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STABILITY AND WEAK–STRONG UNIQUENESS FOR AXISYMMETRIC SOLUTIONS OF THE NAVIER–STOKES EQUATIONS

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Abstract. We consider the Navier–Stokes equations in $\mathbb{R}^3$, in an axisymmetric setting: the data and the solutions only depend on the radial and on the vertical variable. In [7], a unique solution is constructed in a scale invariant function space L^2_0, equivalent to L^2 at finite distance from the vertical axis. We prove here a weak–strong uniqueness result for such solutions associated with data in $L^2 \cap L^2_0$.

1. Introduction

In this note we are interested in the Navier-Stokes equations

$$(NS)\begin{cases} \partial_t v + v\cdot\nabla v - \nu\Delta v &= -\nabla p \quad \text{in} \quad \mathbb{R}^+\times\mathbb{R}^3,\\ \operatorname{div} v &= 0,\\ v_{|t=0} &= v_0, \end{cases}$$

under the assumption that v_0 is a divergence–free, axisymmetric vector field, as defined in Definition 1 below. Let us recall that in the system (NS), $v(t,x,y,z)$ and $p(t,x,y,z)$ are respectively the velocity and the pressure fields of the fluid, and $\nu > 0$ is the viscosity. The velocity is a three-component vector field, and the pressure is a scalar field. The divergence free condition on v represents the incompressibility of the fluid. Here t and (x,y,z) are respectively the time and the space variables, with $t\in\mathbb{R}^+$ and $(x,y,z)\in\mathbb{R}^3$.

Definition 1. Let $(r,\theta,z)\in\mathbb{R}^+\times[0,2\pi)\times\mathbb{R}$ be the cylindrical coordinates in $\mathbb{R}^3$, where $\forall(x,y)\in\mathbb{R}^2$, $x = r\cos\theta$ and $y = r\sin\theta$. A scalar function defined in $\mathbb{R}^3$ is said to be axisymmetric if it does not depend on the angular variable θ, and a vector field is said to be axisymmetric if each of its cylindrical components is.

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We emphasize here the fact that the cylindrical component u^θ of a vector field u is not necessarily zero for u to be axisymmetric. The only assumption is that u^r, u^θ and u^z do not depend on θ. In the following, when in cylindric coordinates, we shall still note the space $L^2(\mathbb{R}^3)$ for the space L^2 with measure $rdrd\theta dz$.

The question of the existence and uniqueness of solutions to (NS) under such assumption on the initial data was addressed by M. Ukhavskii and V. Yudovitch in [16] (with the additional requirement that $v^\theta = 0$). The result proved in [16] states that there is a unique solution to (NS) under axisymmetric assumptions, in the following class:

$$\nabla v \in L^\infty([0,T], L^2(\mathbb{R}^3)), \quad \text{rot } v \in L^\infty([0,T], L^2 \cap L^\infty(\mathbb{R}^3)),$$
$$\frac{\text{rot } v}{r} \in L^\infty([0,T], L^2 \cap L^\infty(\mathbb{R}^3)).$$

Other studies on axisymmetric solutions of the Navier-Stokes equations can be found, in particular the work of O.A. Ladhyzhenskaya [12], as well as the study of G. Ponce, R. Racke, T. Sideris and E. Titi [14]. Recently, in [6]-[7], the aim of S. Ibrahim, M. Majdoub and the author was to obtain solutions in a class of function spaces invariant by the scaling of the equation (and thus with less regularity imposed on the initial data than for the results in [12], [14], [16]): recall that for any real number λ, if v is a solution of the Navier-Stokes equations associated with the initial data v_0, then the same goes for $v_\lambda(t,x,y,z) \stackrel{\text{def}}{=} \lambda v(\lambda^2 t, \lambda x, \lambda y, \lambda z)$, associated with the data $v_{0,\lambda}(x,y,z) \stackrel{\text{def}}{=} \lambda v_0(\lambda x, \lambda y, \lambda z)$. The result in [7] is the following:

Theorem 1 ([7]). *Let u_0 be a divergence free vector field, element of*

$$L^2_0(\mathbb{R}^3) \stackrel{def}{=} \Big\{ f \in \mathcal{S}'(\mathbb{R}^3) \text{ axisymmetric} : \int_{\mathbb{R}^3} |f(x,y,z)|^2 \frac{dxdydz}{\sqrt{x^2+y^2}} < +\infty \Big\}.$$

There exists a unique time $T^* > 0$ and a unique axisymmetric solution u such that for any time $T \in (0,T^*)$, $u \in C^0([0,T], L^2_0(\mathbb{R}^3))$, $t^{1/2}\nabla u \in C^0([0,T], L^2_0(\mathbb{R}^3))$ and $\lim_{t\to 0} t^{1/2}\nabla u = 0$ in $L^2_0(\mathbb{R}^3)$. Moreover, there exists a constant $c > 0$ such that if $\|u_0\|_{L^2_0(\mathbb{R}^3)} \leq c\nu$, then one can choose $T = +\infty$ above.

Remark. The space $L^2_0(\mathbb{R}^3)$ defined above is presented and studied in [7]; we shall refer to [7] whenever special properties of that space are needed (namely in Section 3). We will note

$$\|f\|_{L^2_0(\mathbb{R}^3)} \stackrel{\text{def}}{=} \Big(\int_{\mathbb{R}^+ \times \mathbb{R}} |f(r,z)|^2 \, drdz \Big)^{1/2}.$$

Note that the space $L^2_0(\mathbb{R}^3)$ is invariant by the scaling of the Navier–Stokes equations. Let us also point out that the theorem stated above (and indeed all theorems concerning unique solutions of the Navier–Stokes equations to this day, in three space dimensions: see for instance [1], [3], [4], [11]) is global in time only for small data. It has been known since the early thirties that global solutions for arbitrarily large initial data can be constructed, but those solutions are not known to be unique; that result goes back to the fundamental work of J. Leray (see [13]), and can be stated in the following way: if the initial data is a divergence free element of $L^2(\mathbb{R}^3)$, then there is (at least) a solution v to (NS) which satisfies

$$\forall t \geq 0, \quad \|v(t)\|^2_{L^2(\mathbb{R}^3)} + 2\nu \int_0^t \|\nabla v(s)\|^2_{L^2(\mathbb{R}^3)}\, ds \leq \|v_0\|^2_{L^2(\mathbb{R}^3)}. \tag{1.1}$$

A solution of (NS) satisfying such an energy estimate will be said to be a Leray solution.

Remark. Note that it is possible to construct axisymmetric Leray solutions if the data is axisymmetric (see [7]).

