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APRIORI ESTIMATES OF OSSERMAN-KELLER TYPE

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Abstract. We prove certain pointwise upper bounds in terms of distance to the boundary for subsolutions of equations generalizing (0.1) below. This type of estimates originate from work of Joseph B. Keller and are of importance to the recent study of so called large solutions of such equations initiated by Catherine Bandle and Moshe Marcus.

0. Introduction

Let $f : \mathbb{R} \to (0, \infty)$ be an increasing function, that is, $f(t_1) \leq f(t_2)$ if $t_1 < t_2$, such that $\int^{\infty} F(t)^{-1/2} dt < \infty$, where $F' = f$, and consider the equation

$$\Delta u = f(u) \quad \text{in } \Omega. \tag{0.1}$$

Here, Ω is a domain (nonempty, open and connected) in $\mathbb{R}^n$, $n \geq 2$, and $\Delta = \sum_{j=1}^{n} \partial^2/\partial x_j^2$ is the Laplacian. By a solution u of (0.1) we mean a subharmonic function u in Ω satisfying (0.1) in the sense of distributions in Ω. Of course, by regularity theory, such a solution u is necessarily of type $C^{1,\alpha}$ for every $0 < \alpha < 1$. An example of a nonlinearity f to keep in mind here is $f(u) = e^u$.

By now it has been known for about fifty years that subject to some regularity assumptions on f the solutions u of (0.1) are uniformly bounded from above on compact subsets of Ω. In fact, by the work of Joseph B. Keller [12], see also Robert Osserman [15], there exists a decreasing function $g : (0, \infty) \to \mathbb{R}$ having the limits $\lim_{R \to 0} g(R) = \infty$ and $\lim_{R \to \infty} g(R) = -\infty$ such that

$$u(x) \leq g\left(\text{dist}(x, \partial\Omega)\right), \quad x \in \Omega, \tag{0.2}$$

whenever u is a solution of (0.1). Here $\text{dist}(x, \partial\Omega)$ is the euclidean distance from x to the boundary $\partial\Omega$ of Ω. We emphasize that g is defined only in terms of f and the dimension n of the underlying space and so does not depend on the particular choice of domain Ω. In the literature there seem to

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be some confusion about exactly what regularity needed on f for this above discussed Osserman-Keller result to hold true. By Theorem 3.2, one of the main results of the present paper, the above Osserman-Keller result holds true without any additional regularity assumption on f; in Theorem 3.2 the function f is only assumed to be increasing and satisfy the above growth condition. Furthermore, Theorem 3.2 only requires that u is a subsolution of (0.1) by what we here mean that u is a continuous function in Ω such that $\Delta u \geq f(u)$ holds in $\mathcal{D}'(\Omega)$. We point out that the limit behavior of g is put in a separate theorem, Theorem 3.1.

An important variant on the above theme is when $f : (0,\infty) \to (0,\infty)$ is increasing and satisfies $f(0) = \lim_{u\to 0+} f(u) = 0$. The growth assumption on f here is the same as above, that is, $\int^{\infty} F(t)^{-1/2}dt < \infty$, where $F' = f$. In this case we also require of a solution (subsolution) u of (0.1) that it is nonnegative, that is, $u(x) \geq 0$ for all $x \in \Omega$. The prime example of functions f satisfying these assumptions is $f(u) = u^p$, where $p > 1$. We mention here that some authors have considered f as a function $f : (\tau,\infty) \to (0,\infty)$ for some $\tau \in \mathbb{R}$. This more general case is easily reduced to our case ($\tau = 0$) by a change of dependent variable. The analogue Osserman-Keller result for this setting when $f : (0,\infty) \to (0,\infty)$ is also contained in Theorem 3.2.

Notice that if $u \in C(\Omega)$ is such that $\Delta u \geq f(u)$ in $\mathcal{D}'(\Omega)$, then, in particular, $\Delta u \geq 0$ in $\mathcal{D}'(\Omega)$. It is well-known that distributions u in Ω satisfying $\Delta u \geq 0$ in $\mathcal{D}'(\Omega)$ are in natural correspondence to subharmonic functions in Ω (see Theorem 4.1.8 in [11]). For an arbitrary subharmonic function u the distributional inequality $\Delta u \geq f(u)$ makes sense and can be thought of as a strengthened form of subharmonicity. Assuming in addition that f is convex we can relax the regularity assumption on u above and in Theorem 3.3 prove that (0.2) holds for every subharmonic function u is Ω such that $\Delta u \geq f(u)$ in $\mathcal{D}'(\Omega)$. The passage from Theorem 3.2 to Theorem 3.3 is accomplished by a standard regularization procedure.

A main application of the Osserman-Keller result, which was considered also by J. B. Keller (see Theorem III in [12]), is to prove existence of so called large solutions of equation (0.1). Here, when Ω is bounded, by a large solution of (0.1) we mean a solution u of (0.1) such that $u(x) \to \infty$ as x approaches the boundary $\partial\Omega$ of Ω. Since the pioneering contributions of Catherine Bandle, Matts Essén and Moshe Marcus [2, 4, 5] the study of these large solutions has attracted much attention. In this direction of study some authors have considered the variant of (0.1) where the Laplacian Δ is replaced by the so called p-Laplacian denoted by Δ_p for $1 < p < \infty$. Let us recall the definition of the p-Laplacian. For $u \in W^{1,p}(\Omega)$ and Ω bounded, $\Delta_p u$ is defined in the distributional sense as a distribution in $W^{-1,p'}(\Omega)$,

$1/p + 1/p' = 1$, by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, that is,

$$\langle \Delta_p u, \varphi \rangle = -\int |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx \quad \text{for } \varphi \in \mathcal{D}(\Omega).$$

This definition extends in the obvious way to the case when $u \in W^{1,p}_{\text{loc}}(\Omega)$ and Ω is an arbitrary domain in $\mathbb{R}^n$. Clearly, $\Delta_2 = \Delta$. The p-Laplace version of (0.1) referred to above is

$$\Delta_p u = f(u) \quad \text{in } \Omega. \tag{0.3}$$

In (0.3) we assume that f has monotonicity properties as above and satisfies the growth condition $\int^\infty F(t)^{-1/p} dt < \infty$, where $F' = f$. We emphasize once more that no additional regularity on f is assumed. We say that u is a solution of (0.3) if $u \in C(\Omega) \bigcap W^{1,p}_{\text{loc}}(\Omega)$ and u satisfies (0.3) in the distributional sense. References dealing with large solutions of (0.3) are [3, 10, 13, 14]. In Theorem 3.4 we prove a version of the Osserman-Keller result for (0.3). In this result the function $g = g_p$ in (0.2) also depends on the parameter p, $1 < p < \infty$, and the function u in (0.2) satisfies the slightly more restrictive assumption that $u \in C(\Omega) \bigcap W^{1,p}_{\text{loc}}(\Omega)$ and $\Delta_p u \geq f(u)$ in $\mathcal{D}'(\Omega)$.

