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# UNIFORM DECAY RATES OF SOLUTIONS TO A SYSTEM OF COUPLED PDES WITH NONLINEAR INTERNAL DISSIPATION

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**Abstract.** We consider a coupled system of partial differential equations (PDEs) of hyperbolic/parabolic type, which is a generalization of an established structural acoustic model. A peculiar feature of this model is a high damping term in the boundary condition of the wave component which accounts for lack of uniform stability of the overall system, even in the presence of viscous damping on the entire domain. It has been shown recently that by introducing a comparable static damping in the boundary condition of the wave component, the corresponding feedback system is uniformly stable. In this paper we study the stability properties of the coupled system when the internal damping is subject to nonlinear effects. We provide two main different stability results, which describe decay rates of the solutions to the coupled PDE system via the solution to an appropriate nonlinear ordinary differential equation. Our analysis allows saturation of the nonlinear dissipation term. In this significant case we obtain as well uniform decay estimates of the underlying energy, provided that initial data are measured with a slightly stronger topology. Even though the prime concern of this paper is to deal with coupled structures and the overdamping phenomenon, these results yield, as well, new results on uniform stabilization of a single wave equation.

## 1. Introduction

The present paper studies the stability properties of a coupled system of partial differential equations (PDEs) of different type, which is a (nonlinear) generalization of a mathematical model describing the interactions between a vibrating plate and an enclosed acoustic field [30]. The basic PDE system

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comprises a wave equation on some smooth domain $\Omega \subset \mathbb{R}^3$ and an elastic plate-like equation on a portion of the boundary of $\Omega$, denoted by $\Gamma_0$, where coupling takes place through velocity traces. A technical description of the model under consideration, namely system (2.1), will be given in the next section.

In the last decade linear versions of this model—the "structural acoustic model"—have been an object of intensive experimental and numerical studies, motivated by the problem of reducing noise and vibrations in aircraft cabins, through the implementation of smart material actuators (see, e.g., [15], [6]). These studies revealed a need for a theoretical approach in the analysis of these models, so as to establish well-posedness and regularity results for the uncontrolled system, as well as to insure solvability of related optimal control problems.

In the seminal work [3] positive answers are given with respect to the aforementioned issues, as a key estimate is established for the model under consideration, namely a system of the form (2.1), with $\alpha = 1$ (i.e., Kelvin-Voight damping in the elastic equation); hence $D_0 \equiv 0$ (i.e., no high damping term in the boundary condition of the wave component), and $\beta \equiv 0$, while $g$ is a linear function. The results of [3], and subsequently [19], have in fact pushed forward the theory of Riccati equations with unbounded coefficients, for a completely new class of (linear) abstract dynamics, beside the basic parabolic-like and hyperbolic-like classes (*cf.* [22, 8, 23]). The key feature of this abstract class, which covers systems of coupled PDEs with an analytic component, but which are not themselves analytic, is validity of a so-called *singular estimate* (*cf.* [3, 19, 24, 25], and [33]). Starting from [3], a wealth of results regarding various PDE systems with hyperbolic/parabolic coupling, including thermoelastic problems, have been obtained. The reader is referred to the monograph [19] for a comprehensive treatment, and to the numerous references therein.

Paper [13] considers an important generalization of the structural acoustic model in [3], with a more general (structural) damping of the plate component, represented by term $\rho\mathcal{A}^\alpha v_t$ in the elastic $v$-equation (see (2.3) below). More precisely, the parameter $\alpha$ is allowed to run over the entire range of analyticity of the (free) dynamics, i.e., $\alpha \in [\frac{1}{2}, 1]$ ([14]), in contrast with [3], where $\alpha = 1$.

In order to attain (i) validity of singular estimates for the controlled dynamics and (ii) uniform stability properties of the free system, [13] undertakes a deep and complex analysis of the open-loop dynamics (i.e., system (2.1) with $\beta = 0$ and $g \equiv 0$), yielding positive answers with respect to both goals, provided that, as $\alpha$ decreases from 1 to $\frac{1}{2}$, suitable feedback stabilizers are active into the system. In fact, it turns out that when $D_0$ is an operator of order $s > 1$, the high damping term $D_0 z_t$ (in the boundary condition of the wave component) has a *destabilizing* effect on the overall system (this is referred to as an "overdamping" phenomenon). Consequently, when $D_0$ is an operator of order $s > 1$, the ultimate model displays both viscous damping in the interior of the domain and an additional (comparable) static damping in the boundary condition of the wave component, and it is given by system (2.1) with $g(s) = d_2\, s$, $d_2 > 0$.

The same model with (linear) localized damping has been studied in [12].

The present paper is focused on stability properties of the latter model, in the case when the internal damping is subject to nonlinear effects. As a result, the hyperbolic component of the system is a wave equation with nonlinear (internal) dissipation; see (2.1). More precisely, our aim is to achieve *uniform* decay estimates of the energy $E(\vec{z}, \vec{v}, t)$ of solutions $[\vec{z}, \vec{v}]$ of the coupled PDE system (2.1).

We shall follow the technical approach developed in the fundamental work [21], and successfully adopted later in several papers (see, e.g., [2, 5], [19] and references therein), now thoroughly described in [19]. Namely, we shall use the multiplier method in order to derive appropriate estimates for the energy functional $\int_0^T E(\vec{z}, \vec{v}, t)dt$, rather than differential inequalities with respect to a particular Liapunov function, as in the more classical approach of [18] and references therein. This way, we derive uniform decay estimates for the energy $E(\vec{z}, \vec{v}, t)$ by means of the solution to an associated ordinary differential equation (ODE), under minimal growth conditions on the nonlinear function $g$ (see assumption (H2) of section 2, and successive (H2)'). More precisely,

- as regards regularity properties, it is required that the nonlinear function $g$ be just (monotone and) continuous;
- we do not assume any kind of growth condition on $g$ near the origin;
- we allow maximal growth of $g$ at infinity (up to power five).

Going into more details, our main results for the coupled system (2.1) are given by two different stability results, namely Theorem 2.4 and Theorem 2.8 below. Theorem 2.4 provides uniform decay estimates of the energy $E(\vec{z}, \vec{v}, t)$ of weak solutions of system (2.1). Theorem 2.8 is focused on the important case when the nonlinear term is subject to *saturation*, so that the *lower* growth condition on $g$ at infinity needs to be removed. Then, retaining the properties listed above,

- it is *not* assumed that there exists $m > 0$ such that $s\, g(s) \geq m\, s^2$ for $|s|$ large.

In this case, we are able to obtain uniform decay estimates of the energy $E(\vec{z},\vec{v},t)$ of solutions of system (2.1) depending on *slightly* higher norms of initial data.

The proof of both Theorem 2.4 and Theorem 2.8 proceeds through several steps. The most challenging step is to derive appropriate estimates of the *wave* energy functional (see Proposition 3.3 and Proposition 3.5, respectively). In turn, the proof of Proposition 3.5 (hence, of Theorem 2.8) critically relies on a regularity result for the velocity term $z_t$ of solutions $[\vec{z},\vec{v}]$ to the coupled system (2.1) with slightly smoother initial data, namely Lemma 3.4, which is shown by using linear ([32]) and nonlinear ([31]) interpolation techniques.

We observe that when $g$ is linear, Theorem 2.4 readily reduces to Theorem 1.4.1 (Case 2) of [13]. Besides, it is apparent from their proofs that both results also apply to a simpler PDE model without overdamping term $D_0 z_t$ (hence, with $\beta \equiv 0$) in (2.1).

Furthermore, from Theorems 2.4 and 2.8 we obtain, as by-products, two stabilization results for a single wave equation with nonlinear internal dissipation (and Neumann boundary conditions). Indeed, even though the prime concern of this paper is to deal with *coupled structures* and the *overdamping* phenomenon, our results are new even in the case of a single wave equation. We shall state explicitly a corollary of Theorem 2.8, as Theorem 2.9, which provides uniform decay estimates of the energy (2.27) of solutions to the wave equation (2.26) depending on a little bit higher norm of initial data.

A technical comparison with results which provide decay rates for wave equations with nonlinear damping is given separately in subsection 1.1 below.

Returning to stability properties of interactive structures, two recent papers focused on *strong* (instead of uniform) stability of coupled PDE systems are worth recalling, namely [4] and [20], the latter dealing as well with saturating feedback control laws.

The plan of the paper is the following. In Section 2 we introduce the PDE model under consideration, the corresponding assumptions, and we give precise statements of the paper's main results. Sections 3.1–3.2 contain the proofs of our stability results for the coupled system, namely Theorem 2.4 and Theorem 2.8, respectively. In particular, Section 3.2 is focused on the analysis of the model with saturated feedback laws, and the critical Lemma 3.4 is found here. The proofs of well-posedness and regularity results for the PDE model (2.1), including the basic Theorem 2.3, are collected in Section 4, as an Appendix.

1.1. **Comparison with the literature on stabilization of wave equations.** It is beyond the scope of the present paper to give a comprehensive account of all studies aimed at establishing uniform decay rates of dissipative wave equations. We shall give here few main references on this subject while providing a brief technical comparison.

Almost all papers deal with dissipation subject to either linear or polynomial growth at the origin (in addition to polynomial growth at infinity). In fact, this assumption (at the origin) gives a lot of structure to the estimates which can be carried out by the Liapunov function method. This technique goes back to Haraux (see, e.g., [17]) and has been used later by many authors (see Komornik's monograph [18] and his references).

The first paper dealing with damping not subject to growth assumptions at the origin (while maintaining linear growth at infinity, which is necessary for *boundary* dissipation) is due to Lasiecka and Tataru [21]. The method pursued by these authors relies on the construction of an appropriate concave function which controls the growth of dissipation at the origin. The final result is very general and describes decay rates of solutions to semilinear wave equations via the solution to an appropriate nonlinear ODE.

The method used in [21] was revisited by Liu and Zuazua in [27], where linear wave equations (rather than *semilinear*, as in [21]) with *internal* damping are considered. For this simpler problem the authors of [27] use the same main technical ingredients as in [21], arriving at the same qualitative results with more accurate constants. Indeed, the treatment in [27] relies critically on the construction of a suitable convex function $\phi$—which in fact is the inverse of the (concave) function $h$ in [21]—and the final result is given again in terms of the solution to an ODE (see equation (2.9)). On the other hand, [27] *does* require some growth condition at the origin; see assumption (2.7) therein. This additional condition, together with the fact that a pure wave equation (i.e., no semilinear term) is considered, allows more explicit calculations of the constants appearing in the ODE than in [21], since there are no lower-order terms to be absorbed.

The contribution of Martinez [28] is worth recalling, for his attempt to relax the usual polynomial growth conditions. Unlike most literature, some recent work of Martinez and Vancostenoble deals with nonlinearities which are bounded at infinity [29, 34]. In particular, [29] obtains decay rates of *strong* solutions of a 2-D model for the wave equation.

Comparing the present contribution with the literature, we wish to point out that

- both [21] and [27] assume that the nonlinear function $g$ satisfies $sg(s) \geq ms^2$ for $|s|$ large, while Theorem 2.9 of this paper does not;
- [27] still assumes a growth condition on $g$ at the origin, unlike Theorems 2.4–2.8–2.9;

- contrasted with Theorem 2.9, [29, Theorem 2]
  - assumes that $g$ be a $C^1$ function, with $g'(0) = 0$, which continues to be a kind of growth condition near the origin;
  - requires higher norms of initial data.

We finally note that the case of bounded feedback functions is examined also in a paper by Aassila *et al.* [1], dealing with wave equations with boundary dissipation of memory type. More precisely, [1, Theorem 2.3] provides the energy decay rates for the corresponding *strong* solutions, although the given estimate for the energy $E(t)$, as it appears, suggests that decay rates pertain to *finite energy* solutions, with initial data in $H^1(\Omega) \times L_2(\Omega)$. However, this is not the case. Indeed, the proof of Theorem 2.3 in [1] requires one more derivative of the velocity, namely $u_t \in H^1(\Omega)$. The above result should be contrasted with Theorem 2.9 obtained in this paper, where the regularity of solutions needs to be increased only by (arbitrarily small) $\epsilon$.

