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THE CAUCHY PROBLEM FOR A FIFTH ORDER EVOLUTION EQUATION

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Abstract. In this paper it is shown that the Cauchy problem for a fifth order modification of the Camassa-Holm equation is locally well-posed for initial data of arbitrary size in the Sobolev space $H^s(\mathbb{R})$, $s > 1/4$, and globally well-posed in $H^1(\mathbb{R})$. The proof is based on appropriate bilinear estimates obtained using Fourier analysis techniques.

1. Introduction and results

We consider the following initial value problem

$$\partial_t u - \partial_x^2 \partial_t u + \partial_x^3 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u - \partial_x^5 u = 0, \tag{1.1}$$

$$u(x,0) = \varphi(x), \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}. \tag{1.2}$$

The nonlinear evolution equation (1.1) is a fifth order modification of the equation

$$\partial_t u - \partial_x^2 \partial_t u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u = 0. \tag{1.3}$$

Equation (1.3) was introduced independently by Camassa and Holm [3] and by Fuchssteiner and Fokas [6]. It has since been the subject of extensive studies from algebraic, analytic and geometric points of view (see, e.g., Constantin and Escher [4], Constantin and McKean [5], Holm, Marsden and Ratiu [10], Holm, Kouranbaeva, Marsden, Ratiu and Shkoller [11], McKean [16], Misiołek [17]). In certain respects the CH equation resembles the well-known KdV equation. Like KdV it is bi-hamiltonian, admits soliton-type solutions and an infinite collection of first integrals. Its analytic properties, however, are quite different. This can be seen if one writes the CH equation in a nonlocal form similar to (2.1). Then, there is no third derivative present and therefore there is no smoothing effect similar to the one in KdV. Thus, the KdV techniques developed by Bourgain [1], [2], Kenig, Ponce and Vega,

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[12], [13], [14], [15], and many other authors since the early 90's, are not applicable.

In this work we consider the modified CH equation (1.1). The periodic initial value problem for this equation was investigated by Himonas and Misiołek in [7] and [8]. In [9] they considered the non-periodic case and proved that it is locally well-posed in H^s for $s > 1/2$, and globally well-posed in H^1, if the initial data are sufficiently small. Here, motivated by the work of Kenig, Ponce and Vega [15], we have been able to extend the results in [9] by lowering s to $1/4$ and removing the restriction on the smallness of the initial data, in both the local and the global cases. It has not yet been shown whether the problem is locally well posed for any $s \leq 1/4$.

Following [14], for $s \geq 0$ and $\alpha \in (1/2, 1)$, we define $X^{s,\alpha}$ to be the space of all L^2 functions $u : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ with finite norm

$$|||u|||_{s,\alpha}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} (1+n^2)^s (1 + |\lambda - n^3|)^{2\alpha} |\widehat{u}(n,\lambda)|^2 dnd\lambda.$$

Next, using these spaces we state our results.

Theorem 1.1. *If $s > 1/4$ and $\varphi \in H^s(\mathbb{R})$, then the initial value problem* (1.1)–(1.2) *has a unique local solution in $X^{s,\alpha}$ for appropriate choice of α.*

Theorem 1.2. *If $\varphi \in H^1$, then the initial value problem* (1.1)–(1.2) *has a unique global solution in the space $X^{1,\alpha}$.*

The proofs follow the ideas and techniques developed the last decade in the works of Bourgain (see [1], [2]), Kenig, Ponce and Vega (see [12], [13], [14], [15]), and the references therein.

In Section 2, using Duhamel's formula we convert our IVP to an integral equation. Then we state the basic estimate, which makes the RHS of the integral equation a contraction mapping on a banach space. In Section 3, we state and prove the bilinear estimates needed for the derivation of basic estimate (2.8). Finally, in the appendix we prove a key lemma which is useful in dealing with large data.

2. Proofs of Theorems

First we write equation (1.1) in a convenient form. If we factor out the "Laplacian" $1 - \partial_x^2$ and multiply it by its inverse $(1 - \partial_x^2)^{-1}$, then the initial value problem (1.1)-(1.2) takes the form

$$\begin{cases} \partial_t u + \partial_x^3 u + \frac{1}{2}\partial_x \left(u^2\right) + \left(1 - \partial_x^2\right)^{-1} \partial_x \left[u^2 + \frac{1}{2}\left(\partial_x u\right)^2\right] = 0 \\ u(x,0) = \varphi(x), \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Now we observe that our equation is the sum of two parts: a differential part, which consists of the KdV terms, and a pseudo-differential part, which contains stronger nonlinearities and is nonlocal.

Defining

$$w = \frac{1}{2}\partial_x\left(u^2\right) + \left(1-\partial_x^2\right)^{-1}\partial_x\Big[u^2 + \frac{1}{2}(\partial_x u)^2\Big], \tag{2.2}$$

and solving (2.1) we obtain the integral equation

$$u(x,t) = W(t)\varphi(x) - \int_0^t W(t-\tau)w(x,\tau)d\tau,$$

where we define $W(t) = \exp\left(-t\partial_x^3\right) = \sum_{k=0}^{\infty}\frac{1}{k!}(-t\partial_x^3)^k$.

Then writing $W(t)$ in terms of Fourier transforms, our initial value problem takes the form

$$\begin{aligned} u(x,t) = &\int_{\mathbb{R}} \widehat{\varphi}(n)e^{i(nx+n^3t)}dn \\ &+ i\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\lambda-n^3}\ \Big[e^{i(nx+\lambda t)} - e^{i(nx+n^3t)}\Big]\,\widehat{w}(n,\lambda)dnd\lambda. \end{aligned} \tag{2.3}$$

Now we only expect to have a local solution, that is to say, a solution in a neighborhood of $t=0$. Therefore, we pick a cut-off function $\psi(t)\in C_0^\infty(-1,1)$ such that $0\le\psi\le 1$ and $\psi(t)\equiv 1$ for $|t|<1/2$, and multiply both sides of (2.3) by $\psi(t)$.

Also, since the double integral in (2.3) has a singularity at $\lambda = n^3$, we multiply the integrand by $1-\psi(\lambda-n^3)+\psi(\lambda-n^3)$. Thus, we rewrite (2.3) as

$$\psi(t)u(x,t) = \psi(t)\int_{\mathbb{R}}\widehat{\varphi}(n)e^{i(nx+n^3t)}dn \tag{2.4}$$

$$+\, i\sum_{k=1}^{\infty}\frac{i^k}{k!}t^k\psi(t)\left\{\int_{\mathbb{R}}\int_{\mathbb{R}}\psi(\lambda-n^3)(n^3-\lambda)^{k-1}\widehat{w}(n,\lambda)\ e^{i(nx+\lambda t)}dnd\lambda\right\} \tag{2.5}$$

$$+\, i\psi(t)\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(1-\psi)(\lambda-n^3)}{\lambda-n^3}\ e^{i(nx+\lambda t)}\widehat{w}(n,\lambda)dnd\lambda \tag{2.6}$$

$$-\, i\psi(t)\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(1-\psi)(\lambda-n^3)}{\lambda-n^3}\ e^{i(nx+n^3t)}\widehat{w}(n,\lambda)dnd\lambda. \tag{2.7}$$

Now for any $\delta\in(0,1)$, define $X_\delta^{s,\alpha}$ to be all elements of $X^{s,\alpha}$ which in the t variable are supported in $[-\delta,\delta]$. We then have the following result:

Theorem 2.1. *Let $s \in (1/4, 1/2)$ and $\alpha \in (1/2, 1/2+\beta)$ where*

$$\beta = \frac{1}{3}\min\left\{\frac{s}{3} - \frac{1}{12}, \frac{1}{2} - s\right\}$$

and let T be the map defined by the RHS of (2.4)–(2.7). *Then T is a map from $X^{s,\alpha}_\delta$ to $X^{s,\alpha}$, and in particular there is a $c = c(s,\alpha,\psi) > 0$ such that*

$$|||Tu|||_{s,\alpha} \le c\left(\delta^{2\beta}|||u|||^2_{s,\alpha} + \|\varphi\|_{H^s}\right), \tag{2.8}$$

for all $u \in X^{s,\alpha}_\delta$.