Let us return to the case when $f : (0, \infty) \to (0, \infty)$ is increasing, satisfies $f(0) = \lim_{u \to 0+} f(u) = 0$ and $\int^\infty F(t)^{-1/p} dt < \infty$, where $F(t) = \int_0^t f(s) ds$ for $t \geq 0$. Here two different kinds of behavior occur depending on whether the integral $\int_0^\infty F(t)^{-1/p} dt$ is convergent or not. In the first case when $\int_0^\infty F(t)^{-1/p} dt < \infty$ our results Theorem 3.1 and Theorem 3.4 imply that $u(x) = 0$ if $\operatorname{dist}(x, \partial\Omega) \geq \sqrt[p]{n} R^*$, where R^* is as in Theorem 2.3, whenever u is a solution (or more generally a subsolution) of (0.3). In particular, if Ω contains a ball of radius $R \geq \sqrt[p]{n} R^*$, then every solution of (0.3) has interior zeros in Ω. The zero set of a solution u of (0.3) is in the literature called a dead core. To some extent the study of these dead cores has been motivated by applications in chemical engineering, see [7, 8, 18]. On the contrary, when $\int_0^\infty F(t)^{-1/p} dt = \infty$ and f is continuous, dead cores other than $\emptyset$ and Ω can not exist. This follows by a strong maximum principle of J. L. Vázquez, see [18] or Theorem 1.20 in [8]. To the author's knowledge the above existence of (nontrivial) dead cores for an arbitrary increasing (not necessarily continuous) nonlinearity f is established here for the first time.

This paper contains a detailed study of matters originating from the paper [12] by J. B. Keller. This study is needed to prove the main results in Section 3. The proof of Theorem 3.2 and Theorem 3.4 is accomplished by a comparison of a subsolution of, say, (0.1) to a certain large solution of $\Delta u = f(u)$ in a ball $B(a, R) \subset \Omega$. In this way the proof of (0.2) is reduced to a study of radial large solutions of $\Delta u = f(u)$ in the balls $B(0,R)$ for $0 < R < \infty$.

In Section 2, we study radial solutions of $\Delta_p u = f(u)$. The purpose of this section is twofold. First we establish existence of a maximal large solution of $\Delta_p u = f(u)$ in $B(0,R)$. Second we estimate such a solution in terms of the radius $R > 0$. The results in Section 2 needed in Section 3 are Theorem 2.1, Theorem 2.4 and Theorem 2.5. Theorem 2.1 is concerned with a certain estimate of a radial solution of $\Delta_p u = f(u)$. Theorem 2.4 gives a construction of a large subsolution of (0.3) for $\Omega = B(0,R)$. In Theorem 2.5 we prove existence of a (unique) maximal large solution u_R of (0.3) for $\Omega = B(0,R)$, $0 < R < \infty$. An important property of these maximal large solutions is that the map $R \mapsto u_R$ is leftcontinuous, see Theorem 2.5. In the proof of Theorem 2.5 we first consider the case when f in addition is assumed to be continuous. This case when f is continuous is settled by using ideas from [12]. A basic idea here is to consider the initial value problem (2.2) and study the blow-up $R = R(u_0)$ of this problem as a function of the initial value u_0. The proof of the general case of Theorem 2.5 is accomplished by an approximation argument making use of certain regularizations of f.

We wish to point out that there is some overlap of this paper with the recent interesting paper [16] by Wolfgang Reichel and Wolfgang Walter. This overlap concerns the study of radial solutions in Section 2.

Notation. The notation used is standard. The space of C^∞ test functions with compact support contained in Ω is denoted by $\mathcal{D}(\Omega)$. The space of distributions in Ω is denoted by $\mathcal{D}'(\Omega)$. The action of $u \in \mathcal{D}'(\Omega)$ on $\varphi \in \mathcal{D}(\Omega)$ is written $\langle u, \varphi \rangle$. For $u, v \in \mathcal{D}'(\Omega)$, $u \le v$ in $\mathcal{D}'(\Omega)$ means that $\langle u, \varphi \rangle \le \langle v, \varphi \rangle$ for all $0 \le \varphi \in \mathcal{D}(\Omega)$. In other words, $u \le v$ in $\mathcal{D}'(\Omega)$ means that the difference $\mu = v - u$ is a positive Radon measure in Ω. L^1_{loc}-functions are considered as distributions in the usual way. For $1 \le p \le \infty$, $W^{1,p}(\Omega)$ is the Sobolev space of all functions $u \in L^p(\Omega)$ whose first order distributional derivatives are also in $L^p(\Omega)$. $W^{1,p}_{\mathrm{loc}}(\Omega)$ is the corresponding local function space. $W^{1,p}_0(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$. A subharmonic function u in Ω is a not identically $-\infty$ upper semicontinuous function $u : \Omega \to [-\infty, \infty)$ satisfying the usual mean value inequalities. The euclidean ball with center x and radius r is denoted by $B(x, r)$.

1. Preliminary results

Later it will be convenient to have available precise statements of the following two comparison principles. For the sake of completeness we include some details of proof.

Proposition 1.1. *Let Ω be a domain in $\mathbb{R}^n$. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Let $v, w \in C(\Omega)$ and assume that $\Delta v - f(v) \geq \Delta w - f(w)$ in $\mathcal{D}'(\Omega)$. If $\liminf_{x\to y}(w(x) - v(x)) \geq 0$ for every $y \in \partial\Omega$, then $v \leq w$ in Ω. Here $\infty \in \partial\Omega$ if Ω is unbounded.*

Proof. Let $u = v - w$. Assume that u has a positive maximum at some point x_0 in Ω. A computation shows that $\Delta u \geq 0$ near x_0. Hence, u is subharmonic near x_0, which is a contradiction by the maximum principle. □

Proposition 1.2. *Let Ω be a bounded domain in $\mathbb{R}^n$, $1 < p < \infty$, and let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Let $v, w \in C(\Omega) \bigcap W^{1,p}(\Omega)$ be bounded functions such that*

$$\Delta_p v - f(v) \geq \Delta_p w - f(w) \quad \text{in } \mathcal{D}'(\Omega).$$

If $v \leq w$ on $\partial\Omega$ in the sense that $(v - w)^+ = \max(v - w, 0) \in W_0^{1,p}(\Omega)$, then $v \leq w$ in Ω.

Proof. [Sketch of proof] By assumption we know that

$$\int \left(|\nabla v|^{p-2}\nabla v - |\nabla w|^{p-2}\nabla w\right) \cdot \nabla\varphi dx \leq \int (f(w) - f(v))\, \varphi dx$$

whenever $0 \leq \varphi \in \mathcal{D}(\Omega)$. By an approximation argument using Corollary 9.1.5 in [1], we get

$$\begin{aligned} &\int \left(|\nabla v|^{p-2}\nabla v - |\nabla w|^{p-2}\nabla w\right) \cdot \nabla\,(v - w)^+\, dx \\ &\leq \int (f(w) - f(v))\,(v - w)^+\, dx \leq 0. \end{aligned}$$

We now use that $\nabla(v - w)^+ = \nabla v - \nabla w$ when $v > w$ and $\nabla(v - w)^+ = 0$ when $v \leq w$ (see Theorem 7.8 in [9]). Thus,

$$\int_{v>w} \left(|\nabla v|^{p-2}\nabla v - |\nabla w|^{p-2}\nabla w\right) \cdot (\nabla v - \nabla w)\, dx \leq 0.$$

It is well-known that $(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi - \eta) > 0$ whenever $\xi \neq \eta$, $\xi, \eta \in \mathbb{R}^n$. Hence, $\nabla v = \nabla w$ when $v > w$, which yields $(v - w)^+ = 0$. Thus $v \leq w$. □

The next theorem is concerned with a function Φ_p that will naturally occur later.

