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REGULARITY OF THE ATTRACTOR FOR A COUPLED KLEIN-GORDON-SCHRÖDINGER SYSTEM

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Abstract. It is well known that the coupled Klein-Gordon-Schrödinger system possesses a compact global attractor into a suitable energy space. We prove the asymptotical smoothing effect for this system, i.e., we prove that the attractor is in fact embedded into a smaller energy space.

1. Introduction

This article is concerned with the long time behavior of the solutions (u, ϕ) to a coupled system of evolution equations that reads

$$iu_t + \Delta u + i\nu u + \phi u = f, \tag{1.1}$$

$$\phi_{tt} + \gamma\phi_t - \Delta\phi + \phi - |u|^2 = g. \tag{1.2}$$

The data for these equations are ν, γ that are positive damping parameters and f, g that are time-independent forcing terms. Throughout this article we will assume that f, g belong to L^2. The above-mentioned system is usually referred as the Klein-Gordon-Schrödinger system.

Actually there exists some energy spaces $E_2 \subset E_1$ (that will be specified below) such that if we supplement (1.1)-(1.2) with initial conditions belonging either to E_1 or E_2, then one has a well-posed initial value problem respectively in E_1 or E_2. Moreover, due to the damping parameters, it turns

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out that any solution to (1.1)-(1.2) in E_1 or E_2 enters after a finite transient time into an *absorbing ball* in the corresponding energy space. This means that (1.1)-(1.2) provides a *dissipative* infinite-dimensional dynamical system in either E_1 or E_2 (see [12], [19]).

A major issue in this framework is to study the existence of a *global attractor* for this system. Let us recall that the global attractor for an infinite-dimensional system in E_j ($j = 1$ or 2) is a compact subset of E_j that is invariant by the flow of the solutions and that attracts strongly in E_j all the trajectories when t goes to the infinity. For the Klein-Gordon-Schrödinger system the first result appeared in [4], where the author has proved the existence of a weak global attractor $\mathcal{A}_1$ in the energy space E_1 in the case of bounded domains. Let us recall that a weak global attractor attracts the trajectories only for the weak topology in E_1. Actually, it turned out that this weak attractor is actually a strong one (see [16], [20]). On the other hand, the existence of a strong global attractor $\mathcal{A}_2$ in E_2 was established in [11] and [15]. Going back to the very definition of a global attractor, it is easy to check that $\mathcal{A}_2 \subset \mathcal{A}_1$.

In this article we want to address the issue of the regularity of the attractor. Since (1.1)-(1.2) is a coupled system combining a Schrödinger equation and a wave equation, then there is no finite time smoothing for this system. Nevertheless we prove that the dissipativity provides the system under consideration with an *asymptotical* smoothing effect (see [13]): the dynamical system in E_1 associated to (1.1)-(1.2) possesses a compact global attractor in E_2. It is straightforward to observe that this is equivalent to $\mathcal{A}_1 = \mathcal{A}_2$.

Numerous examples of dissipative evolution equations feature this asymptotical smoothing effect. Beyond the parabolic case, we know for instance that equations like nonlinear Schrödinger system or weakly damped Korteweg-de Vries equations feature this property (see [19] and the references therein). This property holds also for the Zakharov system, that is a different coupling between a nonlinear wave and a Schrödinger equation (see [10]).

The issue of the regularity of the attractor is useful in several aspects. For one, let us pretend that the global attractor describes the permanent regime for the physics of a given dissipative dynamical system (the data for (1.1)-(1.2) being fixed). For the mathematical study (initial value problem, dissipativity...), we first chose an energy space (E_1 or E_2) to deal with. Then we are able to prove the existence of the corresponding global attractors $\mathcal{A}_1$ and $\mathcal{A}_2$. Our main result ($\mathcal{A}_1 = \mathcal{A}_2$) can be understood as follows: the global attractor, that is the object that describes the permanent regime for the physics, does not depend on the mathematical framework. For another, the regularity of the attractor has also some consequences in the numerical analysis of dissipative evolution equations. This is related to the decay of the Fourier modes of the solutions; see [9] and [18] for similar issues about Schrödinger equation.