One natural question one can now ask is the following: if the initial data is axisymmetric, and if it is an element of $L^2_0(\mathbb{R}^3)$ as well as $L^2(\mathbb{R}^3)$, do all axisymmetric Leray solutions coincide with the unique axisymmetric solution given by Theorem 1? The (positive) result is given in the following stability theorem. Let us start by stating the following proposition, which will be proved in Section 3.

Proposition 1. *Under the assumptions of Theorem* 1, *the solution u satisfies moreover*

$$\forall T < T^*, \quad \nabla u \in L^2([0,T], L^2_0(\mathbb{R}^3)).$$

and of $L^2_0(\mathbb{R}^3)$, then the solution u given by Theorem 1 *satisfies also the energy inequality* (1.1).

In the following, we shall use the notation

$$E_T \stackrel{\text{def}}{=} \Big\{ f \in L^\infty([0,T], L^2(\mathbb{R}^3)) \text{ divergence free, } \nabla f \in L^2([0,T], L^2(\mathbb{R}^3)) \Big\},$$

$$E \stackrel{\text{def}}{=} E_\infty,$$

$$E_{0,T} \stackrel{\text{def}}{=} \Big\{ f \in C^0([0,T], L^2_0(\mathbb{R}^3)) \text{ divergence free, } \nabla f \in L^2([0,T], L^2_0(\mathbb{R}^3)) \Big\}.$$

We will also note

$$\|u\|^2_{E_{0,T}} \stackrel{\text{def}}{=} \|u\|^2_{L^\infty([0,T], L^2_0(\mathbb{R}^3))} + \|\nabla u\|^2_{L^2([0,T], L^2_0(\mathbb{R}^3))}.$$

Theorem 2. *Let $u_0 \in L^2_0(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and $v_0 \in L^2(\mathbb{R}^3)$ be two axisymmetric, divergence free vector fields, and let u and v be axisymmetric Leray solutions of (NS) associated respectively with u_0 and v_0; suppose that u is in $E_{0,T}$ for some $T > 0$. Then the function $w \stackrel{def}{=} v - u$ satisfies the following estimate:*

$$\sup_{t\leq T} \|w(t)\|^2_{L^2(\mathbb{R}^3)} + \nu \int_0^T \|\nabla w(t)\|^2_{L^2(\mathbb{R}^3)}\, dt$$
$$\leq \|w_0\|^2_{L^2(\mathbb{R}^3)} \exp\Big(\frac{C}{\nu}\|u\|^2_{E_{0,T}} + \frac{C}{\nu^3}\|u\|^4_{E_{0,T}}\Big),$$

where $w_0 \stackrel{def}{=} v_0 - u_0$. In particular, all axisymmetric Leray solutions associated with data in $L^2_0(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ coincide with the solution u of Theorem 1, for all times $t \leq T$. In particular, if u is a Leray solution associated with $u_0 \in L^2_0(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ such that $u \in E_{0,T}$ for some $T \geq 0$, then all axisymmetric Leray solutions associated with u_0 coincide with u for all times $t \leq T$.

Remark. Note that a stability result is proved in [14], with more smoothness assumptions on the initial data (essentially the assumptions of [16] presented above).

The method we shall follow to prove this result goes back to J. Leray [13], and can be found in the book by W. von Wahl [17], where weak–strong uniqueness results in the $L^3(\mathbb{R}^3)$ case are proved (see also more recently [5] for bidimensional initial data, [7] for the axisymmetric case in special domains of $\mathbb{R}^3$, as well as [9] for the tridimensional equations in Besov spaces). Before going on with the proof of the result, let us comment on the bidimensional case: as an axisymmetric vector field essentially only depends on two variables, r and z, one can wonder at the difference with the bidimensional weak-strong uniqueness result [5]. In fact the axisymmetric case is more complicated since the space in which uniqueness is known is L^2_0 and not L^2, and obviously L^2_0 is a weighted space whose weight degenerates near $r = 0$. We will see in the course of the proofs that this causes many problems which were not present in [5].

Note that the pressure p does not appear in the results; that is due to the fact that if (u, p) satisfies the Navier–Stokes equations, then since u is divergence free, one has

$$p = -\Delta^{-1} \sum_{i,j} \partial_i\partial_j(u^i \otimes u^j). \tag{1.2}$$

Finally, in the whole of this text, we will note by C, or sometimes c, any universal constant which can change from line to line. We will also write $x \lesssim y$ for $x \leq Cy$.

2. Proof of Theorem 2

Let $u_0 \in L^2_0(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and $v_0 \in L^2(\mathbb{R}^3)$ be two axisymmetric, divergence free vector fields, and define associate axisymmetric Leray solutions u and v, where u satisfies moreover the smoothness results of Theorem 1 and Proposition 1: we suppose that v is an element of E, as defined in the introduction, and that u is an element of $E \cap E_{0,T}$ for some $T > 0$.

The method of proof of Theorem 2, following [17], consists simply in considering the function $w \stackrel{\text{def}}{=} u - v$, which is an axisymmetric, divergence free vector field element of E; it satisfies

$$\frac{1}{2}\|w(t)\|^2_{L^2} + \nu \int_0^t \|\nabla w(s)\|^2_{L^2}\, ds = \frac{1}{2}\|u(t)\|^2_{L^2} + \nu \int_0^t \|\nabla u(s)\|^2_{L^2}\, ds \tag{2.1}$$
$$+ \frac{1}{2}\|v(t)\|^2_{L^2} + \nu \int_0^t \|\nabla_2 v(s)\|^2_{L^2} ds - (v(t)|u(t))_{L^2}$$
$$- 2\nu \int_0^t (\nabla v(s)|\nabla u(s))_{L^2}\, ds,$$

where $(\cdot \mid \cdot)_{L^2}$ denotes the scalar product in $L^2(\mathbb{R}^3)$. Now the difficulty is to compute the cross-product terms on the right–hand side of (2.1), due to the lack of regularity of Leray solutions (formal computations would yield the following lemma in a straightforward way).