Proof. For the proof we shall need the following lemma, whose proof is in the appendix. ([14] contains a similar lemma.)

Lemma 2.2. *Let $u \in X^{s,\alpha}$. Then for all $\theta = \theta(t) \in H^\alpha(\mathbb{R})$*

$$|||\theta u|||_{s,\alpha} \lesssim ||\theta||_{H^\alpha}|||u|||_{s,\alpha}. \tag{2.9}$$

Note: Throughout this paper we will use the notation $f \lesssim g$ to mean that for some nonzero constant c we have $f \le cg$. Similarly, we define $f \simeq g$ to mean that for some nonzero constant c, $f = cg$, and $f \sim g$ to mean that $cf \le g \le c^{-1}f$.

Also, we shall need the following important proposition.

Proposition 2.3. *Let s, β, and α be as in Theorem* 2.1. *Then for any $\delta \in (0,1)$ and $f, g \in X^{s,\alpha}_\delta$,*

$$\Big[\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(1+n^2)^s|\widehat{w}_{fg}(n,\lambda)|^2}{(1+|\lambda-n^3|)^{2(1-\alpha)}}\,dnd\lambda\Big]^{\frac{1}{2}} \lesssim \delta^{2\beta}|||f|||_{s,\alpha}\cdot|||g|||_{s,\alpha} \tag{2.10}$$

$$\Big[\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} \frac{(1+n^2)^{\frac{s}{2}}|\widehat{w}_{fg}(n,\lambda)|}{1+|\lambda-n^3|}\,d\lambda\Big)^2 dn\Big]^{\frac{1}{2}} \lesssim \delta^{2\beta}|||f|||_{s,\alpha}\cdot|||g|||_{s,\alpha}, \tag{2.11}$$

where $\widehat{w}_{fg}$ is defined by

$$\widehat{w}_{fg}(n,\lambda) \simeq \Big(n + \frac{2n}{1+n^2}\Big)\widehat{f} * \widehat{g}(n,\lambda) + \frac{n}{1+n^2}\widehat{\partial_x f} * \widehat{\partial_x g}(n,\lambda). \tag{2.12}$$

Note that this gives the same w as in (2.2) if $f = g = u$.

In the next section we will prove Proposition 2.3 and show that it fails for $s < 1/4$. Now we shall use it to prove Theorem 2.1.

Estimate for (2.4). The linear term $\psi(t)\int_{\mathbb{R}}\widehat{\varphi}(n)e^{i(nx+n^3t)}dn$ has Fourier transform

$$\widehat{\varphi}(n)\int \psi(t)e^{it(n^3-\lambda)}dt = \widehat{\varphi}(n)\widehat{\psi}(\lambda - n^3).$$

Therefore,

$$|||(2.4)|||_{s,\alpha}^2 = \int\int(1+n^2)^s(1+|\lambda-n^3|)^{2\alpha}|\widehat{\varphi}(n)\widehat{\psi}(\lambda-n^3)|^2 dnd\lambda$$
$$= \int(1+n^2)^s|\widehat{\varphi}(n)|^2\int\left(1+|\lambda-n^3|\right)^{2\alpha}|\widehat{\psi}(\lambda-n^3)|^2 d\lambda dn = C_\psi||\varphi||_{H^s}^2$$

so that

$$|||(2.4)|||_{s,\alpha} \simeq ||\varphi||_{H^s}.$$

Estimate for (2.5). Using Lemma 2.2, we have

$$|||(2.5)|||_{s,\alpha} \lesssim \sum_{k=1}^{\infty}\frac{1}{k!}\left\|t^k\psi\right\|_{H^\alpha}$$
$$\times\sup_{k\geq 1}\left|\left|\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\psi(\lambda-n^3)(\lambda-n^3)^{k-1}\widehat{w}(n,\lambda)e^{i(nx+\lambda t)}\ dnd\lambda\right|\right|\right|_s.$$

Now, since the support of ψ is in $(-1,1)$,

$$\left\|t^k\psi\right\|_{H^\alpha} \leq \left\|t^k\psi\right\|_{L^2} + \left\|\partial_t(t^k\psi)\right\|_{L^2} \lesssim k,$$

and hence $\sum_{k=1}^{\infty}\frac{1}{k!}\left\|t^k\psi\right\|_{H^\alpha} \lesssim \sum\frac{k}{k!} < \infty$.

And since the fourier transform of the double integral is $\psi(\lambda-n^3)(\lambda-n^3)^{k-1}\widehat{w}(n,\lambda)$, we have

$$|||(2.5)|||_{s,\alpha} \lesssim \sup_{k\geq 1}\Big\{\int_{\mathbb{R}}\int_{\mathbb{R}}(1+n^2)^s(1+|\lambda-n^3|)^{2\alpha}$$
$$\times\left|\psi(\lambda-n^3)(\lambda-n^3)^{k-1}\widehat{w}(n,\lambda)\right|^2 d\lambda dn\Big\}^{\frac{1}{2}}$$
$$\lesssim \sup_{k\geq 1}\Big\{\int_{\mathbb{R}}\int_{|\lambda-n^3|\leq 1}(1+n^2)^s(1+|\lambda-n^3|)^{2\alpha}\left|(\lambda-n^3)^{k-1}\widehat{w}(n,\lambda)\right|^2 d\lambda dn\Big\}^{\frac{1}{2}}$$
$$\lesssim \sup_{k\geq 1}\Big\{\int_{\mathbb{R}}\int_{|\lambda-n^3|\leq 1}(1+n^2)^s|\widehat{w}(n,\lambda)|^2 d\lambda dn\Big\}^{\frac{1}{2}}$$
$$\lesssim \Big\{\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(1+n^2)^s|\widehat{w}(n,\lambda)|^2}{(1+|\lambda-n^3|)^2}d\lambda dn\Big\}^{\frac{1}{2}}.$$

Applying (2.10) to the last quantity gives

$$|||(2.5)|||_{s,\alpha} \lesssim \delta^{2\beta}|||u|||_{s,\alpha}^2.$$

Estimate for (2.6). Lemma 2.2 gives

$$|||(2.6)|||_{s,\alpha}^2 \lesssim \left|\left|\left|\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(1-\psi)(\lambda-n^3)}{\lambda-n^3}\widehat{w}(n,\lambda)e^{i(nx+\lambda t)}dnd\lambda\right|\right|\right|_s^2$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}}(1+n^2)^s(1+|\lambda-n^3|)^{2\alpha}\left|\frac{(1-\psi)(\lambda-n^3)}{\lambda-n^3}\widehat{w}(n,\lambda)\right|^2 d\lambda dn.$$

Using the fact that $1-\psi$ is supported in $|t|>1/2$, we have

$$|||(2.6)|||_{s,\alpha}^2 \lesssim \int\int_{|\lambda-n^3|\geq\frac{1}{2}}(1+n^2)^s(1+|\lambda-n^3|)^{2\alpha}\frac{|\widehat{w}(n,\lambda)|^2}{|\lambda-n^3|^2}d\lambda dn$$
$$\lesssim \int\int\frac{(1+n^2)^s|\widehat{w}(n,\lambda)|^2}{(1+|\lambda-n^3|)^{2(1-\alpha)}}d\lambda dn.$$