Theorem 1.1. *Let $1 < p < \infty$ and $-\infty \leq \tau < \infty$. Let $f : (\tau, \infty) \to (0, \infty)$ be an increasing function and assume that $\int^\infty F(t)^{-1/p} dt < \infty$, where $F' = f$. Put $F(\tau) = \lim_{u\to\tau+} F(u)$. Consider the function Φ_p defined by*

$$\Phi_p(u) = \int_u^\infty \left[\frac{p}{p-1}\big(F(t) - F(u)\big)\right]^{-1/p} dt, \quad \tau < u < \infty.$$

Then Φ_p is a decreasing continuous function on (τ, ∞) having the following limits:

$$\lim_{u\to\infty} \Phi_p(u) = 0, \quad \lim_{u\to\tau} \Phi_p(u) = \infty \quad \textit{if } \tau = -\infty, \quad \textit{and}$$

$$\lim_{u\to\tau} \Phi_p(u) = \int_\tau^\infty \Big[\frac{p}{p-1}\big(F(t) - F(\tau)\big)\Big]^{-1/p} dt \quad \textit{if } \tau \in \mathbb{R}.$$

Proof. By the change of variables $y = F(t) - F(u)$ (see Chapter 8 in [17]) we see that

$$\Phi_p(u) = \int_0^\infty \frac{1}{f\,(F^{-1}(y + F(u)))} \Big(\frac{p-1}{py}\Big)^{1/p} dy, \quad u > \tau. \tag{1.1}$$

Now note that $f(F^{-1})$ is increasing and that $\lim_{x\to\infty} f(F^{-1}(x)) = \infty$. By monotonicity of $f(F^{-1})$ it is clear from (1.1) that Φ_p is decreasing. Continuity and the computation of the limit of $\Phi_p(u)$ as $u \to \infty$ follows by the Lebesgue dominated convergence theorem. To compute the limit of $\Phi_p(u)$ as $u \to \tau$ we appeal to the monotone convergence theorem and obtain that

$$\lim_{u\to\tau} \Phi_p(u) = \int_0^\infty \frac{1}{f(F^{-1}(y + F(\tau)))} \Big(\frac{p-1}{py}\Big)^{1/p} dy. \tag{1.2}$$

If $F(\tau) = -\infty$ (and $\tau = -\infty$) the integral in (1.2) divergent and we have that $\lim_{u\to\tau} \Phi_p(u) = \infty$. Assume next that $F(\tau) > -\infty$. By the change of variables $y = F(t) - F(\tau)$ in (1.2) we see that

$$\lim_{u\to\tau} \Phi_p(u) = \int_\tau^\infty \Big[\frac{p}{p-1}\big(F(t) - F(\tau)\big)\Big]^{-1/p} dt.$$

If also $\tau = -\infty$ this last integral is clearly divergent. □

Remark 1.1. The function Φ_p in Theorem 1.1 is strictly decreasing, that is, $\Phi_p(u_1) > \Phi_p(u_2)$ if $u_1 < u_2$. To prove this, let $u_1 < u_2$ and note that then $F(u_1) < F(u_2)$. Since $f(F^{-1})$ is increasing and non-constant in every neighborhood of ∞ the set

$$\{y \in (0, \infty) : f(F^{-1}(y + F(u_1))) < f(F^{-1}(y + F(u_2)))\}$$

has positive measure. Now (1.1) yields that $\Phi_p(u_1) > \Phi_p(u_2)$.

Remark 1.2. We remark that when $\tau \in \mathbb{R}$ either of the cases

$$\lim_{u\to\tau} \Phi_p(u) < \infty \quad \text{or} \quad \lim_{u\to\tau} \Phi_p(u) = \infty$$

can occur.

2. Radial solutions

In this section we discuss radial solutions of equations involving the p-Laplacian. Recall that if $u(x) = \varphi(|x|)$, where $u \in W^{1,p}_{\mathrm{loc}}(B(0,R) \setminus \{0\})$, then

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = r^{1-n}\left(r^{n-1}|\varphi'(r)|^{p-2}\varphi'(r)\right)' \quad \text{in } B(0,R) \setminus \{0\}, \tag{2.1}$$

where $r = |x|$. The validity of this formula is clear by theory of pullback of distributions, see Section 6.1 in [11]. We emphasize that the parameter p is always assumed to be in the range $1 < p < \infty$ and $n \geq 2$.

Let $-\infty \leq \tau < \infty$. Let $f : (\tau, \infty) \to (0, \infty)$ be an increasing function. In search of radial solutions of $\Delta_p u = f(u)$ one is naturally led to study the initial value problem

$$\begin{aligned} &r^{1-n}\left(r^{n-1}|u'(r)|^{p-2}u'(r)\right)' = f(u(r)), \\ &r \in (0,R), \quad u'(0) = 0, \quad u(0) = u_0, \end{aligned} \tag{2.2}$$

where $u_0 > \tau$. A sufficient regularity of u for this discussion is $u \in C^1[0,R)$ and the equation in (2.2) is to be interpreted in the distributional sense. Multiplying the equation in (2.2) by r^{n-1} and integrating over $(0,r)$ we see that

$$r^{n-1}|u'(r)|^{p-2}u'(r) = \int_0^r t^{n-1} f(u(t))dt.$$

From this we have $u'(r) > 0$ for $r > 0$. Thus,

$$u'(r) = \left(r^{1-n}\int_0^r t^{n-1} f(u(t))dt\right)^{1/(p-1)}, \quad r \in (0,R). \tag{2.3}$$

By integration of (2.3) we arrive at the equivalent integral formulation of (2.2), namely,

$$u(r) = u_0 + \int_0^r \left(s^{1-n}\int_0^s t^{n-1} f(u(t))dt\right)^{1/(p-1)} ds, \quad r \in [0,R). \tag{2.4}$$

Notice also that (2.2) can be simplified to

$$\begin{aligned} &r^{1-n}\left(r^{n-1}u'(r)^{p-1}\right)' = f(u(r)), \\ &r \in (0,R), \quad u'(0) = 0, \quad u(0) = u_0. \end{aligned} \tag{2.5}$$

We will need the following existence and uniqueness result.

Proposition 2.1. *Let $f : (\tau, \infty) \to (0, \infty)$ be an increasing continuous function and $u_0 > \tau$. Then problem (2.2) has a unique local solution u which can uniquely be continued to a solution defined on some right maximal interval $[0,R)$.*

For the proof of Proposition 2.1, we refer to Section 2 in [16] where also more general results can be found. Next we establish existence of a radial solution of $\Delta_p u = f(u)$ in $B(0,R)$ having a given constant boundary value. Existence of such solutions will be used in the proofs of Theorem 2.2 and Theorem 2.5 below.

Proposition 2.2. *Let $f:\mathbb{R}\to[0,\infty)$ be an increasing continuous function. Fix $0<R<\infty$ and $A\in\mathbb{R}$. Then there exists a function $u\in C^1[0,R]$ such that*

$$\begin{aligned} &r^{1-n}\left(r^{n-1}|u'(r)|^{p-2}u'(r)\right)' = f(u(r)),\\ &r\in(0,R),\quad u'(0)=0,\quad u(R)=A. \end{aligned} \tag{2.6}$$

If $A\geq 0$ and $f(0)=0$, then $u\geq 0$.

Proof. [Sketch of proof] By (2.3) the function u must satisfy the integral equation

$$u(r) = A - \int_r^R \left(s^{1-n}\int_0^s t^{n-1} f(u(t))dt\right)^{1/(p-1)} ds,\quad 0\leq r\leq R. \tag{2.7}$$

Consider the operator $T: C[0,R]\to C[0,R]$ defined by

$$(Tu)(r) = A - \int_r^R \left(s^{1-n}\int_0^s t^{n-1} f(u(t))dt\right)^{1/(p-1)} ds,\quad 0\leq r\leq R.$$

Clearly, a fixed point of T is a solution of (2.6). Using the theorem of Ascoli-Arzela it is straightforward to verify that $T: C[0,R]\to C[0,R]$ is a continuous compact operator. Note that $T(K)\subset K$ for $K=\{u\in C[0,R]: B\leq u\leq A\}$ and B suitably chosen. By the Schauder fixed point theorem (see Section 11.1 in [9]) T has a fixed point.