Lemma 1. *Under the assumptions of Theorem* 2, *the functions* u *and* v *satisfy*

$$(v \mid u)_{L^2}(t) + 2\nu \int_0^t (\nabla v \mid \nabla u)_{L^2}(s)\, ds$$
$$= (v_0 \mid u_0)_{L^2} - \int_0^t \Big((v \mid u \cdot \nabla u)_{L^2} + (u \mid v \cdot \nabla v)_{L^2} \Big)(s)\, ds.$$

Before proving that result, let us finish the proof of Theorem 2: we notice that

$$\int_0^t \Big((v \mid u \cdot \nabla u)_{L^2} + (u \mid v \cdot \nabla v)_{L^2} \Big)(s)\, ds = \int_0^t (w \cdot \nabla w \mid u)_{L^2}(s)\, ds,$$

so (2.1) becomes finally, with Lemma 1,

$$\frac{1}{2}\|w(t)\|^2_{L^2} + \nu \int_0^t \|\nabla w(s)\|^2_{L^2}\, ds = \frac{1}{2}\|u(t)\|^2_{L^2} + \nu \int_0^t \|\nabla u(s)\|^2_{L^2}\, ds$$

$$+\frac{1}{2}\|v(t)\|_{L^2}^2+\nu\int_0^t\|\nabla_2 v(s)\|_{L^2}^2\,ds-\int_0^t(w\cdot\nabla w\mid u)_{L^2}(s)\,ds-(u_0\mid v_0)_{L^2}\,.$$

Then estimate (3.10) given in Lemma 3 in Section 3.3, associated with the fact that both u and v satisfy the energy estimate (1.1), yields

$$\begin{aligned}&\|w(t)\|_{L^2}^2+\nu\int_0^t\|\nabla w(s)\|_{L^2}^2\,ds\\&\le\|u_0\|_{L^2}^2+\|v_0\|_{L^2}^2-2\,(u_0\mid v_0)_{L^2}+\frac{C}{\nu}\int_0^t\|\nabla u(s)\|_{L_0^2}^2\|w(s)\|_{L^2}^2\,ds\\&+\ \frac{C}{\nu^3}\int_0^t\|\nabla u(s)\|_{L_0^2}^2\|u(s)\|_{L_0^2}^2\|w(s)\|_{L^2}^2\,ds.\end{aligned}$$

To conclude, a standard conjugation argument yields

$$\begin{aligned}&\|w(t)\|_{L^2}^2+\nu\int_0^t\|\nabla w(s)\|_{L^2}^2\,ds\\&\le\|w_0\|_{L^2}^2\exp\left(\frac{C}{\nu^3}\int_0^t\|\nabla u(s)\|_{L_0^2}^2(\nu^2+\|u(s)\|_{L_0^2}^2)\,ds\right),\end{aligned}$$

and the theorem is proved, up to the proof of Lemma 1, which is achieved in the following section, along with the proof of Proposition 1. The proof of that lemma is of course the core of the paper, and will require a few technical results on axisymmetric vector fields.

3. Proof of some technical results

This section is devoted to the proof of Proposition 1 and Lemma 1, as well as some intermediate technical results.

3.1. **Proof of Proposition 1.** Let us consider an axisymmetric, divergence free vector field u_0 in $L_0^2(\mathbb{R}^3)$ and let us prove that the associate solution given by Theorem 1, satisfies moreover $\nabla u\in L^2([0,T],L_0^2(\mathbb{R}^3))$. To prove that result, we are going to follow the method of [7], otherwise known as "Kato's method" (see the work of T. Kato [10], as well as the book of M. Cannone [1]): we write the solution u in the "mild form"

$$u(t)=S(t)u_0+B(u,u)(t),$$

where $S(t)\stackrel{\text{def}}{=}\exp(t\nu\Delta)$ is the heat semi–group, and

$$B(u,v)(t)\stackrel{\text{def}}{=}-\int_0^t PS(t-s)\text{div}\,(u\otimes v)(s)ds,$$

where P is the Hopf projector onto divergence free vector fields. By a fixed point argument (see [1], [7]), the result will be proved if we show the two following estimates:

$$\|\nabla S(t)u_0\|_{L^2([0,T],L^2_0(\mathbb{R}^3))} \lesssim \|u_0\|_{L^2_0(\mathbb{R}^3)}, \tag{3.1}$$

$$\text{and} \quad \|\nabla B(u,v)\|_{L^2([0,T],L^2_0(\mathbb{R}^3))} \leq \eta \|u\|_{E_{0,T}} \|v\|_{E_{0,T}}, \tag{3.2}$$

where η is a bounded constant independent of u and v. Note that in [7], it is required that η be a bounded function of T going to zero as T goes to zero. That last property is not necessary (it is in fact wrong in general); we refer to [8] for a correction of the proof of the local existence part of Theorem 7 of [7] avoiding that property. The method is classical, all the ingredients for the proof being in [7].

To prove those two estimates, we are going to use some Littlewood–Paley theory (see for instance the book of J.-Y. Chemin [2] for a precise presentation): we consider a function φ, smooth, radial and compactly supported on the ring $\mathcal{C} \stackrel{\text{def}}{=} \{\rho \in \mathbb{R} : \frac{1}{2} \leq |\rho| \leq 2\}$, and such that for any $\rho \in \mathbb{R} \setminus \{0\}$, we have $\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\rho) = 1$. Then the Littlewood–Paley operators are defined, for any function $f \in \mathcal{S}'(\mathbb{R}^3)$, by the means of their Fourier transform $\mathcal{F}$:

$$\forall j \in \mathbb{Z}, \quad \mathcal{F}(\Delta_j f)(\xi) \stackrel{\text{def}}{=} \varphi(2^{-j}|\xi|)\mathcal{F}(f)(\xi).$$

The series $\sum_{j\in\mathbb{Z}} \Delta_j f$ converges towards f in $\mathcal{S}'(\mathbb{R}^3)$, and we recall (see [7] and the references therein) that

$$\|f\|^2_{L^2_0(\mathbb{R}^3)} \sim \sum_{j\in\mathbb{Z}} \|\Delta_j f\|^2_{L^2_0(\mathbb{R}^3)}, \tag{3.3}$$

$$\text{and} \quad \|e^{\nu t\Delta}\Delta_j f\|^2_{L^2_0(\mathbb{R}^3)} \sim e^{-\nu t 2^{2j}} \|\Delta_j f\|^2_{L^2_0(\mathbb{R}^3)}. \tag{3.4}$$

Now the proof of estimate (3.1) is straightforward: it is enough to prove that

$$2^{2j} \int_0^T e^{-c\nu 2^{2j} t} \|\Delta_j u_0\|^2_{L^2_0(\mathbb{R}^3)}\, dt \lesssim c_j^2 \|u_0\|^2_{L^2_0(\mathbb{R}^3)},$$

where (c_j) is a sequence in $\ell^2(\mathbb{Z})$ of norm equal to one. That last inequality being obvious due to (3.3), estimate (3.1) is proved.