## 2. Statement of the problem and main results

The mathematical model under consideration consists of a linear wave equation active within an acoustic chamber, which is then strongly coupled with a linear abstract equation acting only on the elastic, flat wall of the chamber. More precisely, let $\Omega \subset \mathbb{R}^3$ be an open bounded domain ("the acoustic chamber") with boundary $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, where $\Gamma_0$ and $\Gamma_1$ are open, connected, disjoint parts, $\Gamma_0 \cap \Gamma_1 = \emptyset$ in $\mathbb{R}^2$, of positive measure. The subboundary $\Gamma_0$ is flat and is referred to as the elastic or flexible wall. Instead, $\Gamma_1$ is referred to as the rigid or hard wall. The interaction between wave and plate takes place on $\Gamma_0$. We also assume that either $\Omega$ is sufficiently smooth (say, $\Gamma$ is of class $C^2$), or else $\Omega$ is convex. The acoustic medium in the chamber is described by the wave equation in the variable $z$, while $v$ represents the (abstract) deflection of the abstract plate equation on $\Gamma_0$. Then, in this paper, we investigate the stability properties of functions $\vec{z} = [z(t,x), z_t(t,x)]$ and $\vec{v} = [v(t,x), v_t(t,x)]$ which solve the following coupled PDE system:

$$\begin{cases} z_{tt} = c^2 \Delta z - g(z_t) & \text{in } Q = (0,\infty) \times \Omega \\ \dfrac{\partial z}{\partial \nu} + d_1 z = 0 & \text{on } \Sigma_1 = (0,\infty) \times \Gamma_1 \\ \dfrac{\partial z}{\partial \nu} + D_0 z_t + \beta D_0 z = v_t & \text{on } \Sigma_0 = (0,\infty) \times \Gamma_0 \\ z(0,\cdot) = z^0, \quad z_t(0,\cdot) = z^1 & \text{in } \Omega \\ v_{tt} + \mathcal{A} v + \rho \mathcal{A}^\alpha v_t + \rho_1 z_t|_{\Gamma_0} = 0 & \text{on } \Sigma_0 = (0,\infty) \times \Gamma_0 \\ v(0,\cdot) = v^0, \quad v_t(0,\cdot) = v^1 & \text{in } \Gamma_0 \end{cases} \tag{2.1}$$

where the (abstract) linear operators $\mathcal{A}$, $D_0$ are described by assumptions (H0) and (H1) below.

**Assumptions.** The first two assumptions will be held throughout the paper.

(H0) $\mathcal{A}$ (the elastic operator): $L^2(\Gamma_0) \supset \mathcal{D}(\mathcal{A}) \to L^2(\Gamma_0)$ is a positive, self-adjoint operator. Moreover, $\rho > 0$, $\beta \geq 0$, $d_1 > 0$ and $\frac{1}{2} \leq \alpha \leq 1$ are constants.

(H1) $D_0 : L^2(\Gamma_0) \supset \mathcal{D}(D_0) \to L^2(\Gamma_0)$ is a positive, self-adjoint operator, and there exists a constant $r_0$, $0 \leq r_0 \leq 1/4$, and positive constants $\delta_1$, $\delta_2$ such that

$$\delta_1 |z|^2_{\mathcal{D}(\mathcal{A}^{r_0})} \leq (D_0 z, z)_{L^2(\Gamma_0)} \leq \delta_2 |z|^2_{\mathcal{D}(\mathcal{A}^{r_0})} \quad \forall\, z \in \mathcal{D}(\mathcal{A}^{r_0}) \subset \mathcal{D}(D_0^{\frac{1}{2}}). \tag{2.2}$$

Moreover, it is assumed that $H^1(\Gamma_0) \subseteq \mathcal{D}(D_0^{1/2})$; that is,

$$D_0^{\frac{1}{2}} : \text{continuous } H^1(\Gamma_0) \to L^2(\Gamma_0).$$

In particular, for simplicity of exposition, we shall examine only the more challenging case when $D_0$ is the realization of a differential operator of order $s$, with $1 < s \leq 2$. Accordingly, it is assumed that $\beta > 0$; see [13, Sections 1.1–1.2].

**Remark 2.1.** Examples of structurally damped plate-like PDEs (with various boundary conditions) which are modelled by the abstract (uncoupled) $v$-equation in (2.1), namely equation

$$v_{tt} + \mathcal{A}v + \rho \mathcal{A}^\alpha v_t = 0\,, \tag{2.3}$$

are illustrated in [23, Section 3]. It is known [14] that if the parameter $\alpha$ satisfies $\frac{1}{2} \leq \alpha \leq 1$, then the strongly continuous semigroup defined by the map $[v^0, v^1] \longrightarrow [v(t), v_t(t)]$ is, moreover, *analytic* on the space $\mathcal{D}(\mathcal{A}^{1/2}) \times L^2(\Gamma_0)$.

The important case when $\alpha = 1/2$ (square-root damping) is included in the present analysis. Accordingly, $D_0$ is taken of order $s > 1$ (*cf.* [13, Remarks 1.10–1.11–1.12]). The reader is referred to [13, Section 1] for a complete description of the mathematical features of the linearized version of model (2.1), and in particular for a better understanding of the delicate interplay of parameters $r_0$ and $\alpha$.

Regarding the nonlinear function $g$, we shall assume at the outset that

(H2) $g$ is a continuous function on the real line such that
  (i) $g$ is monotone strictly increasing, $g(0) = 0$;
  (ii) $ms^2 \leq s\, g(s) \leq Ms^6$ for $|s| \geq 1$, with $0 < m \leq M$.

**Remark 2.2.** Note that (H2) requires, in addition to the basic monotonicity and continuity properties, that the nonlinearity $g$ satisfies a minimal polynomial growth condition at infinity, whilst we *do not* impose any kind of growth condition near the origin. More precisely, (H2)(ii) allows maximal growth of $g$ at infinity (up to power five, as in the case of a single wave equation with internal dissipation, in order to achieve just *bounded* solutions; see, e.g., [16]). On the other hand, at least linear growth at infinity is required. We shall examine later the stability properties of the PDE model (2.1) under a weaker assumption; see Remark 2.6.

In order to introduce the abstract set-up for the coupled system (2.1), we need to recall some preliminary material from [13]. Mostly for simplicity of notation, we shall normalize the physical constants $c$ and $\rho_1$, by setting $c = 1$, $\rho_1 = 1$.

Let $A_N : L^2(\Omega) \supset \mathcal{D}(A_N) \to L^2(\Omega)$ be the nonnegative, self-adjoint operator defined by

$$A_N h = -\Delta h, \quad \mathcal{D}(A_N) = \left\{ h \in H^2(\Omega) : \frac{\partial h}{\partial \nu}\Big|_{\Gamma_0} = 0\,,\ \left[\frac{\partial h}{\partial \nu} + d_1 h\right]\Big|_{\Gamma_1} = 0 \right\}\,;$$

then $A_N^{-1} \in \mathcal{L}(\Omega)$. Next, let $N_0$ be the Neumann map from $L^2(\Gamma)$ to $L^2(\Omega)$, defined by

$$\psi = N_0 g \Longleftrightarrow \left\{ \Delta\psi = 0 \text{ in } \Omega\,;\ \frac{\partial \psi}{\partial \nu}\Big|_{\Gamma_0} = g,\ \left[\frac{\partial \psi}{\partial \nu} + d_1 \psi\right]\Big|_{\Gamma_1} = 0 \right\}\,. \tag{2.4}$$

It is well known that

$$N_0 \text{ continuous } : L^2(\Gamma_0) \to H^{\frac{3}{2}}(\Omega) \subset \mathcal{D}(A_N^{\frac{3}{4}-\epsilon}),\ \epsilon > 0\,,$$

so that

$$A_N^{\frac{3}{4}-\epsilon} N_0 \text{ continuous} : L^2(\Gamma_0) \to L^2(\Omega)\,.$$

It is also well known that by Green's second theorem, the following trace result holds true (*cf.* [23]):

$$N_0^* A_N h = \begin{cases} h|_{\Gamma_0} & \text{on } \Gamma_0 \\ 0 & \text{on } \Gamma_1 \end{cases} \tag{2.5}$$

for $h \in \mathcal{D}(A_N)$, and the validity of (2.5) may be extended to all $h \in H^1(\Omega) \equiv \mathcal{D}(A_N^{1/2})$, as $\mathcal{D}(A_N^{1/2})$ is dense in $\mathcal{D}(A_N)$.

By using the Green operators introduced above, the coupled PDE problem (2.1) can be rewritten as the following abstract second-order system:

$$z_{tt} + A_N z + \beta A_N N_0 D_0 N_0^* A_N z + A_N N_0 D_0 N_0^* A_N z_t + g(z_t) - A_N N_0 v_t = 0 \tag{2.6a}$$

$$v_{tt} + \mathcal{A}v + \rho\mathcal{A}^{\alpha}v_t + N_0^* A_N z_t = 0 \tag{2.6b}$$

$$z(0) = z^0\,,\ z_t(0) = z^1\,;\ v(0) = v^0\,,\ v_t(0) = v^1\,. \tag{2.6c}$$

We finally define the space

$$Z = \Big\{ h \in D(A_N^{\frac{1}{2}}) \equiv H^1(\Omega): \quad h|_{\Gamma_0} \in \mathcal{D}(D_0^{\frac{1}{2}}) \Big\}, \tag{2.7}$$

endowed with the norm

$$\|h\|_Z^2 := \|A_N^{\frac{1}{2}} h\|_{L_2(\Omega)}^2 + \beta \|D_0^{\frac{1}{2}} N_0^* A_N h\|_{L_2(\Gamma_0)}^2, \tag{2.8}$$

which is needed to describe the state space of problem (2.6a) (here we denoted the function space $Z_\beta$ introduced in [13] simply by $Z$). Then, the function spaces $Y_1$ for the wave component $[z, z_t]$ and $Y_2$ for the plate component $[v, v_t]$ of system (2.6) are given, respectively, by

$$Y_1 := Z \times L^2(\Omega); \qquad Y_2 := \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L^2(\Gamma_0).$$

The state space for problem (2.6) is then

$$Y \equiv Y_1 \times Y_2 = Z \times L^2(\Omega) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L^2(\Gamma_0). \tag{2.9}$$

The appropriate energy function corresponding to the solutions of the coupled system (2.1) is given by

$$\begin{aligned} E(t) := E_z(t) + E_v(t) &:= \|A_N^{\frac{1}{2}} z(t)\|_{L_2(\Omega)}^2 + \beta \|D_0^{\frac{1}{2}} N_0^* A_N z(t)\|_{L_2(\Gamma_0)}^2 \\ &+ \|z_t(t)\|_{L_2(\Omega)}^2 + \|\mathcal{A}^{\frac{1}{2}} v(t)\|_{L_2(\Gamma_0)}^2 + \|v_t(t)\|_{L_2(\Gamma_0)}^2\,, \end{aligned} \tag{2.10}$$

where $E_z(t)$ and $E_v(t)$ denote the wave and plate energy, respectively.

Our first result provides well-posedness of the nonlinear PDE model (2.1) under consideration, along with some regularity properties of the corresponding solutions which will be needed in the proof of the stability results.

**Theorem 2.3.** *Assume hypotheses* (H0), (H1), *and* (H2). (*i*) *With the initial data* $[z^0, z^1, v^0, v^1] \in Y$, *system* (2.1) *has a unique solution* $[\vec{z}, \vec{v}] \in C([0,\infty); Y)$.