Next we prove the last assertion of the proposition. Assume that there exists an $R'\in(0,R]$ such that $u(R')=0$. We prove that then $u(r)=0$ for all $0\leq r\leq R'$. By (2.3), $u'\geq 0$ so that u is increasing. Now $0\leq f(u(r))\leq f(u(R'))=0$ for $0\leq r\leq R'$. By (2.4) we have $u(r)=u(0)$ for $0\leq r\leq R'$. Hence, $u(r)=0$ for $0\leq r\leq R'$. □

The next result is essentially due to J. B. Keller [12].

Theorem 2.1. *Let $f:(\tau,\infty)\to(0,\infty)$ be an increasing function and assume that $\int^\infty F(t)^{-1/p}dt<\infty$, where $F'=f$. Let $u\in C^1[0,R)$ be a solution of (2.2), where $u_0>\tau$. Then*

$$\int_{u(0)}^{u(r)} \left[\frac{p}{p-1}\big(F(t)-F(u(0))\big)\right]^{-1/p} dt \leq r \leq \tag{2.8}$$

$$\leq \sqrt[p]{n} \int_{u(0)}^{u(r)} \Big[\frac{p}{p-1}\big(F(t)-F(u(0))\big)\Big]^{-1/p} dt$$

for all $r \in (0,R)$.

Proof. We follow Keller [12], page 506-507. First notice that (2.3) yields that $u \in C^{1,1}(0,R)$. Also by (2.3) we have that $u'(r)^{p-1} \leq rf(u(r))/n$. Next we note that since $((u')^{p-1})' = (p-1)(u')^{p-2}u''$ in $\mathcal{D}'(0,R)$ (see Theorem 7.8 in [9]) the equation in (2.5) can be written

$$(p-1)u'(r)^{p-2}u''(r) + \frac{n-1}{r}u'(r)^{p-1} = f(u(r)), \quad r \in (0,R). \tag{2.9}$$

Here u'' is to be interpreted in the distributional sense as a function in $L^\infty_{\mathrm{loc}}(0,R)$. Inserting $0 \leq u'(r)^{p-1} \leq rf(u(r))/n$ in (2.9) we see that

$$\frac{1}{n}f(u(r)) \leq (p-1)u'(r)^{p-2}u''(r) \leq f(u(r)) \quad \text{for a.e. } r \in (0,R). \tag{2.10}$$

Now recall that $\int_0^r u'(t)^{p-1}u''(t)dt = u'(r)^p/p$ for $r \in [0,R)$. Indeed, this follows by Theorem 3.1.4 in [11] since $((u')^p)' = p(u')^{p-1}u''$ in $\mathcal{D}'(0,R)$. Multiplying (2.10) by $u'(r)$ and integrating over $(0,r)$, we get

$$\frac{1}{n}\left(F(u(r)) - F(u(0))\right) \leq \frac{p-1}{p}u'(r)^p \leq F(u(r)) - F(u(0)).$$

Taking p-th roots, dividing by $\sqrt[p]{F(u(r)) - F(u(0))}$ and integrating over $(0,r)$, we obtain (2.8). □

Corollary 2.1. *Let* f *and* u *be as in Proposition* 2.1 *and assume in addition that* $\int^\infty F(t)^{-1/p}dt < \infty$, *where* $F' = f$. *Then* $R < \infty$, $u(r) \to \infty$ *as* $r \to R$, *and,* $\Phi_p(u(0)) \leq R \leq \sqrt[p]{n}\Phi_p(u(0))$, *where* Φ_p *is as in Theorem* 1.1.

Proof. Since the right hand side in (2.8) is bounded we must have $R < \infty$. Now since $R < \infty$ and u is increasing, the maximality of R yields that $u(r) \to \infty$ as $r \to R$. Letting $r \to R$ in (2.8) we obtain the estimates asserted by the corollary. □

Our next task is to prove that the blow-up R of (2.2) depends continuously on the initial value u_0. First we need a lemma.

Lemma 2.1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be increasing. Let* $u_j \in C^1[0,R_j)$ *be such that*

$$r^{1-n}\left(r^{n-1}|u_j'(r)|^{p-2}u_j'(r)\right)' = f(u_j(r)), \quad r \in (0,R_j), \quad u_j'(0) = 0$$

for $j = 1,2$. *If* $u_1(0) < u_2(0)$, *then* $u_1(r) < u_2(r)$ *for all* $0 < r < \min(R_1,R_2)$.

Proof. We rewrite the equation to its integral form, that is,

$$u_j(r) = u_j(0) + \int_0^r \Psi^{-1}\Big(s^{1-n}\int_0^s t^{n-1} f(u_j(t))dt\Big)ds, \tag{2.11}$$

where $\Psi(u) = |u|^{p-2}u$ for $u \in \mathbb{R}$, cf. (2.4). If the conclusion of the lemma is not true then there exists a smallest $R > 0$ such that $u_1(R) = u_2(R)$, but by (2.11) this is impossible. □

We inform the reader that a comparison principle somewhat more sophisticated than Lemma 2.1 can be found in Theorem 3 in [16].

Theorem 2.2. *Let $f : (\tau, \infty) \to (0, \infty)$ be an increasing continuous function such that $\int^\infty F(t)^{-1/p}dt < \infty$, where $F' = f$. Denote by $R(u_0)$ the maximum value of R in Proposition* 2.1. *Then $R : (\tau, \infty) \to (0, \infty)$ is a decreasing continuous function and $\Phi_p(u_0) \le R(u_0) \le \sqrt[p]{n}\Phi_p(u_0)$, where Φ_p is as in Theorem* 1.1.

Proof. The monotonicity of R is clear by Lemma 2.1. Also, the estimates of R follow by Corollary 2.1. Let u and u_0 be as in (2.2). We proceed to prove continuity at u_0 from the right. Let $\varepsilon > 0$ be given and choose $A > u(R(u_0) - \varepsilon)$. Let $w \in C^1[0, R(u_0) - \varepsilon]$ be the solution of the problem

$$r^{1-n}\left(r^{n-1}|w'(r)|^{p-2}w'(r)\right)' = f(w(r)), \quad w'(0) = 0, \quad w(R(u_0) - \varepsilon) = A,$$

see Proposition 2.2. By Lemma 2.1 and the uniqueness part of Proposition 2.1, we have $u_0 < w(0)$ and that $u_0 < v_0 < w(0)$ implies $R(u_0) - \varepsilon < R(v_0) \le R(u_0)$.

Next we prove continuity at u_0 from the left. Let v_j be an increasing sequence of positive numbers such that $v_j \to u_0$. It suffices to prove that then $R(v_j) \to R(u_0)$. Denote by u_j the solution of (2.2) having initial data v_j. By Lemma 2.1, we have that $u_j \le u_{j+1} \le u$. Set $\tilde{u}(r) = \lim_{j\to\infty} u_j(r)$ for $0 \le r < \lim_{j\to\infty} R(v_j)$. By a passage to the limit using the integral form (2.4) of (2.2) one verifies that $\tilde{u}$ is a solution of (2.2). It is also clear by construction that $\tilde{u}$ cannot be continued to the right. Thus $R(u_0) = \lim_{j\to\infty} R(v_j)$. □

We wish to mention here also the interesting result in Theorem 6 in [16] which asserts that under the assumptions of Theorem 2.2 the blow-up function R is, in fact, strictly decreasing, that is, $u_1 < u_2$ implies $R(u_1) > R(u_2)$.