This article is organized as follows: In Section 2, we introduce the mathematical framework associated to (1.1)-(1.2). We also outline the known results for this system. We complete this section with a precise statement of our main theorem. Section 3 is devoted to the proof of this main theorem.

2. Statement of the results

2.1. **Previous results.** To begin with, we introduce the mathematical framework associated with (1.1)-(1-2).

Let Ω be either a smooth bounded subdomain of $\mathbb{R}^n, n \leq 3$ or the whole space $\mathbb{R}^n, n \leq 3$. Equations (1.1)-(1.2) are then supplemented with homogeneous Dirichlet boundary conditions if Ω is bounded. We could also consider $\Omega = [0,1]^n, n \leq 3$ with periodic boundary conditions. Let us observe that the unknowns functions (u, ϕ) map $(t, x) \in \mathbb{R} \times \Omega$ into $\mathbb{C} \times \mathbb{R}$.

We supplement (1.1)-(1.2) with initial conditions $(u(0), \phi(0), \phi_t(0))$ that belongs either to

$$E_1 = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega), \tag{2.1}$$

or to

$$E_2 = (H_0^1(\Omega) \cap H^2(\Omega)) \times (H_0^1(\Omega) \cap H^2(\Omega)) \times (H_0^1(\Omega)). \tag{2.2}$$

We now recall from [2], [4], [6], [7], [14], [20], [16] the following statement about the well-posedness of (1.1)-(1.2) in E_1.

Theorem 2.1. *Assume $f, g \in L^2$. Then the initial value problem in E_1 is globally well-posed: there exists a unique solution for* (1.1)-(1-2) *supplemented with initial conditions in E_1 that belongs to $C(\mathbb{R}; E_1)$. Moreover, the solution depends continuously on the initial data.*

This theorem allows us to define a continuous semi-group $S(t) : E_1 \to E_1$ by setting

$$S(t)(u(0), \phi(0), \phi_t(0)) = (u(t), \phi(t), \phi_t(t)). \tag{2.3}$$

Moreover, the system under consideration is dissipative in the following sense:

Theorem 2.2. *There exists a compact global attractor $\mathcal{A}_1$ for $S(t)$ in E_1.*

We could also define a semi-group on E_2 and then state analogous results. For the sake of conciseness, we skip the related statements.

2.2. **The main theorem.** Our main result reads as follows

Theorem 2.3. *Actually $\mathcal{A}_1$ is a compact subset of E_2.*

We would like to point out two comments about this theorem. For one, its proof is totally different from the arguments in [10] and rely on some bootstrapping arguments (see Section 3 below). For another, we would like to emphasize that with minor modifications on this proof one can prove directly the existence and the regularity of the attractor, and then recover the results in [20] and [16].

3. Proof of the main theorem

For the sake of conciseness, we just indicate the proof in the case where $\Omega \subset \mathbb{R}^3$, since in lower dimension the proof is easier and left as an exercise to the reader.

3.1. **Asymptotic smoothing for dissipative wave equation.** Introduce $\Lambda = Id - \Delta$ as the unbounded operator in L^2 whose domain is $H^2(\Omega) \cap H_0^1(\Omega)$. Let a be a nonnegative real number. Define $X_{2a} = C \cap L^\infty(\mathbb{R}, D(\Lambda^a))$ as the set of continuous and bounded function (with respect to the t variable) into the domain of Λ^a. Let us set $H^{2a} = D(\Lambda^a)$. Then we recall from [19] (see Proposition IV.1.3. therein):

Theorem 3.1. *Consider a function $F(t,x)$ that belongs to X_a. Then any solution (ϕ, ϕ_t) in $X_a \times X_{a-1}$ of*

$$\phi_{tt} + \gamma\phi_t - \Delta\phi + \phi = F, \tag{3.1}$$

belongs to $X_{a+1} \times X_a$.