This together with Proposition 2.3 gives

$$|||(2.6)|||_{s,\alpha} \lesssim \delta^{2\beta}|||u|||_{s,\alpha}^2.$$

Estimate for (2.7). We proceed as we did for the linear term. Thus,

$$(2.7) = -i\psi(t)\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(1-\psi)(\lambda-n^3)}{\lambda-n^3}\ e^{i(nx+n^3t)}\widehat{w}(n,\lambda)dnd\lambda \tag{2.13}$$

has Fourier transform:

$$\widehat{(2.7)}(n,\tau) = -i\int_{\mathbb{R}}\psi(t)e^{i(n^3t-\tau t)}dt\int_{\mathbb{R}}\frac{(1-\psi)(\lambda-n^3)}{\lambda-n^3}\widehat{w}(n,\lambda)d\lambda$$
$$= -i\widehat{\psi}(\tau-n^3)\int_{\mathbb{R}}\frac{(1-\psi)(\lambda-n^3)}{\lambda-n^3}\widehat{w}(n,\lambda)d\lambda.$$

$$|||(2.7)|||_{s,\alpha}^2 = \int_{\mathbb{R}}\int_{\mathbb{R}}(1+n^2)^s(1+|\tau-n^3|)^{2\alpha}$$
$$\times\left|\widehat{\psi}(\tau-n^3)\int_{\mathbb{R}}\frac{(1-\psi)(\lambda-n^3)\widehat{w}(n,\lambda)}{\lambda-n^3}d\lambda\right|^2 d\tau dn$$
$$= \int_{\mathbb{R}}(1+|\tau|)^{2\alpha}|\widehat{\psi}(\tau)|^2d\tau\int_{\mathbb{R}}(1+n^2)^s\left|\int_{\mathbb{R}}\frac{(1-\psi)(\lambda-n^3)\widehat{w}(n,\lambda)}{\lambda-n^3}d\lambda\right|^2dn$$
$$\lesssim \int_{\mathbb{R}}(1+n^2)^s\Big[\int_{|\lambda-n^3|>\frac{1}{2}}\frac{|\widehat{w}(n,\lambda)|}{|\lambda-n^3|}d\lambda\Big]^2dn \lesssim \int_{\mathbb{R}}\Big[\int_{\mathbb{R}}\frac{(1+n^2)^{\frac{s}{2}}|\widehat{w}(n,\lambda)|}{1+|\lambda-n^3|}d\lambda\Big]^2dn.$$

Therefore, Proposition 2.3 gives

$$|||(2.7)|||_{s,\alpha} \lesssim \delta^{2\beta}|||u|||_{s,\alpha}^2.$$

This completes the proof of Theorem 2.1. □

So we have shown that T is a map from $X_\delta^{s,\alpha}$ to $X^{s,\alpha}$. But since Tu will not, in general, have t support in $[-\delta,\delta]$, T is not a map from $X_\delta^{s,\alpha}$ to itself. Therefore, we let $\psi_\delta(t)=\psi(t/\delta)$ and define a new map T_δ by

$$T_\delta u = \psi_\delta T u.$$

Since ψ_δ has support in $[-\delta,\delta]$, Theorem 2.1 and Lemma 2.2 tell us that T_δ is a map from $X_\delta^{s,\alpha}$ to itself with

$$|||T_\delta u|||_{s,\alpha} \lesssim ||\psi_\delta||_{H^\alpha}|||Tu|||_{s,\alpha} \lesssim ||\psi_\delta||_{H^\alpha}\left(\delta^{2\beta}|||u|||_{s,\alpha}^2 + \|\varphi\|_{H^s}\right).$$

Furthermore,

$$\begin{aligned}||\psi_\delta||_{H^\alpha}^2 &= \int (1+\lambda^2)^\alpha|\delta\widehat{\psi}(\delta\lambda)|^2 d\lambda = \delta^{1-2\alpha}\int(\delta^2+\tau^2)^\alpha|\widehat{\psi}(\tau)|^2 d\tau\\ &\le \delta^{1-2\alpha}||\psi||_{H^\alpha}^2\end{aligned}$$

so we have proven that, for some constant c,

$$|||T_\delta u|||_{s,\alpha} \quad\le\quad c\delta^{\frac12-\alpha}\left(\delta^{2\beta}|||u|||_{s,\alpha}^2+\|\varphi\|_{H^s}\right). \tag{2.14}$$

Next we show that T_δ is a contraction.

Theorem 2.4. *Let s, β, and α be as in Theorem* 2.1. *Then for all* $0<\delta<1$

$$|||T_\delta u - T_\delta v|||_{s,\alpha} \le c\delta^\epsilon\left(|||u|||_{s,\alpha}+|||v|||_{s,\alpha}\right)|||u-v|||_{s,\alpha}, \tag{2.15}$$

for all $u,v\in X_\delta^{s,\alpha}$ and where $\epsilon = 2\beta+\frac12-\alpha$ and c is the same constant as in (2.14). *Moreover, if we also take*

$$\delta \le (16c^2||\varphi||_{H^s})^{1/(2\alpha-1-2\beta)}, \qquad r=\frac{\delta^{-\epsilon}}{4c},$$

then T_δ is a contraction on the closed ball $B(0,r)=\{u\in X_\delta^{s,\alpha} : |||u|||_{s,\alpha}<r\}$.

Proof. Estimate (2.15) follows from the proof of Theorem 2.1 applied to

$$\begin{aligned}T_\delta u - T_\delta v = \psi_\delta(Tu-Tv) &= \psi_\delta(t)\int_0^t W(t-\tau)\left(w_u(x,\tau)-w_v(x,\tau)\right)d\tau\\ &= \psi_\delta(t)\int_0^t W(t-\tau)w_{fg}(x,\tau)d\tau,\end{aligned}$$

where

$$\begin{aligned}\widehat{w_u - w_v} &= \left(n+\frac{2n}{1+n^2}\right)[\widehat{u}*\widehat{u}-\widehat{v}*\widehat{v}] + \frac{n}{1+n^2}[\widehat{\partial_x u}*\widehat{\partial_x u}-\widehat{\partial_x v}*\widehat{\partial_x v}]\\ &= \widehat{w}_{fg}, \quad \text{with} \quad f=u+v, \text{ and } g=u-v.\end{aligned}$$

Next, assume that $u \in B(0,r)$. Then by (2.14), we have

$$|||T_\delta u|||_{s,\alpha} \leq c\delta^\epsilon |||u|||_{s,\alpha}^2 + c\delta^{\frac{1}{2}-\alpha}||\varphi||_{H^s}.$$

Then since

$$c\delta^\epsilon |||u|||_{s,\alpha}^2 \leq c\delta^\epsilon r^2 = c(4rc)^{-1}r^2 = \frac{r}{4}$$

and

$$\begin{aligned} c\delta^{\frac{1}{2}-\alpha}||\varphi||_{H^s} &= c\delta^{-\epsilon}\delta^{2\beta+1-2\alpha}||\varphi||_{H^s} = 4c^2 r\delta^{2\beta+1-2\alpha}||\varphi||_{H^s} \\ &\leq 4c^2 r(16c^2||\varphi||_{H^s})^{-1}||\varphi||_{H^s} = \frac{r}{4}, \end{aligned}$$

where we used $2\beta + 1 - 2\alpha > 0$. Thus, we have $|||T_\delta u|||_{s,\alpha} \leq \frac{r}{2}$ which shows that T_δ maps $B(0,r)$ into $B(0,r)$. Next, assume that $u, v \in B(0,r)$. Then using (2.15), we have

$$\begin{aligned} |||T_\delta u - T_\delta v|||_{s,\alpha} &\leq c\delta^\epsilon(|||u|||_{s,\alpha} + |||v|||_{s,\alpha})|||u-v|||_{s,\alpha} \\ &\leq 2cr\delta^\epsilon|||u-v|||_{s,\alpha} = \frac{1}{2}|||u-v|||_{s,\alpha}. \end{aligned}$$

Thus, T_δ is a contraction. This completes the proof of the Theorem 2.4. □

Theorem 2.4 and Banach's contraction principle imply that there is a unique $u \in B(0,r)$ which locally solves the IVP (1.1). To complete the proof of Theorem 1.1 we will show that u is unique in $X^{s,\alpha}$. Specifically, we will show that there is $\delta > 0$ (depending on $||\varphi||_{H^s}$) such that if $v \in X^{s,\alpha}$ is a local solution to the IVP, then $u = v$ on $[0, \delta]$.