Now let us prove (3.2). Similarly, we just have to check that

$$2^{2j} \|\Delta_j B(u,v)\|^2_{L^2([0,T],L^2_0(\mathbb{R}^3))} \leq \eta c_j^2 \|u\|^2_{E_{0,T}} \|v\|^2_{E_{0,T}}. \tag{3.5}$$

Since P is continuous on $L^2_0(\mathbb{R}^3)$ (see [7]) we get, still by the spectral localization of Δ_j,

$$2^j \|\Delta_j B(u,v)(t)\|_{L^2_0(\mathbb{R}^3)} \lesssim 2^{2j} c_j \int_0^t e^{-2^{2j} c\nu(t-s)} \|u \otimes v\|_{L^2_0(\mathbb{R}^3)}\, ds$$

$$\lesssim 2^{2j} c_j \int_0^t e^{-2^{2j}c\nu(t-s)} \|u(s)\|^{1/2}_{L^2_0(\mathbb{R}^3)} \|\nabla u(s)\|^{1/2}_{L^2_0(\mathbb{R}^3)} \|v(s)\|^{1/2}_{L^2_0(\mathbb{R}^3)} \|\nabla v(s)\|^{1/2}_{L^2_0(\mathbb{R}^3)} ds.$$

That last inequality is due to the following lemma.

Lemma 2. *There exists a constant C such that the following result holds. Let a and b be two axisymmetric functions, such that a, b, ∇a and ∇b are in $L^2_0(\mathbb{R}^3)$. Then the product ab is also in $L^2_0(\mathbb{R}^3)$, and we have the following estimate:*

$$\|ab\|^2_{L^2_0(\mathbb{R}^3)} \leq C\|a\|_{L^2_0(\mathbb{R}^3)} \|\nabla a\|_{L^2_0(\mathbb{R}^3)} \|b\|_{L^2_0(\mathbb{R}^3)} \|\nabla b\|_{L^2_0(\mathbb{R}^3)}.$$

Supposing that lemma to be true, estimate (3.5) is easily obtained by Young's inequality, and Proposition 1 is proved.

Proof of Lemma 2. It can be seen as a precise version of Lemma 12 in [7] and is in fact simply a bidimensional product rule: the functions a and b being axisymmetric, they are functions of the two variables r and z and the space L^2_0 is precisely the space L^2 with measure $drdz$. So the result follows directly.

3.2. **Proof of Lemma 1.** Let us consider smooth approximations of the vector fields u and v: let (u_n) and (v_n) be two sequences of axisymmetric, divergence free, smooth vector fields, say in $\mathcal{C}^\infty_c(\mathbb{R}^+ \times \mathbb{R}^3)$, bounded respectively in $E \cap E_{0,T}$ and E, such that

$$\begin{aligned} &\nabla v_n(\text{ resp. } \nabla u_n) \text{ converges strongly towards } \nabla v(\text{ resp. } \nabla u) \\ &\text{in } L^2(\mathbb{R}^+, L^2(\mathbb{R}^3)) \end{aligned} \tag{3.6}$$

and additionally

$$\nabla u_n \quad \text{converges strongly towards} \quad \nabla u \quad \text{in} \quad L^2([0,T], L^2_0(\mathbb{R}^3)). \tag{3.7}$$

Since both u and v satisfy the Navier–Stokes equation, taking the scalar product of the equation on v and u respectively with u_n and v_n, and integrating in time yields

$$\int_0^t \Big((\partial_s v \mid u_n)_{L^2} - \nu\,(\Delta v \mid u_n)_{L^2} + (v \cdot \nabla v \mid u_n)_{L^2} \Big)(s)\, ds = 0, \tag{3.8}$$

and

$$\int_0^t \Big((\partial_s u \mid v_n)_{L^2} - \nu\,(\Delta u \mid v_n)_{L^2} + (u \cdot \nabla u \mid v_n)_{L^2} \Big)(s)\, ds = 0. \tag{3.9}$$

It is now a matter of taking the limit in n. The computations are identical to those written in [5], [7] or [9], so we shall skip the details. Lemma 3 stated and proved in the last section, associated with assumptions (3.6)-(3.7), enable us to write

$$\int_0^t \Big(-\nu\, (\Delta v \mid u_n)_{L^2} + (v\cdot\nabla v \mid u_n)_{L^2} \Big)(s)\, ds$$
$$\to \int_0^t \Big(-\nu\, (\Delta v \mid u)_{L^2} + (v\cdot\nabla v \mid u)_{L^2} \Big)(s)\, ds,$$

and similarly for the limit in v_n (noticing that as u is divergence free, $(u\cdot\nabla u \,|\, v_n)_{L^2} = (u\cdot\nabla v_n \,|\, u)_{L^2}$). Then since v solves the Navier-Stokes equation, we have

$$\int_0^t (\partial_s v \mid u)_{L^2}\,(s)\, ds = \int_0^t \Big(\nu\, (\Delta v \mid u)_{L^2} - (v\cdot\nabla v \mid u)_{L^2} \Big)(s)\, ds,$$

which implies that

$$\int_0^t (\partial_s v \mid u_n)_{L^2}\,(s)\, ds \to \int_0^t (\partial_s v \mid u)_{L^2}\,(s)\, ds,$$

and similarly for the limit in v_n. Finally, putting all those limits together, we obtain the following identity:

$$(v(t) \mid u(t))_{L^2} + 2\nu \int_0^t (\nabla v(s) \mid \nabla u(s))_{L^2}\; ds$$
$$= (u_0 \mid v_0)_{L^2} - \int_0^t \Big((v \mid u\cdot\nabla u)_{L^2} + (u \mid v\cdot\nabla v)_{L^2} \Big)(s)\, ds,$$

and Lemma 1 is proved, up to the proof of the following result, used several times above.

3.3. **A continuity lemma.**

Lemma 3. *Let $t \in \mathbb{R}^+$. The application $(a,b,c) \mapsto \int_0^t (div\,(a\otimes b) \mid c)\,(s)\, ds$ is a trilinear form, continuous in $E^2\times E_{0,t}$, with a,b and c axisymmetric and divergence free. More precisely, the following estimate holds:*

$$\int_0^t |(a\cdot\nabla b|c)_{L^2}|\,(s)\, ds \lesssim \int_0^t \|\nabla a\|_{L^2}\|b\|_{L^2}^{1/2}\|\nabla b\|_{L^2}^{1/2}\|c\|_{L^2_0}^{1/2}\|\nabla c\|_{L^2_0}^{1/2}(s)\, ds$$
$$+ \quad \int_0^t \|\nabla b\|_{L^2}\|a\|_{L^2}^{1/2}\|\nabla a\|_{L^2}^{1/2}\|c\|_{L^2_0}^{1/2}\|\nabla c\|_{L^2_0}^{1/2}(s)\, ds$$
$$+ \quad \int_0^t \|\nabla c\|_{L^2_0}\Big(\|a\|_{L^2}\|\nabla b\|_{L^2} + \|b\|_{L^2}^{1/2}\|\nabla b\|_{L^2}^{1/2}\|a\|_{L^2}^{1/2}\|\nabla a\|_{L^2}^{1/2}\Big)\,(s)\, ds.$$