(*ii*) *With the initial data* $[z^0, z^1, v^0, v^1] \in \mathcal{D}(A)$, *where* $\mathcal{D}(A)$ *is the function space given by* (4.7), *the velocity terms of the solution have the following regularity:*

$$z_t \in L^\infty(0,\infty; Z)\,, \quad z_{tt} \in L^\infty(0,\infty; L^2(\Omega)) \tag{2.11a}$$

$$v_t \in L^\infty(0,\infty; \mathcal{D}(\mathcal{A}^{\frac{1}{2}}))\,, \quad v_{tt} \in L^\infty(0,\infty; L^2(\Gamma_0))\,. \tag{2.11b}$$

*In particular,* $z_t \in L^\infty(0,\infty; H^1(\Omega))$, *and consequently*

$$g(z_t) \in L^\infty(0,\infty; H^{-1}(\Omega))\,. \tag{2.12}$$

Before stating our first stability result, i.e., Theorem 2.4 below, we need to introduce some scalar functions which are involved in a crucial way in the (statement and) proof of Theorem 2.4. We recall that by assumption (H2) it follows that there exists a real-valued function $h(x)$ which is defined for $x \geq 0$; it is concave, strictly increasing, with $h(0) = 0$, and it satisfies, for a suitable $N > 0$,

$$h(sg(s)) \geq s^2 + g^2(s) \quad \text{for } |s| \leq N\,. \tag{2.13}$$

Such a function can always be constructed, due to (H2), as explained in [21, p. 510]. With this function we first define

$$\tilde{h}(x) = h\Big(\frac{x}{|Q_{0T}|}\Big), \quad x \geq 0\,, \tag{2.14}$$

where $|Q_{0T}|$ denotes the measure of $Q_{0T} = (0,T) \times Q$. Note that since $\tilde{h}$ is monotone increasing, $cI + \tilde{h}$ is invertible for any constant $c \geq 0$. Then, with $K$ a positive constant, we set

$$p(x) := \big(cI + \tilde{h}\big)^{-1}(Kx)\,, \tag{2.15}$$

which is readily a continuous, positive, strictly increasing function, with $p(0) = 0$. Finally, let

$$q(x) := x - \big(I + p\big)^{-1}(x), \quad x > 0\,. \tag{2.16}$$

We can now present our first stability result, which pertains to the case when the nonlinearity $g$ has at least linear growth at infinity.

**Theorem 2.4.** *Assume hypotheses* (H0), (H1), *and* (H2). *Let* $[\vec{z}, \vec{v}]$ *be the weak solution to the coupled system* (2.1), *guaranteed by part* ($i$) *of Theorem* 2.3. *Then, for the energy* $E(\vec{z}, \vec{v}, t)$, *as defined in* (2.10), *there exists a* $T_0 > 0$ *such that*

$$E(\vec{z}, \vec{v}, t) \leq \mathcal{S}\Big(\frac{t}{T_0} - 1\Big) \quad \textit{for } t > T_0\,, \tag{2.17}$$

*where* $\lim_{t\to+\infty} \mathcal{S}(t) = 0$ *and* $\mathcal{S}(t)$ *is the solution to the ordinary differential equation*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}(t) + q(\mathcal{S}(t)) = 0, \quad \mathcal{S}(0) = E(\vec{z}, \vec{v}, 0) \tag{2.18}$$

(*where* $q$ *is as given in* (2.16)). *Here, the constant* $K$ *in* (2.15) *will depend on* $E(\vec{z}, \vec{v}, 0)$ *and time* $T_0$, *and the constant* $c$ (*in* (2.15)) *depends on the measure of* $Q_{0T_0}$.

**Remark 2.5.** As pointed out already in the Introduction, Theorem 2.4 establishes, as well, uniform stability of a *single* wave equation with internal dissipation subject to assumption (H2). The recent paper [27] derived uniform decay rates for the natural energy of the system, under the same conditions (H2)(ii) on the nonlinearity $g$ at infinity, but assuming, further, a growth condition at the origin.

**Remark 2.6.** If the nonlinear function $g$ is subject to saturation, then it fails to fully satisfy the growth condition in (H2)(ii). This happens, for instance, when $g(y) = \min\{1, C/|y|\}\, y$, $C$ being a given positive constant. Then, since we wish to examine this significant case, we introduce an alternative hypothesis on $g$, namely (H2)' below. Under this weaker condition we are able to show that decay rates of the energy $E(t)$ are still uniform, provided that the initial data are taken in a slightly smoother function space.

Let us assume that—instead of (H2)—the following hypothesis (H2)' holds true, while maintaining the basic assumption (H0)–(H1) on the coupled system (2.1).

(H2)' $g$ is a continuous, nondecreasing function on the real line such that
- (i) $g$ is strictly increasing for $|s| \le 1$, $g(0) = 0$;
- (ii) there exists $M > 0$ such that $s\,g(s) \le M s^6$ for $|s| \ge 1$.

**Remark 2.7.** Note that from assumption (H2)' it follows that there exists a positive constant $m$ such that

$$s\,g(s) \ge m \cdot |s| \ge m \qquad \text{for } |s| \ge 1. \tag{2.19}$$

It is not difficult to verify that Theorem 2.3 still holds true, as its proof exploits critically monotonicity of $g$, whilst the lower bound in the growth condition (H2)(ii) does not play any role therein.

As in the previous case, there exists a concave, strictly increasing function $h(x)$, $x \ge 0$, with $h(0) = 0$, such that (2.13) holds true. Thus, $\tilde{h}$ will have the same meaning as in (2.14). Analogously, for $c \ge 0$ and $K_1$ a positive constant, we define the functions

$$p_1(x) := \big(cI + \tilde{h}\big)^{-1}(K_1 x)\,, \tag{2.20}$$

and

$$q_1(x) := x - \big(I + p_1\big)^{-1}(x), \quad x > 0\,, \tag{2.21}$$

with the same properties as $p$ and $q$ above.

The statement of our second stability result involves a new function space and a new energy. For $\epsilon \in (0,1)$, let us consider the function space

$$Z \cap H^{1+\epsilon}(\Omega) := \Big\{ h \in H^{1+\epsilon}(\Omega) : \quad h|_{\Gamma_0} \in \mathcal{D}(D_0^{\frac{1}{2}}) \Big\}\,,$$

endowed with the norm

$$\|h\|^2_{Z \cap H^{1+\epsilon}(\Omega)} := \|h\|^2_{H^{1+\epsilon}(\Omega)} + \beta \|D_0^{\frac{1}{2}} h|_{\Gamma_0}\|^2_{L^2(\Gamma_0)}\,.$$

With this, we define the following subspace of the energy space $Y$:

$$W_\epsilon = [Z \cap H^{1+\epsilon}(\Omega)] \times H^{\frac{6}{5}\epsilon}(\Omega) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}(1+\epsilon)}) \times \mathcal{D}(\mathcal{A}^{\alpha\epsilon}),\ 0 < \epsilon < \frac{5}{12}. \quad (2.22)$$

Accordingly, let us introduce the higher-level energy $E_1(\vec{z}, \vec{v}, t)$ (for simplicity of notation, we set $E_1(t) \equiv E_1(\vec{z}, \vec{v}, t)$):

$$\begin{aligned} E_1(t) := {} & ||z(t)||^2_{H^{1+\epsilon}(\Omega)} + \beta ||D_0^{\frac{1}{2}} z|_{\Gamma_0}(t)||^2_{L^2(\Gamma_0)} + ||z_t(t)||^2_{H^{\frac{6}{5}\epsilon}(\Omega)} \\ & + ||v(t)||^2_{\mathcal{D}(\mathcal{A}^{(1+\epsilon)/2})} + ||v_t(t)||^2_{\mathcal{D}(\mathcal{A}^{\alpha\epsilon})} . \end{aligned} \quad (2.23)$$

Then, the following stability result—pertaining to the case of nonlinearity subject to saturation—holds true.

**Theorem 2.8.** *Assume hypotheses* (H0), (H1), *and* (H2)'. *Given* $[z^0, z^1, v^0, v^1] \in W_\epsilon$, *for the energy* $E(\vec{z}, \vec{v}, t)$ *of solutions* $[\vec{z}, \vec{v}]$ *of system* (2.1) *there exists a* $T_0 > 0$ *such that*

$$E(\vec{z}, \vec{v}, t) \le \mathcal{S}_1\big(\frac{t}{T_0} - 1\big) \quad \text{for } t > T_0 \,, \quad (2.24)$$

*where* $\lim_{t \to +\infty} \mathcal{S}_1(t) = 0$ *and* $\mathcal{S}_1(t)$ *is the solution to the ordinary differential equation*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}(t) + q_1(\mathcal{S}(t)) = 0, \quad \mathcal{S}(0) = E_1(\vec{z}, \vec{v}, 0) \quad (2.25)$$

(*where* $q_1$ *is as given in* (2.21) *and we recall that* $E_1(\cdot)$ *is defined in* (2.23)). *The constant* $K_1$ *in* (2.20) *will depend on* $E_1(\vec{z}, \vec{v}, 0)$ *and time* $T_0$, *and the constant* $c$ (*in* (2.20)) *depends on the measure of* $Q_{0T_0}$.

As was remarked in the Introduction, from Theorem 2.8 we derive, as a corollary, uniform decay rates for the uncoupled wave component of system (2.1). We shall state explicitly the corresponding result for the following basic model of a wave equation with nonlinear internal dissipation (without overdamping term, i.e., with $D_0 \equiv 0$, hence $\beta \equiv 0$):

$$\begin{cases} z_{tt} = \Delta z - g(z_t) & \text{in} \quad (0, \infty) \times \Omega \\ \dfrac{\partial z}{\partial \nu} + d_1 z = 0 & \text{on} \quad (0, \infty) \times \Gamma_1 \\ \dfrac{\partial z}{\partial \nu} = 0 & \text{on} \quad (0, \infty) \times \Gamma_0 \\ z(0, \cdot) = z^0 \,,\ z_t(0, \cdot) = z^1 & \text{in} \quad \Omega \,. \end{cases} \quad (2.26)$$

From (2.10) we see that the corresponding energy is given, as expected, by

$$\mathcal{E}(t) := \int_\Omega \big(|\nabla z|^2 + |z_t|^2\big)\, dx + d_1 \int_{\Gamma_1} z^2 \, d\sigma \,. \quad (2.27)$$

As before, let us introduce the following higher-level energy $\mathcal{E}_1(\vec{z},t)$ (briefly, $\mathcal{E}_1(t)$), for any $\epsilon \in (0,5/12)$:

$$\mathcal{E}_1(t) := \|z(t)\|^2_{H^{1+\epsilon}(\Omega)} + \|z_t(t)\|^2_{H^{\frac{6}{5}\epsilon}(\Omega)}\,, \qquad 0<\epsilon<\frac{5}{12}. \tag{2.28}$$

The counterpart of Theorem 2.8 is now the following.

**Theorem 2.9.** *Assume hypothesis* (H2)' *and* $d_1 > 0$. *Given* $[z^0, z^1] \in H^{1+\epsilon}(\Omega)\times H^{\frac{6}{5}\epsilon}(\Omega)$, $\epsilon\in(0,\frac{5}{12})$, *for the energy* $\mathcal{E}(t)$ *of solutions of problem* (2.26) *there exists a* $T_0>0$ *such that*

$$\mathcal{E}(t) \le \mathcal{S}_1\big(\frac{t}{T_0}-1\big) \quad \text{for } t>T_0\,, \tag{2.29}$$

*where* $\lim_{t\to+\infty}\mathcal{S}_1(t)=0$ *and* $\mathcal{S}_1(t)$ *is the solution to the ordinary differential equation*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}(t) + q_1(\mathcal{S}(t)) = 0, \quad \mathcal{S}(0)=\mathcal{E}_1(0) \tag{2.30}$$

(*where* $q_1$ *is as given in* (2.21) *and we recall that* $\mathcal{E}_1(\cdot)$ *is defined in* (2.28)). *The constant* $K_1$ *in* (2.20) *will depend on* $\mathcal{E}_1(0)$ *and time* $T_0$, *and the constant* $c$ (*in* (2.20)) *depends on the measure of* $Q_{0T_0}$.

Since the core of the present paper is the stability issue, we postpone the proofs of both well-posedness of system (2.1), and regularity properties of the corresponding solutions, collecting them in the last section, as an Appendix.

## 3. Uniform stability

3.1. **Proof of Theorem 2.4.** The proof of Theorem 2.4 proceeds through several steps. Let us recall that the total energy $E(\vec{z},\vec{v},t)$ of system (2.6) (for simplicity of notation, $E(\vec{z},\vec{v},t)\equiv E(t)$) has been defined by (2.10). We begin, as usual, with the statement of an energy identity, which will be used in a crucial way in the sequel.