First notice that in the proof of 2.4, r increases as δ decreases, and hence by choosing δ sufficiently small, let's say δ_0, we will have $v \in B(0,r)$. Thus by the contraction mapping principle u and v are equal in the neighborhood $[0, \delta_0]$.

More generally, let t_0 be such that $u(-, t_0) = v(-, t_0)$. Then taking $\varphi = u(-, t_0)$ as initial data, Theorem 2.4 yields a solution in the interval $[t_0, t_0 + \delta_0]$, and in particular we must have $u = v$ on this interval as well. (Here the choice of δ_0 depends on $||v(-, t_0)||_{H^s}$ as well as $|||v|||_{s,\alpha}$. However, this is no real restriction, since $||v(-, t_0)||_{H^s} \lesssim |||v|||_{s,\alpha}$.)

This completes the proof, since we can use an appropriate (finite) number of extensions to see that u and v must be equal on the entire interval $[0, \delta]$. □

For Theorem 1.2, we multiply the PDE (1.1) by u, giving

$$0 = u\partial_t u - u\partial_x^2\partial_t u + u\partial_x^3 u + 3u^2\partial_x u - 2u\partial_x u\partial_x^2 u - u^2\partial_x^3 u - u\partial_x^5 u$$

$$= u\partial_t u + \partial_x u \partial_x \partial_t u + \partial_x \Big(- u\partial_x\partial_t u + u\partial_x^2 u - \frac{1}{2}\partial_x u \partial_x u +$$
$$+ u^3 - u^2\partial_x^2 u - u\partial_x^4 u + \partial_x u \partial_x^3 u - \frac{1}{2}\partial_x^2 u \partial_x^2 u\Big).$$

Integrating in the x variable, and using the fact that $u(x,t)$ and all its derivatives go to zero as x goes to infinity, gives

$$0 = \int_{\mathbb{R}} u\partial_t u + \partial_x u \partial_x \partial_t u = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2 + u_x^2.$$

Since the existence time in the local result depends only on the H^s norm of the data, Theorem 1.2 follows from the last equation. □

3. Proof of Proposition 2.3

In the estimates below we shall need the following lemma.

Lemma 3.1. *Let $s \geq 0$ and $h \in X_\delta^{s,\alpha}$ and let $0 < \eta < \alpha - \alpha' < \alpha$. Then*

$$|||h|||_{s,\alpha'} \lesssim \delta^\eta |||h|||_{s,\alpha}$$

Proof. Define $\widehat{H}(n,\lambda) = (1+|n|^2)^{\frac{s}{2}}\widehat{h}(n,\lambda)$. Also, let $\mu = \alpha - \alpha'$, $p = \frac{\alpha}{\alpha-\alpha'} = \frac{\alpha}{\mu}$, $p' = \frac{\alpha}{\alpha'}$. Then we have

$$\begin{aligned} |||h|||_{s,\alpha'} &= ||(1+|\lambda-n^3|)^{\alpha'}\widehat{H}||_{L^2_nL^2_\lambda} \\ &\leq ||(1+|\lambda-n^3|)^{\alpha'}\widehat{H}^{\alpha'/\alpha}||_{L^{2p'}_nL^{2p'}_\lambda}||\widehat{H}^{\mu/\alpha}||_{L^{2p}_nL^{2p}_\lambda} \\ &= ||(1+|\lambda-n^3|)^{\alpha'p'}\widehat{H}^{\alpha'p'/\alpha}||^{1/p'}_{L^2_nL^2_\lambda}||\widehat{H}^{\mu p/\alpha}||^{1/p}_{L^2_nL^2_\lambda} \\ &= ||(1+|\lambda-n^3|)^{\alpha}\widehat{H}||^{1/p'}_{L^2_nL^2_\lambda}||\widehat{H}||^{1/p}_{L^2_nL^2_\lambda}. \end{aligned} \tag{3.1}$$

Let $r = \frac{\mu}{\alpha\eta}$ and $\frac{1}{r} + \frac{1}{q} = \frac{1}{2}$, $\frac{1}{q'} = 1 - \frac{1}{q} = \frac{1}{2} + \frac{1}{r}$. Note that since h has t-support in $[-\delta,\delta]$, H does as well. (To see this, consider the partial Fourier transform $\widehat{H}(n,t)$.) Now apply Holder's inequality and the Haussdorf-Young inequality (with $q' < 2 < q$):

$$\begin{aligned} &||\widehat{H}||_{L^2_nL^2_\lambda} \simeq ||H||_{L^2_tL^2_x} = ||\chi_{[-\delta,\delta]}H||_{L^2_tL^2_x} \leq ||\chi_{[-\delta,\delta]}||_{L^r_t}||H||_{L^q_tL^2_x} \\ &\lesssim \delta^{1/r}||\widehat{H}||_{L^{q'}_\lambda L^2_n} \qquad \text{(Haussdorf-Young)} \\ &\leq \delta^{1/r}||(1+|\lambda-n^3|)^{-\alpha}||_{L^r_\lambda L^\infty_n}||(1+|\lambda-n^3|)^{\alpha}\widehat{H}||_{L^2_\lambda L^2_n} \simeq \delta^{1/r}|||h|||_{s,\alpha}, \end{aligned}$$

where in the last step we used the facts that $\alpha r = \frac{\mu}{\eta} > 1$ to conclude that

$$||(1+|\lambda - n^3|)^{-\alpha}||_{L^r_\lambda L^\infty_n} < \infty.$$

Putting this back in (3.1) proves the lemma, since $\frac{1}{rp} = \frac{\alpha\eta\mu}{\alpha\mu} = \eta$. □

Estimate (2.10). Writing $\widehat{w}_{fg}(n,\lambda) = \widehat{w}_1(n,\lambda) + \widehat{w}_2(n,\lambda)$, where

$$\widehat{w}_1(n,\lambda) \simeq \Big(n + \frac{2n}{n^2+1}\Big)\widehat{f} * \widehat{g}(n,\lambda),$$

and

$$\widehat{w}_2(n,\lambda) \simeq \frac{n}{n^2+1}\widehat{\partial_x f} * \widehat{\partial_x g}(n,\lambda),$$

it suffices to show Proposition 2.3 for $\widehat{w}_1$ and $\widehat{w}_2$ separately. However, we shall only prove it for $\widehat{w}_2$, since $\widehat{w}_1$ is a small perturbation of the KdV term and it can be done along the same lines as in [2]. We can, in fact, prove a stronger version of the proposition. Specifically, we will prove it for the term

$$\widehat{w}(n,\lambda) \simeq \frac{n}{|n|+1}\widehat{\partial_x f} * \widehat{\partial_x g}(n,\lambda). \tag{3.2}$$

Also, we will show only the most interesting case, namely $1/4 < s < 1/2$.