Furthermore, there exists a constant C, independent of t, such that if $a \in E$ and $c \in E_{0,t}$ are two axisymmetric, divergence free vector fields, then we have

$$\begin{aligned}&\Big|\int_0^t (a\cdot\nabla a|c)_{L^2}(s)\,ds\Big| \\ &\leq \frac{\nu}{2}\int_0^t \|\nabla a(s)\|_{L^2}^2\,ds + \frac{C}{\nu^3}\int_0^t \|\nabla c(s)\|_{L_0^2}^2\|a(s)\|_{L^2}^2\|c(s)\|_{L_0^2}^2\,ds.\end{aligned} \tag{3.10}$$

The proof of this result will be achieved in several steps: first, using the fact that the setting is axisymmetric, we are going to compute more precisely the trilinear form appearing in the statement of the lemma, which will enable us to point out its special structure. Then a specific product lemma will yield the result; this result is the core of the paper, and we shall see here why the fact that the vector fields are axisymmetric plays an essential role.

Let us start by giving some notation: if f is a vector field written $f = (f^x, f^y, f^z)$ in cartesian coordinates, then we will note its cylindrical components $f = (f^r, f^\theta, f^z)$, with

$$f^r \stackrel{\text{def}}{=} f^x\cos\theta + f^y\sin\theta \quad \text{and} \quad f^\theta \stackrel{\text{def}}{=} f^x\sin\theta - f^y\cos\theta.$$

Recall that f^r, f^θ and f^z do not depend on θ, and that f^θ is not necessarily zero here.

An easy computation shows that if an axisymmetric vector field f satisfies $\nabla f \in L^2(\mathbb{R}^3)$, then in particular

$$\Big\|\frac{f^r}{r}\Big\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla f\|_{L^2(\mathbb{R}^3)} \quad \text{and} \quad \Big\|\frac{f^\theta}{r}\Big\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla f\|_{L^2(\mathbb{R}^3)}. \tag{3.11}$$

Now let us compute the trilinear form given in Lemma 3: we have, in cylindrical coordinates,

$$a\cdot\nabla b = (a^r\partial_r + \frac{1}{r}a^\theta\partial_\theta + a^z\partial_z)b,$$

so after some computations, one comes up with

$$\begin{aligned}(a\cdot\nabla b|c)_{L^2} \sim &\int_{\mathbb{R}^+\times\mathbb{R}} a^r\partial_r b\cdot c r dr dz + \int_{\mathbb{R}^+\times\mathbb{R}} a^\theta(b^\theta c^r - b^r c^\theta)\,dr dz \\ &+ \int_{\mathbb{R}^+\times\mathbb{R}} a^z\partial_z b\cdot c r dr dz = I + II + III,\end{aligned}$$

and we shall estimate each of those three terms separately.

For the first term, we have by Cauchy–Schwarz,

$$|I| \lesssim \|\partial_r b\|_{L^2}\|a^r c\|_{L^2} \lesssim \|\nabla b\|_{L^2}\|r^{1/2}a^r c\|_{L_0^2}.$$

But Lemma 2 yields

$$\|r^{1/2}a^r c\|_{L^2_0} \lesssim \|r^{1/2}a^r\|_{L^2_0}^{1/2}\|\nabla(r^{1/2}a^r)\|_{L^2_0}^{1/2}\|c\|_{L^2_0}^{1/2}\|\nabla c\|_{L^2_0}^{1/2},$$

and since

$$\|\nabla(r^{1/2}a^r)\|_{L^2_0} \lesssim \|\nabla a^r\|_{L^2} + \left\|\frac{a^r}{r}\right\|_{L^2},$$

the result (3.11) implies that

$$|I| \lesssim \|\nabla b\|_{L^2}\|a\|_{L^2}^{1/2}\|\nabla a\|_{L^2}^{1/2}\|c\|_{L^2_0}^{1/2}\|\nabla c\|_{L^2_0}^{1/2}. \tag{3.12}$$

To estimate II, we write

$$|II| \lesssim \left\|\frac{a^\theta}{r}\right\|_{L^2}\left(\|b^r c\|_{L^2} + \|b^\theta c\|_{L^2}\right),$$

and identical computations to the case I imply that, using again (3.11),

$$|II| \lesssim \|\nabla a\|_{L^2}\|b\|_{L^2}^{1/2}\|\nabla b\|_{L^2}^{1/2}\|c\|_{L^2_0}^{1/2}\|\nabla c\|_{L^2_0}^{1/2}. \tag{3.13}$$

The estimate of III requires some additional computations: let us start by writing

$$\begin{aligned} III \sim \int_{\mathbb{R}^+\times\mathbb{R}} a^z\partial_z b^r c^r \, rdrdz + \int_{\mathbb{R}^+\times\mathbb{R}} a^z\partial_z b^\theta c^\theta \, rdrdz \\ + \int_{\mathbb{R}^+\times\mathbb{R}} a^z\partial_z b^z c^z \, rdrdz = III.1 + III.2 + III.3\,. \end{aligned}$$

Integration by parts yields

$$III.1 \sim -\int_{\mathbb{R}^+\times\mathbb{R}} \partial_z a^z b^r c^r \, rdrdz - \int_{\mathbb{R}^+\times\mathbb{R}} a^z b^r \partial_z c^r \, rdrdz,$$

so as above (in the estimate of I), we get

$$|III.1| \lesssim \|\nabla a\|_{L^2}\|b\|_{L^2}^{1/2}\|\nabla b\|_{L^2}^{1/2}\|c\|_{L^2_0}^{1/2}\|\nabla c\|_{L^2_0}^{1/2} + \Big|\int_{\mathbb{R}^+\times\mathbb{R}} a^z b^r \partial_z c^r \, rdrdz\Big|,$$

and finally

$$|III.1| \lesssim \|\nabla a\|_{L^2}\|b\|_{L^2}^{1/2}\|\nabla b\|_{L^2}^{1/2}\|c\|_{L^2_0}^{1/2}\|\nabla c\|_{L^2_0}^{1/2} + \|\nabla c\|_{L^2_0}\|ra^z b^r\|_{L^2_0}. \tag{3.14}$$