Our next task is to construct a large subsolution of $\Delta_p u = f(u)$ in $B(0, R)$. This construction is accomplished by Theorem 2.4 below.

Theorem 2.3. *Let $-\infty \le \tau < \infty$ and $1 < p < \infty$. Let $f : (\tau, \infty) \to (0, \infty)$ be an increasing function such that $\int^\infty F(t)^{-1/p}dt < \infty$, where $F' = f$. Set*

$R^* = \lim_{u\to\tau}\Phi_p(u)$, where Φ_p is as in Theorem 1.1. Let $0 < R < R^*$ or $R = R^* < \infty$. Define $\varphi : [0,R) \to [\tau,\infty)$ by

$$\varphi^{-1}(x) = \int_{u_0}^{x}\Big[\frac{p}{p-1}\big(F(t)-F(u_0)\big)\Big]^{-1/p}dt, \quad x \geq u_0,$$

where $u_0 \in [\tau,\infty)$ is chosen such that

$$\int_{u_0}^{\infty}\Big[\frac{p}{p-1}\big(F(t)-F(u_0)\big)\Big]^{-1/p}dt = R.$$

Then $\varphi \in C^1[0,R)$, $\big(|\varphi'|^{p-2}\varphi'\big)' = f(\varphi)$ in $\mathcal{D}'(0,R)$, $\varphi'(0) = 0$ and $\varphi(r) \to \infty$ as $r \to R$. If $0 < R < R^*$, then $\varphi > \tau$ on $[0,R)$ whereas $\varphi(0) = \tau$ and $\varphi > \tau$ on $(0,R)$ if $R = R^* < \infty$.

Proof. It is straightforward to see that $\varphi(x) \to \infty$ as $x \to R$. It is also clear that φ is of type C^1 and that

$$\varphi'(x) = \Big[\frac{p}{p-1}\big(F(\varphi(x))-F(u_0)\big)\Big]^{1/p}.$$

By this formula it is clear that $\varphi'(0) = 0$. We proceed to compute $\big(|\varphi'|^{p-2}\varphi'\big)'$. Set $v = |\varphi'|^{p-2}\varphi' = (\varphi')^{p-1}$. It is well-known that

$$v'(x) = \lim_{h\to 0}(v(x+h)-v(x))/h,$$

where the limit is computed in $\mathcal{D}'(0,R)$. It is straightforward to see that $\lim_{h\to 0}(v(x+h)-v(x))/h = f(\varphi(x))$ pointwise whenever $\varphi(x)$ is a point of continuity of f. Thus, since f is increasing and φ is strictly increasing this pointwise limit holds for all but at most countably many $x \in (0,R)$. Next we observe that the difference quotients $(v(x+h)-v(x))/h$ are locally uniformly bounded for $x \in (0,R)$ as $h \to 0$. Indeed, for $x \in K \subset\subset (0,R)$ and h small we have that

$$\Big|\frac{v(x+h)-v(x)}{h}\Big| \leq \text{const.}\Big|\frac{F(\varphi(x+h))-F(\varphi(x))}{h}\Big| \leq \text{const.}$$

An application of the Lebesgue dominated convergence theorem now gives that

$$\int_0^R \frac{v(x+h)-v(x)}{h}\psi(x)dx \to \int_0^R f(\varphi(x))\psi(x)dx$$

for every $\psi \in \mathcal{D}(0,R)$. Thus, $v' = f(\varphi)$ in $\mathcal{D}'(0,R)$. The last assertion of the theorem is clear since $\varphi(x) > \varphi(0) = u_0$ for $x \in (0,R)$. □

We point out that φ in Theorem 2.3 is well-defined by Theorem 1.1 in the sense that the choice of u_0 is always possible.

Lemma 2.2. *Let $n \geq 2$. Let $u \in W^{1,\infty}(B(0,R))$ and $f \in L^1(B(0,R))$ be such that $\Delta_p u \geq f$ in $\mathcal{D}'(B(0,R) \setminus \{0\})$. Then $\Delta_p u \geq f$ in $\mathcal{D}'(B(0,R))$.*

Proof. Fix $0 \leq \varphi \in \mathcal{D}(B(0,R))$, let $0 \leq \chi \leq 1$ be a cut-off function for a neighborhood of the origin and set $\chi_\lambda(x) = \chi(\lambda x)$. By assumption we know that

$$\langle \Delta_p u, (1-\chi_\lambda)\varphi \rangle \geq \int (1-\chi_\lambda)\varphi f \, dx. \tag{2.12}$$

Clearly, the right hand side in (2.12) converges to $\int \varphi f \, dx$ as $\lambda \to \infty$. Notice that since $u \in W^{1,\infty}(B(0,R))$, $\Delta_p u$ satisfies an estimate

$$|\langle \Delta_p v, \psi \rangle| \leq C \int |\nabla \psi| dx$$

for $\psi \in \mathcal{D}(B(0,R))$. Hence,

$$|\langle \Delta_p u, \chi_\lambda \varphi \rangle| \leq C\Big(\lambda^{1-n} \int |\nabla \chi| dx + \int \chi_\lambda |\nabla \varphi| dx\Big) \to 0$$

as $\lambda \to \infty$. Thus, letting $\lambda \to \infty$ in (2.12) we obtain that $\langle \Delta_p u, \varphi \rangle \geq \langle f, \varphi \rangle$. □

Theorem 2.4. *Let f be as in the first paragraph in Section 3. Let $0 < R < \infty$. Let R^* be as in Theorem 2.3. Now define v as follows.*

(1) *If $f : \mathbb{R} \to (0,\infty)$, then we set $v(x) = \varphi(|x|)$ for $x \in B(0,R)$, where φ is as in Theorem 2.3 with $\tau = -\infty$.*
(2) *If $f : (0,\infty) \to (0,\infty)$ and $R \leq R^*$, then we set $v(x) = \varphi(|x|)$ for $x \in B(0,R)$, where φ is as in Theorem 2.3 with $\tau = 0$.*
(3) *If $f : (0,\infty) \to (0,\infty)$ and $R > R^*$, then we set $v(x) = \varphi(|x| - R + R^*)$ for $R - R^* \leq |x| < R$ and $v(x) = 0$ for $|x| < R - R^*$, where φ is as in Theorem 2.3 corresponding to $R = R^*$ and $\tau = 0$ there.*

Then v is in $C^1(B(0,R))$, $\Delta_p v \geq f(v)$ in $\mathcal{D}'(B(0,R))$ and $v(x) \to \infty$ as $|x| \to R$.

Proof. Since $\varphi \in C^1[0,R)$, $\varphi'(0) = 0$, and, $\varphi(0) = 0$ in Part 3, it is straightforward to see that $v \in C^1(B(0,R))$. It is also clear that $v(x) \to \infty$ as $|x| \to R$. By (2.1), we have that

$$\Delta_p v = \big(|v'(r)|^{p-2} v'(r)\big)' + \frac{n-1}{r} |v'(r)|^{p-2} v'(r) \quad \text{in } \mathcal{D}'(B(0,R)\setminus\{0\}), \tag{2.13}$$

where $r = |x|$. We now specialize to Part 1 and Part 2. Since in this case $v' \geq 0$ and $(|v'|^{p-2}v')' = f(v)$ in $\mathcal{D}'(0,R)$ we have by (2.13) that $\Delta_p v \geq f(v)$ in $\mathcal{D}'(B(0,R) \setminus \{0\})$. By Lemma 2.2, $\Delta_p v \geq f(v)$ in $\mathcal{D}'(B(0,R))$.