Proof. This kind of result was previously used in [13], [8]. We indicate below a new trick for proving this well-known result. This argument (learned from [5]) illustrates why to prove the regularity of the attractor is easier for wave equations than for weakly damped Schrödinger equations. The idea is to introduce the change of variable

$$v = \phi + i\Lambda^{-1/2}\phi_t, \tag{3.2}$$

that substitutes to the wave equation (3.1) a Schrödinger-like equation that reads

$$v_t + i\Lambda^{1/2}v = -i\gamma \mathrm{Im} v + i\Lambda^{-1/2}F. \tag{3.3}$$

In the right hand side of (3.3), the first term describes the damping and the second one introduces the regularization due to the operator $\Lambda^{-1/2}$.

First it is classical (see [19] and the references therein) to prove that the linear semi-group $\exp(-tA)$ acting on H^a associated with $v(0) = v_0$ and

$$v_t + i\Lambda^{1/2}v + i\gamma \mathrm{Im} v = 0 \tag{3.4}$$

satisfies the exponential decay

$$||\exp(-tA)|| \leq \exp(-\gamma t). \tag{3.5}$$

Therefore, the only bounded solution (ϕ, ϕ_t) to (3.1) solves the integral equation

$$v(t) = i\int_{-\infty}^{t} e^{(s-t)A}\Lambda^{-1/2}F(s)ds. \tag{3.6}$$

It is now straightforward to establish (due to (3.5) and to the regularizing effect of the operator $\Lambda^{-1/2}$) that for all t

$$||v(t)||_{H^{a+1}} \leq C(\gamma)||F||_{X_a}. \qquad \square \tag{3.7}$$

3.2. **An iteration argument: the first step.** We first state the following result:

Lemma 3.2. *For any $\varepsilon > 0$, there exists C_ε such that*

$$||u\bar{u}||_{H^{\frac{1}{2}-\varepsilon}} \leq C_\varepsilon ||u||^2_{H^1}. \tag{3.8}$$

Proof. Assume first that the space variable x describes the whole space $\mathbb{R}^3$. Introduce then a Littlewood-Paley decomposition of $v = \mathrm{Re}\, u$ as

$$v = \sum_{j=0}^{+\infty} z_j + y_0. \tag{3.9}$$

Here z_j is defined as follows: consider a smooth function Φ whose support is included in $\{\xi; |\xi| \leq 1\}$ and such that $\Phi(\xi) = 1$ for $|\xi| \leq \frac{1}{2}$. Set $\Psi(\xi) = \Phi(\frac{\xi}{2}) - \Phi(\xi)$. z_j is then defined by its Fourier transform by $\hat{z_j}(\xi) = \Psi(\frac{\xi}{2^j})\hat{v}(\xi)$. Observe that $\hat{y}_0(\xi) = \Phi(\xi)\hat{v}(\xi)$.

Setting now $y_{-1} = 0, y_0 = z_{-1}$ and $y_j = \sum_{k\leq j} z_k$, it is well known that $||u||_{H^a}$ and

$$(\sum_j ||z_j||^2_{L^2}(1 + 2^j)^{2a})^{1/2}$$

define equivalent norms on H^a (see [1]). Set now $a = \frac{1}{2} - \varepsilon$.