**Proposition 3.1.** *With respect to the system of equations* (2.6), *the following energy equality holds for any* $T>0$ *and all* $s$, *with* $0\le s\le T$:

$$E(T) + 2\int_s^T \|D_0^{\frac{1}{2}}N_0^*A_N z_t(t)\|^2\,dt + 2\int_s^T (g(z_t),z_t)\,dt + 2\rho\int_s^T \|\mathcal{A}^{\frac{\alpha}{2}}v_t(t)\|^2\,dt = E(s)\,. \tag{3.1}$$

*In particular,* $E(T)\le E(t)$ *for any* $t\in[0,T]$.

**Proof.** It is sufficient to show that (3.1) holds for smooth solutions $[\vec{z},\vec{v}]$. It is readily verified that the computations performed below are justified by the regularity of solutions corresponding to smooth initial data as from part (ii) of Theorem 2.3.

Without loss of generality, we take $s=0$. Running the multiplier $z_t$ on the wave equation (2.6a), we get

$$\frac{1}{2}\frac{d}{dt}\Big(\|z_t\|^2+\|A_N^{\frac12}z\|^2+\beta\|D_0^{\frac12}N_0^*A_Nz\|^2\Big)+\|D_0^{\frac12}N_0^*A_Nz_t\|^2 \\ +(g(z_t),z_t)-(v_t,N_0^*A_Nz_t)=0\,,$$

which yields, after integration between 0 and $T$, the following identity for the wave energy $E_z$, for all $T>0$:

$$E_z(T)+2\int_0^T\|D_0^{\frac12}N_0^*A_Nz_t(t)\|^2\,\mathrm{d}t+2\int_0^T(g(z_t),z_t)\,\mathrm{d}t \\ -2\int_0^T(v_t(t),N_0^*A_Nz_t(t))\,\mathrm{d}t=E_z(0)\,. \tag{3.2}$$

Analogously, we run the multiplier $v_t$ on the plate equation (2.6b), and obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|v_t\|^2+\|\mathcal{A}^{\frac12}v\|^2\right)+\rho\|\mathcal{A}^{\frac{\alpha}{2}}v_t\|^2+(N_0^*A_Nz_t,v_t)=0\,;$$

next, we integrate between 0 and $T$, getting the following equality for the plate energy $E_v$:

$$E_v(T)+2\rho\int_0^T\|\mathcal{A}^{\frac{\alpha}{2}}v_t(t)\|^2\,\mathrm{d}t+2\int_0^T(N_0^*A_Nz_t(t),v_t(t))\,\mathrm{d}t=E_v(0)\,. \tag{3.3}$$

Summing up (3.2) and (3.3) yields the energy identity (3.1). □

Our next goal is to provide estimates of the energy functional on a finite interval. We begin with the simpler analysis of the plate energy functional.

**Proposition 3.2.** *With respect to the plate energy $E_v$ defined in* (2.10)*, the following inequality holds for all $T>0$, with constants $C_1$ and $C_2$ independent of $T$:*

$$\int_0^T E_v(t)\,dt\le C_1E(0)+C_2\int_0^T\|D_0^{\frac12}N_0^*A_Nz_t(t)\|^2\,dt\,. \tag{3.4}$$

**Proof.** We return to the plate equation and run the multiplier $v$ instead, then integrate between 0 and $T$, thus obtaining

$$(v_t(T),v(T))-\int_0^T\|v_t(t)\|^2\,\mathrm{d}t+\int_0^T\|\mathcal{A}^{\frac12}v(t)\|^2\,\mathrm{d}t+\frac{\rho}{2}\|\mathcal{A}^{\frac{\alpha}{2}}v(t)\|^2$$

$$+\int_0^T (N_0^* A_N z_t(t), v(t))\,\mathrm{d}t = (v^1, v^0) + \frac{\rho}{2}\|\mathcal{A}^{\frac{\alpha}{2}} v_0\|^2 . \tag{3.5}$$

After using Schwarz inequality to estimate inner products in (3.5), we get for arbitrary $\epsilon > 0$ and some $c_1$

$$\begin{aligned}
&\int_0^T \|\mathcal{A}^{\frac{1}{2}} v(t)\|^2\,\mathrm{d}t - \int_0^T \|v_t(t)\|^2\,\mathrm{d}t + \frac{\rho}{2}\|\mathcal{A}^{\frac{\alpha}{2}} v(T)\|^2 \\
&\quad \le c_1(E_v(T) + E_v(0)) + \epsilon \int_0^T \|v(t)\|^2\,\mathrm{d}t + C_\epsilon \int_0^T \|N_0^* A_N z_t(t)\|^2\,\mathrm{d}t \\
&\quad \le c_1(E_v(T) + E_v(0)) + \epsilon \|\mathcal{A}^{-\frac{1}{2}}\|^2 \int_0^T \|\mathcal{A}^{\frac{1}{2}} v(t)\|^2\,\mathrm{d}t \\
&\qquad + C_\epsilon \delta_1 \|\mathcal{A}^{-r_0}\|^2 \int_0^T \|D_0^{\frac{1}{2}} N_0^* A_N z_t(t)\|^2\,\mathrm{d}t .
\end{aligned} \tag{3.6}$$

By taking $\epsilon$ sufficiently small in (3.6), namely such that $1 - \epsilon\|\mathcal{A}^{-1/2}\|^2 > 0$, we obtain, with suitable $c_i$,

$$\begin{aligned}
\int_0^T \|\mathcal{A}^{\frac{1}{2}} v(t)\|^2\,\mathrm{d}t &\le c_2(E_v(T) + E_v(0)) \\
&\quad + c_3\Big(\int_0^T \|D_0^{\frac{1}{2}} N_0^* A_N z_t(t)\|^2\,\mathrm{d}t + \int_0^T \|v_t(t)\|^2\,\mathrm{d}t\Big),
\end{aligned} \tag{3.7}$$

which provides an estimate for the plate potential energy.

On the other hand, since $\mathcal{A}^{\frac{\alpha}{2}}$ is a boundedly invertible operator, then from (3.1) it follows that

$$\int_0^T E_v^k(t)\,\mathrm{d}t := \int_0^T \|v_t(t)\|^2\,\mathrm{d}t \le c_4 E(0) , \tag{3.8}$$

which combined with (3.7) yields (3.4), as desired. □

In the following proposition we give a key estimate for the wave energy functional which will provide the desired estimate for the total energy of the system.

**Proposition 3.3.** *With respect to the wave energy $E_z$ defined in* (2.10)*, the following inequality holds for any $T > 0$:*

$$\int_0^T E_z(t)\,dt \le C_3 E(0) + C_4(E(0))\big[I + |Q_{0T}|\tilde{h}\big]\Big(\int_0^T (g(z_t), z_t)\,\mathrm{d}t\Big), \tag{3.9}$$

*with the constant $C_3$ independent of $T$, while $C_4(E(0))$ is a positive constant which increases with $E(0)$ but remains bounded for bounded values of $E(0)$.*

**Proof.** We return to the wave equation (2.6a) and run the multiplier $z$, then integrate between 0 and $T$, thus obtaining

$$\begin{aligned}
&(z_t(T), z(T)) - \int_0^T \|z_t(t)\|^2 \,\mathrm{d}t + \int_0^T \|A_N^{\frac{1}{2}} z(t)\|^2 dt + \beta \int_0^T \|D_0^{\frac{1}{2}} N_0^* A_N z(t)\|^2 dt \\
&+ \frac{1}{2}\|D_0^{\frac{1}{2}} N_0^* A_N z(T)\|^2 + \int_0^T (g(z_t), z)\,\mathrm{d}t - \int_0^T (v_t(t), N_0^* A_N z(t))\,dt \\
&= (z^1, z^0) + \frac{1}{2}\|D_0^{\frac{1}{2}} N_0^* A_N z_0\|^2 . \qquad (3.10)
\end{aligned}$$

We isolate on the left-hand side of (3.10) the difference "potential minus kinetic energy," while applying the Schwarz inequality on the right-hand side:

$$\begin{aligned}
&\int_0^T \|A_N^{\frac{1}{2}} z(t)\|^2 dt + \beta \int_0^T \|D_0^{\frac{1}{2}} N_0^* A_N z(t)\|^2 dt - \int_0^T \|z_t(t)\|^2 dt \\
&\le |(z_t(T), z(T))| + \int_0^T |(v_t, N_0^* A_N z)|\,\mathrm{d}t + \int_0^T |(g(z_t), z)|\,\mathrm{d}t + c_1 E_z(0) \\
&\le c_2 E_z(T) + C_\epsilon \int_0^T \|v_t(t)\|^2\,dt + \epsilon \int_0^T \|N_0^* A_N z(t)\|^2\,dt \\
&\quad + \int_0^T |(g(z_t), z)|\,\mathrm{d}t + c_1 E_z(0) \\
&\le c_2 E_z(T) + c_1 E_z(0) + C_\epsilon \int_0^T \|v_t(t)\|^2\,dt + \epsilon \|N_0^* A_N^{\frac{1}{2}}\|^2 \int_0^T \|A_N^{\frac{1}{2}} z(t)\|^2\,dt \\
&\quad + \int_0^T |(g(z_t), z)|\,\mathrm{d}t . \qquad (3.11)
\end{aligned}$$

By choosing $\epsilon$ sufficiently small, e.g. such that $1 - \epsilon\|N_0^* A_N^{1/2}\|^2 > 1/2$, inequality (3.11) yields

$$\begin{aligned}
&\frac{1}{2}\int_0^T \|A_N^{\frac{1}{2}} z(t)\|^2 dt + \beta \int_0^T \|D_0^{\frac{1}{2}} N_0^* A_N z(t)\|^2 dt - \int_0^T \|z_t(t)\|^2 dt \\
&\le c_1 E_z(0) + c_2 E_z(T) + c_3 \int_0^T \|v_t(t)\|^2\,dt + \int_0^T |(g(z_t), z)|\,\mathrm{d}t . \qquad (3.12)
\end{aligned}$$

Our main task is to estimate the last term on the right-hand side of (3.12). Note that

$$\int_0^T |(g(z_t), z)|\,\mathrm{d}t = \int_0^T \Big| \int_\Omega g(z_t)\, z\,\mathrm{d}x \Big|\,\mathrm{d}t \le \iint_{Q_{0T}} |g(z_t)\, z|\,\mathrm{d}x\,\mathrm{d}t . \qquad (3.13)$$

We need to introduce the measurable subsets

$$Q_1 := \{(t,x) \in Q_{0T} : \ |z_t(t,x)| \geq 1 \ \text{ a.e. } \} , \qquad Q_2 := Q_{0T} \setminus Q_1 , \quad (3.14)$$

and proceed with the estimates on either subset. By using the Sobolev embedding $H^1(\Omega) \subset L^6(\Omega)$ ([11]) and the Hölder inequality (with $p = 6$, $1/p + 1/q = 1$), we get

$$\begin{aligned}\int_{\{x\in\Omega:|z_t|\geq 1\}} |g(z_t)\, z| \, \mathrm{d}x &\leq \Big( \int_{\{x\in\Omega:|z_t|\geq 1\}} |g(z_t)|^q \, \mathrm{d}x \Big)^{\frac{1}{q}} \Big( \int_{\{x\in\Omega:|z_t|\geq 1\}} |z|^p \, \mathrm{d}x \Big)^{\frac{1}{p}} \\ &\leq \Big( \int_{\Omega\cap\{x:\,|z_t|\geq 1\}} |g(z_t)|^{\frac{6}{5}} \, \mathrm{d}x \Big)^{\frac{5}{6}} \|z(t)\|_{L^6(\Omega)} \qquad (3.15) \\ &\leq c_4 \Big( \int_{\Omega\cap\{x:\,|z_t|\geq 1\}} |g(z_t)|\, |g(z_t)|^{\frac{1}{5}} \, \mathrm{d}x \Big)^{\frac{5}{6}} \|z(t)\|_{H^1(\Omega)} \, .\end{aligned}$$