We next consider Part 3. To conclude as above that $\Delta_p v \geq f(v)$ in $\mathcal{D}'(B(0,R))$ we need to prove that $(|v'|^{p-2}v')' = f(v)$ in $\mathcal{D}'(0,R)$. By construction and Theorem 2.3 we know that $(|v'|^{p-2}v')' = f(v)$ in $\mathcal{D}'((0,R) \setminus \{R-R^*\})$. Let $\psi \in \mathcal{D}(0,R)$. Let $0 \leq \chi \leq 1$ be a cut-off function for a neighborhood of the origin and set $\chi_\lambda(r) = \chi(\lambda r)$ for $\lambda > 0$. Let $\tau = R - R^*$. It is clear that

$$\langle (|v'|^{p-2}v')', (1-\chi_\lambda(\cdot - \tau))\psi\rangle = \int f(v(r))\,(1-\chi_\lambda(r-\tau))\,\psi(r)dr. \quad (2.14)$$

It is also clear that the right hand side in (2.14) tends to $\int f(v)\psi dr$ as $\lambda \to \infty$. We write the left hand side in (2.14) as

$$\begin{aligned} &\langle (|v'|^{p-2}v')', (1-\chi_\lambda(\cdot - \tau))\psi\rangle \quad (2.15)\\ &= \langle (|v'|^{p-2}v')', \psi\rangle + \int |v'(r)|^{p-2}v'(r)\lambda\chi'(\lambda(r-\tau))\psi(r)dr \\ &+ \int |v'(r)|^{p-2}v'(r)\chi_\lambda(r-\tau)\psi'(r)dr. \end{aligned}$$

Clearly, the last term on the right hand side in (2.15) tends to 0 as $\lambda \to \infty$. Since v' is continuous, it is straightforward to see that the second term on the right hand side in (2.15) tends to

$$|v'(\tau)|^{p-2}v'(\tau)\psi(\tau)\int \chi'(r)dr = 0$$

as $\lambda \to \infty$. Now letting $\lambda \to \infty$ in (2.14), we get

$$\langle \left(|v'|^{p-2}v'\right)', \psi\rangle = \int f(v)\psi dr.$$

Thus, $(|v'|^{p-2}v')' = f(v)$ in $\mathcal{D}'(0,R)$. □

The next two approximation lemmas are needed in the proof of Theorem 2.5 below.

Lemma 2.3. *Let $f : \mathbb{R} \to (0,\infty)$ be an increasing function. Then there exists a sequence $f_j : \mathbb{R} \to (0,\infty)$, $j \geq 1$, of smooth increasing functions such that*

$$f(x-1/j) \leq f_j(x) \leq f_{j+1}(x) \leq f(x)$$

for $x \in \mathbb{R}$. In particular, if f satisfies the growth condition (3.1)*, then so does every f_j.*

Proof. Let $0 \leq \varphi \in \mathcal{D}(0,1)$, $\int \varphi(x)dx = 1$ and set $\varphi_\lambda(x) = \lambda\varphi(\lambda x)$ for $x \in \mathbb{R}$ and $\lambda > 0$. We now consider the regularizations f_λ of f defined by

$$f_\lambda(x) = f * \varphi_\lambda(x) = \int f(x - y/\lambda)\varphi(y)dy, \quad x \in \mathbb{R}.$$

It is straightforward to verify that the sequence f_j, $j = 1, 2, \ldots$, satisfies the assertions of the lemma. □

Lemma 2.4. *Let $f : (0,\infty) \to (0,\infty)$ be an increasing function. Then there exists a sequence $f_j : (0,\infty) \to (0,\infty)$, $j \geq 1$, of smooth increasing functions such that*

$$f(x/e^{1/j}) \leq f_j(x) \leq f_{j+1}(x) \leq f(x)$$

for $x \in (0,\infty)$. In particular, if f satisfies the conditions $\lim_{x\to 0+} f(x) = 0$ and (3.1), then so does every f_j.

Proof. Set $g(x) = f(e^x)$ for $x \in \mathbb{R}$. Clearly, $g : \mathbb{R} \to (0,\infty)$ is increasing. Let $\{g_j\}$ be an approximating sequence for g as asserted by Lemma 2.3 and let $f_j(x) = g_j(\log x)$ for $x \in (0,\infty)$. It is now straightforward to verify that the f_j's fulfill the requirements of the lemma. □

Theorem 2.5. *Let $0 < R < \infty$, $1 < p < \infty$ and $n \geq 2$. Let f be as in the first paragraph in Section 3. Then in the ball $B(0,R)$ there exists a (unique) large solution $u = u_R$ of the equation $\Delta_p u = f(u)$ which is maximal in the sense that $u_R \geq v$ for every $v \in C(B(0,R)) \bigcap W^{1,p}_{loc}(B(0,R))$ such that $\Delta_p v \geq f(v)$ in $\mathcal{D}'(B(0,R))$. Furthermore, if R_j increases to R, then $u_{R_j}(x)$ decreases to $u_R(x)$ pointwise in $B(0,R)$.*

Proof. [Proof of Theorem 2.5] We first assume in addition that f is continuous. We proceed to show that if $f : (\tau,\infty) \to (0,\infty)$ and $0 < R < R^*$, where R^* is as in Theorem 2.3, then there exists a radial function $u \in C^1(B(0,R))$ such that

$$\Delta_p u = f(u) \quad \text{in } B(0,R), \quad \text{and,} \quad u(x) \to \infty \quad \text{as } |x| \to R. \tag{2.16}$$

By Proposition 2.1 we have for every $u_0 \in (\tau,\infty)$ a solution u of (2.2) defined on some maximal interval $[0, R(u_0))$. By Theorem 2.2 and Theorem 1.1 we know that $u_0 \mapsto R(u_0)$ is a continuous decreasing function such that $\lim_{u_0\to\infty} R(u_0) = 0$ and $\lim_{u_0\to\tau} R(u_0) \geq R^*$. Thus, there exists a solution u of (2.2) with $R(u_0) = R$. Writing $u(x) = u(|x|)$ for $x \in B(0,R)$ we obtain a C^1-function u in $B(0,R)$ satisfying (2.16) except that the equation is only known to be satisfied in $B(0,R) \setminus \{0\}$. By Lemma 2.2, (2.16) holds.

Next, we consider the case when $f : (0,\infty) \to (0,\infty)$ is continuous and $R \geq R^*$. We show that in this case there exists a radial function $u \in C^1(B(0,R))$ satisfying (2.16). By Proposition 2.2 and Lemma 2.2 there exist radial functions u_k such that $\Delta_p u_k = f(u_k)$ in $B(0,R)$ and $u_k = k$ on $\partial B(0,R)$. By Proposition 1.2, $u_k \leq u_{k+1}$. By the first part of the proof we know that there exist large solutions of $\Delta_p u = f(u)$ in all sufficiently small balls. Thus, by Proposition 1.2, the sequence $\{u_k\}$ is bounded from above on compact subsets of $B(0,R)$. We now define u by $u(x) = \lim_{k\to\infty} u_k(x)$ for $x \in B(0,R)$. It is clear that u is radial and that $u(x) \to \infty$ as $|x| \to R$. By (2.4) we have that

$$u_k(r) = u_k(0) + \int_0^r \left(s^{1-n}\int_0^s t^{n-1} f(u_k(t))dt\right)^{1/(p-1)} ds, \quad r \in [0,R). \tag{2.17}$$

A passage to the limit in (2.17) now gives that $u \in C^1(B(0,R))$ and $\Delta_p u = f(u)$ in $B(0,R)\setminus\{0\}$. By Lemma 2.2, (2.16) holds.