We define

$$c_u(n,\lambda) = (1+|n|^2)^{\frac{s}{2}}\left(1+|\lambda - n^3|\right)^{\alpha}\left|\,\widehat{u}(n,\lambda)\right|.$$

Also let $\alpha' = 1/2 - \beta$ and define

$$c'_u(n,\lambda) = (1+n^2)^{\frac{s}{2}}\left(1+|\lambda - n^3|\right)^{\alpha'}|\widehat{u}(n,\lambda)|.$$

Observe that

$$|||u|||^2_{s,\alpha} = \int_{\mathbb{R}}\int_{\mathbb{R}} c_u(n,\lambda)^2 dn d\lambda \quad \text{and} \quad |||u|||^2_{s,\alpha'} = \int_{\mathbb{R}}\int_{\mathbb{R}} c'_u(n,\lambda)^2 dn d\lambda.$$

Using (3.2), we have

$$\begin{aligned}
|\widehat{w}(n,\lambda)| &= \left|\int\int \frac{n}{|n|+1}\widehat{\partial_x f}(n-n_1,\lambda-\lambda_1)\widehat{\partial_x g}(n_1,\lambda_1) dn_1 d\lambda_1\right| \\
&\lesssim \int\int \frac{|n||n-n_1||n_1|}{(|n|+1)(1+|n-n_1|^2)^{\frac{s}{2}}(1+|n_1|^2)^{\frac{s}{2}}} \\
&\quad \times \frac{c'_f(n-n_1,\lambda-\lambda_1)}{(1+|\lambda-\lambda_1-(n-n_1)^3|)^{\alpha'}}\cdot\frac{c'_g(n_1,\lambda_1)}{(1+|\lambda_1-n_1^3|)^{\alpha'}}\; dn_1 d\lambda_1.
\end{aligned}$$

We have

$$
\begin{aligned}
(\text{ LHS of } (2.10)\ ) &\lesssim \left(\iint \frac{(1+n^2)^s |\widehat{w}_{fg}(n,\lambda)|^2}{(1+|\lambda - n^3|)^{2(1-\alpha)}} dn d\lambda \right)^{\frac{1}{2}} \\
&\lesssim \| \int\int \frac{|n||n-n_1|^{1-s}|n_1|^{1-s}}{(1+|n|)^{1-s}} \\
&\times \frac{1}{(1+|\lambda - n^3|)^{1-\alpha}} \ \frac{c_f'(n-n_1, \lambda-\lambda_1)}{(1+|\lambda-\lambda_1-(n-n_1)^3|)^{\alpha'}} \\
&\times \frac{c_g'(n_1,\lambda_1)}{(1+|\lambda_1 - n_1^3|)^{\alpha'}} \ dn_1 d\lambda_1 \|_{L^2_n L^2_\lambda} \\
&= \left\| \int\int Q \cdot \ c_f'(n-n_1,\lambda-\lambda_1) c_g'(n_1,\lambda_1) \ dn_1 d\lambda_1 \right\|_{L^2_n L^2_\lambda},
\end{aligned}
$$

where we define

$$
Q = \frac{|n||n-n_1|^{1-s}|n_1|^{1-s}}{(1+|\lambda-n^3|)^{1-\alpha}(1+|\lambda-\lambda_1-(n-n_1)^3|)^{\alpha'}(1+|\lambda_1-n_1^3|)^{\alpha'}(1+|n|)^{1-s}}.
$$

Note that if we write $Q = Q(s)$, then we have $Q(s) \leq Q(s_1)+Q(s_2)$ whenever $s_1 \leq s \leq s_2$. In particular, the case $s > 1$ has been dealt with in other papers (see [9]), so if we show the case $1/4 < s < 1/2$, it will follow that the proposition holds for any $s > 1/4$.

Next we split the domain of integration. First we notice that, due to symmetry, we need only prove the case $|\lambda - \lambda_1 - (n-n_1)^3| \leq |\lambda_1 - n_1^3|$. To formalize this, define D to be the domain

$$
D = \{(n,\lambda,n_1,\lambda_1) \in \mathbb{R}^4 : |\lambda-\lambda_1-(n-n_1)^3| \leq |\lambda_1 - n_1^3|\}.
$$

Then by a change of variables

$$
\begin{aligned}
&\int_{\mathbb{R}-D}\int \frac{n}{|n|+1} \widehat{\partial_x f}(n-n_1,\lambda-\lambda_1)\widehat{\partial_x g}(n_1,\lambda_1) dn_1 d\lambda_1 \\
&= \int_D \int \frac{n}{|n|+1} \widehat{\partial_x f}(n_1,\lambda_1)\widehat{\partial_x g}(n-n_1,\lambda-\lambda_1) dn_1 d\lambda_1.
\end{aligned}
$$

We also observe that if either $|n_1| < 1$ or $|n-n_1| < 1$, then

$$
|n_1(n-n_1)|^{1-s} \lesssim (1+|n|)^{1-s}
$$

and the problem reduces to the case $s = 1$. Thus, we will only consider the domain

$$
D' = \{(n,\lambda,n_1,\lambda_1) \in \mathbb{R}^4 : |n_1| \geq 1, |n-n_1| \geq 1, |\lambda-\lambda_1-(n-n_1)^3| \leq |\lambda_1 - n_1^3|\}.
$$

Now split D' into A and B where

$$A = \{(n,\lambda,n_1,\lambda_1) \in \mathbb{R}^4 : |n_1| \geq 1, |n-n_1| \geq 1,$$
$$|\lambda - \lambda_1 - (n-n_1)^3| \leq |\lambda_1 - n_1^3| \leq |\lambda - n^3|\},$$
$$B = \{(n,\lambda,n_1,\lambda_1) \in \mathbb{R}^4 : |n_1| \geq 1, |n-n_1| \geq 1,$$
$$|\lambda - n^3| \leq |\lambda_1 - n_1^3|, |\lambda - \lambda_1 - (n-n_1)^3| \leq |\lambda_1 - n_1^3|\}$$

and write $Q_A = Q \cdot \chi_A$ and $Q_B = Q \cdot \chi_B$.

$$\Big\| \int\int Q_A c_f'(n-n_1, \lambda-\lambda_1) c_g'(n_1,\lambda_1)\ dn_1 d\lambda_1 \Big\|_{L^2_n L^2_\lambda}$$
$$\leq \|Q_A\|_{L^2_{n_1} L^2_{\lambda_1} L^\infty_n L^\infty_\lambda} \Big(\int\int c_f'(n-n_1,\lambda-\lambda_1)^2 c_g'(n_1,\lambda_1)^2\ dn_1 d\lambda_1 dn d\lambda \Big)^{1/2}$$
$$= \|Q_A\|_{L^2_{n_1} L^2_{\lambda_1} L^\infty_n L^\infty_\lambda} \|c_f'\|_{L^2_n L^2_\lambda} \|c_g'\|_{L^2_n L^2_\lambda}.$$

For Q_B we first note that

$$\Big\| \int\int Q_B c_f'(n-n_1, \lambda-\lambda_1) c_g'(n_1,\lambda_1)\ dn_1 d\lambda_1 \Big\|_{L^2_n L^2_\lambda}$$
$$= \sup_{\substack{d(n,\lambda)\geq 0 \\ \int d^2 = 1}} \int\int\int\int Q_B \cdot d(n,\lambda)\ c_f'(n-n_1,\lambda-\lambda_1) c_g'(n_1,\lambda_1)\ dn_1 d\lambda_1 dn d\lambda.$$

Then by Holder's inequality:

$$\int\int\int\int Q_B \cdot d(n,\lambda)\ c_f'(n-n_1,\lambda-\lambda_1) c_g'(n_1,\lambda_1)\ dn_1 d\lambda_1 dn d\lambda$$
$$\leq \Big\| \int\int Q_B \cdot d(n,\lambda)\ c_f'(n-n_1,\lambda-\lambda_1) dn d\lambda \Big\|_{L^2_{n_1} L^2_{\lambda_1}} ||c_g'(n_1,\lambda_1)||_{L^2_{n_1} L^2_{\lambda_1}}$$
$$\leq \|Q_B\|_{L^2_n L^2_\lambda L^\infty_{n_1} L^\infty_{\lambda_1}} \Big(\int\int d(n,\lambda)^2 c_f'(n-n_1,\lambda-\lambda_1)^2\ dn_1 d\lambda_1 dn d\lambda \Big)^{\frac{1}{2}} \|c_g'\|_{L^2_n L^2_\lambda}$$
$$= \|Q_B\|_{L^2_n L^2_\lambda L^\infty_{n_1} L^\infty_{\lambda_1}} \|d\|_{L^2_n L^2_\lambda} \|c_f'\|_{L^2_n L^2_\lambda} \|c_g'\|_{L^2_n L^2_\lambda}.$$

By Lemma 3.1

$$\|c_h'\|_{L^2_n L^2_\lambda} = |||h|||_{s,\alpha'} \lesssim \delta^\eta |||h|||_{s,\alpha}$$

for any $\eta \in (0, \alpha - \alpha')$. In particular, we can choose $\eta = \beta = 1/2 - \alpha' < \alpha - \alpha'$.