Since by assumption (H2)(ii) $|s| \geq 1$ implies $|g(s)|^{1/5} \leq M^{1/5}|s|$, then from (3.15) it follows that

$$\begin{aligned}\iint_{Q_1} |g(z_t)\, z| \, \mathrm{d}x \, \mathrm{d}t &\leq c_4 M^{\frac{1}{6}} \int_0^T \Big( \int_{\Omega\cap\{x:\,|z_t|\geq 1\}} |g(z_t)||z_t| \, \mathrm{d}x \Big)^{\frac{5}{6}} \|z(t)\|_{H^1(\Omega)} \, \mathrm{d}t \\ &\leq c_4 M^{\frac{1}{6}} \int_0^T \big[ (g(z_t), z_t) \big]^{\frac{5}{6}} \|z(t)\|_{H^1(\Omega)} \, \mathrm{d}t.\end{aligned}$$

We apply now the Young inequality, with $1/\overline{p} + 1/\overline{q} = 1$, and we require $5/6 \cdot \overline{p} = 1$, hence $\overline{p} = 6/5$, so that $\overline{q} = 6$. Then, for arbitrary positive $\epsilon_1$, we obtain

$$\begin{aligned}&\iint_{Q_1} |g(z_t)\, z| \, \mathrm{d}x \, \mathrm{d}t \leq c_4 M^{\frac{1}{6}} \Big\{ C_{\epsilon_1} \int_0^T (g(z_t), z_t) \, \mathrm{d}t + \epsilon_1 \int_0^T ||z(t)||^6_{H^1(\Omega)} \, \mathrm{d}t \Big\} \\ &\leq c_4 M^{\frac{1}{6}} \Big\{ C_{\epsilon_1} \int_0^T (g(z_t), z_t) \, \mathrm{d}t + \epsilon_1 \sup_{0\leq t\leq T} ||z(t)||^4_{H^1(\Omega)} \int_0^T ||z(t)||^2_{H^1(\Omega)} \, \mathrm{d}t \Big\} \\ &\leq c_4 M^{\frac{1}{6}} \Big\{ C_{\epsilon_1} \int_0^T (g(z_t), z_t) \, \mathrm{d}t + \epsilon_1 [E(0)]^2 \int_0^T \|z(t)\|^2_{H^1(\Omega)} \, \mathrm{d}t \Big\} \, , \qquad (3.16)\end{aligned}$$

where in the last estimate we have used that $||z(t)||^2_{H^1(\Omega)} \leq E(0)$ for all $t \geq 0$, which readily follows from Proposition 3.1.

On the other hand, for $|z_t| \leq 1$ we obtain, by using the Hölder inequality (with $\epsilon_2 > 0$) first, and Jensen inequality next,

$$\iint_{Q_2} |g(z_t)\, z| \, \mathrm{d}x \, \mathrm{d}t \leq C_{\epsilon_2} \iint_{Q_2} |g(z_t)|^2 \, \mathrm{d}x \, \mathrm{d}t + \epsilon_2 \iint_{Q_2} |z|^2 \, \mathrm{d}x \, \mathrm{d}t$$

$$\leq C_{\epsilon_2} \iint_{Q_2} h(g(z_t)\, z_t)\,\mathrm{d}x\,\mathrm{d}t + \epsilon_2 \|A_N^{-\frac{1}{2}}\|^2 \iint_{Q_2} |A_N^{\frac{1}{2}} z|^2\,\mathrm{d}x\,\mathrm{d}t \tag{3.17}$$
$$\leq C_{\epsilon_2} |Q_{0T}| \tilde{h} \left( \iint_{Q_{0T}} g(z_t)\, z_t\,\mathrm{d}x\,\mathrm{d}t \right) + \epsilon_2 \|A_N^{-\frac{1}{2}}\|^2 \iint_{Q_{0T}} |A_N^{\frac{1}{2}} z|^2\,\mathrm{d}x\,\mathrm{d}t,$$

where $h$ is a suitable concave function described by (2.13), and $\tilde{h}$ is defined in terms of $h$ by (2.14). We note again that $\tilde{h}$ is a concave, monotone-increasing function on the positive real line. Summing up (3.16) with (3.17) we can complete the estimate in (3.13) as follows:

$$\int_0^T |(g(z_t), z)|\,\mathrm{d}t \leq c_4 M^{\frac{1}{6}} C_{\epsilon_1} \int_0^T (g(z_t), z_t)\,\mathrm{d}t + c_4 M^{\frac{1}{6}} \epsilon_1 [E(0)]^2 \int_0^T \|A_N^{\frac{1}{2}} z\|^2\,\mathrm{d}t$$
$$+ C_{\epsilon_2} |Q_{0T}| \tilde{h} \Big( \int_0^T (g(z_t), z_t)\,\mathrm{d}t \Big) + \epsilon_2 \|A_N^{-\frac{1}{2}}\|^2 \int_0^T ||A_N^{\frac{1}{2}} z||^2\,\mathrm{d}t\,. \tag{3.18}$$

Let us return to (3.12), and use (3.18) with $\epsilon_1$ and $\epsilon_2$ sufficiently small, e.g. by taking $\epsilon_2 = \frac{1}{4\|A_N^{-\frac{1}{2}}\|^2}$, $\epsilon_1 = \frac{1}{8c_4 M^{\frac{1}{6}} E^2(0)}$, so that $1/2 - \epsilon_1 c_4 M^{1/6} E^2(0) - \epsilon_2 \|A_N^{-1/2}\|^2 = 1/8 > 0$. We note that with this choice the constant $C_{\epsilon_1}$ is increasing with $E(0)$, but remains bounded for bounded values of $E(0)$. We thus obtain, with $c_i$ suitable constants,

$$\frac{1}{8} \int_0^T \|A_N^{\frac{1}{2}} z(t)\|^2 dt + \beta \int_0^T \|D_0^{\frac{1}{2}} N_0^* A_N z(t)\|^2 dt - \int_0^T \|z_t(t)\|^2 dt$$
$$\leq c_1 E_z(0) + c_2 E_z(T) + c_3 \int_0^T \|v_t(t)\|^2\, dt$$
$$+ \big[c_5(E(0)) I + c_6 |Q_{0T}| \tilde{h}\big] \Big( \int_0^T (g(z_t), z_t)\,\mathrm{d}t \Big)$$

(again, $c_5(E(0))$ is increasing with $E(0)$, but remains bounded for bounded values of $E(0)$). Consequently, the following estimate holds true for the wave potential energy functional:

$$\int_0^T \|A_N^{\frac{1}{2}} z(t)\|^2 dt + \beta \int_0^T \|D_0^{\frac{1}{2}} N_0^* A_N z(t)\|^2 dt \leq c_1 E_z(0) + c_2 E_z(T)$$
$$+ c_3 \int_0^T \|v_t(t)\|^2\, dt + \big[c_5(E(0)) I + c_6 |Q_{0T}| \tilde{h}\big] \Big( \int_0^T (g(z_t), z_t)\,\mathrm{d}t \Big)$$
$$+ c_7 \int_0^T \|z_t(t)\|^2 dt\,, \tag{3.19}$$

where $c_i$ still denote suitable positive constants (whose values are however different from above).

To conclude the proof, it remains to estimate the wave kinetic energy functional. Again, we need to distinguish two cases. From Assumption (H2)(ii) it readily follows that

$$\iint_{Q_1} |z_t(t)|^2 \,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{m}\iint_{Q_1} g(z_t)\, z_t \,\mathrm{d}x\,\mathrm{d}t\,. \tag{3.20}$$

On the other hand, by using once more (2.13), we obtain

$$\iint_{Q_2} |z_t(t)|^2 \,\mathrm{d}x\,\mathrm{d}t \le \iint_{Q_2} h(z_t\cdot g(z_t))\,\mathrm{d}x\,\mathrm{d}t \le |Q_{0T}|\tilde h\Big(\iint_{Q_{0T}} g(z_t)\, z_t \,\mathrm{d}x\,\mathrm{d}t\Big) \tag{3.21}$$

which added to (3.20) yields

$$\int_0^T \|z_t(t)\|^2 dt \le \big[m^{-1}I + |Q_{0T}|\tilde h\big]\Big(\int_0^T (g(z_t), z_t)\,\mathrm{d}t\Big)\,. \tag{3.22}$$

By combining (3.19) with (3.22), and by using the property $E(T)\le E(0)$ (which follows from (3.1)) and the estimate (3.8), we obtain

$$\begin{aligned}
&\int_0^T \|A_N^{1/2} z(t)\|^2 dt + \beta\int_0^T \|D_0^{1/2} N_0^* A_N z(t)\|^2 dt + \int_0^T \|z_t(t)\|^2 dt\\
&\le c_1 E_z(0) + c_2 E_z(T) + c_3\int_0^T \|v_t(t)\|^2\,dt\\
&\quad + \big[c_8(E(0))I + c_9|Q_{0T}|\tilde h\big]\Big(\int_0^T (g(z_t), z_t)\,\mathrm{d}t\Big) \qquad (3.23)\\
&\le C_3 E(0) + \max\{c_8(E(0)), c_9\}\big[I + |Q_{0T}|\tilde h\big]\Big(\int_0^T (g(z_t), z_t)\,\mathrm{d}t\Big)\,, \qquad (3.24)
\end{aligned}$$

which finally provides (3.9), as desired. □

**Conclusion of Proof of Theorem 2.4.** Combination of (3.4) with (3.9) yields the following estimate for the energy functional, after using that $\tilde h$ is monotone increasing:

$$\begin{aligned}
\int_0^T E(t)dt \le C_3\, E(0) + C(E(0))\big[I + |Q_{0T}|\tilde h\big]\Big(&\int_0^T \|D_0^{\frac12} N_0^* A_N z_t(t)\|^2\,\mathrm{d}t\\
&+ \int_0^T (g(z_t), z_t)\,\mathrm{d}t + \rho\int_0^T \|\mathcal{A}^{\frac{\alpha}{2}} v_t(t)\|^2\,\mathrm{d}t\Big)\,,
\end{aligned} \tag{3.25}$$

with $C_3$ as from Proposition 3.3 and $C(E(0))$ a positive constant which increases with $E(0)$ but remains bounded for bounded values of $E(0)$.

We now follow [21] (see also [5, 19]). Note as a preliminary that from inequality $E(T)\le E(t)$ (derived in Proposition 3.1) it follows that $TE(T)\le$

$\int_0^T E(t)\,\mathrm{d}t$. Next, by plugging the energy identity (3.1) into the right-hand side of (3.25), we obtain

$$\begin{aligned} TE(T) &\le \int_0^T E(t)dt \\ &\le C_3E(T) + (C(E(0)) + 2C_3)\big[I + |Q_{0T}|\tilde{h}\big]\Big(\int_0^T \|D_0^{\frac{1}{2}} N_0^* A_N z_t(t)\|^2\,dt \\ &+ \int_0^T (g(z_t), z_t)\,\mathrm{d}t + \rho\int_0^T \|\mathcal{A}^{\frac{\alpha}{2}} v_t(t)\|^2\,dt\Big)\,. \end{aligned}$$

Hence, after taking $T$ large enough (namely, $T > C_3$), we achieve

$$\begin{aligned} E(T) &\le C_T(E(0))\,|Q_{0T}|\cdot\big[cI + \tilde{h}\big]\Big(\int_0^T \|D_0^{\frac{1}{2}} N_0^* A_N z_t(t)\|^2\,\mathrm{d}t \\ &\quad + \int_0^T (g(z_t), z_t)\,\mathrm{d}t + \rho\int_0^T \|\mathcal{A}^{\frac{\alpha}{2}} v_t(t)\|^2\,dt\Big) \\ &= C_T(E(0))\,|Q_{0T}|\cdot\big[cI + \tilde{h}\big]\left(\frac{E(0) - E(T)}{2}\right) \qquad (3.26) \\ &\le C_T(E(0))\,|Q_{0T}|\cdot\big[cI + \tilde{h}\big]\,(E(0) - E(T))\,, \qquad (3.27) \end{aligned}$$

where we have set

$$C_T(E(0)) = \frac{C(E(0)) + 2C_3}{T - C_3}\,, \qquad c = \frac{1}{|Q_{0T}|}\,, \tag{3.28}$$

and in (3.26) we have used once more the energy identity (3.1).