Then using the expansion

$$v^2 = \sum_j (y^2_{j+1} - y^2_j) = \sum_j (y_j + y_{j+1})z_{j+1}, \tag{3.10}$$

we have for some numerical constant c

$$||v^2||_{H^a} \leq c \sum_j 2^{ja} ||(y_j + y_{j+1}) z_{j+1}||_{L^2}. \tag{3.11}$$

On the other hand, due to Holder inequality and Sobolev embedding (let us recall that we are working in $\mathbb{R}^3$)

$$||(y_j + y_{j+1}) z_{j+1}||_{L^2} \leq ||y_{j+1} + y_j||_{L^6} ||z_{j+1}||_{L^3} \leq c(\text{Sup}_n ||y_n||_{H^1}) ||z_{j+1}||_{H^{1/2}}. \tag{3.12}$$

Then, in the case $x \in \mathbb{R}^3$, the proof is over, due to $||y_j||_{H^1} \leq ||u||_{H^1}$ and to

$$\sum_j 2^{j(\frac{1}{2}-\varepsilon)} ||z_{j+1}||_{H^{1/2}} \leq C_\varepsilon ||u||_{H^1}. \tag{3.13}$$

On the other hand, when we deal with a bounded domain of $\mathbb{R}^n$ with homogeneous Dirichlet conditions, the prolongation operator defined by $Pu(x) = u(x)$ when $x \in \Omega$ and $Pu(x) = 0$ outside Ω is a bounded operator from $H^1_0(\Omega)$ to $H^1(\mathbb{R}^3)$. Applying the above arguments, we then have for $a \in [0, \frac{1}{2})$

$$||u\bar{u}||_{H^a(\Omega)} \leq ||Pu\bar{P}u||_{H^a(\mathbb{R}^3)} \leq c_a ||Pu||^2_{H^1(\mathbb{R}^3)} \leq C_a ||u||^2_{H^1(\Omega)}. \tag{3.14}$$

This concludes the proof of the lemma. □

Consider now a complete trajectory $(u(t), \phi(t), \phi_t(t))$ that belongs to the global attractor $\mathcal{A}_1$. Then u belongs to $X_1 = C \cap L^\infty(\mathbb{R}, H^1)$. On the other hand, we introduce the splitting $\phi = \phi^* + \psi$, where

$$-\Delta \phi^* + \phi^* = g, \tag{3.15}$$

and

$$\psi_{tt} + \gamma \psi_t - \Delta \psi + \psi = |u|^2. \tag{3.16}$$

On the one hand, ϕ^*, that does not depend on t, belongs to H^2. On the other hand, due to Lemma 3.2 and Theorem 3.1, ψ is a complete and bounded trajectory in $X_{\frac{3}{2}-\varepsilon}$, for any small positive ε. At this stage, we have established

$$(\phi, \phi_t) \in X_{\frac{3}{2}-\varepsilon} \times X_{\frac{1}{2}-\varepsilon}. \tag{3.17}$$

3.3. **An iteration argument: the final step.** Let $(u(t), \phi(t), \phi_t(t))$ be a complete trajectory in $\mathcal{A}_1$. (3.17) gives extra regularity on ϕ. We will now approximate u that is solution to (1.1) using the following argument: let m be a positive integer; consider u^m that solves

$$iu^m_t + \Delta u^m + i\nu u^m + \phi u^m = f, \tag{3.18}$$

supplemented with initial condition at $t = -m$

$$u^m(-m) = 0. \tag{3.19}$$

We now state

Lemma 3.3. *The sequence $\{u^m\}$ belongs to a bounded set of X_2 and there exists C that is independent of m such that for $t \geq -m$*

$$||u(t) - u^m(t)||_{L^2} \leq C\exp(-2\nu(t+m)). \tag{3.20}$$

Application. As a consequence of this lemma it is straightforward (let $m \to +\infty$) to obtain

$$u \in X_2, \tag{3.21}$$

and, going back to the first step (by a bootstrapping argument),

$$(\phi, \phi_t) \in X_2 \times X_1. \tag{3.22}$$

Proof of Lemma 3.3. First, we can easily prove that (3.18)-(3.19) is (locally) well-posed in H^2. We can also prove a uniform bound in H^1 for u^m (this is the proof of the existence of an absorbing set in H^1 for u; see [4]).

