Differential and Integral Equations Volume 16, Number 5, May 2003, Pages 583–585

REMARK ON THE PAPER BY S. DUBOIS "MILD SOLUTIONS TO THE NAVIER-STOKES EQUATIONS AND ENERGY EQUALITY"

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(Submitted by: Y. Giga)

S. Dubois obtains an interesting theorem which states that every mild solution v to the Navier-Stokes equations in the class $C([0,T);L^3(\mathbb{R}^3))$ with the initial data $v_0 \in L^2(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$, div $v_0 = 0$ belongs, necessarily to $C([0,T);L^2(\mathbb{R}^3))$ and fulfills energy equalities

$$\|v(t)\|_{L^2}^2 + 2\int_s^t \|\nabla v(\tau)\|_{L^2}^2 d\tau = \|v(s)\|_{L^2}^2 \quad \text{for all } 0 \le s \le t < T. \tag{0.1}$$

In particular, v is in the Leray-Hopf class. She first establishes a certain bilinear estimate for the evolution $\int_0^t e^{(t-\tau)\Delta} P(v\cdot\nabla v)(\tau)d\tau$, and then investigates behavior near $t \to +0$ of $v(t)$ to apply her estimate to the proof of energy inequalities. See the main result Theorem 2.1 in her paper. The purpose of this note is to show that one can prove validity of energy equalities directly by a local existence theorem of the strong solution with the aid of uniqueness of mild solutions in the class $C([0,T);L^3(\mathbb{R}^3))$. It is essential for our argument that the local existence time of the strong solution can be taken *uniformly* for every *precompact* subset in $L^3(\mathbb{R}^3)$ of the initial data. This was implicitly pointed out by Brezis [1]. Our theorem holds also for $\mathbb{R}^n$ with $n \ge 3$. We follow the same notations and definitions of Dubois' paper except for $n \ge 3$. For simplicity, we denote by $L^n_\sigma(\mathbb{R}^n)$ the solenoidal vector fields in $L^n(\mathbb{R}^n)$.

Theorem 1. *Let $v_0 \in L^2(\mathbb{R}^n) \cap L^n_\sigma(\mathbb{R}^n)$. Suppose that v is a mild solution of (NSI) in the class $C([0,T);L^n(\mathbb{R}^n))$ arising from v_0. Then v is in $C([0,T);L^2(\mathbb{R}^n))$ and fulfills energy equalities* (0.1). *In particular, v is in the Leray-Hopf class on* $(0,T)$*, i.e.,*

$$v \in L^\infty(0,T;L^2(\mathbb{R}^n)) \cap L^2(0,T;\dot{H}^1(\mathbb{R}^n)).$$

Accepted for publication: September 2002.

To prove the above theorem, we need the following time-local existence result for the strong solution.

Lemma 1. (Giga-Miyakawa [3], Kato [4], Brezis [1]). *For any precompact subset K in $L^n_\sigma(\mathbb{R}^n)$ there is $T_* = T_*(K) > 0$ such that for every $a \in L^2(\mathbb{R}^n) \cap K$ we have a unique solution u of (NSE) on $[0, T_*)$ with $u|_{t=0} = a$ having the properties that*

$$u \in C([0, T_*); L^2_\sigma(\mathbb{R}^n) \cap L^n_\sigma(\mathbb{R}^n)), \tag{0.2}$$

$$t^{\frac{1}{2}} \nabla u \in C([0, T_*); L^2(\mathbb{R}^n) \cap L^n(\mathbb{R}^n)), \quad \nabla^2 u \in C((0, T_*); L^2(\mathbb{R}^n) \cap L^n(\mathbb{R}^n)). \tag{0.3}$$

Moreover, such a solution u fulfills the energy identity

$$\|u(t)\|^2_{L^2} + 2 \int_0^t \|\nabla u(\tau)\|^2_{L^2} d\tau = \|a\|^2_{L^2} \quad \text{for all } 0 \le t < T_*. \tag{0.4}$$

Remark. First, Giga-Miyakawa [3] and Kato [4] gave the existence time-interval $[0, T_*)$ of the solution $u(t)$ for each $a \in L^n_\sigma(\mathbb{R}^n)$. Then Brezis [1] proved that T_* can be taken *uniformly* for every *precompact* subset of initial data in $L^n_\sigma(\mathbb{R}^n)$. For more detail, see [5, Proposition 2.4]. It should be noted that T_* can be chosen independently of the size of $\|a\|_{L^2}$.

Concerning uniqueness of the mild solutions of (NSI), we have:

Lemma 2. (Furioli-Lemarié-Rieusset -Terraneo [2], Lions-Masmoudi [6]). *If u and v are mild solutions of (NSI) in $C([0, T); L^n(\mathbb{R}^n))$, then there holds $u \equiv v$ on $[0, T)$.*

Proof of Theorem 1. In Lemma 1, we take $K \equiv \{v(t); t \in (0, T)\}$. Since $v \in C([0, T); L^n(\mathbb{R}^n))$ with div $v = 0$, we see that K is a precompact subset in $L^n_\sigma(\mathbb{R}^n)$. Then it follows from Lemma 1 that there exists a solution u of (NSE) on $[0, T_*)$ with $u|_{t=0} = v_0$ having the properties (0.2), (0.3) and the energy identity (0.4) with a replaced by v_0. By (0.2) and Lemma 2, we see that $v \equiv u$ on $[0, T_*)$. We next solve (NSE) beyond T_*. Indeed, again by Lemma 1, there exists a solution w of (NSE) with $w|_{t=T_*} = v(T_*)$ such that $w \in C([T_*, 2T_*); L^2(\mathbb{R}^n) \cap L^n(\mathbb{R}^n))$ and such that

$$\|w(t)\|^2_{L^2} + 2 \int_{T_*}^t \|\nabla w(\tau)\|^2_{L^2} d\tau = \|v(T_*)\|^2_{L^2} \quad \text{for all } t \in [T_*, 2T_*).$$

By Lemma 2, there holds $v \equiv w$ on $[T_*, 2T_*)$. Then we have that $v \in C([0, 2T_*]; L^2(\mathbb{R}^n))$ and that v fulfills energy equalities (0.1) for $0 \le s \le t \le 2T_*$. Notice that the existence time T_* depends only on K. Repeating this procedure again for $t \geq 2T_*$, within finitely many steps, we reach the conclusion that $v \in C([0,T); L^2(\mathbb{R}^n))$ and that v fulfills energy equalities (0.1) on the whole interval $0 \leq s \leq t \leq T$. This proves Theorem 1.

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