Therefore, since $cI + \tilde{h}$ is invertible, (3.27) provides the desired estimate for the energy at time $T$, namely

$$E(T) + p(E(T)) \le E(0)\,, \tag{3.29}$$

where $p$ is the function defined by (2.15), with

$$K = \frac{1}{C_T(E(0))\,|Q_{0T}|}\,, \tag{3.30}$$

and $c$ as in (3.28). We stress that the constant $K$ in (3.30) is decreasing in $E(0)$. It is readily verified that the result in (3.29) holds as well on intervals $[mT, (m+1)T]$, $m \in \mathbb{N}$, namely

$$E((m+1)T) + p(E((m+1)T)) \le E(mT)\,,$$

where critically $p$ is independent of $m$. The conclusion follows by applying the general result given in [21, Lemma 3.3], here with $s_m = E(mT)$ (see [21, p. 532] and [5, p. 304]). □

3.2. **The case of nonlinearities subject to saturation.** Throughout the present section we assume that the nonlinear function $g$ is subject to the weaker conditions (H2)', instead of (H2), besides to the basic hypotheses (H1)–(H2). In order to show Theorem 2.8, we shall follow verbatim the proof of Theorem 2.4, focusing on those steps where the *lower* bound in the growth assumption (H2)(ii) is exploited so that—under the alternative weaker condition (H2)'—the proof does fail. It will turn out that alternative arguments are called for at some steps in the proof of Proposition 3.3, which will be appropriately restated below, as Proposition 3.5.

As a preliminary, we prove a regularity result of the velocity term $z_t$ of solutions of system (2.1) corresponding to initial data in a slightly smoother function space than the energy space $Y$ in (2.9). This result, stated in Lemma 3.4 below, will play a crucial role in the proof of Proposition 3.5, and consequently in the proof of Theorem 2.8.

Let us consider, for $\theta \in (0, 5/12)$, the function space $W_\theta \subset Y$ introduced in Section 2, which we recall here for reader's convenience:

$$W_\theta = [Z \cap H^{1+\theta}(\Omega)] \times H^{\frac{6}{5}\theta}(\Omega) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}(1+\theta)}) \times \mathcal{D}(\mathcal{A}^{\alpha\theta})\,. \tag{3.31}$$

Then, the following result holds true.

**Lemma 3.4.** *Let $[z, z_t]$ be the first two components of the solution $[\vec{z}, \vec{v}]$ to the coupled system (2.1) with $[\vec{z}(0), \vec{v}(0)] = [z^0, z^1, v^0, v^1] \in W_\theta$. Then, there exists a positive constant $C$ such that the following estimate holds true, uniformly in $t$:*

$$\begin{aligned} ||z_t(t)||_{H^\theta(\Omega)} \le C\Big(&||z(0)||_{H^{1+\theta}(\Omega)} + \beta||D_0^{\frac{1}{2}} z|_{\Gamma_0}(0)||_{L^2(\Gamma_0)} + ||z_t(0)||_{H^{\frac{6}{5}\theta}(\Omega)} \\ &+ ||v(0)||_{\mathcal{D}(\mathcal{A}^{\frac{1}{2}(1+\theta)})} + ||v_t(0)||_{\mathcal{D}(\mathcal{A}^{\alpha\theta})}\Big)\,, \qquad 0 < \theta < \frac{5}{12}\,. \end{aligned} \tag{3.32}$$

**Proof.** Let us consider system (2.1), or its equivalent abstract formulation in the state space $Y$, namely the system

$$y' = Ay\,, \quad y(0) = y_0\,, \tag{3.33}$$

where the (abstract) variables $y$, $y_0$ denote $[z, z_t, v, v_t]$, $[z^0, z^1, v^0, v^1]$, respectively, and the nonlinear operator $A$ is defined by (4.6)–(4.7). Semigroup well-posedness of system (3.33) will be established in Section 4, via maximal monotone operator theory (see Theorem 2.3). In particular, as is well known (*cf.* [10, Theorem 3.1, p. 54]), the solutions $y(\cdot)$ to system (3.33) corresponding to $y_0 \in \mathcal{D}(A)$ satisfy

$$y \in L^\infty(0, \infty; \mathcal{D}(A))\,, \quad \frac{\mathrm{d}}{\mathrm{dt}} y \in L^\infty(0, \infty; Y)\,,$$

and the following two estimates hold true, uniformly in $t$:

$$\begin{aligned} &\|y(t)\|_Y \le \|y_0\|_Y\,, && y_0 \in Y\,; \\ &\|y_t(t)\|_Y \le \|y_0\|_{\mathcal{D}(A)}\,, && y_0 \in \mathcal{D}(A)\,. \end{aligned} \tag{3.34}$$

From the definition (4.7) of $\mathcal{D}(\mathcal{A})$, it is immediately seen that the function space $S$ below is such that $S \subseteq \mathcal{D}(\mathcal{A})$:

$$\begin{aligned} S := \big\{ &y = [z_1, z_2, v_1, v_2] \in Z \times Z \times \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^\alpha) : \\ &z_1 + \beta N_0 D_0 z_1|_{\Gamma_0} + N_0 D_0 z_2|_{\Gamma_0} - N_0 v_2 \in D(A_N), \quad g(z_2) \in L^2(\Omega) \big\}\,. \end{aligned} \tag{3.35}$$

Then, in order to have $g(h) \in L^2(\Omega)$, it is required that $h \in H^{6/5}(\Omega)$, as readily follows on the strength of the Sobolev embedding $H^{6/5}(\Omega) \subset L^{10}(\Omega)$, by using the *upper* bound (up to power 5) in the growth assumption (H2)'(ii) on $g$. Hence (3.35) yields, explicitly,

$$y \in S \iff \begin{cases} y = [z_1, z_2, v_1, v_2] \in Z \times Z \times \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^\alpha) \\ \Delta z_1 \in L^2(\Omega) \\ z_2 \in H^{\frac{6}{5}}(\Omega) \\ \dfrac{\partial z_1}{\partial \nu}\Big|_{\Gamma_1} + d_1 z_1\big|_{\Gamma_1} = 0 \\ \dfrac{\partial z_1}{\partial \nu}\Big|_{\Gamma_0} + D_0 z_2\big|_{\Gamma_0} + \beta D_0 z_1\big|_{\Gamma_0} = v_2 \end{cases}, \tag{3.36}$$

namely

$$\begin{aligned} &y = [z_1, z_2, v_1, v_2] \in S \\ &\iff \begin{cases} z_1 \in H^2(\Omega) \\ z_2 \in Z \cap H^{\frac{6}{5}}(\Omega) \equiv \{ h \in H^{\frac{6}{5}}(\Omega) \,:\, h|_{\Gamma_0} \in \mathcal{D}(D_0^{\frac{1}{2}}) \} \\ v_1 \in \mathcal{D}(\mathcal{A}), \quad v_2 \in \mathcal{D}(\mathcal{A}^\alpha) \\ \dfrac{\partial z_1}{\partial \nu}\Big|_{\Gamma_1} + d_1 z_1\big|_{\Gamma_1} = 0 \\ \dfrac{\partial z_1}{\partial \nu}\Big|_{\Gamma_0} + D_0 z_2\big|_{\Gamma_0} + \beta D_0 z_1\big|_{\Gamma_0} = v_2 \end{cases}. \end{aligned} \tag{3.37}$$

To go from (3.36) to (3.37), we have used that $Z \cap H^2(\Omega) \equiv H^2(\Omega)$, as $H^2(\Omega) \subset Z$. Indeed, if $h \in H^2(\Omega)$, then from trace theory it follows that $h|_{\Gamma_0} \in H^{3/2}(\Gamma_0)$, and consequently $D_0^{1/2} h|_{\Gamma_0} \in H^{(3-s)/2}(\Gamma_0) \subset L^2(\Gamma_0)$, for any $s \in (1, 2]$.

Thus, let $[z, z_t]$ be the first two components of the (weak) solution $[\vec{z}, \vec{v}]$ to the coupled system (2.1), with initial data $[z^0, z^1, v^0, v^1] = y_0$, as from Theorem 2.3. By using the definition (2.9) of $Y$ and the fact $Z \subset H^1(\Omega)$, from (3.34) we derive, simultaneously, the following estimates—uniform in $t$—for the velocity component $z_t$:

$$||z_t(t)||_{L^2(\Omega)} \le ||y_0||_Y \,, \qquad y_0 \in Y \,; \tag{3.38a}$$

$$||z_t(t)||_{H^1(\Omega)} \le ||z_t(t)||_Z \le ||y_0||_S \,, \qquad y_0 \in S \,. \tag{3.38b}$$

By a *nonlinear* interpolation result due to Tartar [31, Theorem 2, p. 474], from (3.38a) and (3.38b) it follows that there exists a constant $C$ such that the following intermediate estimate holds true:

$$||z_t(t)||_{H^\theta(\Omega)} \le C\,||y_0||_{(Y,S)_{\theta,2}} \,, \quad y_0 \in (Y,S)_{\theta,2} \,, \qquad 0<\theta<1 \,, \tag{3.39}$$

where for any $\theta \in (0,1)$ $(Y,S)_{\theta,2}$ denotes the (real) interpolation space between $S$ and $Y$ (*cf.* [32]).

Our next goal is then to show that (3.39) gives, explicitly, (3.32), for $\theta$ sufficiently small. Therefore, we need to characterize the interpolation space $(Y,S)_{\theta,2}$. Let us return to (3.37), and let us consider, first, the function space $W$ given by

$$W := H^2(\Omega) \times [Z \cap H^{\frac{6}{5}}(\Omega)] \times \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^\alpha) \,.$$

Recalling once more that

$$Y = Z \times L^2(\Omega) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L^2(\Gamma_0) \,, \tag{3.40}$$

it is easily shown that for $\theta < \frac{5}{12}$ one has

$$(Y,W)_{\theta,2} = [Z \cap H^{1+\theta}(\Omega)] \times H^{\frac{6}{5}\theta}(\Omega) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}(1+\theta)}) \times \mathcal{D}(\mathcal{A}^{\alpha\theta}),\ 0<\theta<\tfrac{5}{12} \,. \tag{3.41}$$

In fact, when computing the (real) interpolation space between $Z$ and $H^2(\Omega)$, one needs to keep the boundary condition $z_1|_{\Gamma_0} \in \mathcal{D}(D_0^{1/2})$ required to functions $z_1 \in Z$, as the trace $z_1|_{\Gamma_0}$ is well defined for functions $z_1 \in H^{1+\theta}(\Omega)$, for any $\theta \in (0,1)$. In contrast, when computing the (real) interpolation space between $L^2(\Omega)$ and $Z \cap H^{\frac{6}{5}}(\Omega)$, the same boundary condition can be neglected, as long as $\frac{6}{5}\theta < 1/2$, i.e., for $\theta < \frac{5}{12}$. Notice that if instead $\frac{5}{12} \le \theta < 1$, then the trace $z_2|_{\Gamma_0}$ is well defined for functions $z_2 \in H^{\frac{6}{5}\theta}(\Omega)$. In this case we would have

$$(Y,W)_{\theta,2} = [Z\cap H^{1+\theta}(\Omega)]\times[Z\cap H^{\frac{6}{5}\theta}(\Omega)]\times\mathcal{D}(\mathcal{A}^{\frac{1}{2}(1+\theta)})\times\mathcal{D}(\mathcal{A}^{\alpha\theta}),\ \tfrac{5}{12}\le\theta<1.$$

Henceforth, we shall take $\theta \in (0,5/12)$, since it will suffice to obtain the estimate (3.39) in a space $H^\theta(\Omega)$, with $\theta$ (positive but) arbitrarily close to zero.