We now consider the general case. Let $\{f_j\}$ be an approximation of f as asserted by Lemma 2.3 or Lemma 2.4. Let $0 < R_j \le R_{j+1} < R$ be such that $R_j \to R$. Let $u_j \in C^1(B(0,R_j))$ be a radial function such that

$$\Delta_p u_j = f_j(u_j) \quad \text{in } B(0,R_j), \quad u_j(x) \to \infty \quad \text{as } |x| \to R_j. \tag{2.18}$$

By Proposition 1.2, we have that $u_j \ge u_{j+1} \ge v$, where v is as in Theorem 2.4. We now define u by $u(x) = \lim_{j\to\infty} u_j(x)$ for $x \in B(0,R)$. Clearly, $u(x) \to \infty$ as $|x| \to R$.

Our next task is to prove that $u \in C^1(B(0,R))$ and $\Delta_p u = f(u)$. The proof of this assertion goes via (2.20) below. Fix $0 \le r_1 < r_2 < R$. By (2.3) we have that

$$u_j(r_2) - u_j(r_1) = \int_{r_1}^{r_2} \left(s^{1-n}\int_0^s t^{n-1} f_j(u_j(t))dt\right)^{1/(p-1)} ds. \tag{2.19}$$

If the f_j's are as in Lemma 2.3, we have that $f_j(u_j) \ge f(u_j - 1/j) \ge f(u - 1/j)$ and $f_j(u_j) \le f(u_j)$. A passage to the limit in (2.19) yields that

$$\begin{aligned}\int_{r_1}^{r_2} \left(s^{1-n}\int_0^s t^{n-1} f(u(t)-)dt\right)^{1/(p-1)} ds &\le u(r_2) - u(r_1) \\ &\le \int_{r_1}^{r_2} \left(s^{1-n}\int_0^s t^{n-1} f(u(t)+)dt\right)^{1/(p-1)} ds.\end{aligned} \tag{2.20}$$

Similarly, if the f_j's are as in Lemma 2.4, we have that $f_j(u_j) \ge f(u_j e^{-1/j}) \ge f(ue^{-1/j})$ and $f_j(u_j) \le f(u_j)$, and (2.20) follows by a passage to the limit. We point out that when $f : (0,\infty) \to (0,\infty)$, we have written $f(0-) = 0$ in (2.20).

First assume that $f : \mathbb{R} \to (0,\infty)$. By (2.20) the function u is strictly increasing and continuous. Since f is increasing, $f(u(r)-) = f(u(r)+) = f(u(r))$ for all but at most countably many r's. Thus, we have equalities in (2.20) which yields that

$$u(r) = u(0) + \int_0^r \left(s^{1-n}\int_0^s t^{n-1} f(u(t))dt\right)^{1/(p-1)} ds, \quad 0 \le r < R.$$

Hence $u \in C^1(B(0,R))$ and $\Delta_p u = f(u)$ in $B(0,R) \setminus \{0\}$. By Lemma 2.2, $\Delta_p u = f(u)$ in $B(0,R)$.

Next we consider the case when $f : (0,\infty) \to (0,\infty)$. Since u is increasing and $u(r) \to \infty$ as $r \to R$, there exists a $\tau \in [0,R)$ such that $u = 0$ on $(0,\tau)$ and $u > 0$ on (τ,R). Obviously, $f(u(r)-) = f(u(r)+) = f(u(r)) = 0$ for $r \in (0,\tau)$. As in the previous paragraph we see that $f(u(r)-) = f(u(r)+) = f(u(r))$ for all but at most countably many $r \in (\tau,R)$. Hence we have equalities in (2.20) and the conclusion that $u \in C^1(B(0,R))$ and $\Delta_p u = f(u)$ follows as above.

It is easy to see that u constructed as above has the extremal property asserted by the theorem. Indeed, if $\Delta_p v_1 \geq f(v_1)$ in $B(0,R)$, then $\Delta_p v_1 \geq f_j(v_1)$ and by Proposition 1.2 we have that $u_j \geq v_1$ in $B(0,R_j)$. By a passage to the limit we obtain that $u \geq v_1$ in $B(0,R)$.

To prove the last assertion of the theorem it is enough to verify that $\tilde{u}$ defined by $\tilde{u}(x) = \lim_{j\to\infty} u_{R_j}(x)$ for $x \in B(0,R)$ is such that $\Delta_p \tilde{u} = f(\tilde{u})$ in $B(0,R)$. The proof of this last assertion is similar to the above proof that $u \in C^1(B(0,R))$ and $\Delta_p u = f(u)$. The details are omitted. □

Remark 2.3. Clearly u_R is uniquely determined by its extremal property stated in Theorem 2.5. Also, by a rotational argument, if such a u_R exists it must be radial. It is also clear by the same extremal property that $u_R \geq v$ in $B(0,R)$, where v is as in Theorem 2.4. These facts are also clear by the construction of u_R in the proof of Theorem 2.5.

3. Estimates of Osserman-Keller type

We first fix the notation and basic assumptions used in the main results below. Let $n \geq 2$ and $1 < p < \infty$. Let $f : \mathbb{R} \to (0,\infty)$ or $f : (0,\infty) \to (0,\infty)$ be an increasing function. We assume that f satisfies the growth condition

$$\int^{\infty} F(t)^{-1/p} dt < \infty, \quad \text{where } F' = f. \tag{3.1}$$

In the case when $f : (0,\infty) \to (0,\infty)$ we also assume that

$$f(0) = \lim_{u\to 0+} f(u) = 0.$$

In (3.1) the function F can be any primitive function of f. It is clear that the convergence of the integral in (3.1) is independent of this particular choice of F. It is also easy to see that (3.1) is equivalent to $\int^{\infty} (tf(t))^{-1/p} dt < \infty$.

By Theorem 2.5 there exists a (unique) maximal large solution $u = u_R$ in $B(0,R)$ of the equation $\Delta_p u = f(u)$. We now define a function g_p on $(0,\infty)$ by $g_p(R) = u_R(0)$. For $p = 2$ we write $g = g_2$. It is clear that $g_p : (0,\infty) \to \mathbb{R}$

if $f : \mathbb{R} \to (0, \infty)$ and that $g_p : (0, \infty) \to [0, \infty)$ if $f : (0, \infty) \to (0, \infty)$. Our first concern is the limit behavior of g_p.

Theorem 3.1. *Let $n \geq 2$ and $1 < p < \infty$. Let f and g_p be as above. Then g_p is decreasing, leftcontinuous, and has the following limit behavior:*

(1) *If $f : \mathbb{R} \to (0, \infty)$, then $\lim_{R\to\infty} g_p(R) = -\infty$, $\lim_{R\to 0} g_p(R) = \infty$ and*

$$\Phi_p(g_p(R)) \leq R \leq \sqrt[p]{n}\Phi_p(g_p(R)) \tag{3.2}$$

holds for all $R > 0$, where Φ_p is as in Theorem 1.1 with $\tau = -\infty$.

(2) *If $f : (0, \infty) \to (0, \infty)$ and $\int_{0+} F(t)^{-1/p} dt = \infty$, where $F(t) = \int_0^t f(s)ds$, then $g_p(R) > 0$ for all $R > 0$,*

$$\lim_{R\to\infty} g_p(R) = 0, \quad \lim_{R\to 0} g_p(R) = \infty$$

and (3.2) holds for all $R > 0$, where Φ_p is as in Theorem 1.1 with $\tau = 0$.