An identical computation for $III.2$ implies that

$$|III.2| \lesssim \|\nabla a\|_{L^2}\|b\|_{L^2}^{1/2}\|\nabla b\|_{L^2}^{1/2}\|c\|_{L^2_0}^{1/2}\|\nabla c\|_{L^2_0}^{1/2} + \|\nabla c\|_{L^2_0}\|ra^z b^\theta\|_{L^2_0}. \tag{3.15}$$

Finally, since b is axisymmetric and divergence free, we have

$$\partial_z b^z = -\partial_r b^r - \frac{b^r}{r},$$

which implies that

$$\begin{aligned} III.3 \quad &\sim \quad -\int_{\mathbb{R}^+\times\mathbb{R}} a^z \partial_r b^r c^z \, r dr dz - \int_{\mathbb{R}^+\times\mathbb{R}} a^z b^r c^z \, dr dz \\ &\sim \quad \int_{\mathbb{R}^+\times\mathbb{R}} \partial_r a^z b^r c^z \, r dr dz + \int_{\mathbb{R}^+\times\mathbb{R}} a^z b^r \partial_r c^z \, r dr dz, \end{aligned}$$

after integration by parts. So finally we have, using the same estimate as for I in the first integral,

$$|III.3| \lesssim \|\nabla a\|_{L^2} \|b\|_{L^2}^{1/2} \|\nabla b\|_{L^2}^{1/2} \|c\|_{L^2_0}^{1/2} \|\nabla c\|_{L^2_0}^{1/2} + \|\nabla c\|_{L^2_0} \|r a^z b^r\|_{L^2_0}. \quad (3.16)$$

Putting together estimates (3.12) to (3.16), we come up with

$$\begin{aligned} |(a \cdot \nabla b | c)_{L^2}| &\lesssim \|\nabla a\|_{L^2} \|b\|_{L^2}^{1/2} \|\nabla b\|_{L^2}^{1/2} \|c\|_{L^2_0}^{1/2} \|\nabla c\|_{L^2_0}^{1/2} \quad (3.17) \\ &+ \|\nabla b\|_{L^2} \|a\|_{L^2}^{1/2} \|\nabla a\|_{L^2}^{1/2} \|c\|_{L^2_0}^{1/2} \|\nabla c\|_{L^2_0}^{1/2} + \|\nabla c\|_{L^2_0} \left(\|r a^z b^\theta\|_{L^2_0} + \|r a^z b^r\|_{L^2_0} \right). \end{aligned}$$

Now the problem is to compute the terms $\|r a^z b^\theta\|_{L^2_0}$ and $\|r a^z b^r\|_{L^2_0}$: if $\frac{a^z}{r}$ could be estimated in $L^2(\mathbb{R}^3)$, then Lemma 2 would let us conclude: similar estimates to above would indeed enable us to write for instance

$$\begin{aligned} \|r a^z b^\theta\|_{L^2_0}^2 &\lesssim \|r^{1/2} a^z\|_{L^2_0} \|\nabla (r^{1/2} b^\theta)\|_{L^2_0} \|\nabla (r^{1/2} a^z)\|_{L^2_0} \|r^{1/2} b^\theta\|_{L^2_0} \quad (3.18) \\ &\lesssim \|a^z\|_{L^2} \|b^\theta\|_{L^2} \Big(\|\nabla a^z\|_{L^2} + \Big\|\frac{a^z}{r}\Big\|_{L^2}\Big) \Big(\|\nabla b^\theta\|_{L^2} + \Big\|\frac{b^\theta}{r}\Big\|_{L^2}\Big). \end{aligned}$$

However, such an estimate of $\frac{a^z}{r}$ in $L^2(\mathbb{R}^3)$ is not available (see L. Tartar [15] for comments on that subject), so we shall need the following lemma, the proof of which is postponed for a while.

Lemma 4. *Let α and β be two axisymmetric scalar functions, in $L^2(\mathbb{R}^3)$, such that $\nabla\alpha$, $\nabla\beta$ and $\frac{\beta}{r}$ are also in $L^2(\mathbb{R}^3)$. Then the function $r\alpha\beta$ is in $L^2_0(\mathbb{R}^3)$, and*

$$\begin{aligned} \|r\alpha\beta\|_{L^2_0(\mathbb{R}^3)}^2 \lesssim \|\alpha\|_{L^2(\mathbb{R}^3)} \Big(\|\beta\|_{L^2(\mathbb{R}^3)} \|\nabla\alpha\|_{L^2(\mathbb{R}^3)} + \|\alpha\|_{L^2(\mathbb{R}^3)} \Big\|\frac{\beta}{r}\Big\|_{L^2(\mathbb{R}^3)} \Big) \\ \times \ \Big(\|\nabla\beta\|_{L^2(\mathbb{R}^3)} + \Big\|\frac{\beta}{r}\Big\|_{L^2(\mathbb{R}^3)} \Big). \end{aligned}$$

Remark. We see by the lemma that contrary to the computation (3.18) above, it is not necessary for both α and β to satisfy $\frac{\alpha}{r} \in L^2(\mathbb{R}^3)$ and $\frac{\beta}{r} \in L^2(\mathbb{R}^3)$ for the result to hold.

With that result, let us finish the proof of Lemma 3. The continuity result follows immediately, applying Lemma 4 with $\alpha = a^z$ and $\beta = b^r$ or b^θ, since we get from (3.17)

$$|(a\cdot\nabla b|c)_{L^2}| \lesssim \|\nabla a\|_{L^2}\|b\|_{L^2}^{1/2}\|\nabla b\|_{L^2}^{1/2}\|c\|_{L^2_0}^{1/2}\|\nabla c\|_{L^2_0}^{1/2}$$
$$+\ \|\nabla b\|_{L^2}\|a\|_{L^2}^{1/2}\|\nabla a\|_{L^2}^{1/2}\|c\|_{L^2_0}^{1/2}\|\nabla c\|_{L^2_0}^{1/2}$$
$$+\ \|\nabla c\|_{L^2_0}\left(\|a\|_{L^2}\|\nabla b\|_{L^2}+\|b\|_{L^2}^{1/2}\|\nabla b\|_{L^2}^{1/2}\|a\|_{L^2}^{1/2}\|\nabla a\|_{L^2}^{1/2}\right).$$

In the case when $a = b$, that yields

$$|(a\cdot\nabla a|c)_{L^2}| \lesssim \|\nabla a\|_{L^2}^{3/2}\|a\|_{L^2}^{1/2}\|c\|_{L^2_0}^{1/2}\|\nabla c\|_{L^2_0}^{1/2}+\|\nabla c\|_{L^2_0}\|a\|_{L^2}\|\nabla a\|_{L^2},$$

so finally

$$|(a\cdot\nabla a|c)_{L^2}| \leq \frac{\nu}{2}\|\nabla a\|_{L^2}^2+\frac{C}{\nu^3}\|a\|_{L^2}^2\|c\|_{L^2_0}^2\|\nabla c\|_{L^2_0}^2++\frac{C}{\nu}\|a\|_{L^2}^2\|\nabla c\|_{L^2_0}^2,$$

and Lemma 3 is proved.