In order to compute the interpolation space $(Y,S)_{\theta,2}$ between $S$ and $Y$, it remains to take into account the boundary conditions in (3.37). However, for

$\theta < 5/12$ one has $\theta < 1/2$, and consequently the (Neumann-type) boundary conditions in (3.37) can be neglected, since the first component $z_1$ of $y \in (Y,S)_{\theta,2}$ is such that $z_1 \in H^{1+\theta}(\Omega)$, with $1+\theta < 3/2$. In conclusion, from (3.37) and (3.40) we obtain, on the basis of (3.41),

$$(Y,S)_{\theta,2} = [Z \cap H^{1+\theta}(\Omega)] \times H^{\frac{6}{5}\theta}(\Omega) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}(1+\theta)}) \times \mathcal{D}(\mathcal{A}^{\alpha\theta}) \equiv W_\theta\,; \quad (3.42)$$

that is, $(Y,S)_{\theta,2}$ coincides simply with $(Y,W)_{\theta,2} \equiv W_\theta$. Returning to (3.39), (3.42) yields (3.32), as desired. □

Let us recall that the higher-level energy $E_1(t)$ has been defined by (2.23). The following proposition provides the key result in place of the one of Proposition 3.3.

**Proposition 3.5.** *With respect to the wave energy $E_z$ defined in* (2.10)*, the following inequality holds for any $T > 0$, where the constant $C_3$ is independent of $T$ and $C_4(E_1(0))$ denotes a positive constant which increases with $E_1(0)$, but remains bounded for bounded values of $E_1(0)$:*

$$\int_0^T E_z(t)\,dt \le C_3 E(0) + C_4(E_1(0))\big[I + |Q_{0T}|\tilde{h}\big]\Big(\int_0^T (g(z_t), z_t)\,\mathrm{d}t\Big)\,. \quad (3.43)$$

**Proof.** We repeat verbatim the proof of Proposition 3.3, until we arrive at its final part, where an estimate of the kinetic wave energy is established; see (3.20)–(3.22). Here an appropriate change is needed.

In order to provide an estimate for the left-hand side of (3.20), we split $|z_t|^2$ as $|z_t|^{2-\lambda}\,|z_t|^\lambda$, with $\lambda \in (0,1)$ to be specified later, and next apply the Hölder inequality with $p = 1/(1-\lambda)$, so that $1/p + 1/\overline{p} = 1$ with $\overline{p} = 1/\lambda$. We then have

$$\begin{aligned}
&\int_{\{x\in\Omega:|z_t|\ge 1\}} |z_t|^2\,\mathrm{d}x = \int_{\{x\in\Omega:|z_t|\ge 1\}} |z_t|^{2-\lambda}\,|z_t|^\lambda\,\mathrm{d}x \\
&\le \Big(\int_{\{x\in\Omega:|z_t|\ge 1\}} |z_t|^{\frac{2-\lambda}{1-\lambda}}\Big)^{1-\lambda}\Big(\int_{\{x\in\Omega:|z_t|\ge 1\}} |z_t|\,\mathrm{d}x\Big)^\lambda \\
&\le m^{-\lambda}\Big(\int_{\{x\in\Omega:|z_t|\ge 1\}} |z_t|^{2+\frac{\lambda}{1-\lambda}}\Big)^{1-\lambda}\Big(\int_{\{x\in\Omega:|z_t|\ge 1\}} z_t\,g(z_t)\,\mathrm{d}x\Big)^\lambda, \quad (3.44)
\end{aligned}$$

where in the last inequality we have used (2.19). From (2.19) it also follows that there exists a constant $C_m$, which depends on $m$, but does not depend on $t$, such that

$$\Big(\int_{\{x\in\Omega:|z_t|\ge 1\}} z_t\,g(z_t)\,\mathrm{d}x\Big)^\lambda \le C_m \int_{\{x\in\Omega:|z_t|\ge 1\}} z_t\,g(z_t)\,\mathrm{d}x\,,$$

so that (3.44) yields

$$\int_{\{x\in\Omega:|z_t|\geq 1\}} |z_t|^2\,\mathrm{d}x \leq m^{-\lambda} C_m ||z_t||_{L^{2+\sigma}(\Omega)}^{(2+\sigma)(1-\lambda)} \int_\Omega z_t\, g(z_t)\,\mathrm{d}x$$
$$\leq m^{-\lambda} C_m \sup_t ||z_t(t)||_{L^{2+\sigma}(\Omega)}^{(2+\sigma)(1-\lambda)} \int_\Omega z_t\, g(z_t)\,\mathrm{d}x\,, \quad (3.45)$$

where we have set $\sigma := \lambda/(1-\lambda)$. Note that $\sigma$ blows up as $\lambda$ approaches 1. Now, integrating between 0 and $T$, (3.45) implies that

$$\iint_{Q_1} |z_t(t)|^2\,\mathrm{d}x\,\mathrm{d}t \leq m^{-\lambda} C_m \sup_t ||z_t(t)||_{L^{2+\sigma}(\Omega)}^{(2+\sigma)(1-\lambda)} \int_0^T (g(z_t), z_t)\,\mathrm{d}t \quad (3.46)$$

with $Q_1$ as in (3.14). We shall use that there exists a positive constant $C$ such that

$$||z_t||_{L^{2+\sigma}(\Omega)} \leq C ||z_t||_{H^\epsilon(\Omega)} \quad (3.47)$$

for $\epsilon \in (0,1)$. More precisely, $\sigma$ and consequently $\lambda$ can be chosen in order that (3.47) holds true, for any given $\epsilon \in (0,1)$. Thus, an estimate of $||z_t(t)||_{H^\epsilon(\Omega)}$, uniform in $t$, is called for. We see that Lemma 3.4 yields the desired result, provided that initial data are measured with a bit stronger topology, namely by using the higher energy $E_1(0)$, instead of $E(0)$.

Let us return to (3.46), and proceed with the estimate of the right-hand side. By using first (3.47), next (3.32) of Lemma 3.4, we finally obtain that there exists a positive constant, which increases with $E_1(0)$ but remains bounded for bounded values of $E_1(0)$, say $c(E_1(0))$, such that

$$\iint_{Q_1} |z_t(t)|^2\,\mathrm{d}x\,\mathrm{d}t \leq c(E_1(0)) \int_0^T (z_t\, g(z_t))\,\mathrm{d}t\,. \quad (3.48)$$

Therefore, the proof of Proposition 3.3 is continued with (3.22) replaced by

$$\int_0^T \|z_t(t)\|^2 dt \leq \left[c(E_1(0))I + |Q_{0T}|\tilde{h}\right]\Big(\int_0^T (g(z_t), z_t)\,\mathrm{d}t\Big)\,,$$

while (3.23) and (3.24) hold, as well, with the constant $c_8$ depending on $E_1(0)$. In conclusion, (3.43) follows, as desired. □

**Proof of Theorem 2.8.** The proof of Theorem 2.8 is pursued by using the same reasoning as in the *Conclusion of Proof of Theorem 2.4*, with no changes except for the constants $C$ and $C_T$ appearing in the estimates (3.25) and following, which now depend on the new initial energy $E_1(0)$. Consequently, we shall set

$$K_1 = \frac{1}{C_T(E_1(0))\,|Q_{0T}|}\,, \quad (3.49)$$

and $c$ as in (3.28). Accordingly, the function $p$ is changed with $p_1$ defined by (2.20). We stress that critically $p_1$ does not depend on $m$, as a consequence of the uniform estimate (3.32) in Lemma 3.4. The proof is complete. □

## 4. Appendix: Well-posedness and regularity

We begin with the statement of a property which will be needed at several steps in the proof of our main results.

**Lemma 4.1.** *Assume (H2). If $h \in H^1(\Omega)$, then $g(h) \in H^{-1}(\Omega)$.*

**Proof.** Let $h \in H^1(\Omega)$ be given. We must show that there exists a constant $C$ such that, for any $w \in H_0^1(\Omega)$,

$$\Big| \int_\Omega g(h) w \,\mathrm{d}x \Big| \le C ||w||_{H^1(\Omega)} \,. \tag{4.1}$$

Let now $w \in H_0^1(\Omega)$ be arbitrary. By the Sobolev embedding $H^1(\Omega) \subset L^6(\Omega)$, so that $h, w \in L^6(\Omega)$. Let $N > 0$ be such that (H2)(ii) is satisfied for $|s| > N$, and let us introduce the complementary sets $\Omega_1 = \Omega \cap \{x : |g(x)| > N\}$ and $\Omega_2 = \Omega \cap \{x : |g(x)| \le N\}$. Then,

$$\Big| \int_{\Omega_1} g(h) w \,\mathrm{d}x \Big| \le \int_{\Omega_1} |g(h)|\, |w| \mathrm{d}x \le M \int_{\Omega_1} |h|^5\, |w| \mathrm{d}x \tag{4.2}$$

$$\le M \Big( \int_{\Omega_1} |h|^6 \mathrm{d}x \Big)^{\frac{5}{6}} \cdot \Big( \int_{\Omega_1} |w|^6 \mathrm{d}x \Big)^{\frac{1}{6}} = M \|h\|^5_{L^6(\Omega_1)} \|w\|_{L^6(\Omega_1)} \tag{4.3}$$

$$\le c \|h\|^5_{L^6(\Omega_1)} \|w\|_{H^1(\Omega_1)} \tag{4.4}$$

for some constant $c$, after using assumption (H2)(ii) in (4.2), the Hölder inequality in (4.3) (with $1/p + 1/q = 1$ for $p = 6/5$, $q = 6$) and the Sobolev embedding again in (4.4).

On the other hand, since $g$ is a continuous function, then for some constant $k$ we have

$$\Big| \int_{\Omega_2} g(h) w \,\mathrm{d}x \Big| \le \int_{\Omega_2} |g(h)|\, |w| \mathrm{d}x \le k \int_{\Omega_2} |w| \mathrm{d}x$$

$$\le k\, |\Omega_2|^{\frac{1}{2}} \Big( \int_{\Omega_2} |w|^2 \mathrm{d}x \Big)^{\frac{1}{2}} \le k_1 \|w\|_{H^1(\Omega)} \,. \tag{4.5}$$

Combining (4.4) with (4.5) yields (4.1), as desired. □

**Proof of Theorem 2.3.** (i) We invoke here the theory of nonlinear evolution equations (*cf.* [10], [7]). Let us define the dynamics operator $A : Y \supset \mathcal{D}(A) \to Y$ as follows:

$$A\begin{bmatrix} z_1 \\ z_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -A_N z_1 - \beta A_N N_0 D_0 N_0^* A_N z_1 - A_N N_0 D_0 N_0^* A_N z_2 - g(z_2) + A_N N_0 v_2 \\ v_2 \\ -N_0^* A_N z_2 - \mathcal{A} v_1 - \rho \mathcal{A}^\alpha v_2 \end{bmatrix} \tag{4.6}$$

with domain

$$\begin{aligned} \mathcal{D}(A) = \Big\{ & [z_1, z_2, v_1, v_2] \in Z^2 \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]^2 : \\ & A_N[z_1 + \beta N_0 D_0 N_0^* A_N z_1 + N_0 D_0 N_0^* A_N z_2 - N_0 v_2] - g(z_2) \in L^2(\Omega)\,; \\ & \mathcal{A}^{1-\alpha} v_1 + \rho v_2 \in \mathcal{D}(\mathcal{A}^\alpha) \Big\}. \end{aligned} \tag{4.7}$$

Our goal is to show that $-A$ is a maximal monotone operator (see [10]).