Thus, we have shown that

$$\Big[\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(1+n^2)^s |\widehat{w}(n,\lambda)|^2}{(1+|\lambda - n^3|)^{2(1-\alpha)}}\, dn d\lambda \Big]^{\frac{1}{2}}$$

$$\lesssim \delta^{2\beta}|||f|||_{s,\alpha}|||g|||_{s,\alpha}\left(\|Q_A\|_{L^2_{n_1}L^2_{\lambda_1}L^\infty_n L^\infty_\lambda}+\|Q_B\|_{L^2_n L^2_\lambda L^\infty_{n_1}L^\infty_{\lambda_1}}\right).$$

So the proof will be complete if we show that

$$\|Q_A\|_{L^2_{n_1}L^2_{\lambda_1}L^\infty_n L^\infty_\lambda}+\|Q_B\|_{L^2_n L^2_\lambda L^\infty_{n_1}L^\infty_{\lambda_1}}<\infty. \tag{3.3}$$

First we will need some simple integral estimates.

Lemma 3.2. *If* $1/2 < l < 1$, *then*

$$\int_{\mathbb{R}}\frac{dx}{(1+|A-x^2|)^l}\lesssim\frac{1}{(1+|A|)^{l-1/2}} \tag{3.4}$$

$$\int_{\mathbb{R}}\frac{dx}{(1+|x-b|)^l(1+|x-c|)^l}\lesssim\frac{1}{(1+|b-c|)^{2l-1}}. \tag{3.5}$$

Proof. For (3.4) let $a=|\sqrt{A}|$ and note $|A-x^2|\geq|a^2-x^2|$. Hence it suffices to show

$$\int_{\mathbb{R}}\frac{dx}{(1+|a^2-x^2|)^l}\lesssim\frac{1}{(1+a)^{2l-1}}. \tag{3.6}$$

First we compute, using the substitution $x=ay$,

$$\int_{\mathbb{R}}\frac{dx}{|a^2-x^2|^l}=\int_{\mathbb{R}}\frac{a\,dy}{|a^2-a^2y^2|^l}=a^{1-2l}\int_{\mathbb{R}}\frac{dy}{|1-y^2|^l}$$

$$\lesssim a^{1-2l}\Big[\int_{|y|<2}\frac{dy}{|1-y|^l}+\int_{|y|>2}\frac{dy}{|y|^{2l}}\Big]\simeq a^{1-2l}$$

since $l<1$ and $2l>1$. Then (3.6) follows, since $1+|a^2-x^2|\geq|(1+a^2)-x^2|$.

Now observe that

$$(1+|x-b|)(1+|x-c|)\geq 1+|x-b|\cdot|x-c|=1+\left|\left(x-\frac{b+c}{2}\right)^2-\left(\frac{b-c}{2}\right)^2\right|$$

so (3.5) follows from (3.6) by a simple substitution.

Lemma 3.3. *If* $\rho\in(1/2,3/4)$, $\alpha\in(1/2,3/4-\rho/3]$ *and* $\alpha'\in[1/4+\rho/3,1/2)$, *then there exists* $c>0$ *such that for all* $n,\lambda\in\mathbb{R}$:

$$\frac{(1+|\lambda-n^3|)^{2(\alpha-1)}}{(1+|n|)^{2\rho}}\int_A\!\!\int\frac{n^2|n_1(n-n_1)|^{2\rho}dn_1d\lambda_1}{(1+|\lambda_1-n_1^3|)^{2\alpha'}(1+|\lambda-\lambda_1-(n-n_1)^3|)^{2\alpha'}}\leq c,$$

where

$$A=A(n,\lambda)=\{(n_1,\lambda_1)\in\mathbb{R}^2:|\lambda-\lambda_1-(n-n_1)^3|\leq|\lambda_1-n_1^3|\leq|\lambda-n^3|\}.$$

Proof. In A we have

$$|\lambda - n^3 + 3nn_1(n-n_1)| \le |\lambda_1 - n_1^3| + |\lambda - \lambda_1 - (n-n_1)^3| \le 2|\lambda - n^3|$$

and hence $|3nn_1(n-n_1)| \le 3|\lambda - n^3|$. And by (3.5), we have

$$\begin{aligned} &\int \frac{d\lambda_1}{(1+|\lambda_1 - n_1^3|)^{2\alpha'}(1+|\lambda-\lambda_1-(n-n_1)^3|)^{2\alpha'}} \\ &\lesssim \frac{1}{(1+|\lambda - n^3 + 3nn_1(n-n_1)|)^{4\alpha'-1}} \end{aligned}$$

so

$$\begin{aligned} &\frac{(1+|\lambda-n^3|)^{2(\alpha-1)}}{(1+|n|)^{2\rho}} \int_A \int \frac{n^2|n_1(n-n_1)|^{2\rho} dn_1 d\lambda_1}{(1+|\lambda_1 - n_1^3|)^{2\alpha'}(1+|\lambda-\lambda_1-(n-n_1)^3|)^{2\alpha'}} \\ &\lesssim \frac{n^{2-2\rho}(1+|\lambda-n^3|)^{2(\alpha-1+\rho)}}{(1+|n|)^{2\rho}} \int \frac{dn_1}{(1+|\lambda-n^3+3nn_1(n-n_1)|)^{4\alpha'-1}}. \end{aligned}$$

To integrate in n_1, notice

$$\begin{aligned} |\lambda - n^3 + 3nn_1(n-n_1)| &= \frac{1}{4}|4\lambda - 4n^3 - 3n(4n_1^2 - 4nn_1 + n^2) + 3n^3| \\ &= \frac{1}{4}|4\lambda - n^3 - 3n(2n_1 - n)^2|. \end{aligned}$$

So if we make the substitution $N = \sqrt{3n}(2n_1 - n)$, $dN = 2\sqrt{3n}dn_1$ and apply (3.4), we have

$$\begin{aligned} &\frac{n^{2-2\rho}(1+|\lambda-n^3|)^{2(\alpha-1+\rho)}}{(1+|n|)^{2\rho}} \int \frac{dn_1}{(1+|\lambda-n^3+3nn_1(n-n_1)|)^{4\alpha'-1}} \\ &\sim \frac{n^{3/2-2\rho}(1+|\lambda-n^3|)^{2(\alpha-1+\rho)}}{(1+|n|)^{2\rho}} \int \frac{dN}{(1+|4\lambda-n^3-N^2|)^{4\alpha'-1}} \\ &\lesssim \frac{(1+|\lambda-n^3|)^{2(\alpha-1+\rho)}}{(1+|n|)^{4\rho-3/2}(1+|4\lambda-n^3|)^{4\alpha'-3/2}} \\ &\lesssim \frac{\big[(1+|n^3|)(1+|4\lambda-n^3|)\big]^{2(\alpha-1+\rho)}}{(1+|n^3|)^{4\rho/3-1/2}(1+|4\lambda-n^3|)^{4\alpha'-3/2}} \le 1 \end{aligned}$$

since $2\alpha - 2 + 2\rho \le \frac{3}{2} - \frac{2\rho}{3} - 2 + 2\rho = \frac{4\rho}{3} - \frac{1}{2} \le 4\alpha' - \frac{3}{2}$.

