the hyperplane $\left\{\xi^0 = 2\right\}$. It is easy to see that $\pi(\xi) = (2, \widetilde{\pi}\,(\xi))$, where, for every $\xi = (\xi^0, \widetilde{\xi})\in\Sigma\setminus\{0\}$, $\widetilde{\pi}\,(\xi) = \frac{2\widetilde{\xi}}{\xi^0}$. So our local chart is given by $\widetilde{\pi}:\Sigma\setminus\{0\}\to\mathbf{R}^3$. Since $\widetilde{\pi}\,(N) = 0$, we have that

$$\deg\left(P\circ u, S_0\,(u)\,, N\right) = \deg\left(\widetilde{\pi}\circ P\circ u, S_0\,(u)\,, 0\right). \tag{A.9}$$

Finally, we shall prove that $\widetilde{\pi}\circ P\circ u$ and $\widetilde{u}$ are homotopic; then, by the homotopic invariance of degree, we shall get

$$\deg\left(\widetilde{\pi}\circ P\circ u, S_0\,(u)\,, 0\right) = \deg\left(\widetilde{u}, S_0\,(u)\,, 0\right). \tag{A.10}$$

For the reader convenience we write the map P in the form (A.7)

$$P(\xi) = \Big(\frac{\xi^0 - 1}{|\xi - 1|} + 1, \frac{\tilde{\xi}}{|\xi - 1|}\Big)$$

therefore,

$$\widetilde{\pi} \circ P(\xi) = \frac{2\tilde{\xi}}{\xi^0 - 1 + |\xi - 1|}.$$

We consider the homotopy $\Psi : [0,1] \times \overline{S_0(u)} \to \mathbf{R}^3$ defined as follows

$$\Psi(\lambda, x) = \Big((1-\lambda) + \frac{2\lambda}{\left|u(x) - \bar{\xi}\right| + u^0(x) - 1}\Big)\widetilde{u}(x).$$

It is easy to see that $\Psi(0,\cdot) = \widetilde{u}$ and $\Psi(1,\cdot) = \widetilde{\pi} \circ P \circ u$. Then we have only to prove that $0 \notin \Psi([0,1] \times \partial S_0(u))$.

Let $\bar{x} \in \partial S_0(u)$; since $u^0(\bar{x}) = 1$ and $\widetilde{u}(\bar{x}) \neq 0$, we have

$$\Psi(\lambda, \bar{x}) = \Big((1-\lambda) + \lambda\frac{2}{|\widetilde{u}(\bar{x})|}\Big)\widetilde{u}(\bar{x}).$$

Elementary arguments show that

$$\Big((1-\lambda) + \lambda\frac{2}{|\widetilde{u}(\bar{x})|}\Big) \neq 0$$

for every $\lambda \in [0,1]$. By (A.8)-(A.9)-(A.10) the conclusion follows. □

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