Then we subtract (3.18) to (1.1), we multiply the resulting equation by $\bar{u} - \bar{u}^m$ and integrate the imaginary part over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}||u - u^m||_{L^2}^2 + \nu||u - u^m||_{L^2}^2 = 0 \tag{3.23}$$

that leads to (3.20).

It remains to prove an a priori estimate in H^2 for the solution u^m to (3.18)-(3.19). Since $\Delta u^m = iu_t^m +$ (lower order terms), we rather prove below a L^2 estimate for u_t^m. For that purpose, observe that $v = u_t^m$ is solution to

$$iv_t + \Delta v + i\nu v + \phi v + \phi_t u^m = 0. \tag{3.24}$$

Multiply this equation by $\bar{v}$ and integrate the imaginary part of the resulting equation over Ω. We thus obtain

$$\frac{1}{2}\frac{d}{dt}||v||_{L^2}^2 + \nu||v||_{L^2}^2 = -\mathrm{Im}\int_\Omega \phi_t u^m \bar{v}. \tag{3.25}$$

We majorize the right hand side of (3.25) as follows

$$\left|\mathrm{Im}\int_\Omega \phi_t u^m \bar{v}\right| \leq ||v||_{L^2}||\phi_t||_{L^{3^-}}||u^m||_{L^{6^+}}, \tag{3.26}$$

where 3^- (respectively 6^+) are numbers lesser and close to 3 (respectively larger and close to 6) such that the embeddings $H^{\frac{1}{2}-\varepsilon} \subset L^{3^-}$ and $H^{1+\varepsilon} \subset$

L^{6+} hold true. We then have

$$\left|\mathrm{Im}\int_\Omega \phi_t u^m \bar{v}\right| \leq ||v||_{L^2}||\phi_t||_{H^{\frac{1}{2}-\varepsilon}}||u^m||^{\varepsilon}_{H^2}||u^m||^{1-\varepsilon}_{H^1}, \tag{3.27}$$

that leads to (due to (3.17) and due to the boundedness in H^1 for u^m)

$$\left|\mathrm{Im}\int_\Omega \phi_t u^m \bar{v}\right| \leq c(1+||v||^{1+\varepsilon}_{L^2}). \tag{3.28}$$

It is then straightforward to prove that v remains in a bounded set of L^2 for $t \geq -m$ (uniformly in m). This completes the proof of the lemma. □

3.4. **Conclusion.** At this stage (3.21) and (3.22) imply that $\mathcal{A}_1$ is a bounded subset of E_2. The compactness in E_2 can be obtained by the classical Ball argument (see [3]) as follows; consider a sequence (U^n, Φ^n, Φ^n_t) that belongs to $\mathcal{A}_1$. Due to Theorem 2.2, up to a subsequence extraction, this sequence converges strongly in E_1 towards (U, Φ, Φ_t). Since $(\Phi^n - \phi^*, \Phi^n_t)$ remains in a bounded set of $H^3 \times H^2$, by interpolation the convergence of (Φ^n, Φ^n_t) holds for the strong $H^2 \times H^1$ topology. It remains to prove that actually U^n converges towards U strongly in H^2. We already know that we have this convergence for the weak H^2- topology. Then, it remains to prove that

$$||U^n||_{H^2} \to ||U||_{H^2}, \tag{3.29}$$

or, equivalently (going back to (1.1))

$$||U^n_t||_{L^2} \to ||U_t||_{L^2}. \tag{3.30}$$

For that purpose, we make use of the following energy equality that holds true on any trajectory, and that is derived from (3.25)

$$||u_t(t)||^2_{L^2} = ||u_t(0)||^2_{L^2} e^{-2\nu t} - \mathrm{Im}\int_0^t \int_\Omega e^{2\nu(s-t)}\phi_t(s)u(s)\bar{u}_t(s)ds. \tag{3.31}$$

To complete the proof (Ball argument) is standard. We skip the details, referring the reader to [17], [19], [11], [20], [16]. This concludes the proof of the main theorem. □

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