Lemma 3.4. *If $\rho \in (1/2, 3/4)$, $\alpha \in (1/2, 1/2+\beta]$ and $\alpha' \in [1/2-\beta, 1/2)$, where*

$$\beta = \frac{1}{3}\min\Big\{\frac{1}{4} - \frac{\rho}{3}, \rho - \frac{1}{2}\Big\},$$

then there exists $c > 0$ such that for all $n_1, \lambda_1 \in \mathbb{R}$ with $|n_1| \geq 1$:

$$(1+|\lambda_1-n_1^3|)^{-2\alpha'} \int_B \int \frac{(1+|n|)^{-2\rho} n^2 |n_1(n-n_1)|^{2\rho} dn d\lambda}{(1+|\lambda-n^3|)^{2(1-\alpha)}(1+|\lambda-\lambda_1-(n-n_1)^3|)^{2\alpha'}} \leq c,$$

where

$$B = B(n_1,\lambda_1) = \{(n,\lambda) \in \mathbb{R}^2 : |n-n_1| \geq 1, \\ |\lambda - n^3| \leq |\lambda_1 - n_1^3|, |\lambda-\lambda_1-(n-n_1)^3| \leq |\lambda_1-n_1^3|\}.$$

Proof. First note that in B

$$|\lambda_1 - n_1^3 + 3nn_1(n-n_1)| = |\lambda - n^3 - (\lambda-\lambda_1-(n-n_1)^3| \leq 2|\lambda_1 - n_1^3|$$

and hence $3|nn_1(n-n_1)| \leq 3|\lambda_1-n_1^3|$. Also we have by (3.5)

$$\int \frac{d\lambda}{(1+|\lambda-n^3|)^{2(1-\alpha)}(1+|\lambda-\lambda_1-(n-n_1)^3|)^{2\alpha'}} \\ \lesssim \frac{1}{(1+|\lambda_1-n_1^3-3nn_1(n-n_1)|)^{1-4\beta}}.$$

Therefore,

$$(1+|\lambda_1-n_1^3|)^{-2\alpha'} \int_B \int \frac{n^2|n_1(n-n_1)|^{2\rho} dn d\lambda}{(1+|n|)^{2\rho}(1+|\lambda-n^3|)^{2(1-\alpha)}(1+|\lambda-\lambda_1-(n-n_1)^3|)^{2\alpha'}}$$

$$\lesssim I(B') = (1+|\lambda_1-n_1^3|)^{2\rho-2\alpha'} \int_{B'} \frac{dn}{(1+|n|)^{4\rho-2}(1+|\lambda_1-n_1^3-3nn_1(n-n_1)|)^{1-4\beta}},$$

where

$$B' = B'(n_1,\lambda_1) = \{n \in \mathbb{R} : |\lambda_1-n_1^3+3nn_1(n-n_1)| \leq 2|\lambda_1-n_1^3|, |n-n_1| > 1\}.$$

To deal with this integral we split B' into two subsets:

$$B'_1 = \{n \in B' : 3|nn_1(n-n_1)| \leq \frac{1}{2}|\lambda_1 - n_1^3|\}$$

$$B'_2 = \{n \in B' : \frac{1}{2}|\lambda_1-n_1^3| \leq 3|nn_1(n-n_1)| \leq 3|\lambda_1-n_1^3|\}.$$

Case 1. For B'_1 we note that:

$$|\lambda_1 - n_1^3 + 3nn_1(n-n_1)| \geq |\lambda_1 - n_1^3| - |3nn_1(n-n_1)| \geq \frac{1}{2}|\lambda_1 - n_1^3|.$$

And, since $|n| \leq \frac{1}{6}|\lambda_1 - n_1^3|$,

$$I(B'_1) \lesssim (1+|\lambda_1-n_1^3|)^{2\rho-2\alpha'+4\beta-1} \int_{|n|\leq|\lambda_1-n_1^3|} \frac{dn}{(1+|n|)^{4\rho-2}} \\ \lesssim (1+|\lambda_1-n_1^3|)^{2\rho+6\beta-2} \left[(1+|n|)^{3-4\rho}\right]_0^{|\lambda_1-n_1^3|}$$

$$\lesssim (1+|\lambda_1 - n_1^3|)^{6\beta+1-2\rho} \leq 1,$$

where we used the fact that $3-4\rho > 0$ and $6\beta \leq 2\rho - 1$.
Case 2. For B_2' we will use:
Claim: If $c|nn_1(n-n_1)| \geq |\lambda_1 - n_1^3|$, $c > 1$, then

$$\frac{1+|\lambda_1 - n_1^3|}{1+|4\lambda_1 - n_1^3|} \lesssim 1 + min(|n^3|, |n_1^3|). \tag{3.7}$$

Proof.

$$1+|\lambda_1 - n_1^3| \leq 1+|4\lambda_1 - n_1^3| + |3n_1^3| \leq (1+|4\lambda_1 - n_1^3|)(1+|n_1^3|).$$

So (3.7) follows if $|n_1| \leq 4c|n|$. On the other hand, suppose that $|n_1| \geq 4c|n|$. Then

$$2|\lambda_1 - n_1^3| \leq 2c|nn_1(n-n_1)| \leq 2c|nn_1|(|n|+|n_1|) \leq |n_1^3|$$

and

$$|4\lambda_1 - n_1^3| \geq |3n_1^3| - |4\lambda_1 - 4n_1^3| \geq 6|\lambda_1 - n_1^3| - 4|\lambda_1 - n_1^3| = 2|\lambda_1 - n_1^3|.\square$$

The claim implies that:

$$\frac{1}{(1+|n|)^{4\rho-2}} \lesssim \left[\frac{1+|4\lambda_1 - n_1^3|}{1+|\lambda_1 - n_1^3|}\right]^{(4\rho-2)/3} \leq \frac{(1+|4\lambda_1 - n_1^3|)^{1/3}}{(1+|\lambda_1 - n_1^3|)^{(4\rho-2)/3}}$$

and hence

$$I(B_2') \lesssim (1+|\lambda_1 - n_1^3|)^{2\rho/3 - 2\alpha' + 2/3}(1+|4\lambda_1 - n_1^3|)^{1/3} \times \int_{B_2'} \frac{dn}{(1+|\lambda_1 - n_1^3 - 3nn_1(n-n_1)|)^{1-4\beta}}.$$

To integrate in n, notice

$$|\lambda_1 - n_1^3 - 3nn_1(n-n_1)| = \frac{1}{4}|4\lambda_1 - n_1^3 - 3n_1(2n-n_1)^2|.$$

So if we make the substitution $N = \sqrt{3n_1}(2n-n_1)$, $dN = 2\sqrt{3n_1}dn$ and apply (3.4) we have

$$\int \frac{dn}{(1+|\lambda_1 - n_1^3 - 3nn_1(n-n_1)|)^{1-4\beta}} = \int \frac{dN}{2\sqrt{3n_1}(1+|4\lambda_1 - n_1^3 - N^2|)^{1-4\beta}} \lesssim \frac{1}{\sqrt{n_1}(1+|4\lambda_1 - n_1^3|)^{1/2-4\beta}}.$$

Combining this with the previous inequality, we have

$$I(B_2') \lesssim \frac{(1+|\lambda_1 - n_1^3|)^{2\beta+(2\rho-1)/3}}{(1+|n_1^3|)^{1/6}(1+|4\lambda_1 - n_1^3|)^{1/6-4\beta}} \leq 1,$$

where we used (3.7), the fact that $|n_1| \geq 1$ and

$$2\beta + (2\rho - 1)/3 \leq 1/6 - 4\beta < 1/6. \qquad \square$$

Then (3.3) follows from Lemmas 3.3 and 3.4, and hence we have proved (2.10).