**Monotonicity.** If $[\vec{z}, \vec{v}] = [z_1, z_2, v_1, v_2]\,, [\vec{\tilde{z}}, \vec{\tilde{v}}] = [\tilde{z}_1, \tilde{z}_2, \tilde{v}_1, \tilde{v}_2] \in \mathcal{D}(A)$, then

$$\begin{aligned} & \left( A\begin{bmatrix} \vec{z} \\ \vec{v} \end{bmatrix} - A\begin{bmatrix} \vec{\tilde{z}} \\ \vec{\tilde{v}} \end{bmatrix}, \begin{bmatrix} \vec{z} \\ \vec{v} \end{bmatrix} - \begin{bmatrix} \vec{\tilde{z}} \\ \vec{\tilde{v}} \end{bmatrix} \right)_Y \\ &= \left( \begin{bmatrix} z_2 - \tilde{z}_2 \\ -A_N\big[(z_1 - \tilde{z}_1) + \cdots - N_0(v_2 - \tilde{v}_2)\big] - g(z_2) + g(\tilde{z}_2) \\ v_2 - \tilde{v}_2 \\ -N_0^* A_N (z_2 - \tilde{z}_2) - \mathcal{A}(v_1 - \tilde{v}_1) - \rho \mathcal{A}^\alpha (v_2 - \tilde{v}_2) \end{bmatrix}, \begin{bmatrix} z_1 - \tilde{z}_1 \\ z_2 - \tilde{z}_2 \\ v_1 - \tilde{v}_1 \\ v_2 - \tilde{v}_2 \end{bmatrix} \right)_Y \\ &= \big(A_N^{\frac{1}{2}}(z_2 - \tilde{z}_2), A_N^{\frac{1}{2}}(z_1 - \tilde{z}_1)\big) + \beta\big(D_0^{\frac{1}{2}} N_0^* A_N (z_2 - \tilde{z}_2), D_0^{\frac{1}{2}} N_0^* A_N (z_1 - \tilde{z}_1)\big) \\ &\quad - \big(A_N^{\frac{1}{2}}(z_1 - \tilde{z}_1), A_N^{\frac{1}{2}}(z_2 - \tilde{z}_2)\big) - \beta(A_N N_0 D_0 N_0^* A_N (z_1 - \tilde{z}_1), z_2 - \tilde{z}_2) \\ &\quad - (A_N N_0 D_0 N_0^* A_N (z_2 - \tilde{z}_2), z_2 - \tilde{z}_2) + (A_N N_0 (v_2 - \tilde{v}_2), z_2 - \tilde{z}_2) \\ &\quad - (g(z_2) - g(\tilde{z}_2), z_2 - \tilde{z}_2) + \big(\mathcal{A}^{\frac{1}{2}}(v_2 - \tilde{v}_2), \mathcal{A}^{\frac{1}{2}}(v_1 - \tilde{v}_1)\big) \\ &\quad - (N_0^* A_N (z_2 - \tilde{z}_2), v_2 - \tilde{v}_2) - (\mathcal{A}(v_1 - \tilde{v}_1), v_2 - \tilde{v}_2) - \rho(\mathcal{A}^\alpha (v_2 - \tilde{v}_2), v_2 - \tilde{v}_2) \\ &= -\|D_0^{\frac{1}{2}} N_0^* A_N (z_2 - \tilde{z}_2)\|^2 - (g(z_2) - g(\tilde{z}_2), z_2 - \tilde{z}_2) - \rho\|\mathcal{A}^{\frac{\alpha}{2}}(v_2 - \tilde{v}_2)\|^2 \le 0\,, \end{aligned}$$

by using (2.8), (2.9), and finally the monotonicity of $g$ assumed in (H2)(i). Therefore, $-A$ is monotone.

**Maximality.** As is well known (*cf.* [7, Proposition 2.2, p. 23], in order to show that $-A$ is maximal monotone, we need only to prove that $\mathcal{R}(I - A) = Y$ . Given

$[\vec{\varphi}, \vec{\psi}] = [\varphi_1, \varphi_2, \psi_1, \psi_2] \in Y$, we seek to solve the system

$$(I - A)\begin{bmatrix} \vec{z} \\ \vec{v} \end{bmatrix} = \begin{bmatrix} \vec{\varphi} \\ \vec{\psi} \end{bmatrix}, \tag{4.8}$$

or, explicitly (by using (4.6)),

$$\begin{cases} z_1 - z_2 = \varphi_1 \\ z_2 + A_N z_1 + \beta A_N N_0 D_0 N_0^* A_N z_1 + A_N N_0 D_0 N_0^* A_N z_2 + g(z_2) - A_N N_0 v_2 = \varphi_2 \\ v_1 - v_2 = \psi_1 \\ v_2 + N_0^* A_N z_2 + \mathcal{A} v_1 + \rho \mathcal{A}^\alpha v_2 = \psi_2, \end{cases}$$

that is,

$$\begin{cases} z_1 = z_2 + \varphi_1 \\ z_2 + A_N z_2 + (\beta+1) A_N N_0 D_0 N_0^* A_N z_2 + g(z_2) - A_N N_0 v_2 \\ \qquad = \varphi_2 - A_N \varphi_1 - \beta A_N N_0 D_0 N_0^* A_N \varphi_1 \\ v_1 = v_2 + \psi_1 \\ v_2 + N_0^* A_N z_2 + \mathcal{A} v_2 + \rho \mathcal{A}^\alpha v_2 = \psi_2 - \mathcal{A}\psi_1. \end{cases} \tag{4.9}$$

We now use arguments similar to those in [5]. Let us focus on the subsystem consisting of the second and fourth equations of system (4.9), namely

$$\begin{pmatrix} I + A_N + (\beta+1) A_N N_0 D_0 N_0^* A_N & -A_N N_0 \\ N_0^* A_N & I + \mathcal{A} + \rho \mathcal{A}^\alpha \end{pmatrix} \begin{bmatrix} z_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} g(z_2) \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} \varphi_2 - A_N \varphi_1 - \beta A_N N_0 D_0 N_0^* A_N \varphi_1 \\ \psi_2 - \mathcal{A}\psi_1 \end{bmatrix},$$

whose right-hand side readily belongs to $Z' \times [\mathcal{D}(\mathcal{A}^{1/2})]'$. This system can be rewritten as

$$(F + G)\begin{bmatrix} z_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \varphi_2 - A_N \varphi_1 - \beta A_N N_0 D_0 N_0^* A_N \varphi_1 \\ \psi_2 - \mathcal{A}\psi_1 \end{bmatrix}, \tag{4.10}$$

by introducing the operators $F, G : Z \times \mathcal{D}(\mathcal{A}^{1/2}) \to Z' \times [\mathcal{D}(\mathcal{A}^{1/2})]'$ defined as follows:

$$F = \begin{pmatrix} I + A_N + (\beta+1) A_N N_0 D_0 N_0^* A_N & -A_N N_0 \\ N_0^* A_N & I + \mathcal{A} + \rho \mathcal{A}^\alpha \end{pmatrix}, \; G\begin{bmatrix} z_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} g(z_2) \\ 0 \end{bmatrix}. \tag{4.11}$$

Note as a preliminary that $F \in \mathcal{L}(Z \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}), Z' \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]')$, and that $G$ is readily a monotone operator by (H2)(i). We will show the following:

**Proposition 4.2.** *As for the sum of the operators $F$ and $G$ in* (4.11)*, we have*

$$\mathcal{R}(F + G) = Z' \times [\mathcal{D}(\mathcal{A}^{1/2})]'.$$

**Proof.** We wish to apply [7, Corollary 1.3, p. 48] (here with $A = G$, $B = F$), and therefore we need to verify the following properties:

(1) $F$ is a monotone, hemicontinuous operator from $Z \times \mathcal{D}(\mathcal{A}^{1/2})$ to $Z' \times [\mathcal{D}(\mathcal{A}^{1/2})]'$);
(2) $G$ is a maximal monotone operator from $Z \times \mathcal{D}(\mathcal{A}^{1/2})$ to $Z' \times [\mathcal{D}(\mathcal{A}^{1/2})]'$;
(3) $F + G$ is coercive.

It is easily seen that it will be sufficient to show property 2. and that $F$ is coercive; then properties 1. and 3. will follow at once. In fact we have

$$\begin{aligned}
&\left\langle F \begin{bmatrix} z_2 \\ v_2 \end{bmatrix}, \begin{bmatrix} z_2 \\ v_2 \end{bmatrix} \right\rangle_{Z' \times [\mathcal{D}(\mathcal{A}^{1/2})]', Z \times \mathcal{D}(\mathcal{A}^{1/2})} \\
&= \left\langle \begin{bmatrix} z_2 + A_N z_2 + (\beta+1) A_N N_0 D_0 N_0^* A_N z_2 - A_N N_0 v_2 \\ N_0^* A_N z_2 + v_2 + \mathcal{A} v_2 + \rho \mathcal{A}^\alpha v_2 \end{bmatrix}, \right. \\
&\qquad\qquad \left. \begin{bmatrix} z_2 \\ v_2 \end{bmatrix} \right\rangle_{Z' \times [\mathcal{D}(\mathcal{A}^{1/2})]', Z \times \mathcal{D}(\mathcal{A}^{1/2})} \\
&= \|z_2\|^2 + \|A_N^{\frac{1}{2}} z_2\|^2 + (\beta+1) \|D_0^{\frac{1}{2}} N_0^* A_N z_2\|^2 - \langle A_N^{\frac{1}{2}} N_0 v_2, A_N^{\frac{1}{2}} z_2 \rangle + \langle N_0^* A_N z_2, v_2 \rangle \\
&\quad + \|v_2\|^2 + \|\mathcal{A}^{\frac{1}{2}} v_2\|^2 + \rho \|\mathcal{A}^{\frac{\alpha}{2}} v_2\|^2 \\
&\geq \|A_N^{\frac{1}{2}} z_2\|^2 + \beta \|D_0^{\frac{1}{2}} N_0^* A_N z_2\|^2 + \|\mathcal{A}^{\frac{1}{2}} v_2\|^2 = \left\| \begin{bmatrix} z_2 \\ v_2 \end{bmatrix} \right\|^2_{Z \times \mathcal{D}(\mathcal{A}^{1/2})}.
\end{aligned}$$

This shows that $F$ is coercive; hence as a linear operator it is a monotone operator, and 1. holds true; moreover, since $G$ is monotone, then $F + G$ is coercive, as well, so that 3. is satisfied.

It remains to be shown that $G$ is maximal monotone as a mapping from $Z \times \mathcal{D}(\mathcal{A}^{1/2})$ to $Z' \times [\mathcal{D}(\mathcal{A}^{1/2})]'$. Since $g$ is monotone increasing, $g = \partial\Phi(\cdot)$ as a mapping from $Z$ to $Z'$, where $\Phi$ is some proper, convex, lower-semicontinuous functional on $Z$ and $\partial\Phi$ denotes the subgradient of $\Phi$ (*cf.* [9, p. 37]). We finally invoke [7, Theorem 2.1, p. 54] and obtain that $G$ is a maximal monotone operator from $Z \times \mathcal{D}(\mathcal{A}^{1/2})$ to $Z' \times [\mathcal{D}(\mathcal{A}^{1/2})]'$, as desired. □

Thus, solving the subsystem (4.10) yields $[z_2, v_2] \in Z \times \mathcal{D}(\mathcal{A}^{1/2})$, and returning to system (4.9) we then have $z_1 = z_2 + \varphi_1 \in Z$, $v_1 = v_2 + \psi_1 \in \mathcal{D}(\mathcal{A}^{1/2})$. In conclusion, there exists $[\vec{z}, \vec{v}] \in \mathcal{D}(A)$ such that (4.8) holds; hence, $A$ is maximal monotone.

Therefore, with $y = [z, z_t, v, v_t]$, the abstract (nonlinear) system

$$y' = Ay, \quad y(0) = y_0 \tag{4.12}$$

admits a *weak* solution $y(t) = [z(t), z_t(t), v(t), v_t(t)]$, $t \geq 0$, for any initial data $y_0 \in Y$ ([10]).

(ii) In order to establish the additional regularity properties of the solution $y(t)$ to system (4.12) for smoother initial data, we invoke [10, Theorem 3.1, p. 54]. For arbitrary $y_0 \in \mathcal{D}(A)$, we obtain

$$y \in L^\infty(0,\infty;\mathcal{D}(A))\,, \quad \frac{\mathrm{d}}{\mathrm{dt}} y \in L^\infty(0,\infty;Y)\,,$$

which provides, by recalling (2.9), the desired regularity of the velocity terms in (2.11). Finally, the validity of property (2.12) follows from (2.11a) by using Lemma 4.1. □

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