3.4. **Proof of Lemma 4.** We are going to use a dyadic partition of the real axis: let us define a smooth function ψ, supported in $\{r\in\mathbb{R}^+ : a_0<r<b_0\}$, where $0<a_0<b_0$ are such that

$$\sum_{q\in\mathbb{Z}}\psi^2(2^q r)=1\quad \forall r\in\mathbb{R}^+\setminus\{0\},$$

and

$$\forall r\in\mathbb{R}^+,\quad \psi(2^q r)\psi(2^{q'}r)=0\quad\text{if}\quad |q-q'|>2.$$

Then we can write

$$\|r^{1/2}\alpha\beta\|_{L^2(\mathbb{R}^3)}^2\sim\sum_{q\in\mathbb{Z}}\|\psi_q r^{1/2}\alpha\beta\|_{L^2(\mathbb{R}^3)}^2,$$

where we have noted $\psi_q \stackrel{\text{def}}{=} \psi(2^q\cdot)$. Since we have

$$\|\psi_q r^{1/2}\alpha\beta\|_{L^2(\mathbb{R}^3)}\lesssim 2^{-q/2}\|\psi_q\alpha\beta\|_{L^2(\mathbb{R}^3)},$$

it follows that Lemma 4 will be proved if we show that

$$\|\psi_q\alpha\beta\|_{L^2(\mathbb{R}^3)}^2\lesssim 2^q c_q^2\|\alpha\|_{L^2(\mathbb{R}^3)}\Big(\|\beta\|_{L^2(\mathbb{R}^3)}\|\nabla\alpha\|_{L^2(\mathbb{R}^3)}+\|\alpha\|_{L^2(\mathbb{R}^3)}\Big\|\frac{\beta}{r}\Big\|_{L^2(\mathbb{R}^3)}\Big)$$
$$\times\Big(\|\nabla\beta\|_{L^2(\mathbb{R}^3)}+\Big\|\frac{\beta}{r}\Big\|_{L^2(\mathbb{R}^3)}\Big),$$

where (c_q) is a sequence of $\ell^2(\mathbb{Z})$, of norm equal to one.

Writing $\alpha_p \stackrel{\text{def}}{=} \psi_p \alpha$ and $\beta_{p'} \stackrel{\text{def}}{=} \psi_{p'} \beta$, we have, due to the localization of ψ,

$$\|\psi_q \alpha\beta\|^2_{L^2(\mathbb{R}^3)} \lesssim \sum_{\substack{|p-p'|\leq 2\\ |p-q|\leq 2}} \|\psi_q \alpha_p \beta_{p'}\|^2_{L^2(\mathbb{R}^3)},$$

which also reads

$$\|\psi_q \alpha\beta\|^2_{L^2(\mathbb{R}^3)} \lesssim 2^{-q} \sum_{\substack{|p-p'|\leq 2\\ |p-q|\leq 2}} \|\psi_q \alpha_p \beta_{p'}\|^2_{L^2_0(\mathbb{R}^3)}. \tag{3.19}$$

For reasons that will become clearer later on, let us define a new function $\widetilde{\psi}$, smooth, radial and compactly supported, say in $[\frac{a_0}{2}, 2b_0]$, such that $\widetilde{\psi}\psi = \psi$. Then one can replace above the function $\beta_{p'}$ by $\widetilde{\psi}_{p'}\beta_{p'}$, and similarly α_p by $\widetilde{\psi}_p \alpha_p$, and we are led to proving that

$$\sum_{\substack{|p-p'|\leq 2\\ |p-q|\leq 2}} \|\psi_q \widetilde{\alpha}_p \widetilde{\beta}_{p'}\|^2_{L^2_0(\mathbb{R}^3)} \lesssim c_q^2 2^{2q} \|\alpha\|_{L^2(\mathbb{R}^3)} \Big(\|\nabla\beta\|_{L^2(\mathbb{R}^3)} + \Big\|\frac{\beta}{r}\Big\|_{L^2(\mathbb{R}^3)}\Big)$$
$$\times \Big(\|\beta\|_{L^2(\mathbb{R}^3)} \|\nabla\alpha\|_{L^2(\mathbb{R}^3)} + \|\alpha\|_{L^2(\mathbb{R}^3)} \Big\|\frac{\beta}{r}\Big\|_{L^2(\mathbb{R}^3)}\Big), \tag{3.20}$$

where we have noted $\widetilde{\beta}_{p'} \stackrel{\text{def}}{=} \widetilde{\psi}_{p'}\beta_{p'}$ and $\widetilde{\alpha}_p \stackrel{\text{def}}{=} \widetilde{\psi}_p \alpha_p$.

Since $\|\psi_q \widetilde{\alpha}_p \widetilde{\beta}_{p'}\|_{L^2_0(\mathbb{R}^3)} \lesssim \|\widetilde{\alpha}_p \widetilde{\beta}_{p'}\|_{L^2_0(\mathbb{R}^3)}$, Lemma 2 implies that

$$\|\psi_q \widetilde{\alpha}_p \widetilde{\beta}_{p'}\|^2_{L^2_0(\mathbb{R}^3)} \lesssim \|\widetilde{\alpha}_p\|_{L^2_0(\mathbb{R}^3)} \|\nabla\widetilde{\beta}_{p'}\|_{L^2_0(\mathbb{R}^3)} \|\widetilde{\beta}_{p'}\|_{L^2_0(\mathbb{R}^3)} \|\nabla\widetilde{\alpha}_p\|_{L^2_0(\mathbb{R}^3)}.$$