(3) *If $f : (0, \infty) \to (0, \infty)$ and $\int_{0+} F(t)^{-1/p} dt < \infty$, where F is as in Part 2, then $g_p(R) = 0$ for $R \geq \sqrt[p]{n}R^*$, $g_p(R) > 0$ for $0 < R < R^*$, $\lim_{R\to 0} g_p(R) = \infty$ and (3.2) holds for $0 < R < R^*$. Here Φ_p is as in Theorem 1.1 with $\tau = 0$ and $R^* = \lim_{u\to 0} \Phi_p(u)$.*

Proof. The monotonicity and leftcontinuity of g_p is clear by Theorem 2.5. Next we consider Part 1. Substituting $u = u_R$ in (2.8) in Theorem 2.1 we obtain (3.2) by letting $r \to R$. Now by Theorem 1.1 we have that $\lim_{R\to 0} g_p(R) = \infty$ and $\lim_{R\to\infty} g_p(R) = -\infty$.

Let us consider Part 2. Since $u_R \geq v$, where v is as in Part 2 in Theorem 2.4, it is clear that u_R and g_p are strictly positive. The rest of the proof of Part 2 is similar to the proof of Part 1. The details are omitted.

Let us consider Part 3. Assume first that $u_R(0) > 0$. Substituting $u = u_R$ in (2.8) in Theorem 2.1 and passing to the limit as $r \to R$, we obtain that $R \leq \sqrt[p]{n}\Phi_p(u_R(0)) < \sqrt[p]{n}R^*$, where the last strict inequality follows by Remark 1.1. Thus, $g_p(R) = u_R(0) = 0$ for $R \geq \sqrt[p]{n}R^*$. For $0 < R < R^*$ the strict positivity of u_R follows by $u_R \geq v$, where v is as in Part 2 in Theorem 2.4. As above (3.2) follows by a passage to the limit using (2.8). By Theorem 1.1, we have that $\lim_{R\to 0} g_p(R) = \infty$. □

Remark 3.1. Let F be as in Part 2. It is easy to see that $\int_{0+} F(t)^{-1/p} dt < \infty$ is equivalent to $\int_{0+} (tf(t))^{-1/p} dt < \infty$.

Remark 3.2. If f is continuous, then g_p is continuous. We sketch here how this assertion follows by a result in [16]. By Theorem 3.1 it suffices to verify that g_p is rightcontinuous at $R_0 \in (0, \infty)$. Also, since g_p is decreasing, we can assume that $g_p(R_0) > 0$. Let $0 < \varepsilon < g_p(R_0)$. Let u_1 be the solution of (2.2) with initial data $u_1(0) = g_p(R_0) - \varepsilon$ and denote by R_1 the blow-up of u_1. Now $R_1 > R_0$, see the first paragraph after the proof of Theorem 2.2, and, $R_1 > R > R_0$ implies $g_p(R_0) - \varepsilon < g_p(R) < g_p(R_0)$.

Next we state our main results.

Theorem 3.2. *Let $n \geq 2$ $(p = 2)$ and let f and g be as above. Then if v is a continuous function in a domain Ω in $\mathbb{R}^n$ such that*

$$\Delta v \geq f(v) \quad \text{in } \mathcal{D}'(\Omega),$$

then the following estimate holds:

$$v(x) \leq g(\operatorname{dist}(x, \partial\Omega)), \quad x \in \Omega. \tag{3.3}$$

Proof. Let $a \in \Omega$ and let $0 < R < \operatorname{dist}(a, \partial\Omega)$. By Proposition 1.1, we have that $v \leq u_R(\cdot - a)$ in $B(a, R)$. In particular, $v(a) \leq u_R(0) = g(R)$. Letting $R \to \operatorname{dist}(a, \partial\Omega)$ using the leftcontinuity of g the theorem follows. □

Remark 3.3. We point out that when $f : (0, \infty) \to (0, \infty)$ we require in Theorem 3.2 that $v(x) \geq 0$ for all $x \in \Omega$. This requirement is needed for the composition $f(v)$ to make sense. The same remark applies to Theorem 3.3 and Theorem 3.4 below.

Remark 3.4. Taking $v = u_R$ and $\Omega = B(0, R)$ in Theorem 3.2 it is clear that g as above is the smallest function satisfying the assertion of Theorem 3.2. The same observation applies to Theorem 3.4 below.

If f is also convex we have the following stronger version of Theorem 3.2.

Theorem 3.3. *Let $n \geq 2$ $(p = 2)$ and let f and g be as above. Assume further that f is convex. If v is a subharmonic function in a domain Ω in $\mathbb{R}^n$ such that $\Delta v \geq f(v)$ in $\mathcal{D}'(\Omega)$, then (3.3) holds.*

Proof. Let $0 \leq \psi \in \mathcal{D}(B(0,1))$ be a radial test function with $\int \psi(x)dx = 1$ and let $\psi_\varepsilon(x) = \psi(x/\varepsilon)/\varepsilon^n$ for $\varepsilon > 0$. We consider the regularizations v_ε of v defined by

$$v_\varepsilon(x) = v * \psi_\varepsilon(x) = \int v(y)\psi_\varepsilon(x - y)dy$$

for $x \in \Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$. It is well-known that the v_ε's are smooth subharmonic functions and that $v_\varepsilon(x)$ decreases to $v(x)$ as $\varepsilon \to 0$. By Jensen's inequality we have that

$$f(v_\varepsilon(x)) \leq \int f(v(y))\psi_\varepsilon(x - y)dy \leq \int \psi_\varepsilon(x - y)\Delta v(dy) = \Delta v_\varepsilon(x)$$

for $x \in \Omega_\varepsilon$. Theorem 3.2 now yields that $v_\varepsilon(x) \leq g(\operatorname{dist}(x, \partial\Omega_\varepsilon))$. Letting $\varepsilon \to 0$ using the leftcontinuity of g we obtain that $v(x) \leq g(\operatorname{dist}(x, \partial\Omega))$. □

The next theorem is our p-Laplace version of Theorem 3.2.

Theorem 3.4. *Let $n \geq 2$, $1 < p < \infty$ and let f and g_p be as above. Let Ω be a domain in $\mathbb{R}^n$. Then for every $v \in C(\Omega) \bigcap W^{1,p}_{loc}(\Omega)$ such that*

$$\Delta_p v \geq f(v) \quad \textit{in } \mathcal{D}'(\Omega),$$

the following estimate holds:

$$v(x) \leq g_p(\mathrm{dist}(x, \partial\Omega)), \quad x \in \Omega.$$

The proof of Theorem 3.4 follows the same lines as the proof of Theorem 3.2 with the modification that Proposition 1.2 is used instead of Proposition 1.1. The details are omitted.

We conclude this section with a review of two cases in which the function g_p is known. First we consider the equation $\Delta_p u = e^u$. Denote by u_R the maximal solution in the ball $B(0, R)$. We have that $u_R(x) = u_1(x/R) + \log(R^{-p})$. Thus, $g_p(R) = u_R(0) = u_1(0) + \log(R^{-p})$. In the special case $p = n$ an explicit formula for u_R is known, see page 126 in [10].

Next we consider the equation $\Delta_p u = u^q$, where $q > p - 1$. Here we have that $u_R(x) = R^{-p/(q-p+1)} u_1(x/R)$, where u_R is the maximal solution in $B(0, R)$. Thus, $g_p(R) = u_R(0) = u_1(0) R^{-p/(q-p+1)}$. In the special case where $1 < p < n$ and $q = (pn + p - n)/(n - p)$ an explicit formula for u_R is known, see page 126 in [10].

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