Estimate (2.11). To see that (2.11) follows from (2.10) it suffices to show that for any n

$$\int_{\mathbb{R}} \frac{(1+n^2)^{\frac{s}{2}}|\widehat{w}_{fg}(n,\lambda)|}{1+|\lambda - n^3|}\, d\lambda \lesssim \left(\int_{\mathbb{R}} \frac{(1+n^2)^s|\widehat{w}_{fg}(n,\lambda)|^2}{(1+|\lambda - n^3|)^{2(1-\alpha)}}\, d\lambda \right)^{1/2}.$$

More generally we have

$$\int |F(\lambda)| d\lambda \leq ||(1+|\lambda - n^3|)^{-\alpha}||_{L^2_\lambda} ||(1+|\lambda - n^3|)^{\alpha} F(\lambda)||_{L^2_\lambda}$$
$$\simeq \left(\int (1+|\lambda - n^3|)^{2\alpha} |F(\lambda)|^2 d\lambda \right)^{1/2}.$$

This completes the proof of Proposition 2.3. $\square$

Since the restriction on s in Theorem 1.1 comes directly from the restriction on s in Proposition 2.3, it is natural to ask whether Proposition 2.3 holds for any $s < 1/4$.

In particular, recall that it suffices, for the purposes of this paper, to prove Proposition 2.3 for the term

$$\widehat{w}_2(n,\lambda) \simeq \frac{n}{n^2+1} \widehat{\partial_x f} * \widehat{\partial_x g}(n,\lambda),$$

and that we in fact proved it for the more difficult term

$$\widehat{w}(n,\lambda) \simeq \frac{n}{|n|+1} \widehat{\partial_x f} * \widehat{\partial_x g}(n,\lambda).$$

We shall now show, however, that Proposition 2.3, even with the w_2 term, fails if $s < 1/4$. Let N be a positive integer and define $\widehat{f} = \widehat{g} = \chi_S - \chi_{-S}$ where

$$S = \{(n,\lambda) \in \mathbb{R}^2 : |n - N| < 1, |\lambda - 3N^2 n + 2N^3| < N\}$$

and $-S = \{(n,\lambda) : (-n,-\lambda) \in S\}$.

Then $\widehat{\partial_x f} = n\widehat{f} \sim NF$ where $F = \chi_S + \chi_{-S}$. Now note that

$$F * F(n,\lambda) \geq 2\mu\{(n_1,\lambda_1) \in S : (n - n_1, \lambda - \lambda_1) \in -S\}.$$

In particular, let

$$(n,\lambda) \in R = \Big\{(n,\lambda) : \frac{1}{4} < n < \frac{1}{2}, |\lambda - 3N^2 n| < \frac{N}{2}\Big\}$$

and

$$(n_1,\lambda_1) \in S' = \Big\{(n_1,\lambda_1) : |n_1 - N| < \frac{1}{2}, |\lambda_1 - 3N^2 n_1 + 2N^3| < \frac{N}{2}\Big\}.$$

Then $|n_1 - n - N| \le |n_1 - N| + |n| < 1$ and

$$|\lambda_1 - \lambda - 3N^2(n_1 - n) + 2N^3| \le |\lambda_1 - 3N^2 n_1 + 2N^3| + |\lambda + 3N^2 n| < N$$

so that $(n_1 - n, \lambda_1 - \lambda) \in S$, and hence we have shown that

$$F * F \ge 2\mu\{S'\}\chi_R \simeq N\chi_R.$$

Also note that in R, $n \sim 1$ and $\lambda \sim N^2 \pm N \sim N^2$. Therefore,

$$|\widehat{w_2}| = \frac{n}{n^2+1}\widehat{\partial_x f} * \widehat{\partial_x g} \sim \frac{n}{n^2+1}N^2(F*F) \ge \frac{n}{n^2+1}N^3\chi_R \sim N^3\chi_R.$$

$$\begin{aligned}
&\Big(\int\int \frac{(1+n^2)^s|\widehat{w_2}(n,\lambda)|^2}{(1+|\lambda - n^3|)^{2(1-\alpha)}}dnd\lambda\Big)^{1/2} \\
&\gtrsim N^3\Big(\int_R\int \frac{1}{(1+|\lambda - n^3|)^{2(1-\alpha)}}dnd\lambda\Big)^{1/2} \\
&\sim N^3\left(N^{4\alpha-4}\mu(R)\right)^{1/2} \simeq N^{3/2+2\alpha}.
\end{aligned}$$

Now notice that in S,

$$|n^3 - \lambda| \le N + |n^3 - 3N^2 n + 2N^3| = N + (n-N)^2|n+2N| \le N + 4N \sim N.$$

Hence,

$$|||f|||^2_{s,\alpha} = \|(1+n^2)^{s/2}(1+|\lambda - n^3|)^\alpha \widehat{f}\|^2_{L^2_n L^2_\lambda}$$

$$\sim N^{2s}N^{2\alpha}\|\widehat{f}\|^2_{L^2_n L^2_\lambda} = 2N^{2s+2\alpha}\mu(S) \sim N^{1+2s+2\alpha}.$$

So we have shown that Proposition 2.3 can only hold if

$$N^{3/2+2\alpha} \lesssim N^{1+2s+2\alpha}.$$

Therefore, we must have $s \ge 1/4$.

4. Appendix

Proof of Lemma 2.2. We will follow the proof given in [14]. To show

$$\int\int(1+|n|^2)^s(1+|\lambda-n^3|)^{2\alpha}|\widehat{\theta u}(n,\lambda)|^2 dnd\lambda$$
$$\lesssim ||\theta||^2_{H^\alpha}\int\int(1+|n|^2)^s(1+|\lambda-n^3|)^{2\alpha}|\widehat{u}(n,\lambda)|^2 dnd\lambda$$

it suffices to show that for each $n\in\mathbb{R}$,

$$\left\|(1+|\lambda-n^3|)^\alpha\widehat{\theta u}(n,\lambda)\right\|_{L^2}\lesssim||\theta||_{H^\alpha}\left\|(1+|\lambda-n^3|)^\alpha\widehat{u}(n,\lambda)\right\|_{L^2}. \quad (4.1)$$

(We will write $\widehat{u}(\lambda)$ instead of $\widehat{u}(n,\lambda)$ and a instead of n^3.) So consider

$$(1+|\lambda-a|)^\alpha|\widehat{\theta}*\widehat{u}(\lambda)| = (1+|\lambda-a|)^\alpha\left|\int\widehat{\theta}(\lambda-\lambda_1)\widehat{u}(\lambda_1)d\lambda_1\right|$$
$$\leq\int(1+|\lambda_1-a|+|\lambda-\lambda_1|)^\alpha|\widehat{\theta}(\lambda-\lambda_1)\widehat{u}(\lambda_1)|d\lambda_1$$
$$\lesssim\int(1+|\lambda_1-a|)^\alpha|\widehat{\theta}(\lambda-\lambda_1)\widehat{u}(\lambda_1)|d\lambda_1+\int|\lambda-\lambda_1|^\alpha|\widehat{\theta}(\lambda-\lambda_1)\widehat{u}(\lambda_1)|d\lambda_1$$
$$=|\widehat{\theta}|*|(1+|\lambda-a|)^\alpha\widehat{u}|+|\lambda^\alpha\widehat{\theta}|*|\widehat{u}|.$$

Therefore,

$$\left\|(1+|\lambda-n^3|)^\alpha\widehat{\theta u}(n,\lambda)\right\|_{L^2}\lesssim\left\||\lambda^\alpha\widehat{\theta}|*|\widehat{u}|\right\|_{L^2}+\left\||\widehat{\theta}|*|(1+|\lambda-a|)^\alpha\widehat{u}|\right\|_{L^2}$$
$$\leq||\lambda^\alpha\widehat{\theta}||_{L^2}||\widehat{u}||_{L^1}+||\widehat{\theta}||_{L^1}||(1+|\lambda-a|)^\alpha\widehat{u}||_{L^2}.$$

We note that

$$||\widehat{u}||_{L^1}\leq||(1+|\lambda-a|)^\alpha\widehat{u}||_{L^2}||(1+|\lambda-a|)^{-\alpha}||_{L^2}\lesssim||(1+|\lambda-a|)^\alpha\widehat{u}||_{L^2}$$

and similarly

$$||\widehat{\theta}||_{L^1}\lesssim||(1+|\lambda|)^\alpha\widehat{\theta}||_{L^2}$$

which proves (4.1).

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