We also have $\|\widetilde{\alpha}_p\|_{L^2_0(\mathbb{R}^3)} \lesssim \|\alpha_p\|_{L^2_0(\mathbb{R}^3)}$ and $\|\widetilde{\beta}_{p'}\|_{L^2_0(\mathbb{R}^3)} \lesssim \|\beta_{p'}\|_{L^2_0(\mathbb{R}^3)}$. Moreover, one has

$$\|\nabla\widetilde{\beta}_{p'}\|_{L^2_0} \lesssim \|\psi_{p'}\widetilde{\psi}_{p'}\nabla\beta\|_{L^2_0} + 2^{p'}\|\widetilde{\psi}_{p'}\psi'_{p'}\beta\|_{L^2_0} + 2^{p'}\|\widetilde{\psi}'_{p'}\psi_{p'}\beta\|_{L^2_0},$$

where $\psi'_p(r) \stackrel{\text{def}}{=} \psi'(2^p r)$, $\widetilde{\psi}'_p(r) \stackrel{\text{def}}{=} \widetilde{\psi}'(2^p r)$, and an identical result holds for α, so we can write

$$\|\psi_q \widetilde{\alpha}_p \widetilde{\beta}_{p'}\|^2_{L^2_0(\mathbb{R}^3)} \lesssim \Big(\|\psi_{p'}\widetilde{\psi}_{p'}\nabla\beta\|_{L^2_0(\mathbb{R}^3)} + 2^{p'}\|\widetilde{\psi}_{p'}\psi'_{p'}\beta\|_{L^2_0(\mathbb{R}^3)} + 2^{p'}\|\widetilde{\psi}'_{p'}\psi_{p'}\beta\|_{L^2_0(\mathbb{R}^3)}\Big)$$
$$\times\Big(\|\widetilde{\psi}_p\psi_p\nabla\alpha\|_{L^2_0(\mathbb{R}^3)} + 2^p\|\widetilde{\psi}_p\psi'_p\alpha\|_{L^2_0(\mathbb{R}^3)} + 2^p\|\widetilde{\psi}'_p\psi_p\alpha\|_{L^2_0(\mathbb{R}^3)}\Big) \|\alpha_p\|_{L^2_0(\mathbb{R}^3)} \|\beta_{p'}\|_{L^2_0(\mathbb{R}^3)},$$

Now using the fact that ψ' and $\widetilde{\psi}'$ are localized in the same compact set as $\widetilde{\psi}$, and since $\widetilde{\psi}'$ and ψ' are bounded, we get

$$\begin{aligned}
&\|\psi_q\widetilde{\alpha}_p\widetilde{\beta}_{p'}\|^2_{L^2_0(\mathbb{R}^3)}\\
&\lesssim 2^{p+p'}\|\alpha_p\|_{L^2(\mathbb{R}^3)}\Big(\|\psi_{p'}\nabla\beta\|_{L^2(\mathbb{R}^3)} + 2^{p'}\|\widetilde{\psi}_{p'}\beta\|_{L^2(\mathbb{R}^3)} + 2^{p'}\|\psi_{p'}\beta\|_{L^2(\mathbb{R}^3)}\Big)\\
&\times\|\beta_{p'}\|_{L^2(\mathbb{R}^3)}\Big(\|\psi_p\nabla\alpha\|_{L^2(\mathbb{R}^3)} + 2^p\|\widetilde{\psi}_p\alpha\|_{L^2(\mathbb{R}^3)} + 2^p\|\psi_p\alpha\|_{L^2(\mathbb{R}^3)}\Big).
\end{aligned}$$

Then all we need to notice is that

$$2^{p'}\|\widetilde{\psi}_{p'}\beta\|_{L^2(\mathbb{R}^3)} \lesssim \Big\|\widetilde{\psi}_{p'}\frac{\beta}{r}\Big\|_{L^2(\mathbb{R}^3)} \quad\text{and}\quad 2^{p'}\|\psi_{p'}\beta\|_{L^2(\mathbb{R}^3)} \lesssim \Big\|\psi_{p'}\frac{\beta}{r}\Big\|_{L^2(\mathbb{R}^3)},$$

and since $|p-p'|\le 2$, we also have

$$2^p\|\beta_{p'}\|_{L^2(\mathbb{R}^3)} \lesssim \Big\|\psi_{p'}\frac{\beta}{r}\Big\|_{L^2(\mathbb{R}^3)}.$$

so it follows that

$$\begin{aligned}
&\|\psi_q\widetilde{\alpha}_p\widetilde{\beta}_{p'}\|^2_{L^2_0(\mathbb{R}^3)}\\
&\lesssim 2^{p+p'}\|\alpha_p\|_{L^2(\mathbb{R}^3)}\Big(\|\psi_{p'}\nabla\beta\|_{L^2(\mathbb{R}^3)} + \Big\|\psi_{p'}\frac{\beta}{r}\Big\|_{L^2(\mathbb{R}^3)} + \Big\|\widetilde{\psi}_{p'}\frac{\beta}{r}\Big\|_{L^2(\mathbb{R}^3)}\Big)\\
&\times\Big(\|\beta_{p'}\|_{L^2(\mathbb{R}^3)}\|\psi_p\nabla\alpha\|_{L^2(\mathbb{R}^3)} + (\|\widetilde{\psi}_p\alpha\|_{L^2(\mathbb{R}^3)} + \|\psi_p\alpha\|_{L^2(\mathbb{R}^3)})\Big\|\psi_{p'}\frac{\beta}{r}\Big\|_{L^2(\mathbb{R}^3)}\Big).
\end{aligned}$$

Finally we get, using the fact that $2^{p+p'}\sim 2^{2q}$,

$$\begin{aligned}
\sum_{\substack{|p-p'|\le 2\\ |p-q|\le 2}}\|\psi_q\alpha_p\beta_{p'}\|^2_{L^2} &\lesssim 2^{2q}\sum_{\substack{|p-p'|\le 2\\ |p-q|\le 2}}\|\alpha_p\|_{L^2}\Big(\|\psi_{p'}\nabla\beta\|_{L^2}+\Big\|\psi_{p'}\frac{\beta}{r}\Big\|_{L^2}+\Big\|\widetilde{\psi}_{p'}\frac{\beta}{r}\Big\|_{L^2}\Big)\\
&\times\Big(\|\beta_{p'}\|_{L^2}\|\psi_p\nabla\alpha\|_{L^2} + (\|\psi_p\alpha\|_{L^2} + \|\widetilde{\psi}_p\alpha\|_{L^2})\Big\|\psi_{p'}\frac{\beta}{r}\Big\|_{L^2}\Big).
\end{aligned}$$

Finally to obtain (3.20), it is just a matter of using the fact that

$$\|\alpha_p\|_{L^2}\lesssim c_p\|\alpha\|_{L^2},\quad \|\beta_{p'}\|_{L^2}\lesssim c'_{p'}\|\beta\|_{L^2}\quad\text{and}\quad \Big\|\psi_{p'}\frac{\beta}{r}\Big\|_{L^2}\lesssim c''_{p'}\Big\|\frac{\beta}{r}\Big\|_{L^2},$$

where c_p, $c'_{p'}$ and $c''_{p'}$ are sequences of $\ell^2(\mathbb{Z})$ of norm 1, and similarly with ψ replaced by $\widetilde{\psi}$. Lemma 4 is proved.

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